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ASYMPTOTIC ESTIMATES OF FOURIER COEFFICIENTS*

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Abstract. Complex variable techniques are used to estimate the Fourier coefficients of functions expanded in series of Jacobi, Laguerre and Hermite polynomials.

1. Introduction. Let $f(x)$ be a function defined on a real interval (a, b). Throughout the paper we shall assume that $a < b$; furthermore the interval (a, b) may be infinite. Suppose $w(x)$ is a nonnegative function defined on (a, b) such that the quantities $\mu_n = \int_a^b w(x)x^n\,dx$ exist for all $n = 0, 1, 2, \cdots$. It is well known (see, for example, Szegö [7]) that one can construct a sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$, where $p_n(x)$ is of degree n, such that

$$\int_a^b w(x)p_n(x)p_m(x)\,dx = \begin{cases} h_n, & m = n, \\ 0, & m \neq n, \end{cases} \tag{1.1}$$

for $m, n = 0, 1, 2, \cdots$. The "Fourier coefficients" of $f(x)$ are defined by

$$a_n = \frac{1}{h_n}\int_a^b w(x)f(x)p_n(x)\,dx, \tag{1.2}$$

and we shall assume that this integral exists for all $n = 0, 1, 2, \cdots$.

In this paper we shall show first that under certain conditions the real integral in (1.2) may be replaced by a contour integral. We shall then discuss the evaluation of the contour integral when $p_n(x)$ is any of the classical orthogonal polynomials (i.e., Jacobi, Laguerre or Hermite polynomials), and the discussion will be illustrated by considering three examples.

2. Contour integral expression for a_n. Let D denote some unbounded domain of the complex z-plane, where $z = x + iy$, which is such that it contains the open interval (a, b). Equation (1.2) requires only that $f(x)$ be defined on (a, b); let us now assume that the definition of the function $f(x)$ can be extended into D, where we shall denote it by $f(z)$.

DEFINITION 1. Let $q_n(z)$ denote a function such that:

(i) it is analytic in $D - M$, where M is an open interval of the real axis, such that $M \supseteq (a, b)$;

(ii) for $a < x < b$,

$$q_n(x - 0i) - q_n(x + 0i) = 2\pi i w(x)p_n(x).^1 \tag{2.1}$$

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[1] By $q_n(x \pm 0i)$, we mean $\lim_{y \to 0+} q_n(x \pm iy)$.

This definition does not define the function $q_n(z)$ uniquely, since to any $q_n(z)$ satisfying (2.1) we may, for example, add a polynomial and (2.1) will still be satisfied. We shall later impose extra conditions on $q_n(z)$ to make it unique (see Theorem 2, below).

DEFINITION 2. Let C^- denote a simple continuous piecewise smooth contour represented parametrically by

$$x = x(s), \quad y = y(s), \qquad s_a < s < s_b.$$

C^- satisfies the following conditions:

(i) it lies entirely in the intersection of the domain D and the half-plane $\operatorname{Im} z < 0$;

(ii) it is described in the anticlockwise direction, i.e., from a to b;

(iii) $\lim_{s\to s_a} x(s) = a$, $\lim_{s\to s_a} y(s) = 0$, $\lim_{s\to s_b} x(s) = b$, $\lim_{s\to s_b} y(s) = 0$;

(iv) $\lim_{s\to s_a} dy/dx = \lim_{s\to s_b} dy/dx = 0$.

C^+ is defined analogously to be a contour contained entirely in the intersection of D and the half-plane $\operatorname{Im} z > 0$, and described from b to a.

THEOREM 1. *If $f(z)$ is analytic for all $z \in D$, then a_n, given by* (1.2), *may be expressed in the form*

$$a_n = \frac{1}{2\pi i h_n} \int_{\mathscr{C}} q_n(z) f(z)\, dz, \tag{2.2}$$

where $\mathscr{C} = C^+ \cup C^-$.

Proof. We shall start from the contour integral in (2.2) and recover (1.2). Since the integrand is analytic in $D - M$, we may deform the contours C^+ and C^- within $D - M$ to the open interval (a, b) itself. We may then write (2.2) as

$$a_n = \frac{1}{2\pi i h_n} \int_a^b [q_n(x - 0i) - q_n(x + 0i)] f(x)\, dx,$$

from which we obtain (1.2) on making use of (2.1). The convergence of the contour integral follows from the assumption that the integral in (1.2) exists.

In equation (2.2), we have chosen the countour $\mathscr{C}$ so that $f(z)$ does not necessarily have to be analytic at the endpoints a and b, nor does the interval have to be finite. Suppose now that the interval (a, b) is finite and that $f(z)$ is analytic at all points of $[a, b]$. Let $c(a, \delta)$ denote the circular arc $r = a + \delta\, e^{i\theta}$, $0 < \theta < 2\pi$, and let $c(b, \delta)$ denote the circular arc $r = b + \delta\, e^{i\phi}$, $-\pi < \phi < \pi$, where δ is small compared with $(b - a)$. If $\lim_{\delta\to 0} \int q_n(z) f(z) dz = 0$ when the integral is taken over each of $c(a, \delta)$ and $c(b, \delta)$, then the contour $\mathscr{C}$ may be chosen as any simple closed contour in D enclosing the interval $[a, b]$. This formula for a_n is then well known: see, for example, Whittaker and Watson [8, § 15.41]. The more general formulation of Theorem 1 does not appear to have been given explicitly before.

Let us now consider the function $q_n(z)$. In order to determine an explicit representation for this function we shall appeal to the theory of singular integral equations. In the remainder of this paper we shall restrict our choice of (a, b),

$w(x)$ and $p_n(x)$ to be that of the classical orthogonal polynomials. We distinguish three cases:

Case 1. Jacobi polynomials, where $(a, b) = (-1, 1)$;

$$w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \qquad \alpha, \beta > -1; \qquad p_n(x) = P_n^{(\alpha,\beta)}(x).$$

Case 2. Laguerre polynomials, where $(a, b) = (0, \infty)$;

$$w(x) = x^{\alpha} e^{-x}, \qquad \alpha > -1; \qquad p_n(x) = L_n^{(\alpha)}(x).$$

Case 3. Hermite polynomials, where $(a, b) = (-\infty, \infty)$;

$$w(x) = \exp(-x^2); \qquad p_n(x) = H_n(x).$$

THEOREM 2. *Let $M = (a, b)$. If $q_n(z)$ tends to zero as $|z| \to \infty$, then for Cases* 1–3,

$$q_n(z) = \int_a^b \frac{w(t)p_n(t)}{z - t}\,dt, \qquad z \notin [a, b]. \tag{2.3}$$

Proof. This follows immediately from the results given by Muskhelishvili [5]. In Case 1, $w(t)p_n(t)$ is of class H^* on $[-1, 1]$ and the result follows from [5, § 78].

In Case 2, $w(t)p_n(t)$ is of class H on $(0, \infty)$, $\lim_{t\to\infty} w(t)p_n(t) = 0$ and for t large enough $|w(t)p_n(t)| < A/t^{\alpha}$ for any $\alpha > 0$. The result then follows from [5, § 43]. A similar proof holds for Case 3, and the theorem is proved.

From equation (2.3) we find that $q_n(z)$ is a hypergeometric function in Case 1, and a confluent hypergeometric function in the other two cases (see Szegö [7], and Table 1). It is worth noting that for the Chebyshev polynomials of the first and second kinds we can represent $q_n(z)$ in terms of elementary functions. With $(a, b) = (-1, 1)$, $w(x) = (1 - x^2)^{-1/2}$ and $p_n(x) = T_n(x)$ we have

$$q_n(z) = \frac{\pi}{(z^2 - 1)^{1/2}[z + (z^2 - 1)^{1/2}]^n}, \qquad z \notin [-1, 1]. \tag{2.4}$$

Again with $(a, b) = (-1, 1)$, $w(x) = (1 - x^2)^{1/2}$, $p_n(x) = U_n(x)$, then

$$q_n(z) = \frac{2\pi}{[z + (z^2 - 1)^{1/2}]^{n+1}}, \qquad z \notin [-1, 1]. \tag{2.5}$$

In both cases we choose $(z - 1)$ and $(z + 1)$ so that $-\pi < \arg(z \pm 1) < \pi$.

3. Estimates of the coefficients a_n. In a recent paper, Donaldson and Elliott [1] have discussed at some length the evaluation of a contour integral similar to that given in (2.2). They have shown inter alia that the truncation error $R_n(f)$ in Gaussian quadrature rules based on the classical orthogonal polynomials, can be written as

$$R_n(f) = \frac{1}{2\pi i}\int_{\mathscr{C}} f(z)\frac{q_n(z)}{p_n(z)}\,dz, \tag{3.1}$$

where $\mathscr{C} = C^+ \cup C^-$. The discussion given in [1, §§ 8, 9] for the determination of $R_n(f)$ is relevant to the evaluation of a_n. We shall briefly describe the main features of the method here; for further details the reader is referred to [1].

TABLE 1

Asymptotic expressions of $q_n(z)$, for large n

Jacobi Polynomial $P_n^{(\alpha,\beta)}(z)$	$(a,b)=(-1,1)$; $w(x)=(1-x)^\alpha(1+x)^\beta,\quad \alpha,\beta>-1$; $(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)h_n=2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)$	$k=n+(\alpha+\beta+1)/2$
	$q_n(z)=\dfrac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(z+1)^{n+1}}\,{}_2F_1\left(n+1,n+\beta+1;2k+1;\dfrac{2}{1+z}\right)=\dfrac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(z-1)^{n+1}}\,{}_2F_1\left(n+1,n+\alpha+1;2k+1;\dfrac{2}{1-z}\right)$	
J1	z not in neighborhood of $[-1,1]$ $-\pi<\arg(z-1)<\pi$ $-\pi<\arg(z+1)<\pi$	$\dfrac{2^{2n+3(\alpha+\beta+1)/2}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)(z-1)^{(2\alpha-1)/4}(z+1)^{(2\beta-1)/4}}{\Gamma(2n+\alpha+\beta+2)[z+(z^2-1)^{1/2}]^k}$
J2	all z except near $z=-1$; in particular for z near 1 $-\pi<\arg(z-1)<\pi;\quad -\pi<\arg(z+1)<\pi$ $z=\cosh(2\zeta);\quad \alpha\geqq 0$	$\dfrac{2^{(\alpha+\beta+3)/2}\Gamma(n+\alpha+1)}{k^\alpha\Gamma(n+1)}(z-1)^{(2\alpha-1)/4}(z+1)^{(2\beta-1)/4}\zeta^{1/2}K_\alpha(2k\zeta)$
J3	all z except near $z=+1$; in particular for z near -1 $-\pi<\arg(z-1)<\pi,\quad -\pi<\arg(z+1)<\pi$ $z=\cosh(2\zeta),\quad \beta\geqq 0$	$\dfrac{2^{(\alpha+\beta+3)/2}\,e^{-ik\pi}\Gamma(n+\beta+1)}{k^\beta\Gamma(n+1)}(z-1)^{(2\alpha-1)/4}(z+1)^{(2\beta-1)/4}\zeta^{1/2}K_\beta(2k\zeta)$
Laguerre Polynomial $L_n^{(\alpha)}(z)$	$(a,b)=(0,\infty)$; $w(x)=x^\alpha e^{-x},\quad \alpha>-1$; $\Gamma(n+1)h_n=\Gamma(n+\alpha+1)$	$k=n+(\alpha+1)/2$
	$q_n(z)=-\Gamma(n+\alpha+1)U(n+1,1-\alpha;e^{-i\pi}z),\quad 0<\arg z<2\pi$	
L1	$\lvert z\rvert$ bounded $0<\arg z<2\pi,\quad \alpha\geqq 0$	$-\dfrac{2\Gamma(n+\alpha+1)}{k^{\alpha/2}\Gamma(n+1)}(e^{-i\pi}z)^{\alpha/2}\,e^{-z/2}K_\alpha[2(k\,e^{-i\pi}z)^{1/2}]$
L2	$\lvert z\rvert$ unbounded, but z not in neighborhood of $4k$ $0<\arg z<2\pi,\quad 0<\arg(z-4k)<2\pi$ $t=z/4k;\quad \xi=\{(t^2-t)^{1/2}-\log[t^{1/2}+(t-1)^{1/2}]+i\pi/2\}/2$ $\pi/2<\arg\xi<3\pi/2,\quad \alpha>0$	$-\dfrac{4\Gamma(n+\alpha+1)\,e^{-i\pi\alpha/2}}{k^{(\alpha-1)/2}\Gamma(n+1)}z^{(\alpha-1)/2}\,e^{-z/2}\left(\dfrac{z}{z-4k}\right)^{1/4}\xi^{1/2}K_\alpha(4k\xi\,e^{-i\pi})$

L3	$\lvert z\rvert$ unbounded; in particular z near $4k\,e^{i0}$ $0 < \arg z < 2\pi, \quad 0 < \arg(z-4k) < 2\pi$ $t = z/4k, \xi = (t^2 - t)^{1/2} - \log[t^{1/2} + (t-1)^{1/2}]$ $0 < \arg \xi < 3\pi/2$	$(-1)^n 2^{3/2} k^{(\alpha+1)/2}\, e^{-i\pi/2} z^{(\alpha-1)/2}\, e^{-z/2}\left(\frac{z}{z-4k}\right)^{1/4} \xi^{1/2} K_{1/3}(2k\xi\, e^{-i\pi})$
L4	$\lvert z\rvert$ unbounded; in particular z near $4k\,e^{2\pi i}$ $0 < \arg z < 2\pi, \quad 0 < \arg(z-4k) < 2\pi$ $t = z/4k, \quad \xi = (t^2 - t)^{1/2} - \log[t^{1/2} + (t-1)^{1/2}] + i\pi$ $-3\pi/2 < \arg \xi < 0$	$(-1)^n e^{-\pi i(\alpha+1/2)} 2^{3/2} k^{(\alpha+1)/2} z^{(\alpha-1)/2}\, e^{-z/2}\left(\frac{z}{z-4k}\right)^{1/4} \xi^{1/2} K_{1/3}(2k\xi\, e^{\pi i})$
Hermite Polynomial $H_n(z)$	$(a, b) = (-\infty, \infty)$ $w(x) = \exp(-x^2)$	$h_n = \sqrt{\pi}2^n\Gamma(n+1)$ $k = (2n+1)^{1/2}$
	$q_n(z) = e^{\mp i\pi(n+1)/2}\sqrt{\pi}\Gamma(n+1)U\left(\frac{n+1}{2}, \frac{1}{2}, e^{\mp i\pi}z^2\right)$, for $\operatorname{Im} z \begin{Bmatrix} >0 \\ <0 \end{Bmatrix}$ choose $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ sign, $\begin{Bmatrix} 0 < \arg z < \pi \\ -\pi < \arg z < 0 \end{Bmatrix}$	
H1	$\lvert z\rvert$ bounded for $\operatorname{Im} z \begin{Bmatrix} >0 \\ <0 \end{Bmatrix}$ choose $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ sign	$\frac{e^{\mp i\pi(n+1)/2}\pi\Gamma(n+1)}{\Gamma(n/2+1)} \exp[-z^2/2 \pm ikz]$
H2	$\lvert z\rvert$ unbounded, but z not in neighborhoods of $\pm k$ $\operatorname{Im} z > 0, \quad 0 < \arg z < \pi, \quad 0 < \arg(z \pm k) < \pi$ $\operatorname{Im} z < 0, \quad -\pi < \arg z < 0, \quad -\pi < \arg(z \pm k) < 0$	$\frac{\pi\Gamma(n+1)k^{n+1}\exp\{[-z^2 + z(z^2-k^2)^{1/2}]/2\}}{\Gamma(n/2+1)(z^2-k^2)^{1/4}[z+(z^2-k^2)^{1/2}]^{n+1/2}}$

For a given $f(z)$, we attempt to evaluate the integral in (2.2) by suitable deformation of the contour $\mathscr{C}$ away from the interval (a, b), exploiting where possible the singularities of $f(z)$. It is, however, seldom possible to find explicitly the value of the contour integral; in cases where it can be done it is obtained in terms of hypergeometric or confluent hypergeometric functions. For applications in numerical analysis, we are often interested in the behavior of a_n, for large n. If we assume that n is large, then considerable simplification of the analysis is obtained by replacing $q_n(z)$ in (2.2) by a suitably chosen approximation, which is asymptotic to $q_n(z)$ for large n. Such approximations (see Table 1) are given either in terms of elementary functions, or at worst, modified Bessel functions. In any case, the approximations are easier to handle than the exact expressions for $q_n(z)$.

Table 1 has been constructed from the first terms of uniform asymptotic expansions which are to be found in [2] and [6]. In each of the formulas J1–J3 etc. of the table, the conditions under which the asymptotic expression is valid, are first stated and are followed by the appropriate expression for $q_n(z)$. It should be noted that to date no explicit expressions for the error in approximating to $q_n(z)$ by the expressions in the table are available.

In the remainder of this section, we shall illustrate the above discussion by considering, in some detail, three examples.

Example 1. Find the Fourier coefficients a_n when the function $1/(x + \lambda)$, $\lambda > 0$, is expanded in a series of Laguerre polynomials $L_n^{(\alpha)}(x)$ on $(0, \infty)$.

In this case we can obtain an exact expression for a_n in terms of the confluent hypergeometric function. The function $f(z) = 1/(z + \lambda)$ is the obvious extension of $f(x)$ into the complex plane. This function has a simple pole at $z = -\lambda$, with residue 1. From equation (2.2), and using the explicit expression for $q_n(z)$ given in Table 1, we have

$$a_n = -\frac{\Gamma(n+1)}{2\pi i}\int_{\mathscr{C}} \frac{U(n+1, 1-\alpha; e^{-i\pi}z)}{(z+\lambda)}\,dz.$$

Let ρ, r, R be positive numbers such that $\rho + r < \lambda$ and $\lambda + \rho < R$. The contour C^+ is chosen to consist of (i) a line from ∞ to R along the real axis; (ii) a semicircle in $\operatorname{Im} z > 0$ with center at $z = 0$, radius R; (iii) a line segment from $-R$ to $-\lambda - \rho$; (iv) a semicircle in $\operatorname{Im} z > 0$ with center at $-\lambda$, radius ρ; (v) a line segment from $-\lambda + \rho$ to $-r$; (vi) a semicircle in $\operatorname{Im} z > 0$ with center at $z = 0$, radius r; (vii) a line segment from r to 0.

The contour C^- is defined similarly in $\operatorname{Im} z < 0$. It is readily verified that as $R \to \infty$ and $r \to 0$, the only nonzero contribution to the contour integral comes from the two semicircles centered on $z = -\lambda$. By the residue theorem, we have immediately that

$$a_n = \Gamma(n+1)U(n+1, 1-\alpha; \lambda) \quad \text{for } n = 0, 1, 2, \cdots.$$

This result is exact. An asymptotic result valid for n large may be obtained in terms of modified Bessel functions by using equation L1 of the table.

Example 2. Find an estimate for large n of the Fourier–Jacobi coefficients of the function $(c - x)^\phi$, where $c \geqq 1$ but close to 1, and $\phi > -1$ is not an integer.

We choose the continuation of the function $(c - x)^\phi$ to be $f(z) = [(z - c)e^{-i\pi}]^\phi$, where $0 < \arg(z - c) < 2\pi$. The function $f(z)$ is then analytic in the complex plane cut along the positive real axis from c to ∞.

Let ρ, r and R be positive numbers such that $\rho + r < c - 1$, $r < 1$, and $\rho + c < R$. The contour C^- is deformed over $\operatorname{Im} z < 0$ so that it comprises two semicircles of radius r centered at ± 1, a semicircle of radius R with center at the origin, and a semicircle of radius ρ with center at c together with the appropriate straight line segments so that the contour is continuous from -1 to $+1$. The contour C^+ is deformed similarly in $\operatorname{Im} z > 0$. Let AB denote the line segment in $\operatorname{Im} z < 0$ from the point $A(R, 0)$ to the point $B(c + \rho, 0)$; and CD denote the line segment in $\operatorname{Im} z > 0$ from $C(c + \rho, 0)$ to $D(R, 0)$. Since $\phi > -1$ and if we choose $n > \phi$, then the contributions to a_n from the semicircles tend to zero in the limit as we let r and ρ tend to zero and R tend to infinity. The value of a_n is then given by the integrals taken along AB, CD and we find

$$a_n = -\frac{\sin(\pi\phi)}{\pi h_n}\int_c^\infty (x - c)^\phi q_n(x)\, dx.$$

Since c is close to 1, we shall replace $q_n(x)$ by its asymptotic approximation J2. Thus for large n, a_n is approximated by A_n say, where

$$A_n = -\frac{k\sin(\pi\phi)}{\pi 2^{(\alpha+\beta-3)/2}}\int_c^\infty (x - c)^\phi (x - 1)^{(2\alpha-1)/4}(x + 1)^{(2\beta-1)/4}\zeta^{1/2}K_\alpha(2k\zeta)\, dx,$$

provided $\alpha \geqq 0$. Transforming the integral by putting $x = \cosh 2\zeta$, we obtain

$$A_n = -\frac{2^{\phi+3}\, k\sin(\pi\phi)}{\pi}\int_v^\infty [\sinh(\zeta - v)\sinh(\zeta + v)]^\phi(\sinh\zeta)^{\alpha+1/2} \cdot (\cosh\zeta)^{\beta+1/2}\zeta^{1/2}K_\alpha(2k\zeta)\, d\zeta,$$

where $c = \cosh 2v$. It does not seem possible to evaluate this integral explicitly in closed form, but for large k the main contribution to this integral will come from the neighborhood of $\zeta = v$. Since c is assumed to be close to 1, v will be "small" and in the integrand let us replace $\sinh(\zeta \pm v)$ by $(\zeta \pm v)$, $\sinh\zeta$ by ζ and $\cosh\zeta$ by 1. Then A_n will itself be approximated by B_n say, where

$$B_n = -\frac{2^{\phi+5/2}\, k\sin(\pi\phi)^{1/2}}{\pi}\int_v^\infty \zeta^{\alpha+1/2}(\zeta^2 - v^2)^\phi(2k\zeta)^{1/2}K_\alpha(2k\zeta)\, d\zeta.$$

Since $K_\alpha(2k\zeta) = K_{-\alpha}(2k\zeta)$, this integral may be evaluated in closed form (see Erdélyi [4, p. 129, (13)]) to give

$$B_n = \frac{2^{\phi+2}v^{(\phi+\alpha+1)}}{\Gamma(-\phi)k^\phi}K_{\phi+\alpha+1}(2kv).$$

We take this to be the required asymptotic form of a_n for large n, provided $\alpha \geqq 0$ and $n > \phi$. If we let $c \to 1$, we obtain

$$B_n = \frac{2^{\phi+1}\Gamma(\phi + \alpha + 1)}{\Gamma(-\phi)k^{2\phi+\alpha+1}}. \tag{3.2}$$

An explicit representation of a_n can be found in terms of the hypergeometric function. If, in equation (1.2), we use Rodrigues' formula for $P_n^{(\alpha,\beta)}(x)$, and integrate n times by parts, we find

$$
\begin{aligned}
a_n = &\frac{2^n\Gamma(n+\alpha+\beta+1)\Gamma(n-\phi)}{\Gamma(-\phi)\Gamma(2n+\alpha+\beta+1)(c+1)^{n-\phi}} \\
&\cdot {}_2F_1\left(n-\phi, n+\beta+1; 2n+\alpha+\beta+2; \frac{2}{c+1}\right).
\end{aligned}
\tag{3.3}
$$

Putting $c = 1$ in (3.3) we obtain for the function $(1-x)^\phi$ the coefficients

$$
a_n = \left\{\frac{2^{1+\phi}\Gamma(\alpha+\phi+1)}{\Gamma(-\phi)}\right\}\left\{\frac{k\Gamma(n+\alpha+\beta+1)\Gamma(n-\phi)}{\Gamma(n+\alpha+\beta+\phi+2)\Gamma(n+\alpha+1)}\right\}. \tag{3.4}
$$

If we now assume that n (or equivalently k) is large, then on using the result

$$
\frac{\Gamma(n+a)}{\Gamma(n+b)} = n^{a-b}\left[1 + O\left(\frac{1}{n}\right)\right]
$$

in (3.4), we recover (3.2).

Example 3. Estimate for large n, the Fourier–Laguerre coefficients for the function $\exp(-A/x)$, where $A > 0$.

We choose $f(z) = \exp(-A/z)$, which is analytic at all points of the z-plane except at $z = 0$ where it has an essential singularity. To estimate a_n for large n, we first write $a_n = a_n^+ + a_n^-$, where

$$
a_n^+ = \frac{1}{2\pi i h_n}\int_{C^+} q_n(z) f(z)\, dz,
$$

with a_n^- being defined similarly. In order to evaluate this contour integral we shall use the saddle-point method (see, for example, de Bruijn [3]). This may be briefly described as follows. In order to evaluate $I = \int_c g(z)\exp[h(z)]\,dz$, where $g(z)$ is a "slowly varying" function and $h(z)$ frequently depends upon a large parameter, we first determine the "saddle points," which are such that $h'(z) = 0$. Let z_0 be such a point. If β_0 is defined by $|\beta_0| = 1$, $\arg\beta_0 = \pi/2 - [\arg h''(z_0)]/2$, and if we assume that the major contribution to I comes from the neighborhood of this saddle point, then

$$
I = (2\pi)^{1/2}\beta_0 g(z_0)|h''(z_0)|^{-1/2}\exp[h(z_0)],
$$

approximately. If there is more than one saddle point, the contributions from each such point are added together.

To return to our particular problem, we shall first replace $q_n(z)$ by the asymptotic form L3, which is certainly valid for all z in $\operatorname{Im} z > 0$. Then for large n, a_n^+ will be approximated by A_n^+ say, where

$$
A_n^+ = \frac{(-1)^{n+1}2^{1/2}}{\pi k^{(\alpha-1)/2}}\int_{C^+} z^{(\alpha-1)/2}\left(\frac{z}{z-4k}\right)^{1/4}\exp[h(z)]\,dz,
$$

and $h(z)$ is given by

$$
h(z) = -z/2 - A/z + \log[\xi^{1/2}K_{1/3}(2k\xi\, e^{-i\pi})].
$$

The saddle points are given by those values of z for which

$$h'(z) \equiv -\frac{1}{2} + \frac{A}{z^2} + \frac{1}{4k}\left(\frac{z-4k}{z}\right)^{1/2}\left\{\frac{1}{2\xi} + (2k\,e^{-i\pi})\frac{K'_{1/3}(2k\xi\,e^{-i\pi})}{K_{1/3}(2k\xi\,e^{-i\pi})}\right\} = 0.$$

In order to solve this equation, let us assume that $|2k\xi\,e^{-i\pi}|$ is large for k large, and replace the modified Bessel function and its derivative by their asymptotic forms for large argument (see [8, Chap. 17]). We then find

$$h'(z) \equiv -\frac{1}{2} + \frac{A}{z^2} + \frac{1}{2}\left(\frac{z-4k}{z}\right)^{1/2}\left\{1 + O\left(\frac{1}{k^2\xi^2}\right)\right\} = 0.$$

The only solution of this equation that is relevant to our analysis is that given by

$$z_0 = \left(\frac{A^2}{k}\right)^{1/3} e^{i\pi/3} + \frac{A}{3k} + O\left(\frac{1}{k^{5/3}}\right).$$

It is readily verified that the value ξ_0 of ξ corresponding to z_0 is such that $|\xi_0| = O(1/k^{2/3})$, so that our assumption of $|2k\xi\,e^{-i\pi}|$ large when k is large, is justified. Since the main contribution to the integral is assumed to come from the neighborhood of the point z_0, we shall in the remainder of this analysis approximate to $h(z)$ by $H(z)$ say, where

$$H(z) = -\frac{z}{2} - \frac{A}{z} + \frac{1}{2}\log\left(\frac{\pi}{4k}\right) + \frac{i\pi}{2} + 2k\xi.$$

Then

$$H''(z_0) = (3k/2A)[1 + O(1/k^{2/3})],$$

and from consideration of the direction in which C^+ is described, we shall choose $\beta_0 = -i$. Finally we require $\exp[H(z_0)]$. If we replace $(z_0 - 4k)^{1/2}$ by $z_0^{1/2}(1 - 2A/z_0^2)$ in the expression for $H(z_0)$, we find after some algebra that

$$\exp[H(z_0)] = e^{i\pi/2}\left(\frac{\pi}{4k}\right)^{1/2}\frac{\exp[-2A/z_0]}{[(k/A^2)^{1/3}z_0]^{3k}}.$$

Again, for k large, we find that

$$\exp[H(z_0)] = \left(\frac{\pi}{4k}\right)^{1/2}\exp[-3(kA)^{1/3}/2]\exp i[3\sqrt{3}(kA)^{1/3}/2 - k\pi + \pi/2],$$

approximately. If we observe for this particular problem that $a_n = 2\,\mathrm{Re}\,a_n^+$, then on combining these results we obtain

$$a_n = \frac{1}{\sqrt{3}}\left(\frac{A}{k^2}\right)^{(\alpha+1)/3}\exp[-3(kA)^{1/3}/2]\cos[3\sqrt{3}(kA)^{1/3}/2 + \pi(1-2\alpha)/6],$$

approximately for large n.

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A NOTE ON BIORTHOGONAL POLYNOMIALS IN TWO VARIABLES*

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Abstract. The papers of Konhauser, Preiser and Chai are concerned with the investigation of the properties of biorthogonal polynomials in one variable. In this paper we give necessary and sufficient conditions for sets of polynomials in several variables to be biorthogonal. Polynomial expansions for these sets are also determined. In conclusion a necessary and sufficient condition for sets of functions to be biorthogonal is given.

Introduction. The paper of Krall and Sheffer [7] on orthogonal polynomials in two variables and the work of Appell [1] suggested the consideration of two sets of polynomials referred to here as monic and simple polynomial sets. Because of the difficulty in obtaining a suitable basis for polynomials in two variables, a pair of sets of vector-valued functions is defined whose index set is not required to be countable. Upon restricting the index set to be partially ordered, we obtain a necessary and sufficient condition for this pair of sets of functions to be biorthogonal.

1. Biorthogonal polynomials. The results in this section can be extended to polynomials on R^n, but for simplicity they will be given for polynomials restricted to R^2.

DEFINITIONS. The set of polynomials, $\{P_{nm}(x, y)\}$, $n, m = 0, 1, \cdots$, is *monic* if every polynomial P_{nm} has the form

$$P_{nm}(x, y) = x^n y^m + \text{terms of lower degree}.$$

In the above expression, n is the highest power of x and m is the highest power of y.

The set of polynomials, $\{Q_{nm}(x, y)\}$, $n, m = 0, 1, \cdots$, is *simple* if every polynomial Q_{nm} is of degree $n + m$.

The monic polynomials form a basis for the vector space of polynomials in two variables, and in particular we have the following representation.

THEOREM 1.1. *If $\{P_{nm}\}$ is a monic polynomial set, then for any monic polynomial R_{nm},*

$$R_{nm} = P_{nm} + \sum_{i+j=0}^{n+m-1} d(i, j, n, m)P_{ij},$$

where $0 \leqq i \leqq n$ and $0 \leqq j \leqq m$.

DEFINITION. Let $\{P_{nm}\}$ and $\{Q_{nm}\}$ be monic and simple polynomial sets, respectively. The polynomial sets are *biorthogonal* with respect to the bilinear functional $\langle \cdot, \cdot \rangle$ provided

$$\left\langle P_{nm}, Q_{pq} \right\rangle = \begin{cases} 0 & \text{if } |n - p| + |m - q| \neq 0, \\ \neq 0 & \text{if } n = p \text{ and } m = q. \end{cases}$$

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Three examples of biorthogonal sets of monic polynomials and simple polynomials are now mentioned. Only a brief outline of the properties of the polynomials is given since an extensive discussion appears in [1].

Example 1.2. Let

$$F_{mn} = F_2[-m - n, \gamma + m, \gamma' + n, \gamma, \gamma', x, y]$$

and

$$E_{mn} = F_2[\gamma + \gamma' + m + n, -m, -n, \gamma, \gamma', x, y],$$

where F_2 is defined in [1, p. 104]. The Appell functions F_{mn} and E_{mn} are simple polynomials and monic polynomials, respectively, and are biorthogonal with respect to the weight function $\rho(x, y) = x^{\gamma-1}y^{\gamma'-1}$ over the triangular portion of the x, y-plane given by $x \geqq 0$, $y \geqq 0$, $1 - x - y \geqq 0$. It should be noted that an important result of Karlin and McGregor [4] applies to polynomials biorthogonal on this same triangle but with respect to the weight function $x^\alpha y^\beta(1 - x - y)^\gamma$.

Example 1.3. In [1, p. 318], one finds

$$V_{nm} = 2^{n+m}\frac{(n + m)!}{n!m!}x^n y^m F_3\left[\frac{-n}{2}, \frac{-m}{2}\ \frac{1 - n}{2}, \frac{1 - m}{2}, -n - m, 1/y^2, 1/x^2\right]$$

and

$$U_{nm} = \frac{(n + m)!}{n!m!}x^n y^m F_3\left[\frac{n}{2}, \frac{m}{2}, \frac{1 - n}{2}, \frac{1 - m}{2}, 1, \frac{x^2 + y^2 - 1}{y^2}, \frac{x^2 + y^2 - 1}{x^2}\right].$$

On investigating the above Appell functions we see that V_{nm} are monic polynomials and U_{nm} are simple polynomials. The condition of orthogonality is

$$\iint_{1 - x^2 - y^2 \geqq 0} V_{nm}U_{pq}\,dx\,dy = \begin{cases} 0 & \text{if } |n - p| + |m - q| \neq 0, \\ \dfrac{\pi(n + m)!}{n!m!(n + m + 1)} & \text{if } n = p \text{ and } m = q. \end{cases}$$

Example 1.4 The generating functions

$$\exp[ux + vy - (cu^2 - abuv + av^2)/2s]$$

and

$$\exp[u(ax + by) + v(bx + cy) - (au^2 + 2buv + cv^2)/2],$$

where $a > 0$, $c > 0$ and $s = ac - b^2 > 0$, define biorthogonal monic polynomial sets (see [1, p. 370]). The polynomials defined by both generating functions appear as finite linear combinations of products of Hermite polynomials.

We now determine necessary and sufficient conditions for monic polynomials and simple polynomials to be biorthogonal.

THEOREM 1.5. *If* $\{P_{nm}\}$ *and* $\{Q_{nm}\}$ *are biorthogonal sets of monic and simple polynomials, then*

(i) $R(x, y) = \sum_{i+j=0}^{N} c(i, j)P_{ij}(x, y),$

where $R(x, y)$ *is any polynomial of degree* N *and* $c(i,j) = \langle Q_{ij}, R\rangle / \langle Q_{ij}, P_{ij}\rangle$;

(ii) $\langle Q_{pq}, x^r y^s\rangle = \begin{cases} 0 & \text{if } s < q \text{ or } r < p, \\ \neq 0 & \text{if } r = p \text{ and } s = q; \end{cases}$

(iii) *for any polynomial* R *of degree* $N \leqq p + q$, $\langle Q_{pq}, R\rangle = 0$ *if and only if the coefficient of* $x^p y^q$ *is zero*;

(iv) $Q_{nm}(x, y) = \sum_{i+j=n+m} c(i,j,n,m) P_{ij}(x, y)$;

(v) $P_{rs}(x, y) = \sum_{i+j=r+s} d(i,j,r,s) Q_{ij}(x, y)$;

(vi) $\langle P_{nm}, x^r y^s\rangle = 0$ *if* $r + s < n + m$.

Proof. (i) The Fourier representation follows immediately, since the monic polynomials form a basis for the vector space of polynomials in two variables.

(ii) Let r and s be nonnegative integers. By Theorem 1.1, $x^r y^s$ has the representation

$$x^r y^s = P_{rs} + \sum_{i+j=0}^{r+s-1} d(i,j,r,s) P_{ij},$$

with $0 \leqq i \leqq r$ and $0 \leqq j \leqq s$. The inner product of Q_{pq} and $x^r y^s$ is

$$\langle Q_{pq}, x^r y^s\rangle = \langle Q_{pq}, P_{rs}\rangle + \sum_{i+j=0}^{r+s-1} d(i,j,r,s) \langle Q_{pq}, P_{ij}\rangle, \tag{1.1}$$

where $0 \leqq i \leqq r$ and $0 \leqq j \leqq s$. If either $0 \leqq s \leqq q - 1$ or $0 \leqq r \leqq p - 1$, the right-hand side of (1.1) vanishes identically because of the biorthogonality of P_{nm} and Q_{nm}. If $r = p$ and $s = q$, (1.1) reduces to $\langle Q_{pq}, x^r y^s\rangle = \langle Q_{pq}, P_{pq}\rangle$ since the summation on the right-hand side does not contain a term for $(i, j) = (p, q)$.

(iii) We merely point out that this is a direct consequence of condition (ii) and will be used as such later.

(iv) As previously observed,

$$Q_{nm} = \sum_{i+j=0}^{n+m} c(i,j,n,m) P_{ij},$$

and for p and q such that $0 \leqq p + q \leqq n + m$, the inner product of Q_{pq} and Q_{nm} is

$$\langle Q_{pq}, Q_{nm}\rangle = c(p,q,n,m) \langle Q_{pq}, P_{pq}\rangle.$$

By part (iii) of this theorem, the inner product of Q_{rs} with any polynomial of degree $\leqq r + s - 1$ is zero, and therefore

$$\langle Q_{pq}, Q_{nm}\rangle = 0 \quad \text{if } p + q \neq n + m.$$

Thus

$$c(p,q,n,m) = \begin{cases} 0 & \text{if } p + q \neq n + m, \\ \langle Q_{pq}, Q_{nm}\rangle / \langle Q_{pq}, P_{pq}\rangle & \text{if } p + q = n + m, \end{cases}$$

for all nonnegative integers p, q such that $0 \leqq p + q \leqq n + m$, and hence part (iv) is proved.

(v) Using the result in (iv), we obtain the system of equations

$$\begin{bmatrix} c(n+m,0,n+m,0) & \cdots & c(0,n+m,n+m,0) \\ c(n+m,0,n+m-1,1) & \cdots & c(0,n+m,n+m-1,1) \\ \vdots & & \vdots \\ c(n+m,0,0,n+m) & \cdots & c(0,n+m,0,n+m) \end{bmatrix} \begin{bmatrix} P_{n+m,0} \\ \vdots \\ P_{0,n+m} \end{bmatrix} = \begin{bmatrix} Q_{n+m,0} \\ \vdots \\ Q_{0,n+m} \end{bmatrix}.$$

Suppose the rank of the coefficient matrix is less than $n + m + 1$. Then there exist constants (not all zero) such that the $(r + 1)$th row, for some $r \geqq 0$, can be expressed linearly in terms of the other rows; thus

$$Q_{n+m-r,r} - \sum_{i+j=n+m}' d(i,j)Q_{ij} = 0$$

($'$ indicates the term $Q_{n+m-r,r}$ does not appear in the sum). From this equality we obtain

$$\langle P_{n+m-r,r}, Q_{n+m-r,r} \rangle = 0,$$

which contradicts the biorthogonality of the polynomials. Thus, the matrix is nonsingular and the existence of the coefficients for the polynomial expansion is guaranteed.

(vi) Let n and m be nonnegative integers. Then by (v),

$$\langle x^r y^s, P_{nm} \rangle = \sum_{i+j=n+m} d(i,j,n,m) \langle x^r y^s, Q_{ij} \rangle$$

for nonnegative integers r and s. The result now follows from (iii).

THEOREM 1.6. *If* $\{P_{nm}\}$ *and* $\{Q_{nm}\}$ *are monic and simple polynomial sets, respectively, and satisfy* (ii) *and* (vi) *of Theorem* 1.5, *then they are biorthogonal.*

Proof. Case 1. Suppose $n < p$. Then the polynomial P_{nm} is given by

$$P_{nm} = x^n y^m + R_{n+m-1},$$

where R_{n+m-1} is a polynomial of degree $\leqq n + m - 1$ containing no term $x^i y^j$ with $i > n$ or $j > m$ and therefore no term of the form $x^p y^j$ for $j \geqq 0$. The inner product of Q_{pq} and P_{nm} is

$$\langle Q_{pq}, P_{nm} \rangle = \langle Q_{pq}, x^n y^m \rangle + \langle Q_{pq}, R_{n+m-1} \rangle, \tag{1.2}$$

and the right-hand side of (1.2) vanishes by (ii) and (iii) of the previous theorem.

Case 2. If $m < q$, then R_{n+m-1} has no term of the form $x^j y^q$, for $j \geqq 0$, and again by (ii) and (iii), the right side of (1.2) is zero.

Case 3. Suppose $n > p$ and $m \geqq q$. Letting

$$Q_{pq}(x,y) = \sum_{i+j=0}^{p+q} c(i,j,p,q) x^i y^j,$$

we have

$$\langle P_{nm}, Q_{pq} \rangle = \sum_{i+j=0}^{p+q} c(i,j,p,q) \langle P_{nm}, x^i y^j \rangle. \tag{1.3}$$

Since $n + m > p + q \geqq i + j$ for all nonnegative integers i, j such that $0 \leqq i + j \leqq p + q$, by (vi) the right-hand side of (1.3) is zero.

Case 4. Assuming $n = p$, we need only consider $m \geqq q$ since the other possibility has been investigated. For $n = p$ and $m > q$, $n + m > p + q$ so that $\langle P_{nm}, Q_{pq} \rangle = 0$ as argued in the previous case. If $n = p$ and $m = q$, then by (iii), equation (1.2) becomes

$$\langle Q_{nm}, P_{nm} \rangle = \langle Q_{nm}, x^n y^m \rangle.$$

Combining these four cases gives the result

$$\langle P_{nm}, Q_{pq} \rangle = \begin{cases} 0 & \text{if } |n - p| + |m - q| \neq 0, \\ \neq 0 & \text{if } n = p \text{ and } m = q. \end{cases}$$

2. Biorthogonal functions. We now concern ourselves with sets of biorthogonal functions $\{F_\alpha\}_\Omega$ and $\{G_\alpha\}_\Omega$, each function of which maps some nonempty set X into the linear space L. The set Ω is an infinite set of distinct indices of the form $\alpha = (\alpha_1, \cdots, \alpha_n)$, where the α_i belong to a totally ordered set.

Definition. The sets $\{F_\alpha\}_\Omega$ and $\{G_\alpha\}_\Omega$ are *biorthogonal* with respect to the bilinear functional $\langle \cdot , \cdot \rangle$ provided

$$\langle F_\alpha, G_\beta \rangle = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \neq 0 & \text{if } \alpha = \beta. \end{cases}$$

Definition. Let A be a finite subset of Ω. The set of functions $\{f_\alpha\}_A$ is an *associated set* of $\{F_\alpha\}_A$ if for each $\alpha' \in A$

$$F_{\alpha'} = \sum_A c(\alpha, \alpha') f_\alpha, \quad \text{where } c(\alpha', \alpha') \neq 0. \tag{2.1}$$

Assume now that Ω is partially ordered by the relation $\alpha \leqq \beta$ if and only if $\alpha_i \leqq \beta_i$ for all i, $1 \leqq i \leqq n$, and $\{f_\alpha\}_A$ and $\{g_\alpha\}_A$ are the associated sets of $\{F_\alpha\}_A$ and $\{G_\alpha\}_A$ respectively, for finite subsets A of Ω. Furthermore, for each $\gamma \in \Omega$ we assume there exists a finite subset A_γ of Ω for which both of the following hold:

(*) $\gamma \in A_\gamma$ and γ is an upper bound of A_γ;

(**) property (2.1) with $A = A_\gamma$ and

$$G_{\alpha'} = \sum d(\alpha, \alpha') g_\alpha, \quad \text{where } d(\alpha', \alpha') \neq 0, \tag{2.2}$$

are satisfied for each $\alpha' \in A_\gamma$.

Theorem 2.1. *If $\{F_\alpha\}_\Omega$ and $\{G_\alpha\}_\Omega$ are biorthogonal, then for each λ in Ω the following conditions are satisfied*:

$$\langle F_\lambda, g_\alpha \rangle = \begin{cases} 0 & \text{if } \alpha_i < \lambda_i \text{ for some } i,\ 1 \leqq i \leqq n, \\ \neq 0 & \text{if } \lambda = \alpha, \end{cases} \tag{2.3}$$

$$\langle G_\lambda, f_\alpha \rangle = \begin{cases} 0 & \text{if } \alpha_i < \lambda_i \text{ for some } i,\ 1 \leqq i \leqq n, \\ \neq 0 & \text{if } \lambda = \alpha; \end{cases} \tag{2.4}$$

and conversely, if (2.3) *and* (2.4) *are satisfied, then the sets are biorthogonal.*

Proof. (*Necessity*). Let μ be an element of Ω, then there exists a finite subset A of Ω containing the upper bound μ for which (2.2) holds. The biorthogonality of the sets $\{F_\alpha\}_\Omega$ and $\{G_\alpha\}_\Omega$ implies each set is linearly independent and therefore a finite subset of either set will generate the same subspace of L as its corresponding associated set. Thus we write

$$g_\mu = \sum_A b(\alpha, \mu) G_\alpha. \tag{2.5}$$

Let λ be in Ω and suppose $\lambda_k > \mu_k$ for some k, $1 \leqq k \leqq n$. Since λ is not in A, the inner product of F_λ and g_μ is seen from (2.5) to be zero.

Suppose $\lambda = \mu$; then by (**) G_λ may be expressed as

$$G_\lambda = d(\lambda, \lambda) g_\lambda + \sum_{A-\lambda} d(\alpha, \lambda) g_\alpha.$$

To each α in $A - \lambda$ there corresponds a finite subset B_α of Ω containing α as an upper bound and such that $G_{\alpha'} = \sum_{B_\alpha} b(\beta, \alpha') g_\beta$ for each α' in B_α. This implies $g_{\alpha'} = \sum_{B_\alpha} c(\beta, \alpha') G_\beta$ for each α' in B_α, so that

$$G_\lambda = d(\lambda, \lambda) g_\lambda + \sum_B e(\alpha, {}^\lambda) G_\alpha, \tag{2.6}$$

where $\lambda \notin B$ and $B = \bigcup_{A-\lambda} B_\alpha \supseteq A - \lambda$. Since $\lambda \notin B$ and $d(\lambda, \lambda) \neq 0$, on taking the inner product of F_λ and G_λ we find in view of (2.6) that $\langle F_\lambda, g_\lambda \rangle \neq 0$.

(*Sufficiency*). Let λ and β be in Ω; then there exists a subset A of Ω containing β as an upper bound and having the property that

$$G_\beta = \sum_A d(\alpha, \beta) g_\alpha, \quad \text{where } d(\beta, \beta) \neq 0. \tag{2.7}$$

If $\lambda_k < \beta_k$ for some k, $1 \leqq k \leqq n$, then (2.3) and (2.7) show the inner product of F_λ and G_β is zero. If $\lambda_k > \beta_k$ for some k, $1 \leqq k \leqq n$, interchanging the roles of λ and β, F and G, in the above argument leads to the same result.

Assume $\lambda = \beta$ in (2.7), then the inner product of F_λ and G_λ is nonzero, since $d(\lambda, \lambda)$ is not equal to zero and the elements in A are distinct. This completes the proof of the theorem.

Theorem 2.1 is now applied to obtain a multiple basic-set analog of a result due to Konhauser [5]. Let

$$\Omega = \{(n, k_1, \cdots, k_p) | 0 \leqq k_1 + \cdots + k_p \leqq n\},$$

where the solution of the diophantine inequality is over all nonnegative integers, $k_1, \cdots, k_p$.

Notation. Let $t(x)$, $u_1(y_1), \cdots, u_p(y_p)$ be polynomials of degree h in $x, y_1, \cdots, y_p$ respectively; then we shall use $[R_{(n,k_1,\cdots,k_p)}]_\Omega$ to denote a set of polynomials of total degree n in $t(x), u_i(y_i)$, $1 \leqq i \leqq p$, and of degree k_i in $u_i(y_i)$. Similarly, we will let $[S_{(r,s_1,\cdots,s_p)}]_\Omega$ denote a set of polynomials of total degree r in $v(x)$, $w_i(y_i)$, $1 \leqq i \leqq p$, and of degree s_i in $w_i(y_i)$, where the polynomials $v(x)$ and $w_i(y_i)$ are of degree m in x and y_i respectively.

It follows that $R_{(n,k_1,\cdots,k_p)}$ is a polynomial of degree hn in $x, y_1, \cdots, y_p$ together and of degree hk_i in y_i. The polynomial $S_{(r,s_1,\cdots,s_p)}$ is of degree mr in $x, y_1, \cdots, y_p$ together and of degree ms_i in y_i.

Referring to the previous definition of biorthogonal functions, for this particular example we have the condition of biorthogonality of these sets with respect to $\langle \cdot, \cdot \rangle$ described by

$$\langle R_{n,k_1,\cdots,k_p}, S_{r,s_1,\cdots,s_p} \rangle = \begin{cases} 0 & \text{if } |n-r| + \sum_{i=1}^{p} |k_i - s_i| \neq 0, \\ \neq 0 & n = r \text{ and } k_i = s_i \text{ for } 1 \leqq i \leqq p. \end{cases}$$

Let $\bar{k} = (k_1, \cdots, k_p)$ where k_i are nonnegative integers. By the usual iterative procedure, it can be verified that for a given $(n, \bar{k})$ in Ω there exist constants $C(\rho, \sigma_1, \cdots, \sigma_p; n, \bar{k})$ such that

$$R_{n,k_1,\cdots,k_p} = \sum_{\rho=0}^{n} \sum_{\sigma_1+\cdots+\sigma_p=0}^{\rho} C(\rho, \sigma_1, \cdots, \sigma_p; n, \bar{k}) g_{\rho,\sigma_1,\cdots,\sigma_p},$$

where $g_{\rho,\sigma_1,\cdots,\sigma_p} = t(x)^{\rho-\sigma_1-\cdots-\sigma_p} u_1^{\sigma_1}(y_1) \cdots u_p^{\sigma_p}(y_p)$ and $C(\rho, \sigma_1, \cdots, \sigma_p; n, \bar{k}) = 0$ if $\sigma_i > k_i$ for $i = 1, \cdots, p$. Thus for each $\gamma = (n, \bar{k})$ in Ω there exists a finite subset A_γ of Ω, $A_\gamma = \{(m, i_1, \cdots, i_p) | (m, i_1, \cdots, i_p) \leqq \gamma\}$, such that for each $\beta \in A_\gamma$, $R_\beta = \sum_{A_\gamma} C(\alpha, \beta) g_\beta$ with $C(\beta, \beta) \neq 0$. A similar statement for the polynomials $[S_{(r,s_1,\cdots,s_p}]_\Omega$ also holds, and from Theorem 2.1 we have the next result.

THEOREM 2.2. *The polynomial sets* $[R_{n,k_1,\cdots,k_p}]_\Omega$ *and* $[S_{n,k_1,\cdots,k_p}]_\Omega$ *in* $t(x)$, $u_i(y_i)$ *and* $v(x)$, $w_i(y_i)$, *respectively, are biorthogonal with respect to* $\langle \cdot, \cdot \rangle$ *if and only if the sets satisfy the conditions*

$$\langle S_{r,s_1,\cdots,s_p}, t(x)^{n-k_1-\cdots-k_p} u_1^{k_1}(y_1) \cdots u_p^{k_p}(y_p) \rangle$$
$$\begin{cases} =0 & \textit{if } n < r \textit{ or } k_i < s_i \textit{ for some } i, 1 \leqq i \leqq n, \\ \neq 0 & \textit{if } r = n \textit{ and } k_i = s_i \textit{ for all } i, 1 \leqq i \leqq n, \end{cases}$$

and

$$\langle R_{n,k_1,\cdots,k_p}, v(x)^{r-s_1-\cdots-s_p} w_1^{s_1}(y_1) \cdots w_p^{s_p}(y_p) \rangle$$
$$\begin{cases} =0 & \textit{if } r < n \textit{ or } s_i < k_i \textit{ for some } i, 1 \leqq i \leqq n, \\ \neq 0 & \textit{if } r = n \textit{ and } s_i = k_i \textit{ for all } i, 1 \leqq i \leqq n. \end{cases}$$

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ERROR BOUNDS FOR STATIONARY PHASE APPROXIMATIONS*

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Abstract. An error theory is constructed for the method of stationary phase for integrals of the form

$$I(x) = \int_a^b e^{ixp(t)}q(t)\,dt.$$

Here x is a large real parameter, the function $p(t)$ is real, and neither $p(t)$ nor $q(t)$ need be analytic in t. For both finite and infinite ranges of integration, explicit expressions are derived for the truncation errors associated with the asymptotic expansion of $I(x)$. The use of these explicit expressions for the computation of realistic error bounds is illustrated by means of an example.

1. Introduction and summary. The most comprehensive practicable theory of the method of stationary phase for single integrals of the form

$$I(x) = \int_a^b e^{ixp(t)}q(t)\,dt \tag{1.1}$$

appears to be that of Erdélyi [4], [5]. In this integral a, b, and the function $p(t)$ are real, and x is a large real parameter. Erdélyi's first paper concerns the case in which $p(t)$ and $q(t)$ are expansible at a and b in series of fractional powers of $t - a$ and $b - t$. The second paper extends the analysis to singularities of logarithmic type.[1]

The main purpose of the present paper is to supply explicit expressions for the error terms associated with the expansions of [4] from which realistic bounds are readily computable. The derivations of Erdélyi do not lend themselves readily to the construction of error bounds owing to the somewhat artificial nature of the neutralizer functions employed in the analysis. Our approach is based instead on Hardy's theory of generalized integrals [7], [8].

A secondary purpose of the present paper is to facilitate the application of the method of stationary phase to integrals having an infinite range of integration. These integrals can be treated by combination of results given in [4]. In the present account, however, they are analyzed directly, and in certain ways results for infinite integrals are simpler than those for finite integrals. In applications, writers often shun the method of stationary phase for infinite integrals. Instead, the path is deformed into the complex plane and the method of steepest descents invoked; see, for example, [2, Chap. 4], and [3, Chap. 5]. Such deformations presuppose that $p(t)$ and $q(t)$ are analytic functions of the complex variable t, which is not an

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[1] For a brief history of the method of stationary phase, see Jones [10] and another paper by Erdélyi [6], and for an important correction to [5], see [11].

inherent feature of the method nor one which is needed in the present theory or that of Erdélyi.

The paper is arranged as follows. Assumptions are listed and discussed in § 2, and the main theorem stated in § 3. This theorem gives the expansion of the integral (1.1) for large x, complete with explicit error terms. The proof follows in §§ 4 and 5. In § 6 the asymptotic nature of the expansion is discussed, including the derivation of a stationary phase analogue of Watson's lemma. The same section also indicates how bounds for the error terms can be obtained. The concluding section, § 7, contains an illustrative example concerning the Anger–Weber functions.

2. Assumptions. The following conditions and notations are similar to those adopted for Laplace's method in [12] and [13]. In (1.1) the limits a and b are independent of the positive parameter x, a being finite and b ($>a$) finite or infinite. The functions $p(t)$ and $q(t)$ are independent of x, $p(t)$ being real and $q(t)$ real or complex. They have the properties:

(i) In (a, b), $p^{(m+1)}(t)$ and $q^{(m)}(t)$ are continuous, m being a nonnegative integer, and $p'(t) > 0$.

(ii) As $t \to a$ from the right,

$$p(t) \sim p(a) + \sum_{s=0}^{\infty} p_s(t-a)^{s+\mu}, \qquad q(t) \sim \sum_{s=0}^{\infty} q_s(t-a)^{s+\lambda-1}, \tag{2.1}$$

where the coefficients p_0 and q_0 are nonzero, and μ and λ are constants such that

$$\mu > 0, \quad (m+1)\mu + 1 > \operatorname{Re}\lambda > 0. \tag{2.2}$$

Moreover, the first of these expansions is differentiable $m + 1$ times and the second m times.

(iii) When $p(b) \equiv \lim_{t\to b-} \{p(t)\}$ is finite, each of the functions

$$P_s(t) \equiv \left\{\frac{1}{p'(t)}\frac{d}{dt}\right\}^s \frac{q(t)}{p'(t)}, \qquad s = 0, 1, \cdots, m, \tag{2.3}$$

tends to a finite limit as $t \to b-$.

(iv) When $p(b) = \infty$, $\lim_{t\to b-} \{q(t)/p'(t)\} = 0$ and each of the integrals

$$\int e^{ixp(t)} P_s(t) p'(t)\, dt, \qquad s = 0, 1, \cdots, m,$$

converges at $t = b$ uniformly for all sufficiently large x.

Remarks. (a) Cases in which x is a negative parameter, or $p'(t)$ is negative, can be included by changing the sign of i throughout. Cases in which $p'(t)$ has zeros in (a, b), that is, cases in which the integral (1.1) has interior stationary points, are treatable by subdividing the range at the stationary points and intermediate points. Similar subdivisions may also be made when b is finite and $p(t)$ and $q(t)$ have expansions at b in fractional powers of $b - t$ of the type (2.1).

(b) Condition (iii) is fulfilled in the common case in which $p^{(m+1)}(t)$ and $q^{(m)}(t)$ are continuous at b and $p'(b) \neq 0$.

(c) Condition (iv), with $s = 0$, implies that the original integral converges uniformly at its upper limit.

3. Main result. In consequence of condition (i) there is a one-to-one relationship between t and the variable v, defined by

$$v = p(t) - p(a). \tag{3.1}$$

In terms of this variable the integral (1.1) transforms into

$$\int_a^b e^{ixp(t)} q(t)\, dt = e^{ixp(a)} \int_0^{p(b)-p(a)} e^{ixv} f(v)\, dv, \tag{3.2}$$

in which

$$f(v) = q(t)/p'(t) = P_0(t). \tag{3.3}$$

Again, condition (i) shows that $f(v)$ and its first m derivatives are continuous when $0 < v < p(b) - p(a)$. For small v, $t - a$ and $f(v)$ can be expanded in asymptotic series of the form

$$t - a \sim \sum_{s=1}^{\infty} c_s v^{s/\mu}, \qquad f(v) \sim \sum_{s=0}^{\infty} a_s v^{(s+\lambda-\mu)/\mu}. \tag{3.4}$$

The coefficients c_s and a_s depend on p_s and q_s, and may be found by standard procedures for reverting series. In particular,[2]

$$c_1 = \frac{1}{p_0^{1/\mu}}, \qquad c_2 = -\frac{p_1}{\mu p_0^{1+(2/\mu)}},$$

$$a_0 = \frac{q_0}{\mu p_0^{\lambda/\mu}}, \qquad a_1 = \left\{\frac{q_1}{\mu} - \frac{(\lambda+1)p_1 q_0}{\mu^2 p_0}\right\}\frac{1}{p_0^{(\lambda+1)/\mu}}.$$

THEOREM 1. *Assume the conditions and notation of* § 2 *and the present section, and let n be a nonnegative integer satisfying*[3]

$$m\mu - \lambda \leqq n < (m+1)\mu - \lambda + 1 \qquad (\lambda \textit{ real}), \tag{3.5}$$

or

$$m\mu - \operatorname{Re}\lambda < n < (m+1)\mu - \operatorname{Re}\lambda + 1 \qquad (\lambda \textit{ complex}). \tag{3.6}$$

Then

$$\begin{aligned}\int_a^b e^{ixp(t)} q(t)\, dt = {} & e^{ixp(a)} \sum_{s=0}^{n-v} \exp\left\{\frac{(s+\lambda)\pi i}{2\mu}\right\} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \\ & - e^{ixp(b)} \sum_{s=0}^{m-1} P_s(b)\left(\frac{i}{x}\right)^{s+1} + \delta_{m,n}(x) - \varepsilon_{m,n}(x)\end{aligned} \tag{3.7}$$

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[2] With the assumed conditions, p_0 is necessarily positive. Corresponding expressions for c_3 and a_2 are given in [13].

[3] The inequalities (2.2) guarantee that at least one value of n can be found.

if $p(b) < \infty$, or

$$(3.8)\qquad \int_a^b e^{ixp(t)}q(t)\,dt = e^{ixp(a)} \sum_{s=0}^{n-\nu} \exp\left\{\frac{(s+\lambda)\pi i}{2\mu}\right\} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} + \delta_{m,n}(x)$$

if $p(b) = \infty$. Here $\nu = 0$ when $n = m\mu - \lambda$, and $\nu = 1$ in all other cases.[4] *The error terms are given by*

$$(3.9)\qquad \begin{aligned} \varepsilon_{m,n}(x) = e^{ixp(a)} \sum_{s=0}^{n-1} \exp\left\{\frac{(s+\lambda)\pi i}{2\mu}\right\} &\frac{\Gamma\{(s+\lambda)/\mu\}}{\Gamma\{(s+\lambda-m\mu)/\mu\}} \\ \times \Gamma\left\{\frac{s+\lambda-m\mu}{\mu}, ixp(a) - ixp(b)\right\} &\frac{a_s}{x^{(s+\lambda)/\mu}}, \end{aligned}$$

and

$$(3.10)\qquad \delta_{m,n}(x) = \left(\frac{i}{x}\right)^m \int_a^b e^{ixp(t)} Q'_{m,n}(t)\,dt,$$

where[5]

$$(3.11)\qquad Q_{m,n}(t) = P_{m-1}(t) - \sum_{s=0}^{n-1} \frac{\Gamma\{(s+\lambda)/\mu\}}{\Gamma\{(s+\lambda+\mu-m\mu)/\mu\}} \frac{a_s}{\{p(t)-p(a)\}^{(m\mu-s-\lambda)/\mu}}.$$

In (3.9) the incomplete gamma function takes its principal value, that is,

$$(3.12)\qquad \Gamma(\alpha, z) = \int_z^\infty e^{-t} t^{\alpha-1}\,dt,$$

where the path does not intersect the negative real axis, and $t^{\alpha-1}$ has its principal value.

4. Preliminary lemmas.

LEMMA 1. *When $x > 0$ and $\operatorname{Re}\alpha > 0$,*

$$(4.1)\qquad \lim_{\eta\to 0+} \int_0^\infty e^{-\eta v} e^{ixv} v^{\alpha-1}\,dv = \frac{e^{\alpha\pi i/2}\Gamma(\alpha)}{x^\alpha}.$$

This may be proved by rotation of the path of integration until it coincides with the ray $\operatorname{ph} v = \tan^{-1}(x/\eta)$, the deformation being easily justified by means of Cauchy's theorem. On the new path set

$$v = \tau/(\eta - ix),$$

so that τ is real and ranges from 0 to ∞. Then

$$\int_0^\infty e^{-\eta v} e^{ixv} v^{\alpha-1}\,dv = \frac{1}{(\eta-ix)^\alpha}\int_0^\infty e^{-\tau}\tau^{\alpha-1}\,d\tau = \frac{\Gamma(\alpha)}{(\eta-ix)^\alpha}.$$

Passage to the limit yields (4.1).

[4] As usual, empty sums are understood to be zero.

[5] In the case $m = 0$, $P_{-1}(t)$ is defined to be $\int q(t)\,dt$, consistent with (2.3).

The left-hand side of (4.1) is a *generalized integral* in the sense of Hardy [7], [8]. Its importance lies in assigning a meaning to Euler's integral

$$\int_0^\infty e^{ixv}v^{\alpha-1}\,dv \tag{4.2}$$

when Riemann's definition is inapplicable owing to divergence at the upper limit.

In the case $0 < \operatorname{Re}\alpha < 1$, (4.2) converges and equals $e^{\alpha\pi i/2}\Gamma(\alpha)/x^\alpha$. In other words, Hardy's generalization is consistent with the ordinary definition. This is a special case of the following result, included in Hardy's theory.[6]

LEMMA 2. *If $\phi(v)$ is sectionally (piecewise) continuous in $(0, \infty)$ and $\int_0^\infty \phi(v)\,dv$ converges, then $\int_0^\infty e^{-\eta v}\phi(v)\,dv$ converges for every positive number η and tends to $\int_0^\infty \phi(v)\,dv$ as $\eta \to 0+$.*

Lastly we shall need the following result, which is provable in a manner similar to Lemma 1.

LEMMA 3. *When $x > 0$ and $\beta > 0$,*

$$\lim_{\eta\to 0+}\int_\beta^\infty e^{-\eta v}\,e^{ixv}v^{\alpha-1}\,dv = \frac{e^{\alpha\pi i/2}}{x^\alpha}\Gamma(\alpha, -ix\beta). \tag{4.3}$$

5. Proof of Theorem 1. Write

$$\beta \equiv p(b) - p(a),$$

and for each nonnegative integer n, define $\phi_n(v)$ by

$$f(v) = \sum_{s=0}^{n-1} a_s v^{(s+\lambda-\mu)/\mu} + \phi_n(v), \qquad 0 < v < \beta, \tag{5.1}$$

so that

$$\phi_n(v) \sim a_n v^{(n+\lambda-\mu)/\mu} + a_{n+1}v^{(n+1+\lambda-\mu)/\mu} + \cdots, \qquad v \to 0+, \tag{5.2}$$

and

$$\phi_n^{(j)}(v) = f^{(j)}(v) - \sum_{s=0}^{n-1}\frac{\Gamma\{(s+\lambda)/\mu\}}{\Gamma\{(s+\lambda-j\mu)/\mu\}}a_s v^{(s+\lambda-\mu-j\mu)/\mu}, \qquad j = 0, 1, \cdots, m. \tag{5.3}$$

Assume first that β is finite. Since

$$f^{(j)}(v) = P_j(t), \tag{5.4}$$

conditions (i) and (iii) of § 2 show that $f(v)$, $\phi_n(v)$, and their first m derivatives are continuous when $v \in (0, \beta]$. With η denoting an arbitrary positive number, we have

$$\begin{aligned}\int_0^\beta & e^{-\eta v+ixv}f(v)\,dv \\ &= \sum_{s=0}^{n-1} a_s \int_0^\infty e^{-\eta v+ixv}v^{(s+\lambda-\mu)/\mu}\,dv + E_n(\eta, x) - F_n(\eta, x),\end{aligned} \tag{5.5}$$

[6] [7, p. 51]. It is important to notice that $\int_0^\infty \phi(v)\,dv$ need not converge absolutely at either limit.

where

$$E_n(\eta, x) = \int_0^\beta e^{-\eta v + ixv} \phi_n(v)\, dv, \tag{5.6}$$

and

$$F_n(\eta, x) = \sum_{s=0}^{n-1} a_s \int_\beta^\infty e^{-\eta v + ixv} v^{(s+\lambda-\mu)/\mu}\, dv. \tag{5.7}$$

Letting $\eta \to 0$ and applying Lemmas 1 and 2, we obtain[7]

$$\int_0^\beta e^{ixv} f(v)\, dv = \sum_{s=0}^{n-1} \exp\left\{\frac{(s+\lambda)\pi i}{2\mu}\right\} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} + E_n(x) - F_n(x), \tag{5.8}$$

where

$$E_n(x) = \lim_{\eta \to 0+} E_n(\eta, x), \qquad F_n(x) = \lim_{\eta \to 0+} F_n(\eta, x).$$

By integrating by parts, we have

$$E_n(\eta, x) = \frac{\phi_n(0)}{\eta - ix} - \frac{e^{-\eta\beta + ix\beta}}{\eta - ix}\phi_n(\beta) + \frac{1}{\eta - ix}\int_0^\beta e^{-\eta v + ixv} \phi_n'(v)\, dv. \tag{5.9}$$

Since $\phi_n^{(m)}(v)$ is continuous in $(0, \beta]$ the process of partial integration may be repeated $m - 1$ times. The conditions of § 2 show that the expansion (3.4) for $f(v)$ may be differentiated m times, hence the same is true of (5.2). In consequence of the conditions (3.5) and (3.6), all needed derivatives of $\phi_n(v)$ at $v = 0$ vanish, except possibly $\phi_n^{(m-1)}(v)$; accordingly

$$\begin{aligned} E_n(\eta, x) = {} & \frac{\phi_n^{(m-1)}(0)}{(\eta - ix)^m} - e^{-\eta\beta + ix\beta} \sum_{j=0}^{m-1} \frac{\phi_n^{(j)}(\beta)}{(\eta - ix)^{j+1}} \\ & + \frac{1}{(\eta - ix)^m} \int_0^\beta e^{-\eta v + ixv} \phi_n^{(m)}(v)\, dv. \end{aligned} \tag{5.10}$$

We propose to let $\eta \to 0$ in this result to obtain

$$E_n(x) = \left(\frac{i}{x}\right)^m \phi_n^{(m-1)}(0) - e^{ix\beta} \sum_{j=0}^{m-1} \phi_n^{(j)}(\beta) \left(\frac{i}{x}\right)^{j+1} + \left(\frac{i}{x}\right)^m \int_0^\beta e^{ixv} \phi_n^{(m)}(v)\, dv. \tag{5.11}$$

This step is justifiable by Lemma 2, provided that the integral in (5.11) converges. Now from the differentiated forms of (5.2) we see that as $v \to 0, \phi_n^{(m)}(v)$ is $O\{v^{(n + \mathrm{Re}\,\lambda - \mu - m\mu)/\mu}\}$ or $O\{v^{(1/\mu)-1}\}$ according as $n > m\mu - \mathrm{Re}\,\lambda$ or $n = m\mu - \lambda$. In either event, the integral in question converges absolutely. Thus (5.11) is established.

Next, consider $F_n(x)$. Application of Lemma 3 to (5.7) gives

$$F_n(x) = \sum_{s=0}^{n-1} a_s \frac{e^{(s+\lambda)\pi i/(2\mu)}}{x^{(s+\lambda)/\mu}} \Gamma\left(\frac{s+\lambda}{\mu}, -ix\beta\right).$$

[7] In applying Lemma 2, $\phi(v)$ is taken to be $e^{ixv} f(v)$ when $0 < v \leqq \beta$ and 0 when $v > \beta$.

From (3.12) we derive by repeated integrations by parts,

$$\Gamma(\alpha, z) = e^{-z}z^{\alpha-1}\sum_{j=0}^{m-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-j)}\frac{1}{z^j} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}\Gamma(\alpha-m, z).$$

Hence

$$\begin{aligned} F_n(x) = e^{ix\beta}\sum_{s=0}^{n-1}\sum_{j=0}^{m-1} a_s \frac{\Gamma\{(s+\lambda)/\mu\}}{\Gamma\{(s+\lambda-j\mu)/\mu\}}\left(\frac{i}{x}\right)^{j+1}\beta^{(s+\lambda-\mu-j\mu)/\mu} \\ + e^{-ixp(a)}\varepsilon_{m,n}(x), \end{aligned} \tag{5.12}$$

where $\varepsilon_{m,n}(x)$ is defined by (3.9).

We now substitute in (5.11) by means of (5.3), with $v = \beta$, and subtract (5.12). The double sum cancels, and we are left with

$$\begin{aligned} E_n(x) - F_n(x) = -e^{ix\beta}\sum_{j=0}^{m-1} f^{(j)}(\beta)\left(\frac{i}{x}\right)^{j+1} + \left(\frac{i}{x}\right)^m \phi_n^{(m-1)}(0) \\ + e^{-ixp(a)}\{\delta_{m,n}(x) - \varepsilon_{m,n}(x)\}, \end{aligned} \tag{5.13}$$

where

$$\delta_{m,n}(x) = e^{ixp(a)}\left(\frac{i}{x}\right)^m \int_0^\beta e^{ixv}\phi_n^{(m)}(v)\,dv. \tag{5.14}$$

From (2.3), (3.11), and (5.3), it is verifiable that

$$\phi_n^{(m)}(v) = Q'_{m,n}(t)/p'(t), \tag{5.15}$$

and thence that (5.14) agrees with (3.10). From (3.5), (3.6) and (5.2) it is seen that $\phi_n^{(m-1)}(0)$ vanishes unless λ is real and $n = m\mu - \lambda$, in which event it equals $(m-1)!a_n$. Combination of (3.2), (5.4), (5.8) and (5.13) yields the first of the desired results (3.7).

The proof of (3.8) is similar. First, we observe that when $0 \leqq s \leqq m-1$,

$$\int e^{ixp(t)}P_s(t)p'(t)\,dt = \frac{e^{ixp(t)}}{ix}P_s(t) - \frac{1}{ix}\int e^{ixp(t)}P_{s+1}(t)p'(t)\,dt.$$

By hypothesis, as $t \to b-$, both integrals converge and $p(t) \to \infty$. Therefore $P_s(t) \to 0$. In terms of v this implies that $\int^\infty e^{ixv}f^{(s)}(v)\,dv$ converges when $s \leqq m$, and $f^{(s)}(v) \to 0$ when $s \leqq m-1$.

In (5.5) and (5.6), β is replaced by ∞ and the term $F_n(\eta, x)$ is absent. The convergence of the integrals is assured by Lemma 2 and the convergence of $\int_0^\infty e^{ixv}f(v)\,dv$. From (5.1) and the fact that $f(v) \to 0$ as $v \to \infty$ (condition (iv)), it follows that $e^{-\eta v + ixv}\phi_n(v) \to 0$ as $v \to \infty$. Hence (5.9) becomes

$$E_n(\eta, x) = \frac{\phi_n(0)}{\eta - ix} + \frac{1}{\eta - ix}\int_0^\infty e^{-\eta v + ixv}\phi_n'(v)\,dv.$$

Similarly, in place of (5.10) we have

$$E_n(\eta, x) = \frac{\phi_n^{(m-1)}(0)}{(\eta - ix)^m} + \frac{1}{(\eta - ix)^m}\int_0^\infty e^{-\eta v + ixv}\phi_n^{(m)}(v)\,dv.$$

In (5.3) with $j = m$, all powers of v in the sum have negative real part: this is a consequence of (3.5) and (3.6). Hence $\int_0^\infty e^{ixv}\phi_n^{(m)}(v)\,dv$ converges, and by application of Lemma 2 we obtain

$$E_n(x) = \left(\frac{i}{x}\right)^m \phi_n^{(m-1)}(0) + \left(\frac{i}{x}\right)^m \int_0^\infty e^{ixv}\phi_n^{(m)}(v)\,dv;$$

compare (5.11). Combination of this result with (5.8) (again with $\beta = \infty$) leads to (3.8).

The proof of Theorem 1 is complete.

6. Asymptotic properties and error bounds. From (3.10) we immediately have

$$|\delta_{m,n}(x)| \leqq \frac{1}{x^m}\int_a^b |Q'_{m,n}(t)|\,dt = \frac{\mathscr{V}_{a,b}\{Q_{m,n}(t)\}}{x^m}, \tag{6.1}$$

where $\mathscr{V}_{a,b}$ denotes the total variation of the function within the braces over the closure of the given interval. Another bound for $\delta_{m,n}(x)$, which has the advantage of being $O(x^{-m-1})$ for large x, may be found by partial integration of (3.10), using the identity

$$Q'_{m,n}(t)/p'(t) = Q_{m+1,n}(t). \tag{6.2}$$

Thus

$$|\delta_{m,n}(x)| \leqq [|Q_{m+1,n}(a)| + |Q_{m+1,n}(b)| + \mathscr{V}_{a,b}\{Q_{m+1,n}(t)\}]x^{-m-1}, \tag{6.3}$$

provided that the right-hand side is finite: in contrast to (6.1), finiteness is not guaranteed by the conditions adopted in §§ 2 and 3.

Now consider the other error term. For fixed α we have

$$\Gamma(\alpha, z) \sim e^{-z}z^{\alpha-1}$$

as $z \to \infty$ in a sector which includes $|\arg z| \leqq \pi$; see [1, § 6.5]. Application to (3.9) shows that

$$\varepsilon_{m,n}(x) = O(x^{-m-1}), \qquad x \to \infty. \tag{6.4}$$

Relations (6.1), (6.3), and (6.4) confirm the asymptotic nature of the expansions (3.7) and (3.8).

An interesting special case of (3.8) is obtained on taking $a = 0$, $b = \infty$, $m = \infty$, $p(t) = t^\mu$, $q(t) = \mu t^{\mu-1}g(t^\mu)$, and subsequently replacing t^μ by t.

THEOREM 2. *Assume that*

(i) *$g(t)$ is infinitely differentiable in* $(0, \infty)$;

(ii)

$$g(t) \sim \sum_{s=0}^{\infty} g_s t^{(s+\lambda-\mu)/\mu}, \qquad t \to 0+, \tag{6.5}$$

where Re λ *and* μ *are positive, this asymptotic expansion being differentiable any number of times;*

(iii) *Each of the integrals*

$$\int e^{ixt} g^{(s)}(t)\, dt, \qquad s = 0, 1, \cdots,$$

converges at $t = \infty$ *uniformly for sufficiently large* x.[8]

Then for large positive values of x, *the asymptotic expansion of the integral*

$$\int_0^\infty e^{ixt} g(t)\, dt$$

is obtained by substituting (6.5) *and integrating formally term by term in Hardy's generalized sense.*

Although this theorem can be derived from existing results, for example [9], it does not appear to have been emphasized in the literature. This is somewhat surprising since it is the natural analogue for Fourier integrals of Watson's lemma for Laplace integrals.

Let us return to Theorem 1 and consider the actual calculation of bounds for the error terms. For $\delta_{m,n}(x)$ the inequality (6.1) (or (6.3)) may be used as it stands. For $\varepsilon_{m,n}(x)$ we make the simplifying assumption *that* λ *is real.* We may then apply the inequality

$$(6.6) \qquad |\Gamma(\alpha, \pm iy)| \leqq 2y^{\alpha - 1}, \qquad \alpha < 1, \quad y > 0,$$

which is itself establishable by deforming the path in (3.12) until it lies along the imaginary axis and then integrating by parts. Thus we have

$$(6.7) \qquad |\varepsilon_{m,n}(x)| \leqq \frac{2}{x^{m+1}} \sum_{s=0}^{n-1} \frac{\Gamma\{(s + \lambda)/\mu\}}{|\Gamma\{(s + \lambda - m\mu)/\mu\}|} \frac{|a_s|}{\{p(b) - p(a)\}^{(m\mu + \mu - s - \lambda)/\mu}}.$$

In the case $p(b) = \infty$, this bound vanishes and only the error term $\delta_{m,n}(x)$ survives.

7. Example. As a simple example of the calculation of error bounds, consider the functions of Anger and Weber with equal argument and order:

$$\mathbf{J}_x(x) = \frac{1}{\pi} \int_0^\pi \cos(xt - x \sin t)\, dt, \qquad \mathbf{E}_x(x) = \frac{1}{\pi} \int_0^\pi \sin(xt - x \sin t)\, dt;$$

see [14, § 10.1]. In the notation of §§ 2 and 3 we have

$$\pi\{\mathbf{J}_x(x) + i\mathbf{E}_x(x)\} = \int_0^\pi e^{ixp(t)}\, dt,$$

where

$$(7.1) \qquad p(t) = t - \sin t = \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \cdots.$$

Thus $a = 0$, $b = \pi$, $q(t) = 1$, $\mu = 3$, and $\lambda = 1$.

[8] Convergence at $t = 0$ is not required, however, except for $s = 0$.

The definition (2.3) gives

$$P_s(t) = \frac{1}{2^{s+1}}\left(\frac{1}{\sin^2 \frac{1}{2}t}\frac{d}{dt}\right)^s \frac{1}{\sin^2 \frac{1}{2}t}. \tag{7.2}$$

In particular,

$$P_0(t) = \frac{1}{2\sin^2 \frac{1}{2}t}, \qquad P_1(t) = -\frac{\cos \frac{1}{2}t}{4\sin^5 \frac{1}{2}t}, \qquad P_2(t) = \frac{5 - 4\sin^2 \frac{1}{2}t}{16 \sin^8 \frac{1}{2}t}. \tag{7.3}$$

By reversion of (7.1) we have for small $v \equiv p(t)$,

$$t = (6v)^{1/3} + \tfrac{1}{60}(6v) + \tfrac{1}{1400}(6v)^{5/3} + \tfrac{1}{25200}(6v)^{7/3} + \tfrac{43}{17248000}(6v)^3 + \cdots;$$

compare [14, § 8.21]. Differentiation yields

$$f(v) = \frac{dt}{dv} = \sum_{s=0}^{\infty} a_s v^{(s-2)/3};$$

compare the second of (3.4). In this expansion the coefficients a_s of odd suffix vanish, and

$$a_0 = \tfrac{1}{3}6^{1/3}, \quad a_2 = \tfrac{1}{60}6, \quad a_4 = \tfrac{1}{840}6^{5/3}, \quad a_6 = \tfrac{1}{10800}6^{7/3}, \quad a_8 = \tfrac{129}{17248000}6^3, \cdots.$$

Using the asymptotic results of § 6, we derive from (3.7)

$$\pi\{\mathbf{J}_x(x) + i\mathbf{E}_x(x)\} \sim \sum_{s=0}^{\infty} e^{(2s+1)\pi i/6}\Gamma(\tfrac{2}{3}s + \tfrac{1}{3})\frac{a_{2s}}{x^{(2s+1)/3}} - e^{\pi i x}\sum_{s=0}^{\infty} P_s(\pi)\left(\frac{i}{x}\right)^{s+1}. \tag{7.4}$$

Explicitly, the first few terms on the right-hand side are given by

$$e^{\pi i/6}\Gamma\left(\frac{1}{3}\right)\frac{1}{3}\left(\frac{6}{x}\right)^{1/3} + i\frac{1}{60}\left(\frac{6}{x}\right) + e^{5\pi i/6}\Gamma\left(\frac{2}{3}\right)\frac{1}{1260}\left(\frac{6}{x}\right)^{5/3} - e^{\pi i/6}\Gamma\left(\frac{1}{3}\right)\frac{1}{24300}\left(\frac{6}{x}\right)^{7/3}$$
$$-i\frac{129}{8624000}\left(\frac{6}{x}\right)^3 + \cdots + i\,e^{\pi i x}\left(-\frac{1}{2x} + \frac{1}{16x^3} + \cdots\right).$$

To evaluate error bounds, we note that the condition (3.5) requires

$$3m - 1 \leqq n < 3m + 3.$$

Therefore from (3.7),

$$\begin{aligned}\pi\{\mathbf{J}_x(x) + i\mathbf{E}_x(x)\} = \sum_{s=0}^{n-1} e^{(s+1)\pi i/6}\Gamma(\tfrac{1}{3}s + \tfrac{1}{3})\frac{a_s}{x^{(s+1)/3}} \\ - e^{\pi i x}\sum_{s=0}^{m-1} P_s(\pi)\left(\frac{i}{x}\right)^{s+1} + \delta_{m,n}(x) - \varepsilon_{m,n}(x),\end{aligned} \tag{7.5}$$

where m is an arbitrary nonnegative integer, and $n = 3m$, $3m + 1$, or $3m + 2$.

Suppose, for example, that $m = 0$ and $n = 1$, that is, the expansion comprises the single term $e^{\pi i/6}\Gamma(\frac{1}{3})\frac{1}{3}(6/x)^{1/3}$. Equations (3.11) and (7.3) give

$$Q_{1,1}(t) = \frac{1}{2\sin^2 \frac{1}{2}t} - \frac{2}{\{6(t - \sin t)\}^{2/3}}. \tag{7.6}$$

The bound (6.3) yields

$$|\delta_{0,1}(x)| \leqq \left[\frac{1}{10} + \left\{\frac{1}{2} - \frac{2}{(6\pi)^{2/3}}\right\} + \mathscr{V}_{0,\pi}(Q_{1,1})\right]\frac{1}{x}.$$

Also, from (6.7) we derive

$$|\varepsilon_{0,1}(x)| \leqq 4(6\pi)^{-2/3}x^{-1}.$$

Numerical calculation shows that

$$2(6\pi)^{-2/3} = 0.28, \qquad \mathscr{V}_{0,\pi}(Q_{1,1}) = 0.12,$$

to two decimal places, and thence

$$|\delta_{0,1}(x)| \leqq 0.44x^{-1}, \qquad |\varepsilon_{0,1}(x)| \leqq 0.56x^{-1}.$$

That the combined bound $1.00x^{-1}$ for $|\delta_{0,1}(x)| + |\varepsilon_{0,1}(x)|$ is quite realistic can be seen by comparing it with the first neglected terms of the asymptotic expansion, given by

$$\frac{i}{10x} - \frac{i\,e^{\pi i x}}{2x}.$$

Similar results can be obtained for higher error terms, provided that in calculating the functions $Q_{m,n}(t)$ from expressions of the type (7.6), sufficient precision is maintained to overcome numerical cancellation for small t.

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ON THE EXISTENCE OF ANALYTIC SOLUTIONS OF SYSTEMS OF EQUATIONS IF THE JACOBIAN IS ZERO*

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Abstract. The well-known implicit function theorem states that the equations $f_i(z_1, \cdots, z_k; w) = 0$, $f_i(0;0) = 0$, $i = 1, \cdots, k$, have a unique solution $z_j(w)$, $j = 1, \cdots, k$, in a neighborhood of $w = 0$ if the Jacobian of the analytic functions f_i with respect to the variables z_j is nonzero at $z_j = w = 0$. This is a sufficient, but not a necessary condition for the equations $f_i = 0$ to have an analytic solution. In this article weaker conditions will be given. As has been noted, the solution is unique if the Jacobian is nonzero. This is in general not true if only the weaker conditions are satisfied. It is possible in that case that more than one analytic solution exists. In the proof use is made of the concept of formal power series.

A theorem of Artin[1] can also be used to give sufficient conditions under which a system of equations has analytic solutions. In the proof of his general theorem use was made of the techniques of algebraic geometry, whereas in the proof presented here only function theoretic aspects appear.

1. Introduction. All variables used in this article are supposed to be complex, unless stated differently. It is assumed that the reader is familiar with the theory of analytic functions and formal power series. Formal power series are treated, for instance, in [2].

In the following we will use formal power series (f.p.s.) as well as convergent power series (c.p.s.). Though this is not strictly necessary, we will use the symbol $\triangleq$ for the equality of two f.p.s., implying that all the corresponding coefficients coincide; for the equality of two c.p.s. the usual symbol $=$ will be used. The negation of $\triangleq$ will be denoted by $\not\triangleq$. For example, if $\sum_{j=1}^{\infty} a_j z^j$ is a f.p.s., then

$$\sum_{j=1}^{\infty} a_j z^j \triangleq 0 \Leftrightarrow a_j = 0, \quad j = 1, 2, \cdots .$$

The symbol $\triangleq$ is also used if a f.p.s. is denoted by a letter. So $\zeta \triangleq \sum_{j=1}^{\infty} a_j z^j$ means that the f.p.s. is denoted by ζ. If $\sum_{j=1}^{\infty} a_j z^j$ is convergent, then ζ will be equal to the sum of this convergent series.

For the sake of completeness we repeat the following basic theorem, which can also be found in [2].

THEOREM 1. *Let*

$$f_i(z_1, \cdots, z_k; w) \triangleq \sum_{j_1, \cdots, j_{k+1}=0}^{\infty} a_{ij_1\cdots j_{k+1}} z_1^{j_1} \cdots z_k^{j_k} w^{j_{k+1}}, \qquad i = 1, \cdots, k,$$

be f.p.s. with $a_{i0\cdots 0} = 0$, $i = 1, \cdots, k$.

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[1] I thank the referee for drawing my attention to this theorem, which was published as theorem (1.2) in [1]. This theorem—in the context of this paper—can be formulated as follows. Suppose that we try to find $\bar{z}(w) = (\bar{z}_1(w), \cdots, \bar{z}_k(w))$, where $\bar{z}_i(w) = \sum_{j=1}^{\infty} a_{ij} w^j$ are formal power series, which solves $f(z; w) = 0$, that is, $f(\bar{z}; w) = 0$. The formal power series $\bar{z}_i(w)$ are substituted in $f(.; w)$ and the result is a formal power series in w of which the coefficients must be zero. This gives recurrence relations for a_{ij}. If an integer N exists in such a way that a_{ij}, $j \geqq N$, are uniquely determined by these recurrence relations, then the formal power series $\bar{z}_i(w)$ have positive radii of convergence.

If the $(k \times k)$ *matrix with* (i, j)*-th element* $a_{i0\cdots 010\cdots 0}$, *the number* 1 *being the* $(j+1)$*-th index, is nonsingular, then the equations*

$$f_i(z_1, \cdots, z_k; w) \triangleq 0, \qquad i = 1, \cdots, k, \tag{1.1}$$

have one and only one solution of the kind

$$z_i \triangleq \sum_{j=1}^{\infty} b_{ij} w^j, \qquad i = 1, \cdots, k. \tag{1.2}$$

In this solution $b_{i0} = 0$, $i = 1, \cdots, k$, *and the other coefficients are uniquely determined by recurrence relations. (The f.p.s.* (1.2) *are said to be a solution of* (1.1) *if the f.p.s. obtained by formal substitution of* (1.2) *in* (1.1) *are all zero.)*

If no danger of confusion exists $(z_1, \cdots, z_k)$ will be abbreviated as z. This notation will also be used with other variables.

2. The main theorem and its proof.

THEOREM 2 (the main theorem). *Suppose the functions* $f_j(z; w)$, $j = 1, \cdots, k$, *with* $z = (z_1, \cdots, z_k)$, *are analytic in a neighborhood* Ω_1 *of* $z = w = 0$ *and have the following convergent power series in* Ω_1:

$$f_j(z_1, \cdots, z_k; w) = \sum_{i_1, \cdots, i_{k+1} = 0}^{\infty} a_{ji_1 \cdots i_{k+1}} z_1^{i_1} \cdots z_k^{i_k} w^{i_{k+1}}, \qquad j = 1, \cdots, k, \tag{2.1}$$

and suppose formal power series

$$\zeta_i \triangleq \sum_{j=1}^{\infty} b_{ij} w^j, \qquad i = 1, \cdots, k, \tag{2.2}$$

are given, which satisfy

$$f_j(\zeta; w) \triangleq 0, \qquad j = 1, \cdots, k, \tag{2.3}$$

where the formal power series $f_j(\zeta; w)$ *has been obtained by formal substitution of* (2.2) *in the convergent power series at the right-hand side of* (2.1).

Let $\det[\partial f(\zeta; w)/\partial z]$ *be the formal power series which arises if the series* (2.2) *are substituted in the Jacobian* $\det[\partial f(z; w)/\partial z]$ *of the functions* f *with respect to* z. *If*

$$\det\left[\frac{\partial f(\zeta; w)}{\partial z}\right] \not\triangleq 0, \tag{2.4}$$

then the formal power series (2.2) *have positive radii of convergence and hence the functions* $\zeta_i(w) = \sum_{j=1}^{\infty} b_{ij} w^j$, $i = 1, \cdots, k$, *analytic in a neighborhood of* $w = 0$, *satisfy* $f_i(\zeta(w); w) \equiv 0$, $i = 1, \cdots, k$.

Remark. From (2.2) and (2.3) it follows that $f_i(0; 0) = 0$, $i = 1, \cdots, k$, or $a_{i0\cdots 0} = 0$, and from (2.4) it follows that $f_i(z; w) \not\equiv 0$, $i = 1, \cdots, k$.

Proof of Theorem 2. We assume that for at least one i, infinitely many b_{ij}, $j = 1, 2, \cdots$, are nonzero, otherwise the f.p.s. (2.2) are all polynomials in w and then the convergence is evident.

Suppose first that the coefficient of w^0 (the constant term) of the f.p.s. represented by the left-hand side (l.h.s.) of (2.4) is not equal to zero. Then it is easily verified that

$$\left.\frac{\partial(f_1, \cdots, f_k)}{\partial(z_1, \cdots, z_k)}\right|_{z=w=0} \neq 0, \tag{2.5}$$

and the implicit function theorem can be applied which states that $f_j(z_1, \cdots, z_k; w) = 0$, $j = 1, \cdots, k$, uniquely determine $z_1, \cdots, z_k$ as analytic functions of w in a neighborhood of $w = 0$. Hence, according to Theorem 1, $\zeta_1, \cdots, \zeta_k$ represent these functions. Now Theorem 2 has been proved.

For the remainder of the proof we assume that the l.h.s. of (2.5) is zero. Because this remainder is rather lengthy, it will be split up into four stages, to be denoted by I, II, III and IV. The aims of these stages are, respectively:

Stage I: to reduce the case $\partial f_i(0;0)/\partial z_j = 0$, $i,j = 1, \cdots, k$, to the case $\partial \bar{f}_i(0;0)/\partial \bar{z}_j \neq 0$ for at least one i and j by means of proper changes of variables $(z_i \to \bar{z}_i, f_i \to \bar{f}_i)$.

Stage II: to give the functional matrix $[\partial \bar{f}/\partial \bar{z}]$ a specific form.

Stage III: to reduce the original system (k equations, $k + 1$ variables) to a new system ($k - 1$ equations, k variables).

Stage IV: to serve as the concluding part.

I. We distinguish two cases:

(i) $\partial f_i(0;0)/\partial z_j = 0$ for all $i,j = 1, \cdots, k$;

(ii) for at least one i and j: $\partial f_i(0;0)/\partial z_j \neq 0$.

If case (ii) occurs, a bar is added to all symbols except w, Stage I is omitted and the proof is continued at Stage II, where, for notational convenience it is supposed that $\partial f_1(0;0)/\partial z_1 \neq 0$. For the time being we assume that case (i) applies.

Suppose that

$$\zeta_i \triangleq b_{i\mu_i} w^{\mu_i} + b_{i\mu_i+1} w^{\mu_i+1} + \cdots, \qquad i = 1, \cdots, k, \tag{2.6}$$

with $b_{i\mu_i} \neq 0$ and $b_{ij} = 0$ for $j < \mu_i$ $(\mu_i \geqq 1)$, and define

$$h_{ij} \triangleq \frac{\partial f_i}{\partial z_j}(\zeta; w) \triangleq h_{ij\nu_{ij}} w^{\nu_{ij}} + h_{ij\nu_{ij}+1} w^{\nu_{ij}+1} + \cdots, \qquad i,j = 1, \cdots, k, \tag{2.7}$$

with $h_{ij\nu_{ij}} \neq 0$. Possibly $\zeta_i \triangleq 0$ for some i; in this case we take $\mu_i = \infty$. However, for at least one i we have $\mu_i < \infty$. An analogous situation prevails for h_{ij}. Possibly some $h_{ij} \triangleq 0$, in which case we again take $\nu_{ij} = \infty$. However, from (2.4) it follows that for each i at least one j exists for which ν_{ij} is finite.

Taking $\mu = \min\{\mu_i\}$, we introduce new variables z_i^*, $i = 1, \cdots, k$, and a neighborhood Ω_2 of $z^* = w = 0$ in such a way that Ω_2 is mapped into Ω_1 under the mapping

$$z_i = (q_i + z_i^*) w^\mu, \qquad i = 1, \cdots, k, \quad w = w, \tag{2.8a}$$

where $q_i = b_{i\mu_i}$ if $\mu_i = \mu$, otherwise $q_i = 0$. Also new f.p.s. ζ_i^* are introduced:

$$\zeta_i \triangleq (q_i + \zeta_i^*) w^\mu, \qquad i = 1, \cdots, k, \tag{2.8b}$$

where q_i has been defined above. The new f.p.s. ζ_i^* have the behavior

$$\begin{aligned} \zeta_i^* &\triangleq b_{i\mu_i} w^{\mu_i - \mu} + \cdots \quad \text{if } \mu_i > \mu, \\ \zeta_i^* &\triangleq b_{i\mu_i+1} w + \cdots \quad \text{if } \mu_i = \mu. \end{aligned} \tag{2.9}$$

Functions $\tilde{f}_j(z^*; w), j = 1, \cdots, k$, will now be defined in Ω_2 as

$$\tilde{f}_j(z_1^*, \cdots, z_k^*; w) = f_j((q_1 + z_1^*)w^\mu, \cdots, (q_k + z_k^*)w^\mu; w)$$

$$(2.10) \qquad = \sum_{i_1, \cdots, i_{k+1} = 0}^{\infty} a_{ji_1 \cdots i_{k+1}} w^{i_{k+1} + \mu(i_1 + \cdots + i_k)} \prod_{m=1}^{k} (q_m + z_m^*)^{i_m}, \qquad j = 1, \cdots, k.$$

The functions $\tilde{f}_j(z^*; w), j = 1, \cdots, k$, are analytic functions in Ω_2. A neighborhood $\Omega_1^* \subset \Omega_2$ of $z^* = w = 0$ exists in such a way that

$$(2.11) \qquad \tilde{f}_j(z_1^*, \cdots, z_k^*; w) = \sum_{i_1, \cdots, i_{k+1} = 0}^{\infty} \tilde{a}_{ji_1 \cdots i_{k+1}} z^{*i_1} \cdots z_k^{*i_k} w^{i_{k+1}}, \qquad j = 1, \cdots, k.$$

For fixed j, each coefficient in this expansion is a function of finitely many $a_{ji_1 \cdots i_{k+1}}$ and finitely many b_{it}. It follows that when ζ_i^*, $i = 1, \cdots, k$, are substituted in (2.11), we obtain

$$(2.12) \qquad \tilde{f}_j(\zeta^*; w) \triangleq f_j(\zeta; w) \triangleq 0, \qquad j = 1, \cdots, k.$$

Define

$$(2.13) \qquad m = \min_j \left[\min \left\{ i_{k+1} + \mu \sum_{t=1}^{k} i_t | a_{ji_1 \cdots i_{k+1}} \neq 0 \right\} \right],$$

which is the lowest power of w which appears on the right-hand side (r.h.s.) of (2.10). A reasoning given in Appendix A shows that

$$(2.14) \qquad m > \mu.$$

The functions $\tilde{f}_j(z^*; w)$ can be divided by w^m. The quotient-functions, to be denoted by $f_j^*(z^*; w)$, are again analytic functions in Ω_1^*, where they have the expansions

$$(2.15) \qquad f_j^*(z_1^*, \cdots, z_k^*; w) = \sum_{i_1, \cdots, i_{k+1} = 0}^{\infty} a_{ji_1 \cdots i_{k+1}}^* z_1^{*i_1} \cdots z_k^{*i_k} w^{i_{k+1}}, \qquad j = 1, \cdots, k,$$

with $a_{ji_1 \cdots i_{k+1}}^* = \tilde{a}_{ji_1 \cdots i_k i_{k+1} + m}$. From (2.12) it is clear that

$$(2.16) \qquad f_j^*(\zeta^*; w) \triangleq 0, \qquad j = 1, \cdots, k.$$

Moreover, it is easily proved that

$$(2.17) \qquad h_{ij}^* \overset{\text{def}}{\triangleq} \frac{\partial f_i^*}{\partial z_j^*}(\zeta_1^*, \cdots, \zeta_k^*; w) \triangleq h_{ij\nu_{ij}} w^{\nu_{ij} - m + \mu} + \cdots, \qquad i,j = 1, \cdots, k.$$

Because

$$(2.18) \qquad \frac{\partial(f_1^*, \cdots, f_k^*)}{\partial(z_1^*, \cdots, z_k^*)} = \frac{\partial(f_1, \cdots, f_k)}{\partial(z_1, \cdots, z_k)} w^{k(\mu - m)},$$

it follows from (2.4) that

$$(2.19) \qquad \det \left[\frac{\partial f^*}{\partial z^*}(\zeta^*; w) \right] \not\triangleq 0.$$

To summarize: originally we dealt with the functions f_j and the variables z_j, w and ζ_j; after the transformations (2.8), new functions f_j^* were obtained and the corresponding variables became z_j^*, w and ζ_j^*. The functions f_j^* have the same properties as the functions f_j; they are analytic in a neighborhood Ω_1^* of $z^* = w = 0$, where they have the c.p.s. (2.15). If the f.p.s. ζ_i^*, $i = 1, \cdots, k$, given in (2.9), are substituted in the c.p.s. (2.15), then the relations (2.16) are valid. The analogue of (2.4) is (2.19).

If $v_{ij} - m + \mu \geqq 1$ for all i and j, then

$$\frac{\partial f_i^*}{\partial z_j^*}(0;0) = 0 \tag{2.20}$$

for all i and j, and the whole process of performing new transformations of the kind (2.8) is repeated, until we have, for at least one i and one j,

$$h_{ij}^{*\cdots*} \triangleq h_{ijv_{ij}} + h_{ijv_{ij}+1}w + \cdots, \tag{2.21}$$

with $h_{ijv_{ij}} \neq 0$. For the sake of economy, we will use bars in place of the repeated index $* \cdots *$. Thus instead of $h_{ij}^{*\cdots*}$ we now write $\bar{h}_{ij}$. Suppose that after the last transformation we deal with the functions $\bar{f}_j$ and the variables $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_k)$, w and $\bar{\zeta} = (\bar{\zeta}_1, \cdots, \bar{\zeta}_k)$. Then we know that $\bar{f}_j(\bar{z}; w)$, $j = 1, \cdots, k$, are analytic functions in a neighborhood $\bar{\Omega}_1$ of $\bar{z} = w = 0$, and

$$\bar{f}_j(\bar{\zeta}; w) \triangleq 0, \qquad j = 1, \cdots, k, \tag{2.22}$$

where

$$\bar{\zeta}_i = \sum_{j=1}^{\infty} \bar{b}_{ij} w^j, \qquad i = 1, \cdots, k. \tag{2.23}$$

In addition we know that

$$\det\left[\frac{\partial \bar{f}}{\partial \bar{z}}(\bar{\zeta}; w)\right] \not\triangleq 0, \tag{2.24}$$

and that (2.21) is valid for at least one i and one j. In order to simplify the notation we take $i = j = 1$.

II. We perform new coordinate transformations of the form

$$\bar{z}_1^* = \bar{f}_1(\bar{z}; w); \qquad \bar{z}_i^* = z_i, \quad i = 2, \cdots, k, \tag{2.25}$$

and

$$\bar{\zeta}_1^* \triangleq \bar{f}_1(\bar{\zeta}; w) \triangleq 0, \qquad \bar{\zeta}_i^* \triangleq \bar{\zeta}_i, \quad i = 2, \cdots, k. \tag{2.26}$$

From (2.21), with $i = j = 1$, it follows directly that $\partial \bar{f}_1(\bar{z}; w)/\partial \bar{z}_1 \neq 0$ at $\bar{z} = w = 0$, and hence the implicit function theorem can be applied to the first equation of (2.25), which results in $\bar{z}_1$ being an analytic function of $\bar{z}_1^*, \bar{z}_2, \cdots, \bar{z}_k$ and w in a neighborhood of $\bar{z}_1^* = \bar{z}_2 = \cdots = \bar{z}_k = w = 0$;

$$\bar{z}_1 = \bar{g}(\bar{z}_1^*, \bar{z}_2, \cdots, \bar{z}_k; w) = \bar{g}(\bar{z}^*; w). \tag{2.27}$$

From the lemma in Appendix B it follows that

$$\bar{\zeta}_1 \triangleq \bar{g}(\bar{\zeta}_1^*, \bar{\zeta}_2, \cdots, \bar{\zeta}_k; w) \triangleq \bar{g}(\bar{\zeta}^*; w). \tag{2.28}$$

In (2.27) and (2.28) we have written $\bar{z}^* = (\bar{z}_1^*, \cdots, \bar{z}_k^*)$ and $\bar{\zeta}^* = (\bar{\zeta}_1^*, \cdots, \bar{\zeta}_k^*)$. Because $\bar{f}_1(\bar{g}(\bar{z}^*; w), \bar{z}_2^*, \cdots, \bar{z}_k^*; w) = \bar{z}_1^*$, and hence

$$\frac{\partial \bar{f}_1}{\partial \bar{z}_1^*} = \frac{\partial \bar{f}_1}{\partial \bar{z}_1} \cdot \frac{\partial \bar{g}}{\partial \bar{z}_1^*} = 1, \tag{2.29}$$

it follows that

$$\frac{\partial \bar{g}}{\partial \bar{z}_1^*}(\bar{z}^*; w)|_{\bar{z}^* = w = 0} \neq 0. \tag{2.30}$$

This result will be needed later on.

We now introduce the functions

$$\bar{f}_j^*(\bar{z}_1^*, \cdots, \bar{z}_k^*; w) \stackrel{\text{def}}{=} \bar{f}_j(\bar{g}(\bar{z}_1^*, \cdots, \bar{z}_k^*; w), \bar{z}_2^*, \cdots, \bar{z}_k^*; w), \qquad j = 1, \cdots, k, \tag{2.31}$$

which are analytic in some neighborhood $\bar{\Omega}_1^*$ of $\bar{z}^* = w = 0$. For $j = 1$, definition (2.31) can be simplified to

$$\bar{f}_1^*(\bar{z}_1^*, \cdots, \bar{z}_k^*; w) = \bar{z}_1^*. \tag{2.32}$$

With the aid of (2.22), (2.23) and (2.25) it is easily verified that

$$\bar{f}_j^*(\bar{\zeta}^*; w) \triangleq 0, \qquad j = 1, \cdots, k. \tag{2.33}$$

It will now be proved that if $\bar{\zeta}^*$ is substituted in $\partial(\bar{f}_1^*, \cdots, \bar{f}_k^*)/\partial(\bar{z}_1^*, \cdots, \bar{z}_k^*)$, the resulting f.p.s., to be denoted by ρ, is not equal to zero. Because

$$\frac{\partial(\bar{f}_1^*, \cdots, \bar{f}_k^*)}{\partial(\bar{z}_1^*, \cdots, \bar{z}_k^*)} = \frac{\partial(\bar{f}_1, \cdots, \bar{f}_k)}{\partial(\bar{z}_1, \cdots, \bar{z}_k)} \cdot \frac{\partial(\bar{z}_1, \cdots, \bar{z}_k)}{\partial(\bar{z}_1^*, \cdots, \bar{z}_k^*)}, \tag{2.34}$$

it follows that the f.p.s. ρ is the product of two other f.p.s. in w:

$$\rho \triangleq \det\left[\frac{\partial \bar{f}}{\partial \bar{z}}(\bar{\zeta}; w)\right] \cdot \frac{\partial \bar{g}}{\partial \bar{z}_1^*}(\bar{\zeta}^*; w). \tag{2.35}$$

Both of these f.p.s. are $\not\triangleq 0$; the first f.p.s. according to (2.24); the second f.p.s. starts with a nonzero constant according to (2.30). All f.p.s. in w constitute a ring without null-divisors [2] and hence $\rho \not\triangleq 0$.

III. Let us now consider the functions

$$\bar{\bar{f}}_j(\bar{z}_2^*, \cdots, \bar{z}_k^*; w) \stackrel{\text{def}}{=} \bar{f}_j^*(0, \bar{z}_2^*, \cdots, \bar{z}_k^*; w), \qquad j = 2, \cdots, k. \tag{2.36}$$

These functions are analytic in a neighborhood of $\bar{z}_2^* = \cdots = \bar{z}_k^* = w = 0$. Moreover,

$$\bar{\bar{f}}_j(\bar{\zeta}_2^*, \cdots, \bar{\zeta}_k^*; w) \triangleq 0, \qquad j = 2, \cdots, k. \tag{2.37}$$

Because $\partial \bar{f}_1^*/\partial \bar{z}_i^* = \delta_{1,i}$, where $\delta_{1,i}$ denotes the Kronecker symbol,

$$\frac{\partial(\bar{f}_1^*, \cdots, \bar{f}_k^*)}{\partial(\bar{z}_1^*, \cdots, \bar{z}_k^*)} = \frac{\partial(\bar{f}_2^*, \cdots, \bar{f}_k^*)}{\partial(\bar{z}_2^*, \cdots, \bar{z}_k^*)}, \tag{2.38}$$

and it follows that

$$\det\left[\frac{\partial\bar{\bar{f}}}{\partial\bar{z}^*}(\bar{\zeta}_2^*, \cdots, \bar{\zeta}_k^*; w)\right] \mathrel{\hat{\neq}} 0, \tag{2.39}$$

where $[\partial\bar{\bar{f}}/\partial\bar{z}^*]$ denotes the functional matrix of the $(k-1)$ functions $\bar{\bar{f}}_2, \cdots, \bar{\bar{f}}_k$ with respect to $\bar{z}_2^*, \cdots, \bar{z}_k^*$.

To resume, we are now given $k-1$ functions,

$$\bar{\bar{f}}_j(\bar{z}_2^*, \cdots, \bar{z}_k^*; w), \qquad j = 2, \cdots, k, \tag{2.40}$$

analytic in a neighborhood of $\bar{z}_2^* = \cdots = \bar{z}_k^* = w = 0$, and we are also given $k-1$ f.p.s.; $\bar{\zeta}_2^*, \cdots, \bar{\zeta}_k^*$. If these f.p.s. are substituted in (2.40), then (2.37) is obtained, and if these f.p.s. are substituted in the Jacobian of the functions $\bar{\bar{f}}_2, \cdots, \bar{\bar{f}}_k$ with respect to $\bar{z}_2^*, \cdots, \bar{z}_k^*$, a f.p.s. results which is $\mathrel{\hat{\neq}} 0$. Hence our original problem (k analytic functions and k f.p.s.) has been reduced, after a finite number of steps, to an analogous problem with $k-1$ analytic functions and $k-1$ f.p.s.

IV. In this way we continue: the whole process, described in this proof so far, is repeated until one of the three following situations arises:

(i) t functions and t f.p.s. with $1 \leqq t < k$ remain and these t f.p.s. are all polynomials in w (that is of each of the f.p.s. only finitely many coefficients are nonzero). The convergence of these t f.p.s. is evident.

(ii) t functions,

$$g_j(s_1, \cdots, s_t; w), \qquad j = 1, \cdots, t, \tag{2.41}$$

and t f.p.s.; σ_i, $i = 1, \cdots, t$, remain, which satisfy $g_j(\sigma_1, \cdots, \sigma_t; w) \mathrel{\hat{=}} 0$, $j = 1, \cdots, t$. The Jacobian

$$\frac{\partial(g_1, \cdots, g_t)}{\partial(s_1, \cdots, s_t)} \tag{2.42}$$

is nonzero at $s_1 = \cdots = s_t = w = 0$, and the implicit function theorem can be applied. In connection with Theorem 1, this theorem states that $\sigma_1, \cdots, \sigma_t$ are analytic functions of w in the neighborhood of $w = 0$.

(iii) One function, given by

$$g(s; w), \tag{2.43}$$

and one f.p.s. σ remain. This f.p.s. satisfies

$$g(\sigma; w) \mathrel{\hat{=}} 0, \quad \frac{\partial g}{\partial s}(\sigma; w) \mathrel{\hat{=}} 0. \tag{2.44}$$

Now Stage I of this proof is applied to the function g and the variables w and σ, which results in

$$\bar{g}(\bar{\sigma}; w) \mathrel{\hat{=}} 0, \tag{2.45}$$

$$\frac{\partial\bar{g}}{\partial\bar{\sigma}}(\bar{\sigma}, w) \mathrel{\hat{\neq}} 0, \tag{2.46}$$

where $\bar{g}$ and $\bar{\sigma}$ are the transformed function and variable respectively. Because $\bar{\sigma}$ is a f.p.s. which starts with a linear term in w, it is easily seen from (2.46) that

$$\frac{\partial \bar{g}}{\partial \bar{\sigma}}(0,0) \neq 0. \tag{2.47}$$

Now Theorem 1 can be applied to (2.45) and (2.47), which states that $\bar{\sigma}$ is an analytic function of w in a neighborhood of $w = 0$.

In each of the three situations a straightforward calculation, in which we perform the inverse transformations, shows that the f.p.s. $\zeta_1, \cdots, \zeta_k$ have positive radii of convergence, which had to be proved.

3. Some examples. Consider the functions

$$f_j(z_1, z_2; w) = 2z_1^j - z_2^j + w^2 = 0, \qquad j = 1, 2. \tag{3.1}$$

It is easily seen that the Jacobian of the equations (3.1) with respect to z_1 and z_2 at $z_1 = z_2 = w = 0$ is equal to zero and hence the implicit function theorem cannot be applied. We will show that Theorem 2 can be applied in order to prove that (3.1) has analytic solutions. By the elementary procedure of elimination it can be seen that (3.1) has two different analytic solutions $z_1(w)$ and $z_2(w)$.

The existence of two analytic solutions of (3.1) is also obtained with the aid of Theorem 2. To this end we substitute the f.p.s.

$$\zeta_1 \triangleq \sum_{j=1}^{\infty} \alpha_j w^j, \quad \zeta_2 \triangleq \sum_{j=1}^{\infty} \beta_j w^j \tag{3.2}$$

in (3.1) and try to determine α_j and β_j in such a way that $f_i(\zeta_1, \zeta_2; w) \triangleq 0$, $i = 1, 2$. Substitution of (3.2) in (3.1) and equating these f.p.s. to zero yields

$$2\sum_{j=1}^{\infty} \alpha_j w^j - \sum_{j=1}^{\infty} \beta_j w^j + w^2 \triangleq 0, \tag{3.3}$$

$$2\left(\sum_{j=1}^{\infty} \alpha_j w^j\right)^2 - \left(\sum_{j=1}^{\infty} \beta_j w^j\right)^2 + w^2 \triangleq 0. \tag{3.4}$$

In order that $f_1(\zeta_1, \zeta_2; w) \triangleq 0$, it follows from (3.3) that

$$2\alpha_j - \beta_j + \delta_{j,2} = 0, \qquad j = 1, 2, 3, \cdots, \tag{3.5}$$

where $\delta_{k,i}$ denotes the Kronecker symbol. In the same way, it follows from (3.4) that

$$2\alpha_1^2 - \beta_1^2 + 1 = 0, \qquad \text{(coefficient of } w^2\text{)}, \tag{3.6}$$

$$4\alpha_1\alpha_2 - 2\beta_1\beta_2 = 0, \qquad \text{(coefficient of } w^3\text{)}, \tag{3.7}$$

$$4\alpha_1\alpha_3 + 2\alpha_2^2 - 2\beta_1\beta_3 - \beta_2^2 = 0, \quad \text{(coefficient of } w^4\text{)}, \tag{3.8}$$

and so on. The quantities α_1 and β_1 can be solved from (3.5) for $j = 1$ and (3.6) (two different solutions: $\beta_1 = 2\alpha_1 = \pm\sqrt{2}$). Once α_1 and β_1 are known, α_2 and β_2 can be solved from (3.5) for $j = 2$ and (3.7). The determinant of the required system matrix equals $-4\beta_1 + 4\alpha_1$, which is nonzero because $2\alpha_1 = \beta_1 \neq 0$. Once $\alpha_1, \beta_1, \alpha_2$ and β_2 are known, α_3 and β_3 can be obtained from (3.5) for $j = 3$

and (3.8); these equations again constitute a linear system of which the system matrix equals the one we dealt with when determining α_2 and β_2.

In general, to determine α_p and β_p we need the coefficients of w^{t+p-1} in $f_t(\zeta_1, \zeta_2; w)$ as well as α_i, β_i, $i = 1, \cdots, p-1$. We make use of the following lemma.

LEMMA 1. *In the coefficient of* w^{t+p-1} *in* $f_t(\zeta_1, \zeta_2; w)$ *only terms appear which are functions of* $\alpha_i, \beta_i, i = 1, \cdots, p-1$, *except one term, viz.*, $t(2\alpha_1^{t-1}\alpha_p - \beta_1^{t-1}\beta_p)$.

Proof. All terms of the coefficient of w^{t+p-1} are of the kind

$$(3.9) \qquad \frac{(\mu_1 + \cdots + \mu_q)!}{\mu_1! \cdots \mu_q!}\{2\alpha_{i_1}^{\mu_1} \cdots \alpha_{i_q}^{\mu_q} - \beta_{i_1}^{\mu_1} \cdots \beta_{i_q}^{\mu_q}\},$$

with

$$(3.10) \qquad \sum_{j=1}^{q} \mu_j i_j = t + p - 1, \quad \sum_{j=1}^{q} \mu_j = t,$$

for all possible positive integers q (here 1 or 2). Quantities i_k and μ_k are positive integers with $i_k \neq i_{k'}$, if $k \neq k'$. Suppose it is possible that $i_j \geqq p$ for some j, say $j = q$. Then it follows from (3.10) that

$$t - \mu_q = \sum_{j=1}^{q-1} \mu_j \leqq \sum_{j=1}^{q-1} i_j\mu_j = t + p - 1 - i_q\mu_q,$$

from which

$$p - 1 \leqq i_q - 1 \leqq \frac{p-1}{\mu_q}.$$

These latter inequalities are only possible for $\mu_q = 1$ and $i_q = p$, which proves the lemma.

It is now clear that α_p and β_p can be determined from two linear equations, of which the system matrix is nonsingular.

In order to prove that the f.p.s. (3.2), of which the coefficients are now known, have positive radii of convergence, it remains to be shown that (2.4) is valid, or that[2]

$$(3.11) \qquad \det \begin{bmatrix} 2 & -1 \\ 4\zeta_1 & 2\zeta_2 \end{bmatrix} \triangleq 4(\zeta_2 - \zeta_1) \not\triangleq 0.$$

This is true because the coefficient of w in the f.p.s. $(\zeta_2 - \zeta_1)$ equals $(\beta_1 - \alpha_1)$, which is not equal to zero. Now Theorem 2 can be applied and hence the f.p.s. (3.2), with known coefficients, have positive radii of convergence and are analytic solutions of (3.1) in the neighborhood of $w = 0$.

Let us now consider another example, a generalization of (3.1), in which the analyticity of the solution is less trivial:

$$(3.12) \quad f_t(z_1, \cdots, z_n; w) = 2\sum_{j=1}^{n-1} \{(-1)^{j-1}z_j^t\} + (-1)^{n-1}z_n^t + (-1)^n w^n = 0, \qquad t = 1, \cdots, n.$$

[2] Note that when the theorem of Artin is applied in order to prove that the system has analytic solutions, this last part is not necessary.

The Jacobian of these equations with respect to the variables $z_1, \cdots, z_n$ at $z_1 = \cdots = z_n = w = 0$ is equal to zero and hence we will apply Theorem 2 in order to prove that (3.12) has an analytic solution. To this end we will substitute the f.p.s.

$$\zeta_i \triangleq \sum_{j=1}^{\infty} \alpha_{ij} w^j, \qquad i = 1, \cdots, n, \tag{3.13}$$

in (3.12) and show that the coefficients α_{ij} can be determined in such a way that $f_t(\zeta_1, \cdots, \zeta_n; w) \triangleq 0, t = 1, \cdots, n.$

The coefficient of w^t in $f_t(\zeta_1, \cdots, \zeta_n; w)$, which must be zero, equals

$$2 \sum_{j=1}^{n-1} (-1)^{j+1}\alpha_{j1}^t + (-1)^{n-1}\alpha_{n1}^t + (-1)^n \cdot \delta_{t,n} = 0, \qquad t = 1, \cdots, n. \tag{3.14}$$

The equations (3.14) constitute a system of n nonlinear equations with n unknowns $\alpha_{i1}, i = 1, \cdots, n$. With the aid of some trigonometrical manipulations, it can be shown that

$$\alpha_{i1} = \frac{4}{\sqrt[n]{4n}} \sin^2 \frac{i\pi}{2n}, \qquad i = 1, \cdots, n, \tag{3.15}$$

is a solution of (3.14).

To show that we can also determine α_{i2}, $i = 1, \cdots, n$, we consider the coefficient of w^{t+1} in $f_t(\zeta_1, \cdots, \zeta_n; w)$, $t = 1, \cdots, n$;

$$t\left\{2 \sum_{j=1}^{n-1} \{(-1)^{j-1}\alpha_{j1}^{t-1}\alpha_{j2}\} + (-1)^{n-1}\alpha_{n1}^{t-1}\alpha_{n2}\right\} + (-1)^n\delta_{t,n-1} = 0, \qquad t = 1, \cdots, n. \tag{3.16}$$

Equations (3.16) constitute a system of n linear equations with the unknowns α_{i2}, $i = 1, \cdots, n$. Because the system matrix, to be denoted by A, which has as (i,j)th element

$$i(2 - \delta_{j,n})(-1)^{j-1}\alpha_{j1}^{i-1} \tag{3.17}$$

has a Vandermonde character and is nonsingular on account of $\alpha_{i1} \neq \alpha_{j1}$ for $i \neq j$, the unknowns α_{i2} can be solved.

In this way we continue. To determine α_{i3}, $i = 1, \cdots, n$, we consider the coefficient of w^{t+2} in $f_t(\zeta_1, \cdots, \zeta_n; w)$, $t = 1, \cdots, n$;

$$\begin{aligned} &t\left\{2 \sum_{j=1}^{n-1} (-1)^{j-1}\alpha_{j1}^{t-1}\alpha_{j3} + (-1)^{n-1}\alpha_{n1}^{t-1}\alpha_{n3}\right\} + (-1)^n\delta_{t,n-2} \\ &\quad + g_t(\alpha_{11}, \cdots, \alpha_{n1}; \alpha_{12}, \cdots, \alpha_{n2}) = 0, \qquad t = 1, \cdots, n, \end{aligned} \tag{3.18}$$

where g_t are known functions of known variables, which can be proved with techniques similar to those used in Lemma 1. The equations (3.18) again constitute a system of n linear equations with the unknowns $\alpha_{i3}, i = 1, \cdots, n$. which can be solved because the system matrix of this system equals the matrix A, which is nonsingular.

By proceeding this way, we find that all coefficients α_{ij} can be determined. In order to prove that the f.p.s. found have positive radii of convergence, it must be shown that (2.4) is valid, which in our case requires that det $[B] \ncong 0$, where B is the $n \times n$ matrix of which the (i, j)th element is given by

$$(2 - \delta_{j,n})i(-1)^{j-1}\zeta_j^{i-1}. \tag{3.19}$$

Det $[B]$ is a f.p.s. in w, and it is easily shown that the coefficient of $w^{n(n-1)/2}$ equals the determinant of the matrix A. Because A is nonsingular it follows that the coefficient of $w^{n(n-1)/2}$ is not equal to zero, which proves det $[B] \ncong 0$.

Hence we have found that the system of equations (3.12) has at least one analytic solution $z_i(w)$ in the neighborhood of $w = 0$. Because $z_1, z_3, z_5, \cdots$ appear symmetrically in equations (3.12), as do $z_2, z_4, z_6, \cdots$, it is clear that the solutions of z_i and z_{i+2} $(i \leqq n - 3)$ can be interchanged. These solutions are not equal (the coefficients of the first term of the expansions, α_{i1} and $\alpha_{i+2,1}$, are different) and this shows that the equations (3.12) have more than one analytic solution.

Appendix A. In this appendix we will prove the inequality (2.14). The quantity m has been defined in (2.13). Suppose that this minimum m is achieved for $i_1 = \cdots = i_k = 0$ and some i_{k+1} (which is equal to m) and some j, say j^*. The power series for $\tilde{f}_{j^*}$ in (2.10) has the term $\alpha_{j^*0\cdots 0m}w^m$, with $\alpha_{j^*0\cdots 0m} \neq 0$.

Since for this j^* quantity m is achieved for $i_1 = \cdots = i_k = 0$, m is also achieved for this j^* and an $i_t \neq 0$ for at least one t with $1 \leqq t \leqq k$. Otherwise it would be impossible that $\tilde{f}_{j^*}(\zeta^*; w) \triangleq 0$, which is shown as follows. Suppose the contrary: the minimum m is only achieved for $j = j^*$, $i_1 = \cdots = i_k = 0$, $i_{k+1} = m$. In this case

$$\tilde{f}_{j^*}(z^*; w) = a_{j^*0\cdots 0m}w^m + \text{higher powers of } w. \tag{A.1}$$

Because $\tilde{f}_{j^*}(\zeta^*; w) \triangleq 0$, the coefficient of w^m must be zero in the f.p.s. $\tilde{f}_{j^*}(\zeta^*; w)$. However, if we substitute the f.p.s. $\zeta^* = (\zeta_1^*, \cdots, \zeta_k^*)$ in (A.1), the coefficient of w^m in the f.p.s. $\tilde{f}_{j^*}(\zeta^*; w)$ is $a_{j^*0\cdots 0m}$, which is nonzero.

So we know that for $j = j^*$ the minimum m is achieved for an $i_t \neq 0$ with $1 \leqq t \leqq k$. This holds for all j with $1 \leqq j \leqq k$ and hence $m \geqq \mu$. Suppose next that $m = \mu$ and that this is the case if $j = j^*$. Apart from the possibility ($j = j^*$, $i_1 = \cdots = i_k = 0$, $i_{k+1} = m$) this can only be realized for ($j = j^*$, $i_i = \cdots = i_{t-1} = i_{t+1} = \cdots = i_k = i_{k+1} = 0$, $i_t = 1$) for some t with $1 \leqq t \leqq k$. However, this implies that $a_{j^*0\cdots 010\cdots 0} \neq 0$, where the number 1 is the $(t + 1)$th index. In Stage I of the proof, where this Appendix is used, $\partial f_{j^*}(z; w)/\partial z_t = 0$ at $z = w = 0$, or $a_{j^*0\cdots 010\cdots 0} = 0$, and a contradiction has been obtained. Hence $m > \mu$.

Appendix B.

LEMMA. *We are given a function* $f(z_1, \cdots, z_k; w)$, $f(0; 0) = 0$, *represented by the (absolutely) convergent power series*

$$f = \sum_{i_1,\cdots,i_{k+1}=0}^{\infty} a_{i_1\cdots i_{k+1}} z_1^{i_1} \cdots z_k^{i_k} w^{i_{k+1}} \tag{B.1}$$

in a neighborhood Ω *of* $z_1 = \cdots = z_k = w = 0$. *Hence* f *is analytic in* Ω. *We suppose that* $a_{10\cdots 0} \neq 0$. *According to the implicit function theorem, the equation* $f(z_1, \cdots, z_k; w) = 0$ *can be solved with respect to* z_1 *in terms of the remaining* *variables*:

$$z_1 = g(z_2, \cdots, z_k; w) = \sum_{i_2,\cdots,i_{k+1}=0}^{\infty} c_{i_2\cdots i_{k+1}} z_2^{i_2} \cdots z_k^{i_k} w^{i_{k+1}}, \tag{B.2}$$

where g is an analytic function in some domain containing the point $z_2 = \cdots = z_k = w = 0$. *In the power series expansion of g on the right-hand side of* (B.2), $c_{0\cdots 0} = 0$.

If f.p.s. are given,

$$\zeta_i \triangleq \sum_{j=1}^{\infty} b_{ij} w^j, \qquad i = 1, \cdots, k, \tag{B.3}$$

which satisfy $f(\zeta_1, \cdots, \zeta_k; w) \triangleq 0$, *then*

$$\zeta_1 \triangleq g(\zeta_2, \cdots, \zeta_k; w). \tag{B.4}$$

Proof. Suppose that $g(\zeta_2, \cdots, \zeta_k; w)$ is given by the f.p.s.

$$g(\zeta_2, \cdots, \zeta_k; w) \triangleq \sum_{i=1}^{\infty} d_i w^i. \tag{B.5}$$

Then (B.4) states that $b_{1i} = d_i$ for all $i \geqq 1$. Suppose (B.4) is not true and hence $b_{1i} \neq d_i$ for some i. Define

$$N = \min\{i | b_{1i} \neq d_i, i \geqq 1\}, \tag{B.6}$$

$$\bar{\zeta}_i = \sum_{j=1}^{N} b_{ij} w^j, \qquad i = 2, \cdots, k. \tag{B.7}$$

The functions $\bar{\zeta}_i$, $2 \leqq i \leqq k$, are analytic in w. By the implicit function theorem, the function $\bar{\zeta}_1(w)$ is uniquely defined by

$$f(\bar{\zeta}_1(w), \bar{\zeta}_2(w), \cdots, \bar{\zeta}_k(w); w) \equiv 0, \tag{B.8}$$

and is an analytic function of w in a neighborhood of $w = 0$. Now $\bar{\zeta}_1(w)$ satisfies

$$\bar{\zeta}_1(w) = g(\bar{\zeta}_2(w), \cdots, \bar{\zeta}_k(w); w) \overset{\text{def}}{=} \sum_{i=1}^{\infty} \bar{b}_{1i} w^i. \tag{B.9}$$

It is easily seen that $\bar{b}_{1i} = d_i$ for $1 \leqq i \leqq N$. If we can prove that $\bar{b}_{1i} = b_{1i}$, $1 \leqq i \leqq n$, then a contradiction has been obtained, because from (B.6) it follows that $b_{1N} \neq d_N = \bar{b}_{1N}$. Thus (B.4) is true.

It remains to be proved that $\bar{b}_{1i} = b_{1i}$ for $1 \leqq i \leqq N$. For that purpose consider

$$f_1(z_1; w) \overset{\text{def}}{\triangleq} f(z_1, \zeta_2, \cdots, \zeta_k; w) \triangleq \sum_{i,j=0}^{\infty} e_{ij} z_1^i w^j, \tag{B.10}$$

$$\bar{f}_1(z_1; w) \overset{\text{def}}{=} f(z_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_k; w) = \sum_{i,j=0}^{\infty} \bar{e}_{ij} z_1^i w^j, \tag{B.11}$$

It is clear that $e_{10} = \bar{e}_{10} = a_{10\cdots 0} \neq 0$ and that $e_{ij} = \bar{e}_{ij}$ for all i and those j which satisfy $0 \leqq j \leqq N$.

The f.p.s. $\zeta_1 \triangleq \sum_{i=1}^{\infty} b_{1i}w^i$ is uniquely determined by $f_1(\zeta_1; w) \triangleq 0$ and the c.p.s. $\bar{\zeta}_1 = \sum_{i=1}^{\infty} \bar{b}_{1i}w^i$ is uniquely determined by $\bar{f}_1(\bar{\zeta}_1; w) \equiv 0$. Hence the coefficients b_{1j} are functions of the coefficients e_{ij}, and $\bar{b}_{1j}$ are functions of $\bar{e}_{ij}$.

Let us consider the dependence of b_{1j} on the coefficients e_{ij} in detail. By substitution of $\zeta_1 \triangleq \sum_{i=1}^{\infty} b_{1i}w^i$ in the f.p.s. (B.10) we get a new f.p.s. in w of which all coefficients are zero. The coefficient of w gives

$$e_{10} \cdot b_{11} + e_{01} = 0, \tag{B.12}$$

from which it follows that b_{11} is uniquely determined as a function of e_{10} and e_{01}. The coefficient of w^2 gives

$$e_{10} \cdot b_{12} + e_{20} \cdot b_{11}^2 + e_{11} \cdot b_{11} + e_{02} = 0,$$

from which it follows that b_{12} is uniquely determined as a function of e_{10}, e_{11}, e_{02} and b_{11} and hence of e_{10}, e_{11}, e_{02} and e_{01}. In this way we continue. It is easily seen that b_{1m} is a function of b_{1j}, $1 \leqq j \leqq m - 1$, and e_{ij}, $0 \leqq i \leqq m$, $0 \leqq j \leqq m$. Hence the coefficients $b_{11}, b_{12}, \cdots, b_{1N}$ are all uniquely determined by the quantities e_{ij}, $0 \leqq i \leqq N$, $0 \leqq j \leqq N$.

The coefficients $\bar{b}_{11}, \bar{b}_{12}, \cdots, \bar{b}_{1N}$ depend on $\bar{e}_{ij}$, $0 \leqq i \leqq N$, $0 \leqq j \leqq N$, in exactly the same way as the $b_{11}, b_{12}, \cdots, b_{1N}$ depend on e_{ij}. Because $e_{ij} = \bar{e}_{ij}$ for all i and $j \leqq N$, we have proved that $b_{1i} = \bar{b}_{1i}$, $i = 1, \cdots, N$.

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ON GENERALIZED HEAT POLYNOMIALS*

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Abstract. The concept of a heat polynomial defined earlier by Rosenbloom and Widder is extended and the polynomials are characterized using integrals of Poisson type. One particular class of polynomials is examined in detail and leads to expansions of solutions of the problem

$$u_{xx} = u_t, \quad u(x, 0) = f(x),$$

which are different from those obtained by Rosenbloom and Widder. In the last section some applications of polynomials to heat equation problems are given. In most results, much use is made of the close relationship between these polynomials and Hermite polynomials.

1. Introduction. Following Rosenbloom and Widder [7], we say that $u(x, t)$ is in H on a region D if it has continuous second partial derivatives on D and there satisfies the heat equation

$$u_t = u_{xx}. \tag{1.1}$$

Of particular interest are the polynomials defined by the generating function

$$\sum_{n=0}^{\infty} v_n(x, t)\frac{z^n}{n!} = \exp[zx + z^2 t]. \tag{1.2}$$

These polynomials, appearing in Appell's work [1], were called *heat polynomials* by Rosenbloom and Widder who studied their properties in great detail. One can write

$$v_n(x, t) = (-t)^{n/2} H_n\left(\frac{x}{2\sqrt{-t}}\right), \qquad H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}, \tag{1.3}$$

where $H_n(x)$ is the Hermite polynomial of degree n.

In this paper we broaden this concept by saying that $u(x, t)$ is a *generalized heat polynomial*, abbreviated GHP, if it is a polynomial in x and t which is in H for $-\infty < x < \infty$, $-\infty < t < \infty$. It is not difficult to characterize such polynomials, and this is the subject of § 2. In § 3 we introduce a special class of such polynomials which are simply related to the heat polynomials $v_n(x, t)$. They bear a relationship to $v_n(x, t)$ similar to that of the Hermite polynomials to x^n in the sense that expansions of nonanalytic functions are possible (at $t = 0$). Finally in § 4 we give examples of the application of polynomials to the solution of some problems of heat conduction.

2. Generalized heat polynomials. We begin with a characterization of generalized heat polynomials. Define

$$k(x, t) = (4\pi t)^{-1/2} e^{-x^2/(4t)}. \tag{2.1}$$

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THEOREM 2.1. *The following two conditions are each necessary and sufficient that $u(x, t)$ be a* GHP.

(a) *There exists a polynomial $P(x)$ so that for $-\infty \times x \times \infty$, $0 < t$,*

$$u(x, t) = k(x, t) * P(x) = \int_{-\infty}^{+\infty} k(x - y, t)P(y)\, dy.$$

(b) *There exists a polynomial $Q(x)$ so that for $-\infty < x < \infty$, $t < 0$,*

$$u(x, t) = \int_{-\infty}^{+\infty} k(y + ix, -t)Q(y)\, dy.$$

Moreover, $P(x) = u(x, 0)$, $Q(x) = u(ix, 0)$.

Proof. Let $u(x, t)$ be a GHP. Then it is easy to see that for $-\infty < a < t < b < \infty$,

$$\int_{-\infty}^{+\infty} k(y, b - t)|u(y, t)|\, dy = \int_{-\infty}^{+\infty} k(y, 1)|u(\sqrt{b - t}\, y, t)|\, dy \leqq M(a, b),$$

where $M(a, b)$ is a constant depending on a, b. A result of Rosenbloom and Widder [7; p. 248] is that

$$u(x, t) = \sum_{n=0}^{\infty} a_n v_n(x, t), \qquad n!a_n = D_x^n u(0, 0).$$

Since $u(x, t)$ is a polynomial in x, $a_n = 0$ for $n > N$ for some N. Also, [7; pp. 222 and 227],

$$v_n(x, t) = k(x, t) * x^n, \qquad t > 0, \tag{2.2}$$

and

$$v_n(x, t) = \int_{-\infty}^{+\infty} k(y + ix, -t)(iy)^n\, dy, \qquad t < 0, \tag{2.3}$$

so that in the former case,

$$u(x, t) = \sum_{n=0}^{N} a_n v_n(x, t) = k(x, t) * P(x),$$

where $P(x) = \sum_{n=0}^{N} a_n x^n = u(x, 0)$ and in the latter case,

$$u(x, t) = \int_{-\infty}^{+\infty} k(y + ix, -t)Q(y)\, dy,$$

where $Q(x) = \sum_{n=0}^{N} a_n(ix)^n = u(ix, 0)$.

Conversely, let (a) hold. Then $u(x, t)$ is in H for $-\infty < x < \infty$, $0 < t$ from [5; p. 181]. Moreover, by a change of variable,

$$u(x, t) = \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-y^2} P(x - y\sqrt{4t})\, dy,$$

from which we can easily see that $u(x, t)$ is a polynomial in x, t and that $u(x, 0+) = P(x)$. For part (b), let the representation in (b) hold. Then we can apply the formula in (2.3) to get, for $-\infty < x < \infty$, $t < 0$,

$$u(x, t) = \sum_{n=0}^{N} a_n v_n(x, t),$$

where we assume that

$$Q(x) = \sum_{n=0}^{N} a_n (ix)^n.$$

Thus $u(x, t)$ is a polynomial in x and t and is in H for $-\infty < x < \infty$, $-\infty < t < \infty$ since this is true for $v_n(x, t)$. Also

$$u(x, 0) = \sum_{n=0}^{N} a_n v_n(x, 0) = \sum_{n=0}^{N} a_n x^n = Q(-ix).$$

This completes the proof of the theorem.

From (2.2), we see that $v_n(x, t)$ is the special case corresponding to $P(x) = x^n$. Another case of interest occurs when $P(x) = H_n(x/2)$. Then the corresponding GHP is

$$Z_n(x, t) = k(x, t) * H_n(x/2) = k(x, t) * v_n(x, -1)$$
$$= v_n(x, t-1) = (1-t)^{n/2} H_n\left(\frac{x}{2\sqrt{1-t}}\right)$$

using the fact that $v_n(x, t)$ satisfies the Huygens property (see [7; p. 249] and § 4 of this paper). Although this analysis is valid only for $t > 0$, it is easy to see that it holds for all t by analytic continuation.

3. A new class of GHP. In this section we will concentrate on the GHP $Z_n(x, t)$ introduced at the end of § 2. The reason for this choice is based on the following argument. If a solution of the heat equation (1.1) with $u(x, 0) = f(x)$ is represented by

$$u(x, t) = \sum_{n=0}^{\infty} a_n v_n(x, t),$$

then

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} a_n v_n(x, 0) = \sum_{n=0}^{\infty} a_n x^n$$

and thus a typical result (Theorem 5.5 of [7]) has $f(x)$ an entire function of some class. With the choice made here, we will obtain instead

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} a_n H_n(x/2). \tag{3.1}$$

The theory of expansions in Hermite polynomials does not require analyticity so that different results can be expected.

THEOREM 3.1. *Let $k(x, r)f(x)$ be in $L(-\infty, \infty)$ for every $r < 2$. Then the series*

$$\sum_{n=0}^{\infty} a_n Z_n(x, t), \qquad 2^n n! a_n = \int_{-\infty}^{+\infty} k(y, 1) H_n(y/2) f(y)\, dy$$

converges to a function $Z(x, t)$ in H on $-\infty < x < \infty$, $0 < t < 1$. Moreover $Z(x, 0+) = f(x)$ for almost all x and, in particular, at points of continuity.

Proof. The series in question can be written as

$$\sum_{n=0}^{\infty} a_n (1 - t)^{n/2} H_n\left(\frac{x}{2\sqrt{1 - t}}\right) \tag{3.2}$$

and in this form is a recognizable series from the theory of Abel summability of Hermite series. Thus this series under the given conditions converges for $0 < t < 1$ and moreover the sum of the series, $Z(x, t)$, can be written as [3; pp. 450 and 453]

$$Z(x, t) = \int_{-\infty}^{+\infty} k(x - y, t) f(y)\, dy.$$

Now we obtain the conclusions that $Z(x, t)$ is in H for $-\infty < x < \infty, 0 < t < 1$, and $Z(x, 0+) = f(x)$ almost everywhere with an application of known results, for example, [5; pp. 181 and 189].

It is not true under the general conditions of this theorem that

$$f(x) = \sum_{n=0}^{\infty} a_n Z_n(x, 0) = \sum_{n=0}^{\infty} a_n H_n(x/2). \tag{3.3}$$

For example, let $f(x) = x\, e^{x^2/8}$. Then the conditions of the theorem hold, but the series in (3.3) does not converge for any $x \neq 0$, a result mentioned by Szegö [8; p. 243]. To correct this situation, we impose slightly stronger conditions.

THEOREM 3.2. *Let $k(x, 2)f(x)$ be in $L(-\infty, \infty)$. Then the series* (3.2) *converges at $t = 0$ and*

$$Z(x, 0+) = Z(x, 0) = f(x)$$

at points in a neighborhood of which $f(x)$ is of bounded variation and at which

$$2f(x) = f(x+) + f(x-).$$

Proof. Clearly the conditions of Theorem 3.1 are satisfied and hence $Z(x, t)$ is in H for $-\infty < x < \infty, 0 < t < 1$, and $Z(x, 0+) = f(x)$ for almost all x and, in particular, at points where $2f(x) = f(x+) + f(x-)$. Also the series in (3.3) will represent the expansion of $f(x)$ in the series of Hermite polynomials $H_n(x/2)$. From known results [8; p. 240], this series will converge to $f(x)$ if $f(x)$ is of bounded variation in a neighborhood of x and $2f(x) = f(x+) + f(x-)$. Thus $Z(x, 0+) = Z(x, 0) = f(x)$. This completes the proof.

The conditions of these theorems are severe on the behavior of $f(x)$ in the neighborhood of $\pm\infty$. We can relax these by a simple device.

THEOREM 3.3. *Let $f(x)$ be integrable in every finite interval and let $f(x) = O(e^{cx^2})$ as $x \to \pm\infty$ for some $c > 0$. Then there exists an $a > 0$ such that the series*

$$\sum_{n=0}^{\infty} a^{-n/2} c_n v_n(x, t - a), \qquad 2^n n! c_n = \int_{-\infty}^{+\infty} k(y, 1) H_n\left(\frac{y}{2}\right) f(\sqrt{a} y)\, dy$$

converges to a function $v(x,t)$ *in* H *for* $-\infty < x < \infty$, $0 < t < a$. *Also* $v(x,0+) = v(x,0) = f(x)$ *at the points indicated in Theorem* 3.2.

Proof. We observe that $f(\sqrt{a}x) = O(e^{cax^2})$ and that if we choose $0 < a < (8c)^{-1}$, then $f(\sqrt{a}x)$ satisfies the conditions of both Theorems 3.1 and 3.2. Thus there exists $u(x,t)$ in H in the strip $0 < t < 1$ defined by the series in Theorem 3.1, with the coefficient (replacing a_n by c_n)

$$2^n n! c_n = \int_{-\infty}^{+\infty} k(y,1) H_n(y/2) f(\sqrt{a}y)\,dy.$$

Also $u(x,0+) = u(x,0) = f(\sqrt{a}x)$ at the points mentioned. Let

$$v(x,t) = u\left(\frac{x}{\sqrt{a}}, \frac{t}{a}\right).$$

Then $v(x,t)$ is in H in the strip $0 < t < a$ and $v(x,0+) = v(x,0) = f(x)$ at suitable points. Moreover,

$$v(x,t) = \sum_{n=0}^{\infty} c_n Z_n\left(\frac{x}{\sqrt{a}}, \frac{t}{a}\right)$$

and this can be simplified since

$$Z_n\left(\frac{x}{\sqrt{a}}, \frac{t}{a}\right) = a^{-n/2} v_n(x, t-a).$$

4. Applications. Our first application will be to a problem initiated by Appell [1] and partially solved by a number of authors. We state the problem as follows. Given an entire function $h(x)$, what are sufficient conditions on $h(x)$ that there exist a $T > 0$ and a function $u(x,t)$ in H for $-\infty < x < \infty$, $0 < t < T$ with $u(x,T) = h(x)$? In effect then, $h(x)$ is to be the result of a cooling process from some temperature $u(x,0)$. Using heat polynomials, we will essentially obtain the solution found by Blackman [2] and Oseen [6].

THEOREM 4.1. *Let* $h(z)$ *be an entire function for which for some* $\alpha > 0$,

$$|h(x+iy)| \leqq K(\beta)\, e^{\alpha y^2}$$

for $-\infty < y < \infty$, *and* $|x| \leqq \beta$ *for any* $\beta > 0$. *Then there exists a* $T > 0$ *and a function* $u(x,t)$ *in* H *in the strip* $0 < t < T$ *with* $u(x,T) = h(x)$.

Proof. Set $h(x+iy) = h(2i\sqrt{T}w) = g(w)$, $T > 0$ to be chosen and $w = (1/2\sqrt{T})(y - ix) = \sigma + iv$. Then

$$|g(w)| \leqq K(\beta) \exp[4\alpha T \sigma^2].$$

Now choose T so that $4\alpha T < 1/2$, and we apply a result of Hille [4] on the expansion of functions in Hermite polynomials in the complex plane. This theorem asserts the conclusion that for any w in the complex plane,

$$g(w) = \sum_{n=0}^{\infty} a_n H_n(w), \qquad 2^n n! a_n = \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-t^2} H_n(t) g(t)\,dt,$$

and for every $\beta > 0$, there is a constant $A(\beta)$ so that

$$|a_n| \leqq A(\beta)(2^n n!)^{-1/2} e^{-\beta\sqrt{n}}. \tag{4.1}$$

Letting $y = 0$, we may write

$$h(x) = \sum_{n=0}^{\infty} a_n H_n\left(\frac{-ix}{2\sqrt{T}}\right) = \sum_{n=0}^{\infty} a_n(-T)^{-n/2} v_n(x, T).$$

We now form the function

$$u(x, t) = \sum_{n=0}^{\infty} a_n(-T)^{-n/2} v_n(x, t). \tag{4.2}$$

Clearly $u(x, T) = h(x)$. From the inequality [7; p. 226],

$$|v_n(x, t)| \leqq \left(1 + \frac{t}{\delta}\right)^{1/2} [2n(t + \delta)]^{n/2} e^{-n/2} e^{x^2/4\delta}$$

holding for $0 \leqq t < \infty$, $-\infty < x < \infty$, and any $\delta > 0$, we conclude that the series in (4.2) is dominated by

$$K\left(1 + \frac{t}{\delta}\right)^{1/2} e^{x^2/4\delta} \sum_{n=0}^{\infty} \left(\frac{(t + \delta)}{T}\right)^{n/2} e^{-\beta\sqrt{n}}. \tag{4.3}$$

Since $\delta > 0$ is arbitrary, it follows that the series in (4.2) converges for $0 \leqq t < T$. A similar argument using the inequality in [7; p. 227] for $v_n(x, t)$ with $t < 0$ shows that the series converges also for $-T < t < 0$. We conclude [7; p. 233] that $u(x, t)$ is in H in the strip $|t| < T$ and moreover $u(x, T) = h(x)$. The function $u(x, t)$ in (4.2) is thus a solution to the problem. It is of interest to observe that $h(x)$ can thus be considered the result of cooling from an initial temperature $u(x, 0)$ which is an entire function (we use here the fact that $v_n(x, 0) = x^n$).

As a second application, we will examine solutions of (1.1) which are analytic in both x and t (see also [9]). It has long been known that any solution of (1.1) in the strip $-\infty < x < \infty$, $0 < t < r$ has the form

$$u(x, t) = \sum_{n=0}^{\infty} g^{(n)}(t)\frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} h^{(n)}(t)\frac{x^{2n+1}}{(2n + 1)!}, \tag{4.4}$$

so that any solution is entire in x and infinitely differentiable (not necessarily analytic) in t. In addition,

$$u(0, t) = g(t), \quad u_x(0, t) = h(t). \tag{4.5}$$

THEOREM 4.2. *Let $u(x, t)$ be a solution of* (1.1) *in the strip $0 < t < r$ with $u(0, t), u_x(0, t)$ analytic for t in $(0, r)$. Then for any t_0 in $(0, r)$,*

$$u(x, t) = \sum_{n=0}^{\infty} g^{(n)}(t_0)\frac{v_{2n}(x, t - t_0)}{(2n)!} + \sum_{n=0}^{\infty} h^{(n)}(t_0)\frac{v_{2n+1}(x, t - t_0)}{(2n + 1)!}, \tag{4.6}$$

holding in the strip $|t - t_0| < a$ for some $a > 0$.

Proof. We use the representation

$$v_n(x, t) = n! \sum_{2k \leqq n} \frac{x^{n-2k}}{(n - 2k)!} \frac{t^k}{k!},$$

which comes from (1.3). Now the first series in (4.4) becomes

$$u_1(x,t) = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} g^{(k)}(t_0)\frac{(t-t_0)^{k-n}}{(k-n)!}\frac{x^{2n}}{(2n)!}.$$

The convergence of this double series is a consequence of the fact that $|g^{(k)}(t_0)| \leqq Mk!/a^k$ for some constants M and a with $a > 0$. Also absolute convergence holds for $-\infty < x < \infty$, $|t - t_0| < a$. Then

$$\begin{aligned} u_1(x,t) &= \sum_{k=0}^{\infty} g^{(k)}(t_0)\sum_{n=0}^{k}\frac{x^{2n}}{(2n)!}\frac{(t-t_0)^{k-n}}{(k-n)!} \\ &= \sum_{k=0}^{\infty} g^{(k)}(t_0)\frac{v_{2k}(x,t-t_0)}{(2k)!}, \end{aligned}$$

and this is the first series in the theorem. The second follows in the same way. This proves the theorem.

It is a consequence of [7; p. 250] that the conclusion of Theorem 4.2 implies that $u(x, t)$ satisfies a *Huygens principle* expressed by

$$(4.7) \qquad u(x,t_2) = k(x,t_2-t_1) * u(x,t_1) = \int_{-\infty}^{+\infty} k(x-y,t_2-t_1)u(y,t_1)\,dy$$

for any $-\infty < x < \infty$, $t_0 - a < t_1 < t_2 < t_0 + a$. This relationship merely expresses the desirable physical property that the temperature $u(x, t_2)$ can be obtained from a knowledge of $u(x, t)$ at a prior time $t = t_1$. A partial converse is also true. If (4.7) holds in the stated interval, then the same holds for $u(x, t + t_0)$ in the interval $|t| < a$ and hence, from [7; p. 250],

$$u(x,t+t_0) = \sum_{n=0}^{\infty} a_n v_n(x,t)$$

for $-\infty < x < \infty$, $|t| < a$, or equivalently

$$u(x,t) = \sum_{n=0}^{\infty} a_n v_n(x,t-t_0)$$

for $-\infty < x < \infty$, $|t - t_0| < a$. Moreover,

$$u(0,t) = \sum_{n=0}^{\infty} a_n v_n(0,t-t_0) = \sum_{n=0}^{\infty} a_n\frac{(2n)!}{n!}(t-t_0)^n,$$

so that $u(0, t)$ is analytic at $t = t_0$. Similarly $u_x(0, t)$ is analytic at $t = t_0$. We have thus proven the following result.

THEOREM 4.3. *Let $u(x, t)$ be in H in the strip $0 < t < r$. A necessary and sufficient condition that $u(0, t)$, $u_x(0, t)$ be analytic at $t = t_0$ is that $u(x, t)$ satisfy the Huygens principle expressed by* (4.7) *in some strip $|t - t_0| < a$, $a > 0$.*

Unfortunately we may not in general extend this result to having $u(0, t)$, $u_x(0, t)$ analytic in $(0, r)$ and have at the same time $u(x, t)$ satisfying (4.7) in the strip $0 < t < r$. Rosenbloom and Widder show that the function

$$u(x,t) = k(x,t+a)$$

with $(4a)^{-1} = 1 + i$ is in H for $-\infty < x < \infty$, $-\infty < t < \infty$. Moreover it is clear that, since a is not on the real axis, $u(0, t)$ and $u_x(0, t)$ are analytic for each t in $(-\infty, \infty)$. However $u(x, t)$ does not satisfy the Huygens principle for $-\infty < x < \infty$, $-\infty < t < \infty$, [7; p. 242].

Actually the above argument shows that the Huygens principle in the strip $0 < t < r$ implies the analyticity of $u(0, t)$, $u_x(0, t)$ for the same t. It is the converse of this which is not true in general.

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AN EXPANSION IN ULTRASPHERICAL POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS*

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Abstract. It is shown that the following expansion has nonnegative coefficients:

$$x^p \frac{P_k^{(n,p)}(2x^2 - 1)}{P_k^{(n,p)}(1)} = \sum_{s=0}^{p+2k} \alpha_s \frac{C_s^{(m/2)-1}(x)}{C_s^{(m/2)-1}(1)}$$

for $n, m, p = 0, 1, 2, \cdots$ and $2 \leqq m \leqq n + 2$. The proof involves harmonic analysis on the unitary group.

By geometric reasoning, we show that the following expansion has nonnegative coefficients:

$$x^p \frac{P_k^{(n,p)}(2x^2 - 1)}{P_k^{(n,p)}(1)} = \sum_{s=0}^{p+2k} \alpha_s \frac{C_s^{(m/2)-1}(x)}{C_s^{(m/2)-1}(1)}$$

for $n, m, p = 0, 1, 2, \cdots$ and $2 \leqq m \leqq n + 2$. We will use the notation of Szegö [7] for Jacobi and ultraspherical polynomials, and the notation of [4] for harmonic analysis on compact groups.

The idea is this: if H is a closed subgroup of a compact group G, and f is a continuous positive definite function on G, then $f|H$ is a positive definite function on H. Further, if f is bi-invariant for some closed subgroup K of G, then $f|H$ is bi-invariant for $K \cap H$. If $H/K \cap H$ is multiplicity-free, then any bi-invariant positive-definite function on H is a nonnegative linear combination of the spherical functions of $H/K \cap H$ (see [4, p. 105]).

Let $U(n)$ be the unitary group on $C^n, n \geqq 2$. A typical element is an $n \times n$ unitary matrix $u = (u_{ij})_{i,j=1}^n$. Let K be the subgroup $\{u \in U(n) : u_{11} = 1\} \simeq U(n-1)$. For the irreducible representation of $U(n)$ with highest weights $(j, 0, \cdots, 0, -k), j, k \geqq 0$, the spherical function for $U(n)/K$ is

$$\phi_{jk}(u) = \begin{cases} [P_k^{(n-2,j-k)}(1)]^{-1} P_k^{(n-2,j-k)}(2|u_{11}|^2 - 1) u_{11}{}^{j-k} & \text{for } j \geqq k, \\ [P_j^{(n-2,k-j)}(1)]^{-1} P_j^{(n-2,k-j)}(2|u_{11}|^2 - 1) \overline{u_{11}}{}^{k-j} & \text{for } j < k. \end{cases}$$

(Note: to verify that these are the spherical functions, it is enough to check their homogeneity and harmonicity properties; see Boyd [3], [4, Chap. 10], or Ikeda [5].)

For the subgroup H we will take $SO(m)$ (the rotation group on R^m) with $m \leqq n$, embedded into $U(n)$ by (symbolically)

$$H = \begin{bmatrix} SO(m) & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

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We see that $H \cap K \simeq SO(m-1)$. The spherical functions for $SO(m)/SO(m-1)$ are well known (see for example [4, p. 109]) to be

$$\psi_s(g) = [C_s^{(m/2)-1}(1)]^{-1} C_s^{(m/2)-1}(g_{11}),$$

$g \in SO(m)$, $s = 0, 1, 2, \cdots$. (For $m = 2$ the limiting case $\psi_s(g) = T_s(g_{11}) = \cos s\theta$, where $\cos\theta = g_{11}$, is not spherical for $s > 0$ since it decomposes into $e^{is\theta}$ and $e^{-is\theta}$, but in our application the cosine series suffices.)

We now restrict ϕ_{jk} to $H \simeq SO(m)$ to obtain

$$g_{11}^{j-k} \frac{P_k^{(n-2,j-k)}(2g_{11}^2 - 1)}{P_k^{(n-2,j-k)}(1)} = \sum_{s=0}^{j+k} \alpha_s \frac{C_s^{(m/2)-1}(g_{11})}{C_s^{(m/2)-1}(1)}$$

with $\alpha_s \geqq 0$, by our previous remarks. Replace g_{11} by x, n by $n+2$, j by $k+p$ to obtain the previously stated expansion.

Schoenberg [6] and Askey [1] have previously used the restriction idea in similar situations. See Askey's survey paper [2, pp. 64–85] for more examples of expansions with nonnegative coefficients.

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CERTAIN RATIONAL FUNCTIONS WHOSE POWER SERIES HAVE POSITIVE COEFFICIENTS. II*

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Abstract. Dunkl's recent expression of a certain Jacobi polynomial times a simple polynomial as a sum of ultraspherical polynomials with nonnegative coefficients is translated into a result between Jacobi polynomials of the same argument and then applied to prove that

$$\frac{1}{\{(1-r)(1-s)(1-t)[(1-r)(1-s)(1+t)+(1-r)(1+s)(1-t)+(1+r)(1-s)(1-t)]\}^{(\alpha+1)/2}}$$

has nonnegative power series coefficients for $\alpha = 0, 1, 2, \cdots$.

1. Introduction. A fascinating class of problems was initiated when Friedrichs and Lewy conjectured

$$\frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)} = \sum_{k,m,n=0}^{\infty} A_{k,m,n} r^k s^m t^n, \tag{1.1}$$

$$A_{k,m,n} > 0, \quad k, m, n = 0, 1, \cdots,$$

and Szegö not only proved (1.1), but extended it to the following.

THEOREM A. *Let* $f(x) = (x - x_1)\cdots(x - x_k)$. *If* $\alpha \geqq -\frac{1}{2}$, *then*

$$\frac{1}{[f'(1)]^{\alpha+1}} = \sum A^{\alpha}_{n_1,\cdots,n_k} x_1^{n_1} \cdots x_k^{n_k}, \tag{1.2}$$

with $A^{\alpha}_{n_1,\cdots,n_k} \geqq 0$.

Szegö's proof used Bessel functions; another proof which uses Jacobi and Laguerre polynomials was given by Askey and Gasper [3]. The next result of this type was

$$\frac{1}{[(1-r)(1-s)(2+t)+(1-r)(2+s)(1-t)+(2+r)(1-s)(1-t)]^{\alpha+1}}$$

$$= \sum_{k,m,n=0}^{\infty} B^{\alpha}(k,m,n) r^k s^m t^n, \tag{1.3}$$

with $B^{\alpha}(k, m, n) > 0$ when $\alpha \geqq [-5 + (17)^{1/2}]/2 = \alpha_0$ unless $\alpha = \alpha_0$ and $k = m = n = 1$, when the coefficient is zero. See [4].

Equation (1.3) was motivated by consideration of Laguerre polynomials, since the coefficients B^{α} are a positive multiple of

$$\int_0^{\infty} L_k^{\alpha}(x) L_m^{\alpha}(x) L_n^{\alpha}(x) x^{\alpha} e^{-2x}\, dx, \tag{1.4}$$

and the coefficients in (1.2) are a positive multiple of

$$\int_0^{\infty} L_k^{\alpha}(x) L_m^{\alpha}(x) L_n^{\alpha}(x) x^{\alpha} e^{-3x}\, dx \tag{1.5}$$

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when $f(x)$ has three roots. From this point of view, it is easy to see that (1.3) is much deeper than (1.2), a fact which is far from clear when just these two expansions are considered. A third result can now be obtained, in as yet an imperfect form.

THEOREM 1. *If* $\alpha = 0, 1, 2, \cdots$, *then*

$$\frac{1}{\{(1-r)(1-s)(1-t)[(1-r)(1-s)(1+t)+(1-r)(1+s)(1-t)+(1+r)(1-s)(1-t)]\}^{(\alpha+1)/2}} = \sum C^{\alpha}(k,m,n) r^k s^m t^n,$$

with $C^{\alpha}(k, m, n) \geqq 0$.

Theorem 1 is probably true for $\alpha \geqq 0$, but the proof rests on an interesting recent result of Dunkl about Jacobi polynomials [5], and his proof is group theoretic and thus only valid for integer values of the parameters.

2. Dunkl's expansion. The Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$, can be defined by

$$(2.1)\qquad (1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}], \qquad \alpha, \beta > -1.$$

Gegenbauer polynomials, $C_n^{\lambda}(x)$, are connected with Jacobi polynomials by

$$(2.2)\qquad \frac{C_n^{\lambda}(x)}{C_n^{\lambda}(1)} = \frac{P_n^{(\lambda-1/2,\lambda-1/2)}(x)}{P_n^{(\lambda-1/2,\lambda-1/2)}(1)},$$

where

$$(2.3)\qquad C_n^{\lambda}(1) = (2\lambda)_n/n!,$$

$$(2.4)\qquad P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!,$$

and $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a)$. There is a second connection given by

$$(2.5)\qquad \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-1/2)}(2x^2-1)}{P_n^{(\alpha,-1/2)}(1)},$$

$$(2.6)\qquad \frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,1/2)}(2x^2-1)}{P_n^{(\alpha,1/2)}(1)}.$$

See Szegö [8] for all of the results on orthogonal polynomials which are stated without reference. Dunkl [5] proved

$$(2.7)\qquad x^{\beta}\frac{P_n^{(\alpha,\beta)}(2x^2-1)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{2n+\beta} \alpha_k \frac{C_k^{(m/2)-1}(x)}{C_k^{(m/2)-1}(1)},$$

with $\alpha_k \geqq 0$ when $\alpha, \beta, m = 0, 1, \cdots$, and $2 \leqq m \leqq \alpha + 2$. The usual convention

$$C_n^0(\cos\theta)/C_n^0(1) = \cos n\theta$$

is taken, even though $C_n^0(1)$ and $C_n^0(\cos\theta)$ are zero, $n = 1, 2, \cdots$. The essential case in (2.7) is $m = \alpha + 2$; the others follow from this case and Gegenbauer's formula

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \sum_{k=0}^{n} \alpha_{k,n} \frac{C_k^\mu(x)}{C_k^\mu(1)}, \tag{2.8}$$

$$\alpha_{k,n} \geqq 0 \quad \text{when } \lambda > \mu \geqq 0 \quad [1].$$

Use of $m = \alpha + 2$ and (2.2) in (2.7) gives

$$x^\beta \frac{P_n^{(\alpha,\beta)}(2x^2 - 1)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{2n+\beta} \alpha_k \frac{P_k^{((\alpha-1)/2,(\alpha-1)/2)}(x)}{P_k^{((\alpha-1)/2,(\alpha-1)/2)}(1)}. \tag{2.9}$$

Consider separately the cases in which β is even and β is odd. When β is even, (2.5) and the change of variables $2x^2 - 1 = y$ gives

$$\left(\frac{1+y}{2}\right)^j \frac{P_n^{(\alpha,2j)}(y)}{P_n^{(\alpha,2j)}(1)} = \sum_{k=0}^{n+j} \alpha_{2k} \frac{P_k^{((\alpha-1)/2,-1/2)}(y)}{P_k^{((\alpha-1)/2,-1/2)}(1)}, \tag{2.10}$$

$$\alpha, j = 0, 1, \cdots, \quad \alpha_{2k} \geqq 0, \quad k = 0, 1, \cdots, n + j.$$

When β is odd, (2.6) and a simple calculation gives

$$\left(\frac{1+y}{2}\right)^j \frac{P_n^{(\alpha,2j+1)}(y)}{P_n^{(\alpha,2j+1)}(1)} = \sum_{k=0}^{n+j} \alpha_{2k+1} \frac{P_k^{((\alpha-1)/2,1/2)}(y)}{P_k^{((\alpha-1)/2,1/2)}(1)}, \tag{2.11}$$

$$\alpha, j = 0, 1, \cdots, \quad \alpha_{2k+1} \geqq 0, \quad k = 0, 1, \cdots, n + j.$$

These results are new for $j > 0$; for $j = 0$, (2.10) was proven in [1] and (2.11) was proven in [2].

3. Power series with nonnegative coefficients. Dunkl's result in the form (2.10) leads to a new nonnegative integral of Jacobi polynomials.

THEOREM 2. *If* $\alpha, j = 0, 1, \cdots$, *then*

$$\int_{-1}^{1} P_n^{(\alpha,2j)}(y) P_m^{(\alpha,2j)}(y) P_k^{(\alpha,2j)}(y)(1 + y)^{3j}(1 - y)^{(\alpha-1)/2}(1 + y)^{-1/2}\, dy \geqq 0, \tag{3.1}$$

$$k, m, n = 0, 1, \cdots.$$

Proof. Multiply (2.10) with n replaced by n, m, k, and use

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x)(1 - x)^\alpha(1 + x)^\beta\, dx \geqq 0, \tag{3.2}$$

$$\alpha \geqq \beta \geqq -\tfrac{1}{2} \quad [6],$$

and Dunkl's result that the coefficients in (2.10) are nonnegative.

Let $y = 1 - (z/j)$, change variables in (3.1), and use

$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - (2x/\beta)) = L_n^\alpha(x), \tag{3.3}$$

to obtain

$$\int_0^\infty L_n^\alpha(z)L_m^\alpha(z)L_k^\alpha(z)z^{(\alpha-1)/2}\,e^{-3z/2}\,dz \geqq 0, \tag{3.4}$$
$$\alpha = 0, 1, \cdots, \quad k, m, n = 0, 1, \cdots.$$

The details of a similar argument are given in [3, § 5].

The polynomials $L_n^\alpha(x)$ are orthogonal,

$$\int_0^\infty L_n^\alpha(x)L_m^\alpha(x)x^\alpha\,e^{-x}\,dx = 0, \qquad m \neq n, \tag{3.5}$$

$$\int_0^\infty [L_n^\alpha(x)]^2x^\alpha\,e^{-x}\,dx = \Gamma(n+\alpha+1)/\Gamma(n+1), \tag{3.6}$$

and can be obtained from the generating function

$$\frac{e^{-xr/(1-r)}}{(1-r)^{\alpha+1}} = \sum_{n=0}^\infty L_n^\alpha(x)r^n. \tag{3.7}$$

Multiply (3.4) by $r^ns^mt^k$, sum, and then integrate using

$$\int_0^\infty t^{\beta-1}\,e^{-ct}\,dt = \Gamma(\beta)c^{-\beta} \tag{3.8}$$

to obtain

$$\frac{\Gamma((\alpha+1)/2)}{\left\{(1-r)^2(1-s)^2(1-t)^2\left[\frac{r}{1-r}+\frac{s}{1-s}+\frac{t}{1-t}+\frac{3}{2}\right]\right\}^{(\alpha+1)/2}} = \sum_{k,m,n=0}^\infty \int_0^\infty L_n^\alpha(z)L_m^\alpha(z)L_k^\alpha(z)z^{(\alpha-1)/2}\,e^{-3z/2}\,dz\,r^ns^mt^k; \tag{3.9}$$

so these coefficients are nonnegative for $\alpha = 0, 1, \cdots$. The left-hand side can be simplified to

$$\frac{\Gamma((\alpha+1)/2)2^{(\alpha+1)/2}}{\{(1-r)(1-s)(1-t)[(1-r)(1-s)(1+t)+(1-r)(1+s)(1-t)+(1+r)(1-s)(1-t)]\}^{(\alpha+1)/2}}. \tag{3.10}$$

This proves Theorem 1, which was stated in the Introduction.

If an analytic proof of Dunkl's result could be found, it would probably extend to $\alpha \geqq 0$, α real, and not just $\alpha = 0, 1, \cdots$. This was shown in [2] in the case $j = 0$. If Dunkl's result could be proven for all real $\alpha \geqq 0$, then Theorem 1 would also be true for all $\alpha \geqq 0$.

Using the orthogonality of $L_n^\alpha(x)$, we see that

$$x^{-(\alpha+1)/2}L_n^\alpha(x)L_m^\alpha(x) \sim \sum_{k=0}^\infty \beta_kL_k^\alpha(x), \quad \beta_k \geqq 0, \quad \alpha = 0, 1, \cdots,$$

or

$$\varphi_n^\alpha(x)\varphi_m^\alpha(x) \sim \sum_{k=0}^\infty \beta_k\varphi_k^\alpha(x), \qquad \varphi_k^\alpha(x) = x^{-(\alpha+1)/2}L_k^\alpha(x). \tag{3.11}$$

Statement (3.11) can be iterated, but it doesn't lead to anything useful. However, (3.2) could have been iterated to obtain

$$\int_{-1}^{1} P_{n_1}^{(\alpha,\beta)}(x) \cdots P_{n_k}^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx \geqq 0, \qquad \alpha \geqq \beta \geqq -\tfrac{1}{2}, \tag{3.12}$$

as was pointed out in [3], and this leads to

$$\int_{0}^{\infty} L_{n_1}^{\alpha}(x) \cdots L_{n_k}^{\alpha}(x)x^{(\alpha-1)/2}\,e^{-kx/2}\,dx \geqq 0, \qquad \alpha = 0, 1, \cdots. \tag{3.13}$$

This in turn leads to a generalization to k variables of Theorem 1, which will be left to the reader.

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SOME ABSOLUTELY MONOTONIC AND COMPLETELY MONOTONIC FUNCTIONS*

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Abstract. The functions $(1 - r)^{-2|\lambda|}(1 - 2xr + r^2)^{-\lambda}$ are shown to be absolutely monotonic, or equivalently, that their power series have nonnegative coefficients for $-1 \leqq x \leqq 1$. One consequence is a simple proof of Kogbetliantz's theorem on positive Cesàro summability for ultraspherical series, [7].

1. Ultraspherical polynomials and absolute monotonicity. Certain power series occur so often that their coefficients acquire names. One such is

$$(1.1) \qquad \frac{1}{(1 - 2xr + r^2)^\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) r^n,$$

where $C_n^\lambda(x)$ is a polynomial of degree n. These polynomials are either called Gegenbauer polynomials, since Gegenbauer discovered most of their deep properties, or ultraspherical polynomials, because they can be used to construct the spherical harmonics on spheres of arbitrary dimension. For $\lambda > -\frac{1}{2}$, $C_n^\lambda(x)$ are orthogonal:

$$\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x)(1 - x^2)^{\lambda - 1/2}\, dx = 0, \qquad m \neq n,$$

$$\int_{-1}^{1} [C_n^\lambda(x)]^2 (1 - x^2)^{\lambda - 1/2}\, dx = \frac{\pi^{1/2}(2\lambda)_n \Gamma(\lambda + \frac{1}{2})}{(n + \lambda) n! \Gamma(\lambda)} = h_n^\lambda,$$

with $(a)_n = \Gamma(n + a)/\Gamma(a) = a(a + 1) \cdots (a + n - 1)$. When $\lambda = 0$ we will substitute

$$\lim_{\lambda \to 0} \frac{n + \lambda}{2\lambda} C_n^\lambda(\cos\theta) = \begin{cases} \frac{1}{2}, & n = 0, \\ \cos n\theta, & n = 1, 2, \cdots. \end{cases}$$

In this section we shall always assume $-1 \leqq x \leqq 1$, and will not repeat this condition except in the statement of Theorem 1.

With the formal expansion

$$(1.2) \qquad \begin{aligned} f(x) &\sim \sum_{n=0}^{\infty} a_n C_n^\lambda(x), \\ a_n &= \frac{1}{h_n^\lambda} \int_{-1}^{1} f(x)(1 - x^2)^{\lambda - 1/2}\, dx, \end{aligned}$$

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we associate the Poisson sum

$$f_r(x) = \sum_{n=0}^{\infty} a_n r^n C_n^\lambda(x) = \int_{-1}^{1} f(y) P_r(x, y)(1 - y^2)^{\lambda - 1/2}\, dy,$$

where

$$P_r(x, y) = \sum_{n=0}^{\infty} r^n C_n^\lambda(x) C_n^\lambda(y)/h_n^\lambda.$$

The integrated form of the addition formula,

$$C_n^\lambda(\cos\theta) C_n^\lambda(\cos\varphi) = \frac{\int_0^\pi C_n^\lambda(\cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\chi) C_n^\lambda(1)(\sin\chi)^{2\lambda-1}\, d\chi}{\int_0^\pi (\sin\chi)^{2\lambda-1}\, d\chi}, \qquad \lambda > 0,$$

[2, 3.15.1 (20)] gives

$$f_r(\cos\theta) \tag{1.3}$$
$$= \frac{\int_0^\pi \int_0^\pi f(\cos\varphi) P_r(\cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\chi, 1)(\sin\varphi)^{2\lambda}(\sin\chi)^{2\lambda-1}\, d\varphi\, d\chi}{\int_0^\pi (\sin\chi)^{2\lambda-1}\, d\chi}.$$

Since

$$P_r(x, 1) = \frac{\lambda(1 - r^2)}{(1 - 2xr + r^2)^{\lambda+1}}, \qquad \lambda > 0, \tag{1.4}$$

it is clear that the operator $T_r f(x) = f_r(x)$ is a positive operator (see [1, (2.5)] and references given there). For $\lambda = 0$ and $\lambda = \frac{1}{2}$, Fejér [3], [4] proved the deeper result that certain Cesàro means, $(C, 1)$ for $\lambda = 0$ and $(C, 2)$ for $\lambda = \frac{1}{2}$, were already positive. Later Kogbetliantz [7] proved that the $(C, 2\lambda + 1)$ means are positive, but this result has not become well known. Szegö only mentioned this result in passing in [10], and he clearly would have included it if a simple proof had existed, since he mentions that the result has important consequences. One of these consequences is an analogue of the Pólya–Fejér theorem that the Fourier transform of an even convex function (or cosine series with convex coefficients) is non-negative. This will be given in a later paper.

Before giving an elementary proof of Kogbetliantz's result let us recall the definition of the Cesàro means of order γ. For the formal series $\sum_{n=0}^{\infty} a_n$ define

$$\sigma_n^\gamma = \frac{n!}{(\gamma + 1)_n} \sum_{k=0}^{n} \frac{(\gamma + 1)_{n-k}}{(n - k)!} a_k,$$

and call $\{\sigma_n^\gamma\}$ the Cesàro means of order γ. We shall only be concerned with positivity, and so may instead consider

$$\tau_n^\gamma = \sum_{k=0}^{n} \frac{(\gamma + 1)_{n-k}}{(n - k)!} a_k.$$

The τ_n^γ may be defined by the following alternative procedure: Set

$$f(r) = \sum_{n=0}^{\infty} a_n r^n$$

and consider

$$(1-r)^{-\gamma-1}f(r)=\sum_{n=0}^{\infty}\frac{(\gamma+1)_n r^n}{n!}\sum_{n=0}^{\infty}a_n r^n=\sum_{n=0}^{\infty}\tau_n^{\gamma}r^n.$$

Thus to prove Kogbetliantz's theorem that the $(C, 2\lambda+1)$ means of (1.2) are positive if $f(x)\geqq 0$, it is sufficient to prove that

$$g_\lambda(r)=\frac{1-r^2}{(1-r)^{2\lambda+2}(1-2xr+r^2)^{\lambda+1}} \tag{1.5}$$

has nonnegative power series coefficients, or to phrase it another way, $g_\lambda(r)$ is an absolutely monotonic function.

Since

$$g_\lambda(r)=g_0(r)\cdot\frac{1}{(1-r)^{2\lambda}(1-2xr+r^2)^{\lambda}},$$

the product of absolutely monotonic functions is absolutely monotonic, and $g_0(r)$ is absolutely monotonic because of Fejér's theorem on the positivity of the $(C, 1)$ means for Fourier series, it is sufficient to prove that the second factor is absolutely monotonic.

THEOREM 1. *For* $-1\leqq x=\cos\theta\leqq 1$, *and* $\lambda>0$, *the functions* $(1-r)^{-2\lambda}\cdot(1-2xr+r^2)^{-\lambda}$ *are absolutely monotonic.*

Proof. Let

$$\begin{aligned} h(r)&=\log\,[(1-r)^{-2}(1-2xr+r^2)^{-1}]\\ &=-2\log(1-r)-\log(1-2r\cos\theta+r^2).\end{aligned}$$

Then

$$\begin{aligned} h'(r)&=\frac{2}{1-r}+\frac{2\cos\theta-2r}{1-2r\cos\theta+r^2}=\frac{2}{1-r}+\frac{e^{-i\theta}}{1-r\,e^{-i\theta}}+\frac{e^{i\theta}}{1-r\,e^{i\theta}}\\ &=\sum_{n=0}^{\infty}[2+e^{i(n+1)\theta}+e^{-i(n+1)\theta}]r^n\\ &=\sum_{n=0}^{\infty}[2+2\cos(n+1)\theta]r^n.\end{aligned}$$

Therefore $h'(r)$ is absolutely monotonic, and since $h(0)=0$, so is $h(r)$. But

$$\frac{1}{(1-r)^{2\lambda}(1-2xr+r^2)^{\lambda}}=e^{\lambda h(r)}=\sum_{n=0}^{\infty}\frac{\lambda^n[h(r)]^n}{n!},$$

and this completes the proof.

COROLLARY 1 (Fejér [5]). $(1-r)^{-1}(1-2xr+r^2)^{-\lambda}$ *has positive power series coefficients for* $0<\lambda\leqq\frac{1}{2}$, $-1<x\leqq 1$.

Proof. Use Theorem 1 and $(1-r)^{-1}=(1-r)^{-1+2\lambda}(1-r)^{-2\lambda}$.

In addition to Corollary 1, Fejér [5] obtained the same conclusion for $\lambda=-\frac{1}{2}$, and Szegö [9] extended this result to $-\frac{1}{2}<\lambda<0$. A stronger theorem follows.

THEOREM 2. *For* $-1\leqq x\leqq 1$ *and* $\lambda<0$, *the functions* $(1-r)^{-2|\lambda|}(1-2xr+r^2)^{-\lambda}$ *are absolutely monotonic.*

The only change in the proof from the proof of Theorem 1 is that $h'(r)$ is now

$$h'(r) = \sum_{n=0}^{\infty} [2 - 2\cos(n+1)\theta] r^n.$$

COROLLARY 2 (Szegö [9]). $(1-r)^{-1}(1-2xr+r^2)^{\lambda}$ *has positive power series coefficients for* $0 < \lambda \leqq \frac{1}{2}$, $-1 \leqq x \leqq 1$.

A further application of Theorem 1 to Jacobi series is given in [1].

Both Theorems 1 and 2 are best possible, in the sense that the averaging factor $(1-r)^{-2|\lambda|}$ cannot be replaced by $(1-r)^{-k}$ for $k < 2|\lambda|$ and still retain nonnegative coefficients for $-1 \leqq x \leqq 1$. To see this let $x = -1$ in Theorem 1 and $x = 1$ in Theorem 2 and observe that the coefficient of r is negative for $k < 2|\lambda|$.

2. Hankel transforms and complete monotonicity. The results in § 1 suggest the following analogue.

THEOREM 3. *If λ is real, then the functions* $x^{-2|\lambda|}(x^2+1)^{-\lambda}$ *are completely monotonic functions for* $x > 0$, *that is*,

$$\frac{1}{x^{2|\lambda|}(x^2+1)^{\lambda}} = \int_0^{\infty} e^{-xt}\, d\mu_{\lambda}(t), \qquad x > 0, \quad d\mu_{\lambda}(t) \geqq 0.$$

Proof. Theorem 3 follows easily from the following theorem of Schoenberg.

THEOREM A. *A function $f(x)$, $x \geqq 0$, with $f(0) = 1$ has the property that $[f(x)]^{\lambda}$ is completely monotonic for $x \geqq 0$ and all $\lambda > 0$ if and only if*

$$f(x) = \exp\left\{-\int_0^x g(t)\, dt\right\},$$

where $g(t)$ is a completely monotonic function.

For let

$$f_{\varepsilon}(x) = \frac{\varepsilon^{2|\lambda|}(\varepsilon^2+1)^{\lambda}}{(x+\varepsilon)^{2|\lambda|}[(x+\varepsilon)^2+1]^{\lambda}} = \exp\left\{-|\lambda|\int_0^x g_{\varepsilon}(t)\, dt\right\}$$

so that

$$g_{\varepsilon}(x) = 2\int_0^{\infty} e^{-(x+\varepsilon)t}\left[1 + \frac{|\lambda|}{\lambda}\cos t\right] dt, \qquad x > -\varepsilon,$$

which is clearly completely monotonic. Thus

$$\frac{f_{\varepsilon}(x-\varepsilon)}{\varepsilon^{2|\lambda|}(\varepsilon^2+1)^{\lambda}} = \frac{1}{x^{2|\lambda|}(x^2+1)^{\lambda}}$$

is completely monotonic for $x > \varepsilon$ for each $\varepsilon > 0$, and so for $x > 0$.

An elementary direct proof can also be given. By the Hausdorff–Bernstein–Widder theorem [11] it is sufficient to show that

$$(-1)^n \frac{d^n}{dx^n}[x^2(x^2+1)^{\pm 1}]^{-\lambda} \geqq 0, \qquad x > 0, \quad \lambda > 0. \tag{2.1}$$

(2.1) is obvious for $n = 0$. Assume that it holds for $1, 2, \cdots, n$, and let $f(x) = x^2(x^2 + 1)^{\pm 1}$. Then

$$\begin{aligned}(-1)^{n+1}\frac{d^{n+1}}{dx^{n+1}}[f(x)]^{-\lambda} &= (-1)^n\lambda\frac{d^n}{dx^n}\left[[f(x)]^{-\lambda}\frac{f'(x)}{f(x)}\right] \\ &= (-1)^n\lambda\sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}[f(x)]^{-\lambda}\frac{d^k}{dx^k}\left[\frac{f'(x)}{f(x)}\right] \\ &= \lambda\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{d^{n-k}}{dx^{n-k}}[f(x)]^{-\lambda}(-1)^k\frac{d^k}{dx^k}\left[\frac{f'(x)}{f(x)}\right].\end{aligned}$$

The first factor in the sum is positive by the hypothesis of induction, so it is sufficient to prove

$$(-1)^k\frac{d^k}{dx^k}\left[\frac{f'(x)}{f(x)}\right] \geqq 0,$$

or that $f'(x)/f(x)$ is completely monotonic. But

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\log f(x) = \frac{2}{x} \pm \frac{2x}{x^2+1} = 2\int_0^\infty e^{-xt}[1 \pm \cos t]\,dt.$$

The necessity of Schoenberg's condition can be used to show that $x^{-k|\lambda|}(x^2 + 1)^{-\lambda}$ is not completely monotonic for all $\lambda > 0$ if $k < 2$. However this is not the whole story, since it is not unlikely that $x^{-\lambda}(x^2 + 1)^{-\lambda}$ is completely monotonic for $\lambda \geqq 1$. Results of this type, which only hold for λ with a positive lower bound, are harder to prove, since there is no analogue of Schoenberg's theorem in this case.

The most promising method seems to be the use of various asymptotic techniques. Kogbetliantz used Darboux's method to prove that (1.5) is absolutely monotonic, and J. Fields has proved Theorem 3 for $\lambda > 0$ using fairly complicated estimates. It seems likely that Fields' methods can be used to prove the above conjecture, as well as answer the more general question of when $x^{-\mu}(x^2 + 1)^{-\lambda}$ is completely monotonic.

If $\lambda = \alpha + k$, $0 < \alpha \leqq 1$, $k = 0, 1, \cdots$, then $x^{-\mu}(x^2 + 1)^{-\lambda}$ is completely monotonic for $\mu = 2\alpha + k$, $0 < \alpha \leqq \frac{1}{2}$; $\mu = 1 + k$, $\frac{1}{2} \leqq \alpha \leqq 1$. This is Theorem 8 in [1]. The proof uses Theorem 3 of this paper,

$$\int_0^\infty e^{-xt}(1 - \cos t)\,dt = \frac{1}{x(x^2+1)},$$

and the inequality

$$\int_0^x t^\alpha J_\alpha(t)\,dt \geqq 0, \qquad 0 < \alpha \leqq \tfrac{1}{2}.$$

The connection of these problems with Hankel transforms is given in [1].

An even more general problem is suggested by an old result of Hamburger [6], which was brought to our attention by I. J. Schoenberg. He proved

$$\frac{\Gamma(2n+1)}{2^n x(x^2+1)\cdots(x^2+n^2)} = \int_0^\infty e^{-xt}[1 - \cos t]^n\,dt.$$

This raises the question of when

$$f(x) = \frac{1}{x^{c_0}(x^2 + 1)^{c_1} \cdots (x^2 + n^2)^{c_n}}$$

is completely monotonic. The connection with Cesàro summability of Hankel transforms disappears in this more general problem, so the correct state of affairs is far from clear.

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CONVERGENCE AND EVALUATION OF SUMS OF RECIPROCAL POWERS OF EIGENVALUES OF BOUNDARY VALUE PROBLEMS NONLINEAR IN THE EIGENVALUE PARAMETER*

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Abstract. In this paper the trace equations

$$\sum_{i=1}^{\infty} \lambda_i^{-p} = \int_0^1 K_p(x, x)\, dx \tag{1}$$

arising in the Hilbert–Schmidt theory of Fredholm integral equations are extended to integral equations of the form

$$\phi(x) = \lambda \int_0^1 K(x, y, \lambda)\phi(y)\, dy, \tag{2}$$

in which the kernel $K(x, y, \lambda)$ is a rather general function of λ. Attention is focused on equations that correspond to differential boundary value problems via the Green's function. The underlying boundary value problem may be nonlinear in λ in the differential equation, in the boundary conditions, or in both. Three theorems are proved, each of which asserts the convergence of

$$\sum_{i=1}^{\infty} \lambda_i^{-p} \tag{3}$$

for p sufficiently large, under relatively mild hypotheses on the coefficients appearing in the boundary value problem. Finally, a procedure is established whereby (3) can sometimes be evaluated in terms of the Taylor coefficients of a certain function, but without the necessity for the repeated integration implied by (1).

1. Introduction. In the standard theory of homogeneous Fredholm integral equations,

$$\phi(x) = \lambda \int_0^1 K(x, y)\phi(y)\, dy, \tag{1.1}$$

the kernel K is Lebesgue square integrable on $[0, 1] \times [0, 1]$, and is independent of λ. If $\{\lambda_1\}$ is the sequence of eigenvalues of (1.1), then (see Dunford–Schwartz [3, pp. 1033–4])

$$\sum_{i=1}^{\infty} \lambda_i^{-p} = \int_0^1 K_p(x, x)\, dx \tag{1.2}$$

is valid if p is an integer greater than unity and under additional restrictions if $p = 1$ as well. The right-hand side of (1.2) is the pth trace of the kernel K, and K_p denotes the pth iterated kernel. Equation (1.2) has important applications, among which is its usefulness in estimating the lower eigenvalues of (1.1).

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Our purpose is to establish results analogous to (1.2) for a more general class of problems of the form

$$\phi(x) = \lambda \int_0^1 K(x, y, \lambda)\phi(y)\, dy, \tag{1.3}$$

in which the kernel depends on λ. The problems to be considered are derivable from boundary value problems which are nonlinear in λ either in the differential equation, boundary conditions, or both. The kernel $K(x, y, \lambda)$ in (1.3) is related to a Green's function. We will take advantage of this fact in obtaining improved results by requiring certain regularity conditions on the coefficients in the differential operators. We will also establish a relation between the sum $\sum_{i=1}^{\infty} \lambda_i^{-p}$ and the Taylor coefficients of a certain function to be defined later. In some cases this permits evaluation of $\sum_{i=1}^{\infty} \lambda_i^{-p}$ without the calculation of iterated integrals.

Some work has been done on convergence and evaluation of these series. Müller [8] proves the absolute convergence of $\sum_{i=1}^{\infty} \lambda_i^{-p}$ and proves a formula for its evaluation in the case of a very special kernel which is a polynomial in λ. Goodwin [5] provides a generalization of equation (1.1) to kernels nonlinear in λ for the cases $p = 1$ and $p = 2$. Tamarkin [9] extends the Fredholm theory to integral equations nonlinear in λ via the method of infinite determinants of von Koch. Hille and Tamarkin [6] have done extensive work on convergence of the sums $\sum_{i=1}^{\infty} \lambda_i^{-p}$ using infinite determinants; their kernels are independent of λ, and are not in general Green's functions.

In §§ 2–3 we define the class of problems to be considered and review certain background material needed in the rest of the paper. Section 4 contains the theorems concerning the growth of the eigenvalues and the convergence of $\sum_{i=1}^{\infty} \lambda_i^{-p}$. By means of examples, we illustrate the use of these theorems, and show that in a certain sense they are the best obtainable. In § 5 we investigate the relation between the sums $\sum_{i=1}^{\infty} \lambda_i^{-p}$ and the Green's function for a slightly restricted class of problems, thereby generalizing (1.2).

2. Notation and assumptions. We shall consider primarily the eigenvalue problem

$$Mu = N_\lambda u, \tag{2.1}$$

$$U_j(u, \lambda) = \sum_{i=0}^{m-1} [f_{ij}(\lambda)u^{(i)}(0) + g_{ij}(\lambda)u^{(i)}(1)] = 0, \qquad j = 1, \cdots, m, \tag{2.2}$$

where M and N_λ are ordinary differential operators of orders m and n, respectively, with $0 \leqq n \leqq m - 1$. The nonsingular operator M is of the form

$$Mu = \sum_{j=0}^{m} p_{m-j}(x)u^{(j)}(x), \qquad (p_0(x) > 0) \tag{2.3}$$

where p_{m-j} is continuous on $[0, 1]$. Further,

$$N_\lambda u = \sum_{i=0}^{s} \lambda^i N_i u, \tag{2.4}$$

where N_i is a differential operator of order n or less with coefficients continuous for $x \in [0, 1]$ and independent of λ; λ is a complex parameter. We assume that the order of N_i is exactly n for at least one i. We also assume that $N_s \neq 0$ and that $s \geqq 1$ so that N_λ actually does depend on λ. Finally, we assume that the operators U_j form a linearly independent set for each fixed complex λ and that the coefficients f_{ij} and g_{ij} are entire functions of λ.

We will also consider the more general problem

$$Mu = \sigma N_\lambda u, \qquad U_j(u, \lambda) = 0, \tag{2.5}$$

where λ is fixed. Denote by $\{\phi_i(x, \sigma, \lambda)\}$, $i = 1, \cdots, m$, the fundamental set for $Mu = \sigma N_\lambda u$ satisfying

$$\phi_i^{(j-1)}(x, \sigma, \lambda)|_{x=0} = \delta_{ij}, \qquad j = 1, \cdots, m, \tag{2.6}$$

where δ_{ij} is the Kronecker delta. Clearly $\phi_i^{(j-1)}(x, \sigma, \lambda)$ is an entire function of σ for fixed x and λ, an entire function of λ for fixed x and σ, and continuous in (x, σ, λ) together.

Let

$$\Delta(\sigma, \lambda) = \begin{vmatrix} U_1(\phi_1, \lambda) & \cdots & U_1(\phi_m, \lambda) \\ \vdots & & \vdots \\ U_m(\phi_1, \lambda) & \cdots & U_m(\phi_m, \lambda) \end{vmatrix}, \tag{2.7}$$

where $\phi_i = \phi_i(x, \sigma, \lambda)$. We shall assume throughout that $\Delta(0, \lambda)$ and $\Delta(1, \lambda)$ do not vanish identically. In this case, each of these functions has a countable number of zeros accumulating at ∞ only. Note that λ is a zero of $\Delta(1, \lambda)$ if and only if λ is an eigenvalue of (2.1) and (2.2). Let $\lambda_1, \lambda_2, \cdots$ be the enumerated set of all nonzero roots of $\Delta(1, \lambda) = 0$, written according to algebraic multiplicity.

If μ is a zero of $\Delta(0, \lambda)$, then μ is an eigenvalue of

$$Mu = 0, \qquad U_j(u, \lambda) = 0, \tag{2.8}$$

and conversely. Let $\mu_1, \mu_2, \cdots$ be the enumerated set of all nonzero roots of $\Delta(0, \lambda) = 0$, written according to algebraic multiplicity. Note that $\phi_i(x, 0, \lambda)$ is independent of λ; if the f_{ij} and g_{ij} are polynomials of degree not greater than p, then $\Delta(0, \lambda)$ is a polynomial of degree not exceeding pm. If $\sigma = 0$ is not an eigenvalue of (2.5) with λ fixed, then the inhomogeneous problem

$$Mu = f, \qquad U_j(u, \lambda) = 0 \tag{2.9}$$

has a Green's function $M(x, y, \lambda)$. Clearly $M(x, y, \lambda)$ is meromorphic in λ with poles at the eigenvalues μ_i.

In terms of the Green's function $M(x, y, \lambda)$, the original boundary value problem (2.1), (2.2) is equivalent to the integral equation

$$u(x) = \int_0^1 M(x, y, \lambda)(N_\lambda u)(y)\, dy. \tag{2.10}$$

Following Buscham [1], we apply N_λ to both sides of (2.10) and let

$$v(x) = (N_\lambda u)(x), \tag{2.11}$$

thereby obtaining

$$v(x) = \int_0^1 [(N_\lambda)_1 M](x, y, \lambda)v(y)\,dy. \tag{2.12}$$

The subscript on N_λ indicates that it operates on M as a function of its first variable. Equation (2.12) has the same eigenvalues as (2.10) (see Buscham [1]). We will use the notation

$$K(x, y, \lambda) = [(N_\lambda)_1 M](x, y, \lambda) \tag{2.13}$$

to denote the kernel of (2.12); more briefly, we will refer to this function simply as K_λ. The corresponding Fredholm function $\mathscr{D}_{K_\lambda}(\sigma)$ is given by

$$\mathscr{D}_{K_\lambda}(\sigma) = 1 + \sum_{i=1}^{\infty} \frac{(-\sigma)^i}{i!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(x_1, x_1, \lambda) & \cdots & K(x_1, x_i, \lambda) \\ \vdots & & \vdots \\ K(x_i, x_1, \lambda) & \cdots & K(x_i, x_i, \lambda) \end{vmatrix} dx_1 \cdots dx_i. \tag{2.14}$$

It can be shown that

$$\Delta(\sigma, \lambda) = \Delta(0, \lambda)\mathscr{D}_{K_\lambda}(\sigma), \tag{2.15}$$

provided $K(x, y, \lambda)$ is suitably defined along $x = y$ if it is discontinuous there.

Equation (2.15) is a consequence of the following facts. Suppose we have a system

$$Mu = \lambda Nu, \qquad U_i(u) = 0, \qquad i = 1, \cdots, m, \tag{2.16}$$

where M is of order m, N is of order n and $m > n$. Assume that $\lambda = 0$ is not an eigenvalue of (2.16). If $M(x, y)$ is the Green's function for M, and if

$$K(x, y) = (N_1 M)(x, y), \tag{2.17}$$

then the determinant $\Delta(\lambda)$ and the Fredholm function of K differ by a multiplicative constant; that is,

$$\Delta(\lambda) = A\mathscr{D}_K(\lambda), \qquad A \neq 0. \tag{2.18}$$

We construct our Green's function in the manner of Coddington–Levinson [2, Chap. 7]. Equation (2.18) is true provided $K(x, y)$ is defined along the diagonal by a limit process from the triangle $x < y$ if it is discontinuous there. The result is plausible, since, as before, the eigenvalues of the system (2.16) are also the eigenvalues of the equation (see Buscham [1])

$$v(x) = \lambda \int_0^1 K(x, y)v(y)\,dy, \tag{2.19}$$

where $v = Nu$. However, this does not constitute a proof of (2.18).

If we set $\lambda = 0$ in (2.18), then using the fact that

$$\mathscr{D}_K(0) = 1, \tag{2.20}$$

we find that

$$\Delta(\lambda) = \Delta(0)\mathscr{D}_K(\lambda). \tag{2.21}$$

Equation (2.15) is a restatement of (2.21) with σ as the "eigenvalue parameter" (instead of λ) and with λ as a "fixed parameter".

Equation (2.18) is well known, but its proof appears not to be readily available in the literature. Therefore we will outline a proof here, although we do not believe that the results of this section are new.

Let $G(x, y, \lambda)$ be the Green's function for the operator $(M - \lambda N)$ with boundary conditions $U_i(u) = 0$. Note that $G(x, y, 0) = M(x, y)$. First, we must show that

$$(N_1 G)(x, y, \lambda) = R_K(x, y, \lambda), \tag{2.22}$$

where the kernel K is given in (2.17), and where $R_K(x, y, \lambda)$ denotes the resolvent of K and is given (for small $|\lambda|$) by

$$R_K(x, y, \lambda) = \sum_{j=1}^{\infty} \lambda^{j-1} K_j(x, y). \tag{2.23}$$

In (2.23), K_j is the jth iterate of K.

We prove (2.22) by first establishing that

$$G(x, y, \lambda) = G(x, y, 0) + \lambda \int_0^1 G(x, z, 0) R_K(z, y, \lambda)\, dz. \tag{2.24}$$

Let us denote the right side of (2.24) by $\tilde{G}(x, y, \lambda)$. Note that if f is continuous, then

$$\begin{aligned}(M - \lambda N) \int_0^1 \tilde{G}(x, y, \lambda) f(y)\, dy &= f(x) + \lambda \int_0^1 R_K(x, y, \lambda) f(y)\, dy \\ &\quad - \lambda \int_0^1 K(x, y) f(y)\, dy \\ &\quad - \lambda \int_0^1 \int_0^1 K(x, z) R_K(z, y, \lambda) f(y)\, dz\, dy \\ &= f(x).\end{aligned} \tag{2.25}$$

The last step follows by a standard equality in the theory of integral equations (see Mikhlin [7, p. 47, eq. # 4]).

Now $w(x)$, defined by

$$w(x) = \int_0^1 \tilde{G}(x, y, \lambda) f(y)\, dy, \tag{2.26}$$

satisfies the boundary conditions $U_i(w) = 0$; this follows from the properties of $G(x, y, 0)$ and from the fact that

$$w(x) = \int_0^1 G(x, y, 0) r(y)\, dy, \tag{2.27}$$

with

$$r(y) = f(y) + \lambda \int_0^1 R_K(y, z) f(z)\, dz. \tag{2.28}$$

Hence $w(x)$ given by (2.26) satisfies the system

$$(M - \lambda N)w = f, \qquad U_i(w) = 0, \qquad i = 1, \cdots, m. \tag{2.29}$$

Hence, since $\lambda = 0$ is not an eigenvalue of (2.16), the solution of (2.29) is unique, so

$$\int_0^1 G(x, y, \lambda) f(y)\, dy = w(x) = \int_0^1 \tilde{G}(x, y, \lambda) f(y)\, dy. \tag{2.30}$$

Hence $G(x, y, \lambda) = \tilde{G}(x, y, \lambda)$, except perhaps at points on the line $x = y$, where these functions will be discontinuous if $n = m - 1$. If all functions involved are defined on the line $x = y$ by a limit process from the triangle $y > x$, then indeed $G = \tilde{G}$ even on this diagonal.

We obtain (2.22) from (2.24) by applying the operator N_1 to both sides of (2.24), and again using equation # 4 in Mikhlin [7, p. 47].

Having established (2.22), we proceed to the proof of (2.21). We will show that

$$\frac{\mathscr{D}_K'(\lambda)}{\mathscr{D}_K(\lambda)} = -\int_0^1 (N_1 G)(x, x, \lambda)\, dx \tag{2.31}$$

and

$$-\int_0^1 (N_1 G)(x, x, \lambda)\, dx = \frac{\Delta'(\lambda)}{\Delta(\lambda)}. \tag{2.32}$$

Here, $\mathscr{D}_K(\lambda)$ is the Fredholm function given essentially by (2.14). The equality of the logarithmic derivatives of $\Delta(\lambda)$ and $\mathscr{D}_K(\lambda)$ will yield (2.18).

The first inequality (2.31) is well known (see Tricomi [11, p. 72]) provided we replace $(N_1 G)$ in (2.31) by R_K. This is where (2.22) is needed. The second equality (2.32) is perhaps not so trivial. To prove it, we must differentiate $\Delta(\lambda)$. Let $S(x, y, \lambda)$ denote the Green's function for the initial value problem

$$(M - \lambda N)v = f, \tag{2.33}$$

$$v^{(j)}(0) = 0, \qquad j = 0, \cdots, m - 1. \tag{2.34}$$

Let the integral operator S_λ be defined by

$$S_\lambda u(x) = \int_0^1 S(x, y, \lambda) u(y)\, dy. \tag{2.35}$$

If $\{\phi_i(x, \lambda)\}$ is a fundamental set of solutions for $(M - \lambda N)u = 0$ satisfying $\{\phi_i^{(j-1)}(0, \lambda)\} = \delta_{ij}$ as in (2.6), then we note that

$$M(\partial \phi_i / \partial \lambda) - \lambda N(\partial \phi_i / \partial \lambda) = N \phi_i \tag{2.36}$$

and that $\partial \phi_i / \partial \lambda$ vanishes along with its first $(m - 1)$ derivatives at $x = 0$. Hence,

$$\partial \phi_i / \partial \lambda = S_\lambda N \phi_i. \tag{2.37}$$

We shall use the notation $U_i S(\cdot, y, \lambda)$ to indicate that U_i acts on $S(x, y, \lambda)$ as a function of x; the result is clearly independent of x.

We have

$$\Delta'(\lambda) = \begin{vmatrix} U_1\dfrac{\partial\phi_1}{\partial\lambda} & \cdots & U_1\dfrac{\partial\phi_m}{\partial\lambda} \\ U_2\phi_1 & \cdots & U_2\phi_m \\ \vdots & & \vdots \\ U_m\phi_1 & \cdots & U_m\phi_m \end{vmatrix} + \cdots + \begin{vmatrix} U_1\phi_1 & \cdots & U_1\phi_m \\ \vdots & & \vdots \\ U_{m-1}\phi_1 & \cdots & U_{m-1}\phi_m \\ U_m\dfrac{\partial\phi_1}{\partial\lambda} & \cdots & U_m\dfrac{\partial\phi_m}{\partial\lambda} \end{vmatrix}. \tag{2.38}$$

We replace $\partial/\partial\lambda$ by $S_\lambda N$ in (2.38). Then by a series of interchanges we can bring the jth row in the jth determinant in (2.38) up to the top of the determinant, leaving the other rows in their original relative order. Thus the new determinant and the old one are related by a factor of $(-1)^j$. The sum of determinants can then be consolidated so as to obtain

$$-\Delta'(\lambda) = \int_0^1 \begin{vmatrix} 0 & N\phi_1(y,\lambda) & \cdots & N\phi_m(y,\lambda) \\ U_1S(\cdot,y,\lambda) & U_1\phi_1 & \cdots & U_1\phi_m \\ \vdots & \vdots & & \vdots \\ U_mS(\cdot,y,\lambda) & U_m\phi_1 & \cdots & U_m\phi_m \end{vmatrix} dy. \tag{2.39}$$

The validity of (2.39) can also be verified by expanding the determinant using cofactors of the first column. Now we note that if $m > n + 1$, then $N_1S(x, x, \lambda) = 0$ since $N_1S(x, y, \lambda) \equiv 0$ if $y > x$ and since the latter function must be continuous on $[0, 1] \times [0, 1]$. If $m = n + 1$, we still define $N_1S(x, x, \lambda) = N_1S(x, x^+, \lambda) = 0$. If we replace 0 in the determinant in (2.39) by $N_1S(x, x, \lambda)$, we obtain (2.32) (see Coddington–Levinson [2, p. 204, prob. 12]).

3. The order of an entire function and consequences. We are interested in investigating the convergence or divergence of the series $\sum_{i=1}^{\infty} |1/\lambda_i|^p$ and $\sum_{i=1}^{\infty} |1/\mu_i|^q$. This depends on the density of λ_i and μ_i about ∞, which in turn is related to the order of the functions $\Delta(0, \lambda)$ and $\Delta(1, \lambda)$. In this section we will review this concept and also point out how these sums are related to the Taylor coefficients of $\Delta(0, \lambda)$ and $\Delta(1, \lambda)$ via Newton's formula.

Following Titchmarsh [10, Chap. 8], we say that the entire function f is of order $\rho \geqq 0$ if and only if

$$f(z) = f(re^{i\theta}) = O(e^{r^{\rho+\varepsilon}}) \tag{3.1}$$

for each $\varepsilon > 0$, but for no negative value of ε. If we write

$$|f(re^{i\theta})| \leqq Me^{r^{\rho+\varepsilon}}, \tag{3.2}$$

then M may depend on ε. It is easy to verify that if f_i has order ρ_i for $i = 1, 2$, then the functions $(f_1 + f_2)$ and $(f_1 \cdot f_2)$ have finite orders not exceeding $\max(\rho_1, \rho_2)$. It is also easy to show that this estimate may be too high.

If f is of order ρ, and if $z_1, z_2, \cdots$ are the nonzero roots of the equation

$$f(z) = 0, \tag{3.3}$$

written according to multiplicity, then the series

$$\sum_{i=1}^{\infty} |1/z_i|^{\rho+\varepsilon} \tag{3.4}$$

converges for each $\varepsilon > 0$.

Let $h + 1$ be the smallest integer larger than ρ. Then f may be represented as an infinite product. If

$$E(z, h) = \begin{cases} 1 - z & \text{for } h = 0, \\ (1 - z)\exp\left[z + \dfrac{z^2}{2} + \cdots + \dfrac{z^h}{h}\right] & \text{for } h = 1, 2, \cdots, \end{cases} \tag{3.5}$$

then

$$f(z) = Cz^p e^{Q(z)} \prod_{i=1}^{\infty} E[(z/z_i), h], \tag{3.6}$$

where p is a nonnegative integer and $Q(z)$ is a polynomial of degree not exceeding h. The infinite product here is not necessarily the canonical product discussed in Titchmarsh [10, p. 250]. An explanation of this concept is not needed in this paper.

If

$$|z_1| \leqq |z_2| \leqq \cdots, \tag{3.7}$$

then for each $\varepsilon > 0$, there exists an $A = A(\varepsilon)$ so that

$$A \cdot j^{1/(\rho+\varepsilon)} < |z_j|, \qquad j = 1, 2, \cdots. \tag{3.8}$$

The lower the order, the faster will the quantity $|z_j|$ tend to ∞. Equation (3.8) thus indicates how the order ρ affects the growth of the quantities $|z_j|$. It follows that for each $\varepsilon > 0$,

$$|z_j| \cdot j^{-1/(\rho+\varepsilon)} \to +\infty \quad \text{as } j \to \infty. \tag{3.9}$$

From the infinite product and Taylor series representations of f, we can find the sums

$$\sum_{i=1}^{\infty} [1/z_i]^k, \qquad k = h + 1, h + 2, \cdots, \tag{3.10}$$

as explicit functions of the Taylor series coefficients of f. Let

$$g(z) = z^{-p} f(z). \tag{3.11}$$

Taking the logarithmic derivative of $g(z)$ from (3.6) and expanding it in a Taylor series in the region $|z| < |z_1|$, we obtain

$$\begin{aligned} g'(z)/g(z) = {} & [Q_1 + \cdots + hQ_h z^{h-1}] \\ & - z^h \sum_{i=1}^{\infty} (1/z_i)^{h+1} - z^{h+1} \sum_{i=1}^{\infty} (1/z_i)^{h+2} - \cdots, \end{aligned} \tag{3.12}$$

where (see (3.6))

$$Q(z) = \sum_{i=0}^{h} Q_i z^i. \tag{3.13}$$

Let

$$f(z) = z^p \sum_{i=0}^{\infty} a_i z^i, \qquad a_0 \neq 0. \tag{3.14}$$

If we set

$$g'(z) = -\left[\sum_{i=0}^{\infty} c_{i+1} z^i\right] g(z), \tag{3.15}$$

then

$$\sum_{i=1}^{\infty} [1/z_i]^k = c_k \quad \text{for } k = h+1, h+2, \cdots, \tag{3.16}$$

provided, of course, that f has zeroes away from the origin.

We can provide a formula for c_k in terms of the Taylor coefficients of $f(z)$ in (3.14). By expanding both sides of (3.15) in powers of z, and comparing coefficients, we obtain, after slight rearrangement,

$$\begin{aligned}
a_0 c_1 \qquad\qquad\qquad &= -a_1 \\
a_1 c_1 + a_0 c_2 \qquad\qquad &= -2a_2 \\
a_2 c_1 + a_1 c_z + a_0 c_3 \qquad &= -3a_3 \\
\vdots \qquad\qquad\qquad &\quad\ \vdots \\
a_{k-1} c_1 + a_{k-2} c_2 + \cdots + a_0 c_k &= -k a_k.
\end{aligned} \tag{3.17}$$

Solving for c_k by means of determinants, we obtain

$$c_k = \frac{(-1)^k}{a_0^k} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ 2a_2 & a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & & a_0 \\ k a_k & a_{k-1} & a_{k-2} & \cdots & a_1 \end{vmatrix}, \qquad k = 1, 2, \cdots. \tag{3.18}$$

Turnbull [12, p. 74] gives a similar formula for polynomials. He attributes (3.18) to Newton.

Clearly a necessary and sufficient condition that f have no zeroes away from the origin is that the determinants c_k vanish for all $k \geqq h + 1$. In this case, $f(z) = Cz^p e^{Q(z)}$.

4. Theorems on the order of the determinant $\Delta(1, \lambda)$. Henceforth we will assume that $t \geqq 0$ is the maximum order of the functions f_{ij} and g_{ij}. Note that the order of $\Delta(0, \lambda)$ cannot exceed t, and therefore $\sum_{i=1}^{\infty} |1/\mu_i|^{t+\varepsilon}$ converges for each $\varepsilon > 0$. Of course, if all f_{ij} and g_{ij} are polynomials, then $t = 0$, and the set $\{\mu_i\}$ is finite.

The main purpose of this section is to establish three theorems giving upper bounds on the order of $\Delta(1, \lambda)$.

THEOREM 1. *The function* $\Delta(1, \lambda)$ *has a finite order not exceeding* $\max(s, t)$.

Proof. Let $S(x, y)$ be the Green's function for the initial value problem

$$Mu = f, u^{(j)}(0) = 0, \qquad j = 0, 1, \cdots, m-1. \tag{4.1}$$

If $\phi_i(x, \sigma, \lambda)$ is the function given in § 2, set

$$\phi_i(x) = \phi_i(x, 0, \lambda), \tag{4.2}$$

since the latter function does not really depend on λ. Then we maintain that

$$\begin{aligned} \phi_i(x, \sigma, \lambda) &= \phi_i(x) + \sigma \int_0^1 S(x, y) N_\lambda \phi_i(y)\, dy \\ &\quad + \sigma \sum_{j=1}^{\infty} \sigma^j \int_0^1 \int_0^1 S(x, z) T_j(z, y, \lambda) N_\lambda \phi_i(y)\, dz\, dy, \end{aligned} \tag{4.3}$$

where

$$T(x, y, \lambda) = [(N_\lambda)_1 S](x, y). \tag{4.4}$$

To establish (4.3) we can proceed in the following way. In the first place, the convergence of the series in (4.3) is a consequence of certain inequalities given later. For now, denote the right side of (4.3) by the symbol $\Phi_i(x, \sigma, \lambda) = \Phi_i$. One can then show by direct computation that $w = \Phi_i$ satisfies (as a function of x)

$$Mw = \sigma N_\lambda w, \qquad w^{(j-1)}(0) = \delta_{ij}, \qquad j = 1, \cdots, m-1, \tag{4.5}$$

where δ_{ij} is the Kronecker delta. We take here a slightly different point of view. The right side Φ_i, of (4.3) may be written

$$\Phi_i(x, \sigma, \lambda) = \phi_i(x) + \sigma \int_0^1 \left[S(x, y) + \sigma \int_0^1 S(x, z) R_{T_\lambda}(z, y, \sigma)\, dz \right] \phi_i(y)\, dy,$$

where R_{T_λ}, the resolvent kernel, is given by

$$R_{T_\lambda}(x, y, \sigma) = \sum_{j=1}^{\infty} \sigma^{j-1} T_j(x, y, \lambda). \tag{4.6}$$

However, $Q(x, y, \sigma, \lambda)$, defined by

$$Q(x, y, \sigma, \lambda) = S(x, y) + \sigma \int_0^1 S(x, z) R_{T_\lambda}(z, y, \sigma)\, dz,$$

is precisely the Green's function for

$$Mu - \sigma N_\lambda u = f, u^{(j)}(0) = 0, \qquad j = 0, 1, \cdots, m-1. \tag{4.7}$$

This fact is proved in a manner entirely analogous to (2.24); the proof merely involves changing the "boundary" conditions in the discussion preceding (2.24) to "initial" conditions. Using the fact that Q is the Green's function for (4.7), we obtain

$$(M - \sigma N_\lambda)\Phi_i(x, \sigma, \lambda) = (M - \sigma N_\lambda)\phi_i(x) + \sigma(M - \sigma N_\lambda)\int_0^1 Q(x, z, \sigma, \lambda) N_\lambda \phi_i(z)\,dz$$
$$= -\sigma N_\lambda \phi_i(x) + \sigma N_\lambda \phi_i(x)$$
$$= 0.$$

Also, since

$$\frac{\partial^i S}{\partial x^i}(x, y)\bigg|_{x=0} = 0 \quad \text{for } i = 0, \cdots, m-1, \tag{4.8}$$

the function Φ_i on the right side of (4.3) must satisfy (see (4.2) and (2.6))

$$\frac{\partial^{(j-1)}\Phi_i(x, \sigma, \lambda)}{\partial x^{(j-1)}}\bigg|_{x=0} = \phi_i^{(j-1)}(0) = \delta_{ij}, \qquad j = 0, \cdots, m-1.$$

Hence Φ_i must satisfy the same differential equation and initial conditions as $\phi_i(x, \sigma, \lambda)$, namely (4.5); hence the two functions are equal, and (4.3) is verified.

We will now concern ourselves with convergence of (4.3), and with bounds on the function $\Delta(1, \lambda)$.

Choose $a > 0$ so that for all $i = 1, \cdots, m$:

$$|\phi_i(x)| \leqq a, \tag{4.9}$$

$$|N_\lambda \phi_i(x)| \leqq a[|\lambda|^s + 1], \tag{4.10}$$

$$|T(x, y, \lambda)| \leqq a[|\lambda|^s + 1], \tag{4.11}$$

$$\left|\frac{\partial^q}{\partial x^q} S(x, y)\right| \leqq a, \qquad q = 0, \cdots, m-1. \tag{4.12}$$

Since $S(x, y)$ and $T(x, y, \lambda)$ are Volterra kernels, we may show by standard techniques (see Mikhlin [7, p. 16]) that

$$\begin{aligned} &\left|\frac{d^q}{dx^q}\int_0^1\int_0^1 S(x, z) T_p(z, y, \lambda) N_\lambda \phi_i(y)\,dz\,dy\right| \\ &\qquad \leqq \frac{a^{p+2}}{(p+1)!}[|\lambda|^s + 1]^{p+1}, \qquad q = 0, \cdots, m-1. \end{aligned} \tag{4.13}$$

The estimate (4.13) guarantees convergence of the series in (4.3) for all σ and λ, and guarantees that the series may be differentiated up to $(m-1)$ times in the variable x. The estimate (4.13) will help us prove a bound on the order of $\Delta(1, \lambda)$. However, this estimate (4.13) is certainly crude; in view of regularity conditions which S and T_λ frequently possess, as we shall see later, the inequality (4.13) is quite unsatisfactory to demonstrate the convergence of

$$\sum_{i=1}^{\infty} |1/\lambda_i|^p \tag{4.14}$$

for $t < p < s$, when it happens that $t < s$.

For convenience, let $\psi_p(x, \lambda)$ be the coefficient of σ^p in (4.3), where $p = 0, 1, \cdots$. We suppress the dependence of ψ_p on i so as to simplify the notation. Assume that

$$|f_{ij}(\lambda)| + |g_{ij}(\lambda)| \leqq b(\varepsilon) \exp [|\lambda|^{t+\varepsilon}] \tag{4.15}$$

for each $\varepsilon > 0$, and for $i = 0, \cdots, m-1$ and $j = 1, \cdots, m$. For convenience, set

$$\tilde{\phi}_i = \phi_i(x, 1, \lambda). \tag{4.16}$$

Then, using (4.3) and the estimate (4.13), we obtain

$$|U_j(\tilde{\phi}_i)| = \left| \sum_{q=0}^{m-1} f_{qj}(\lambda)\phi_i^{(q)}(0, 1, \lambda) + g_{qj}(\lambda)\phi_i^{(q)}(1, 1, \lambda) \right| \tag{4.17}$$

$$\leqq b(\varepsilon)[\exp |\lambda|^{t+\varepsilon}] \sum_{p=0}^{\infty} \sum_{q=0}^{m-1} |\psi_p^{(q)}(0, \lambda)| + |\psi_p^{(q)}(1, \lambda)| \tag{4.18}$$

$$\leqq 2mb(\varepsilon)[\exp |\lambda|^{t+\varepsilon}] \sum_{p=0}^{\infty} \max_{\substack{x \in [0,1] \\ q=0,\cdots,m-1}} |\psi_p^{(q)}(x, \lambda)| \tag{4.19}$$

$$\leqq 2mb(\varepsilon)[\exp |\lambda|^{t+\varepsilon}] \sum_{p=0}^{\infty} \frac{a^{p+1}}{p!}[|\lambda|^s + 1]^p \tag{4.20}$$

$$= 2amb(\varepsilon)[\exp |\lambda|^{t+\varepsilon}] \exp [a(|\lambda|^s + 1)]. \tag{4.21}$$

It follows easily that

$$|\Delta(1, \lambda)| \leqq m![2amb(\varepsilon)]^m \exp (am) \cdot \exp [m|\lambda|^{t+\varepsilon} + am|\lambda|^s], \tag{4.22}$$

from which it follows that the order of $\Delta(1, \lambda)$ does not exceed $\max(s, t)$, thus completing the proof.

Theorem 1 is a "best" one in the sense that there are systems for which the order of $\Delta(1, \lambda)$ is exactly $\max(s, t)$.

Example 1. Consider the system

$$\begin{aligned} u^{(n)}(x) &= i\lambda^s u^{(n-1)}(x), \\ u^{(j)}(0) &= i\lambda^s u^{(j-1)}(0), \qquad j = 1, \cdots, n-1, \\ u(0) \exp i\lambda^t &= u(1), \end{aligned}$$

where n, s, t are integers such that $n \geqq 1$, $s \geqq 1$, $t \geqq 0$, and $s \neq t$. If $n = 1$, the boundary condition obtained by setting $j = 1$ is no restriction. The above system is entirely equivalent to the system

$$\begin{aligned} u'(x) &= i\lambda^s u(x), \\ u(0) \exp i\lambda^t &= u(1). \end{aligned}$$

For the latter system, $\Delta(1, \lambda) = [\exp i\lambda^t] - [\exp i\lambda^s]$, so that the order of $\Delta(1, \lambda)$ is exactly $\max(s, t)$. The eigenvalues satisfy

$$\lambda^t - \lambda^s = 2p\pi, \qquad p = 0, \pm 1, \pm 2, \cdots$$

and we will show further that $\sum_{i=1}^{\infty} |1/\lambda_i|^{\max(s,t)}$ diverges. Let us assume that $t > s$

for convenience. Pick one root σ_p of

$$\lambda^t - \lambda^s = 2p\pi, \qquad p = 1, 2, \cdots.$$

Now $|\sigma_p| \to +\infty$ as $p \to \infty$. For large $|\lambda|$,

$$\left| \frac{1}{\lambda^t} - \frac{1}{\lambda^t - \lambda^s} \right| \leqq \frac{A}{|\lambda|^{2t-s}},$$

so setting $\lambda = \sigma_p$ for large p, we find that

$$\left| \frac{1}{\sigma_p^t} - \frac{1}{2p\pi} \right| \leqq \frac{A}{|\sigma_p|^{2t-s}}.$$

Therefore, the inequality

$$\frac{1}{2p\pi} - \frac{A}{|\sigma_p|^{2t-s}} \leqq \frac{1}{|\sigma_p|^t}$$

yields divergence of $\sum_{p=1}^{\infty} |1/\sigma_p^t|$; to see this, we recall that $2t - s > t$, so that for all p sufficiently large, $1/2p\pi < (A + 1)/|\sigma_p|^t$. The divergence of $\sum_{p=1}^{\infty} 1/p$ then yields the desired result. By symmetry, $\sum_{i=1}^{\infty} |1/\lambda_i|^s$ diverges if $s > t$. In any case, $\sum_{i=1}^{\infty} |1/\lambda_i|^{\max(s,t)}$ diverges.

Theorem 1 can also give poor estimates on the order of $\Delta(1, \lambda)$, as one might expect.

Example 2. Consider

$$u' = i\lambda^s u,$$

$$\{[\exp i\lambda^s] + p(\lambda)\} \quad u(0) = u(1),$$

where $p(\lambda)$ is any polynomial. Here, $\Delta(1, \lambda) = p(\lambda)$, so $\Delta(1, \lambda)$ has order zero; Theorem 1 states that the order of $\Delta(1, \lambda)$ does not exceed s. This example also shows us that sometimes only a finite number of eigenvalues occur.

Examples 1 and 2 are at the opposite ends of the spectrum of possibilities; in general, the cancellations occurring in the second example will not happen. Hence the first example is considered to be more "typical".

We shall see that while the order of the functions f_{ij} and g_{ij}, and also the degree s of the polynomial operator N_λ, are working against us, the difference between m and n will be to our advantage. This is no surprise, since the greater $m - n$ is, the greater the regularity properties of $K(x, y, \lambda)$ [see (2.13)] and $T(x, y, \lambda)$, provided the coefficients of $u^{(i)}$ in $N_j u$ also share suitable regularity properties.

The following theorem will have interest when $t = 0$, or when $\Delta(0, \lambda)$ is a polynomial, for here it shows the possibility of convergence of $\sum_{i=1}^{\infty} |1/\lambda_i|^q$ when $q < s$; in the important case $s = 1$, $t = 0$, it yields absolute convergence and the possibility of evaluation of $\sum_{i=1}^{\infty} 1/\lambda_i$.

THEOREM 2. *Let k be a positive integer satisfying*

$$k \leqq m - n - 1.$$

Let the coefficient of $u^{(i)}$ in the expression for $N_j u$ be k times continuously differentiable for each i in $0 \leqq i \leqq n$, and for each j in $0 \leqq j \leqq s$. Then the order of $\Delta(1, \lambda)$ does *not exceed*

$$\max(t, s/(k+1)).$$

Remark. Note that in Theorem 2 we may take $k = m - n - 1$, provided that the coefficients in $N_j u$ are sufficiently smooth. In this event, the conclusion of Theorem 2 becomes

$$\operatorname{order} \Delta(1, \lambda) \leqq \max(t, s/m - n)).$$

Proof. The quantities $\psi_p(x, \lambda)$ and $b(\varepsilon)$ of the previous theorem shall retain their significance. We note that

$$\left(\frac{\partial^j}{\partial x^j} T\right)(x, y, \lambda) \tag{4.23}$$

is continuous in x and y together for $0 \leqq j \leqq k - 1$, and that each of these functions vanishes for $y \geqq x$. For $j = k$, $\partial^j T/\partial x^j$ is piecewise continuous in $[0, 1]^2$, with a possible simple jump across $x = y$. For $k = 1$ and $x \geqq y$,

$$T(x, y, \lambda) = \int_y^x \left(\frac{\partial T}{\partial u}\right)(u, y, \lambda)\, du, \tag{4.24}$$

and for $k \geqq 2$ and $x \geqq y$,

$$T(x, y, \lambda) = \int_y^x \int_y^{x_1} \cdots \int_y^{x_{k-1}} \left(\frac{\partial^k}{\partial x_k^k} T\right)(x_k, y, \lambda)\, dx_k \cdots dx_1. \tag{4.25}$$

Choose $a > 0$ so large that

$$\left| \left(\frac{\partial^k}{\partial x^k} T\right)(x, y, \lambda) \right| \leqq a[|\lambda|^s + 1] \tag{4.26}$$

for all $(x, y) \in [0, 1]^2$ and for all complex λ. For $k \geqq 2$ and $x \geqq y$,

$$|T(x, y, \lambda)| \leqq a[|\lambda|^s + 1] \int_y^x \int_y^{x_1} \cdots \int_y^{x_{k-1}} dx_k \cdots dx_1.$$

Evidently, for $k \geqq 1$ and $x \geqq y$,

$$|T(x, y \cdot \lambda)| \leqq a[|\lambda|^s + 1](x - y)^k/k!. \tag{4.27}$$

Let

$$J(x, y) = \begin{cases} \dfrac{(x - y)^k}{k!}, & x \geqq y, \\ 0, & \text{otherwise}. \end{cases} \tag{4.28}$$

The iterated kernels of J are given by

$$J_p(x, y) = \begin{cases} \dfrac{(x - y)^{p(k+1)-1}}{[p(k+1) - 1]!}, & x \geqq y, \\ 0, & \text{otherwise}, \end{cases} \tag{4.29}$$

a result that can be established by induction. Certainly, (4.29) is valid for $p = 1$.

We show that if (4.29) is valid for p, then it is valid for $p + 1$. Now

$$J_{p+1}(x, y) = \int_y^x \frac{(x-z)^k}{k!} \frac{(z-y)^{p(k+1)-1}}{[p(k+1)-1]!} dz \quad \text{for } x \geqq y, \tag{4.30}$$

and

$$J_{p+1}(x, y) = 0 \quad \text{otherwise}.$$

Let $u = z - y$ in (4.30). Then, for $x \geqq y$,

$$J_{p+1}(x, y) = \int_0^{x-y} \frac{u^{p(k+1)-1}}{[p(k+1)-1]!} \frac{(x-y-u)^k}{k!} du. \tag{4.31}$$

The integral in (4.31) is a convolution. If

$$v_i(\xi) = \xi^i, \qquad i = 1, 2, \cdots \tag{4.32}$$

then (see Tricomi [11, p. 25])

$$\left(\frac{v_i}{i!} * \frac{v_j}{j!}\right)(\xi) = \left(\frac{v_{i+j+1}}{(i+j+1)!}\right)(\xi). \tag{4.33}$$

(The asterisk in (4.33) denotes a convolution.) The asserted result (4.29) follows by setting $\xi = x - y$, $i = p(k + 1) - 1$ and $j = k$ in (4.33). Now

$$|T_p(x, y, \lambda)| \leqq a^p[|\lambda|^s + 1]^p J_p(x, y), \tag{4.34}$$

so for $q = 0, \cdots, m - 1$,

$$\begin{aligned} |\psi_{p+1}^{(q)}(x, \lambda)| &= \left| \int_0^1 \int_0^1 \frac{\partial^q}{\partial x^q} S(x, z) T_p(z, y, \lambda) N_\lambda \phi_i(y) \, dz \, dy \right| \\ &\leqq a^2[|\lambda|^s + 1] \int_0^1 \int_0^1 |T_p(z, y, \lambda)| \, dz \, dy \\ &\leqq a^{p+2}[|\lambda|^s + 1]^{p+1} \int_0^1 \int_0^z \frac{(z-y)^{p(k+1)-1}}{[p(k+1)-1]!} dy \, dz \\ &\leqq a^{p+2}[|\lambda|^s + 1]^{p+1} \frac{1}{[p(k+1)+1]!}. \end{aligned} \tag{4.35}$$

From (4.19) and (4.35),

$$|U_j(\tilde{\phi}_i)| \leqq 2ab(\varepsilon)m[\exp|\lambda|^{t+\varepsilon}] \left[1 + \sum_{p=0}^{\infty} \frac{a^{p+1}[|\lambda|^s + 1]^{p+1}}{[p(k+1)+1]!}\right]. \tag{4.36}$$

Since the function

$$a[|\lambda|^s + 1] \exp[a(|\lambda|^s + 1)]^{1/(k+1)} \tag{4.37}$$

dominates the infinite series in (4.36), it follows that $\Delta(1, \lambda)$ has finite order not exceeding $\max(t, s/(k + 1))$.

The results given are again "best" results, as seen from the following examples.

Example 3. Let m and n be even integers with $m > n$. Consider

$$u^{(m)} = \lambda u^{(n)},$$
$$u^{(i)}(0) = u^{(i)}(1) = 0, \qquad i = 0, 2, 4, \cdots, m-2.$$

The functions $\{\sin j\pi x | j = 1, 2, \cdots\}$ are eigenfunctions and $\{(j\pi)^{m-n} | j = 1, 2, \cdots\}$ is the set of absolute values of the associated eigenvalues. Here $\max[t, s/(k+1)] = 1/(m-n)$, since $t = 0$ and $s = 1$, and

$$\sum_{j=1}^{\infty} |1/\lambda_j|^{1/(m-n)+\varepsilon}$$

converges only if $\varepsilon > 0$.

Example 4. Let m, n, and t be nonnegative integers and let $m - n$ be a positive even integer. Consider

$$u^{(m)}(x) = \lambda^s u^{(n)}(x), \qquad 0 \leqq x \leqq 1,$$
$$u^{(m-h)}(0) = \lambda^s u^{(n-h)}(0), \qquad h = 1, \cdots, n,$$
$$u^{(j)}(0) = 0, \qquad j = 1, 3, 5, \cdots, m-n-1,$$
$$u^{(k)}(1) + \cos \lambda^t u^{(k)}(0) = 0, \qquad k = 0, 2, 4, \cdots, m-n-2.$$

Let $r = s/(m-n)$ for convenience. In this case we take $k = m - n - 1$ in Theorem 2, so that $\max[t, s/(k+1)] = \max(t, r)$. Assume that $r > t$ for simplicity. We will show that the series $\sum_{i=1}^{\infty} |\lambda_i|^{-r}$ diverges. The original system is equivalent to the reduced system

$$u^{(m-n)}(x) = \lambda^s u(x), \qquad 0 \leqq x \leqq 1,$$
$$u^{(j)}(0) = 0, \qquad j = 1, 3, 5, \cdots, (m-n-1),$$
$$u^{(k)}(1) + (\cos \lambda^t) u^{(k)}(0) = 0, \qquad k = 0, 2, 4, \cdots, (m-n-2),$$

as a simple integration will show. A solution of the reduced system is the function $\cos \lambda^r x$, where λ is a solution of the equation

$$0 = \cos \lambda^r + \cos \lambda^t = 2 \cos (\lambda^r + \lambda^t)/2 \cos (\lambda^r - \lambda^t)/2.$$

Let σ_p be any positive root of the equation

$$\lambda^r + \lambda^t = (2p-1)\pi, \qquad p = 1, 2, \cdots.$$

Note that $\cos[(\lambda^r + \lambda^t)/2] = 0$ when $\lambda = \sigma_p$. The divergence of $\sum_{p=1}^{\infty} \sigma_p^{-r}$, and hence divergence of $\sum_{i=1}^{\infty} |\lambda|^{-r}$, follows from the inequality

$$1/\sigma_p^r \geqq 1/(\sigma_p^r + \sigma_p^s) = 1/(2p-1)\pi.$$

If $t > r$, one can obtain similar results, so in any case,

$$\sum_{i=1}^{\infty} \left| \frac{1}{\lambda_i} \right|^{\max[t, s/(k+1)]}$$

diverges.

One may obtain unreasonably high estimates on the order of $\Delta(1, \lambda)$ using Theorem 2, just as in the case of Theorem 1.

Example 5. Let s be a positive integer and let $r = s/2$. Consider

$$u'' = \lambda^{2r}u, \qquad u'(0) = 0,$$
$$u(0)[\cos \lambda^r + p(\lambda)] = u(1),$$

where p is a polynomial. In this case $\Delta(1, \lambda) = p(\lambda)$, provided we choose as a fundamental set $u(x) = \cos \lambda^r$, $v(x) = (\sin \lambda^r x)/\lambda^r$. Obviously the order of $\Delta(1, \lambda)$ is zero, while the estimate provided by Theorem 2 is $\max [t, s/(k + 1)] = r$.

It is not possible to say very much about the asymptotic behavior of the eigenvalues of problem (2.1)–(2.2) under the generality assumed. It is not at all surprising that $\Delta(1, \lambda)$ can be *any* entire function, possibly one of infinite order. Equation (3.9) is the only asymptotic result for the eigenvalues of (2.1)–(2.2), unless less general systems are considered. We may not generally even set $\varepsilon = 0$ in (3.9), as the previous examples show.

We now give another result of the same type as Theorem 2; this time, we place restrictions on M instead of on N_λ in order to improve the bound on the order of $\Delta(1, \lambda)$.

THEOREM 3. *Let*

$$Mu = \sum_{j=0}^{m} p_{m-j}(x)u^{(j)}(x), \tag{4.38}$$

and suppose that $p_{m-j} \in C^{(j)}[0, 1]$ *for* $j = 0, \cdots, m$. *Then the order of* $\Delta(1, \lambda)$ *does not exceed*

$$\max [t, s/(m - n)].$$

Proof. Under our assumptions, the adjoint M^* of M exists, and[1]

$$S(x, y) = \sum_{i=1}^{m} \phi_i(x)\overline{\zeta_i(y)} \tag{4.39}$$

for $x \geqq y$, where $\phi_i(x)$ is given by (4.2), and where the ζ_i are solutions of the adjoint equation (see problems 19 and 21, Coddington and Levinson [2, p. 101])

$$M^*\zeta_i = 0. \tag{4.40}$$

Hence $\zeta_i \in C^{(m)}[0, 1]$ and it follows that

$$\frac{\partial^{p+q}}{\partial x^p\, \partial y^q} S(x, y) \tag{4.41}$$

is continuous on all of $[0, 1]^2$ provided that $p + q \leqq m - 2$.

For $p + q$ so restricted, $\partial^{p+q}S/\partial x^p\, \partial y^q$ vanishes along $x = y$. Let $k = m - n - 1$. If $k = 1$ and $x \geqq y$, then

$$T(x, y, \lambda) = \int_x^y \frac{\partial T}{\partial u}(x, u, \lambda)\, du. \tag{4.42}$$

If $k \geqq 2$, and $x \geqq y$, then

$$T(x, y, \lambda) = \int_x^y \int_x^{y_1} \cdots \int_x^{y_{k-1}} \left(\frac{\partial^k}{\partial y_k^k} T\right)(x, y_k, \lambda)\, dy_k \cdots dy_1. \tag{4.43}$$

[1] Assume $p_0 \equiv 1$ without loss of generality.

Hence, for $x \geqq y$ and $k \geqq 1$, and for $a > 0$ and sufficiently large,

$$|T(x, y, \lambda)| \leqq a[|\lambda|^s + 1](x - y)^k/k!. \tag{4.44}$$

The rest of the proof follows as before.

For a given system, there are often several possibilities in the way we can choose M and N_λ. Depending on how we choose M and N_λ, and on which theorem we use, we may obtain different estimates for the order of $\Delta(1, \lambda)$. If some of the functions $p_j(x)$ (where $0 \leqq m - j \leqq n$) do not have the required regularity conditions of Theorem 3, the offending terms can be incorporated into $N_\lambda u$ without harming the estimates.

5. Evaluation of traces from the Fredholm function $\mathscr{D}_{K_\lambda}(\lambda)$. In this section we will examine the relation between $\sum_{i=1}^{\infty} \lambda_i^{-p}$ and the Fredholm function $\mathscr{D}_{K_\lambda}(\lambda)$. By doing this, we develop the analogue of (1.2) for the equation (1.3). For small values of p, the equations sought sometimes yield a practical means of evaluating these sums, provided one of the previous theorems is applicable. At this stage it is convenient to restrict slightly the class of problems under investigation; instead of (2.1) and (2.2), we will consider

$$Mu = \lambda N_\lambda u, \qquad U_j(u, \lambda) = 0, \tag{5.1}$$

where M, N_λ, U_j, $\phi_i(x, \sigma, \lambda)$, $\Delta(\sigma, \lambda)$, m, n, s, t, μ_i, $M(x, y, \lambda)$ and $K(x, y, \lambda)$ are as before, with the exception that now we may assume $s \geqq 0$ instead of $s \geqq 1$. The eigenvalues λ_i of (5.1) taken according to algebraic multiplicity are given as solutions of

$$\Delta(\lambda, \lambda) = 0. \tag{5.2}$$

One can use the function $\mathscr{D}_{K_\lambda}(\lambda)$ to compute the sums $\sum_{i=1}^{\infty} [1/\lambda_i]^q$, but again the usefulness of the procedure is limited to special types of problems. Evidently, the equation $Mu = 0$ must be solvable; that is, $K(x, y, \lambda)$ must be computable. Formulas connecting the coefficients of λ^i in the Taylor expansion of

$$\left[\frac{d}{d\lambda}\mathscr{D}_{K_\lambda}(\lambda)\right] \Big/ \mathscr{D}_{K_\lambda}(\lambda)$$

about $\lambda = 0$ with these sums possess a complication; since $\mathscr{D}_{K_\lambda}(\lambda)$ is meromorphic and not generally entire, we will shortly see that one can find only the sums

$$\sum_{i=1}^{\infty} \left[\frac{1}{\lambda_i}\right]^{p+1} - \sum_{i=1}^{\infty} \left[\frac{1}{\mu_i}\right]^{p+1} \tag{5.3}$$

directly in terms of the said Taylor coefficients of its logarithmic derivative. However, the μ_i are zeroes of $\Delta(0, \lambda)$, so use of (3.18) yields explicit expressions for the series above involving the μ_i, provided $Mu = 0$ is solvable.

We shall presently develop the abovementioned formulas. We assume that $M(x, y, 0)$ exists, where $M(x, y, \lambda)$ is the Green's function for (2.9). From (2.15),

$$-\frac{\frac{d}{d\lambda}\Delta(\lambda, \lambda)}{\Delta(\lambda, \lambda)} + \frac{\frac{d}{d\lambda}\Delta(0, \lambda)}{\Delta(0, \lambda)} = -\frac{\frac{d}{d\lambda}\mathscr{D}_{K_\lambda}(\lambda)}{\mathscr{D}_{K_\lambda}(\lambda)}, \tag{5.4}$$

so (using equation (3.12)) the expression (5.3) is precisely the Taylor coefficient of λ^p in the expansion of the function on the right side of equation (5.4), provided $(p+1)$ is larger than the orders of $\Delta(\lambda, \lambda)$ and $\Delta(0, \lambda)$. Let

$$\mathscr{D}_{K_\lambda}(x, y, \lambda) = -\sum_{i=0}^{\infty} \frac{(-\lambda)^i}{i!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(x, y, \lambda) & K(x, y_1, \lambda) & \cdots & K(x, y_i, \lambda) \\ K(y_1, y, \lambda) & K(y_1, y_1, \lambda) & & \cdot \\ \vdots & \vdots & & \vdots \\ K(y_i, y, \lambda) & K(y_i, y_1, \lambda) & \cdots & K(y_i, y_i, \lambda) \end{vmatrix} dy_1 \cdots dy_i.$$

By a general formula[2] of Fredholm, modified by our notation, (see Fredholm [4, pp. 379–381])

$$\text{(5.5)} \quad \frac{\frac{d}{d\lambda}\mathscr{D}_{K_\lambda}(\lambda)}{\mathscr{D}_{K_\lambda}(\lambda)} = -\int_0^1 \frac{\partial}{\partial\lambda}[\lambda K(x, x, \lambda)]\, dx + \lambda \int_0^1 \int_0^1 \frac{\mathscr{D}_{K_\lambda}(x, y, \lambda)}{\mathscr{D}_{K_\lambda}(\lambda)} \frac{\partial}{\partial\lambda}[\lambda K(y, x, \lambda)]\, dx\, dy.$$

Now

$$\text{(5.6)} \quad \frac{\mathscr{D}_{K_\lambda}(x, y, \lambda)}{\mathscr{D}_{K_\lambda}(\lambda)} = -\sum_{j=0}^{\infty} \lambda^j K_{j+1}(x, y, \lambda),$$

so

$$\text{(5.7)} \quad \frac{-\frac{d}{\partial\lambda}\mathscr{D}_{K_\lambda}(\lambda)}{\mathscr{D}_{K_\lambda}(\lambda)} = \sum_{j=0}^{\infty} \lambda^j \int_0^1 K_{j+1}(x, x, \lambda)\, dx + \lambda \int_0^1 \frac{\partial}{\partial\lambda} K(x, x, \lambda)\, dx + \sum_{j=0}^{\infty} \lambda^{j+2} \int_0^1 \int_0^1 K_{j+1}(x, y, \lambda) \frac{\partial}{\partial\lambda} K(y, x, \lambda)\, dx\, dy.$$

Next we expand $K(x, y, \lambda)$ in powers of λ, i.e.,

$$\text{(5.8)} \quad K(x, y, \lambda) = {}_0K(x, y) + \lambda\, {}_1K(x, y) + \cdots,$$

and substitute this expression into (5.7). We will partially verify below that all necessary rearrangements and term by term integrations of series are valid. Proceeding formally, it is possible to collect terms in (5.7) according to powers of λ and thereby obtain, after considerable manipulation, an equation of the form

$$\text{(5.9)} \quad \frac{-\frac{d}{d\lambda}\mathscr{D}_{K_\lambda}(\lambda)}{\mathscr{D}_{K_\lambda}(\lambda)} = \sum_{p=0}^{\infty} a_p \lambda^p,$$

[2]Fredholm's notation is somewhat deceiving; his kernel $f(x, y)$ is actually dependent upon a hidden parameter. It is by varying this parameter that he computes the differential of his function D_f, although he does not say so explicitly.

where as we will show shortly,[3] for $p \geqq 1$,

$$
\begin{aligned}
a_p = {} & \int_0^1 {}_0K_{p+1}(x,x)\,dx + (p+1)\int_0^1 {}_pK(x,x)\,dx \\
& + \sum_{j=1}^{p-1} \sum_{\substack{i_1+\cdots+i_{j+1}=p-j \\ i_1,i_2,\cdots \geqq 0}} \int_0^1 \cdots \int_0^1 {}_{i_1}K(x,z_1)\,{}_{i_2}K(z_1,z_2) \\
& \qquad \cdots {}_{i_{j+1}}K(z_j,x)\,d(z_1,\cdots,z_j,x) \\
& + \sum_{j=0}^{p-2} \sum_{h=0}^{p-j-2} \sum_{\substack{i_1+\cdots+i_{j+1}=h \\ i_1,i_2,\cdots \geqq 0}} (p-j-h-1)\int_0^1 \cdots \int_0^1 {}_{i_1}K(x,z_1) \\
& \cdot {}_{i_2}K(z_1,z_2)\cdots {}_{i_{j+1}}K(z_j,y)\,{}_{p-j-h-1}K(y,x)\,d(z_1,\cdots,z_j,x,y).
\end{aligned} \tag{5.10}
$$

Combining this result with (5.4), we obtain

$$\sum_{i=1}^{\infty}\left[\frac{1}{\lambda_i}\right]^{p+1} - \sum_{i=1}^{\infty}\left[\frac{1}{\mu_i}\right]^{p+1} = a_p, \tag{5.11}$$

where a_p is given by (5.10). The (standard) notation ${}_iK_j$ denotes the jth iterated kernel of ${}_iK$. Clearly, if ${}_iK(x,y) \geqq 0$ for each nonnegative integer $i \leqq p$, then existence of eigenvalues or poles is assured if (5.11) holds for that p. In the case in which $K(x, y, \lambda) = {}_0K(x, y)$, the equations (5.11) reduce to

$$\sum_{i=1}^{\infty}\left[\frac{1}{\lambda_i}\right]^{p+1} = \int_0^1 {}_0K_{p+1}(x,x)\,dx, \tag{5.12}$$

as we should expect.

In general, (5.11) holds if $p + 1$ is larger than the orders of $\Delta(\lambda, \lambda)$ and $\Delta(0, \lambda)$. It is clear that equations (5.11) hold for $p + 1 > \max(s + 1, t)$; we must recall that λN_λ is a polynomial of order $s + 1$ in the variable λ. Equations (5.11) are valid also for $p + 1 > \max(t, (s + 1)/(k + 1))$, where k is the number given in Theorem 2. If Theorem 3 is applicable, then we have validity of (5.11) for $p + 1 > \max[t, (s + 1)/(m - n)]$. Since the μ_i are zeroes of $\Delta(0, \lambda)$, expansion of this function in powers of λ and use of formula (3.16) yields expressions for the sums $\sum_{i=1}^{\infty}[1/\mu_i]^{p+1}$, valid for $p + 1 > t$.

For $p = 0, 1$, and 2, equations (5.11) reduce to

$$\sum_{i=1}^{\infty}\left[\frac{1}{\lambda_i}\right] - \sum_{i=1}^{\infty}\left[\frac{1}{\mu_i}\right] = \int_0^1 {}_0K(x,x)\,dx, \tag{5.13}$$

$$\sum_{i=1}^{\infty}\left[\frac{1}{\lambda_i}\right]^2 - \sum_{i=1}^{\infty}\left[\frac{1}{\mu_i}\right]^2 = 2\int_0^1 {}_1K(x,x)\,dx + \int_0^1 {}_0K_2(x,x)\,dx, \tag{5.14}$$

$$
\begin{aligned}
\sum_{i=1}^{\infty}\left[\frac{1}{\lambda_i}\right]^3 - \sum_{i=1}^{\infty}\left[\frac{1}{\mu_i}\right]^3 = {} & 3\int_0^1 {}_2K(x,x)\,dx \\
& + 3\int_0^1\int_0^1 {}_1K(x,y)\,{}_0K(y,x)\,dx\,dy \\
& + \int_0^1 {}_0K_3(x,x)\,dx.
\end{aligned} \tag{5.15}
$$

[3] If $j = 0$, the integrand in the second series in (5.10) is ${}_{i_1}K(x,y)\,{}_{p-h-1}K(y,x)$.

Equation (5.13) is valid if $t < 1$, $s < m - n - 1$, and if $p_j \in C^{(m-j)}[0, 1]$, where the p_j are given in Theorem 3. Equation (5.14) is valid if $t < 2$, $s < 2(m - n) - 1$ and if $p_j \in C^{(m-j)}[0, 1]$.

If Theorem 3 is inapplicable, then assume that the hypothesis on N_λ in Theorem 2 is valid. Equation (5.13) holds if $t < 1$ and $s < k$. Equation (5.14) holds if $t < 2$ and $s < 2k + 1$. Here $k \leqq m - n - 1$.

If $m = n + 1$, the appearance of the term $\int_0^1 {}_pK(x, x)\,dx$ on the right side of (5.10) may cause concern, since ${}_pK(x, x)$ seems to vary depending on how we define $K(x, x, \lambda)$ if $K(x, y, \lambda)$ has a jump across the line $y = x$. This is, in fact, only partially true; the values of ${}_pK(x, x)$ are independent of how we define $K(x, x, \lambda)$ provided $p > s$, as we shall see shortly when we examine the structure of $K(x, y, \lambda)$. This is, of course, good reason why we should expect that the formulas (5.11) are generally not valid when $p \leqq s$ unless further assumptions are made.

An explanation of (5.10) is in order here. Let the first double series in (5.10) be written as $\sum_{j=1}^{p-1} b_j^{(p)}$, where the $b_j^{(p)}$ are sums of integrals over the variables $i_1, \cdots, i_{j+1}$. Similarly, let the second double series in (5.10) be written as $\sum_{j=0}^{p-2} c_j^{(p)}$. The latter series vanishes, of course, for $p = 0$ or $p = 1$. Let

$$(5.16) \qquad b_0^{(p)} = \int_0^1 {}_pK(x, x)\,dx, \qquad b_p^{(p)} = \int_0^1 {}_0K_{p+1}(x, x)\,dx;$$

then the right side of (5.10) is precisely

$$(5.17) \qquad \sum_{j=0}^{p} b_j^{(p)} + \sum_{j=0}^{p-2} c_j^{(p)} + p\int_0^1 {}_pK(x, x)\,dx.$$

First we note that

$$(5.18) \qquad p\int_0^1 {}_pK(x, x)\,dx$$

in (5.17) is precisely the coefficient of λ^p in the MacLaurin expansion of

$$(5.19) \qquad \lambda\int_0^1 \frac{\partial}{\partial\lambda}K(x, x, \lambda)\,dx$$

in (5.7).

Next we will show that $\sum_{j=0}^{p} b_j^{(p)}$ in (5.17) is precisely the coefficient of λ^p in the MacLaurin expansion of

$$(5.20) \qquad \sum_{j=0} \lambda^j \int_0^1 K_{j+1}(x, x, \lambda)\,dx$$

in (5.7). If we write

$$(5.21) \qquad K(x, z_1, \lambda)K(z_1, z_2, \lambda)\cdots K(z_j, y) = \sum_{h=0}^{\infty} f_h^{(j)}(x, y, z_1, \cdots, z_j)\lambda^h,$$

then

$$(5.22) \qquad f_h^{(j)}(x, y, z_1, \cdots, z_j) = \sum_{\substack{i_1+\cdots+i_{j+1}=h \\ i_1,\cdots,i_{j+1}\geqq 0}} {}_{i_1}K(x, z_1)\,{}_{i_2}K(z_1, z_2)\cdots {}_{i_{j+1}}K(z_j, y).$$

Let

$$g_h^{(j)}(x, y) = \int_0^1 \cdots \int_0^1 f_h^{(j)}(x, y, z_1, \cdots, z_j)\, d(z_1, \cdots, z_j). \tag{5.23}$$

Then

$$b_j^{(p)} = \int_0^1 g_{p-j}^{(j)}(x, x)\, dx. \tag{5.24}$$

Note that the integral of the left side of (5.21) over the square $0 \leqq z_1, z_2, \cdots, z_j \leqq 1$ is precisely $K_{j+1}(x, y, \lambda)$ and that

$$K_{j+1}(x, y, \lambda) = \sum_{h=0}^{\infty} \lambda^h g_h^{(j)}(x, y). \tag{5.25}$$

Justification of the term by term integration needed to obtain (5.25) will be done later.

We maintain that

$$\sum_{j=0}^{\infty} \lambda^j \int_0^1 K_{j+1}(x, x, \lambda)\, dx$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} \lambda^{j+h} \int_0^1 g_h^{(j)}(x, x)\, dx \tag{5.26}$$

$$= \sum_{p=0}^{\infty} \sum_{j=0}^{p} \lambda^p \int_0^1 g_{p-j}^{(j)}(x, x)\, dx \tag{5.27}$$

$$= \sum_{p=0}^{\infty} \lambda^p \sum_{j=0}^{p} b_j^{(p)}. \tag{5.28}$$

Equation (5.26) follows from (5.25). If we set $p = j + h$ and eliminate h in the right side of (5.26), we get (5.27). The last equality follows from (5.24). Finally, the rearrangement of terms in (5.27) will be justified later when we examine the structure of $K(x, y, \lambda)$.

By (5.26)–(5.28), we see that the coefficient of λ^p in the MacLaurin expansion of the first series on the right side of (5.7) is precisely $\sum_{j=0}^{p} b_j^{(p)}$.

It remains to be shown that the series $\sum_{j=0}^{p-2} c_j^{(p)}$ in (5.17) is the coefficient of λ^p in the MacLaurin expansion of the last series on the right side of (5.7). Here we will be brief. From (5.25),

$$\begin{aligned} &\lambda^{j+2} \int_0^1 \int_0^1 K_{j+1}(x, y) \frac{\partial}{\partial \lambda} K(y, x, \lambda)\, dx\, dy \\ &= \lambda^{j+2} \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} (i + 1)\lambda^{h+i} \int_0^1 \int_0^1 g_h^{(j)}(x, y)\, {}_{i+1}K(y, x)\, dx\, dy. \end{aligned} \tag{5.29}$$

Letting $k = h + i + 2$ in the right side of (5.29), and eliminating i, we obtain the equality of the right side of (5.29) and of

$$\lambda^j \sum_{k=2}^{\infty} \sum_{h=0}^{k-2} \lambda^k (k - h - 1) \int_0^1 \int_0^1 g_h^{(j)}(x, y)\, {}_{k-h-1}K(y, x)\, dx\, dy. \tag{5.30}$$

We sum (5.29) and (5.30) from $j = 0$ to $j = +\infty$, set $p = j + k$, and eliminate the variable k from the summation. The result will be that $\sum_{j=0}^{p-2} c_j^{(p)}$ in (5.17) is the coefficient of λ^p in the MacLaurin expansion of the last term (series) on the right side of (5.7). Again, certain term by term integrations and rearrangements would have to be justified.

Hence we have the relation between (5.7) and (5.17), namely that (5.17) is the coefficient of λ^p in the MacLaurin expansion of the right side of (5.7).

We now will justify the term by term integration needed to obtain (5.25) and the rearrangement of terms in (5.27).

In Coddington–Levinson [2, p. 204, prob. 12], we note the formula given (in our notation) for $M(x, y, \lambda)$, the Green's function for (2.9) (see (4.1)–(4.2)):

$$\Delta(0,\lambda)M(x,y,\lambda) = \begin{vmatrix} S(x,y) & \phi_1(x) \cdots \phi_m(x) \\ U_1[S(\cdot,y),\lambda] & U_1[\phi_1,\lambda] \cdots U_1[\phi_m,\lambda] \\ \vdots & \vdots \qquad\quad \vdots \\ U_m[S(\cdot,y),\lambda] & U_m[\phi_1,\lambda] \cdots U_m[\phi_m,\lambda] \end{vmatrix}. \tag{5.31}$$

(Coddington–Levinson's variable l is our σ.) In (5.31), the operator $U_j[\cdot,\lambda]$ operates on $S(x, y)$ as a function of x; the result is independent of x as the notation $U_j[S(\cdot, y), \lambda]$ indicates. If we operate on (5.31) with $(N_\lambda)_1$ and expand in cofactors via the first row, then we get

$$K(x,y,\lambda) = H(x,y,\lambda) + \sum_{i=1}^{m} \frac{X_i(x,\lambda)Y_i(y,\lambda)}{p(\lambda)}, \tag{5.32}$$

where

$$H(x,y,\lambda) = [(N_\lambda)_1 S](x,y), \tag{5.33}$$

$$p(\lambda) = \Delta(0,\lambda), \tag{5.34}$$

and

$$X_i(x,\lambda) = (N_\lambda\phi_i)(x,0,\lambda) = (N_\lambda\phi_i)(x). \tag{5.35}$$

Here, $[-Y_i(y, \lambda)]$ is the determinant obtained by replacing the ith column of $\Delta(0, \lambda)$ by

$$\begin{matrix} U_1[S(\cdot,y),\lambda] \\ \vdots \\ U_m[S(\cdot,y),\lambda]. \end{matrix}$$

If we let

$$H(x,y,\lambda) = \sum_{i=0}^{s} \lambda^i \,{}_iH(x,y), \tag{5.36}$$

then

$$\sum_{i=0}^{s} |\lambda^i|\,|{}_iH(x,y)| \tag{5.37}$$

is piecewise continuous on $[0, 1]^2$, with a possible simple jump across the line $y = x$. Similarly, if we let

$$X_i(x, \lambda) = \sum_{j=0}^{s} X_{ij}(x)\lambda^j, \tag{5.38}$$

then

$$\sum_{j=0}^{s} |X_{ij}(x)|\,|\lambda^j| \tag{5.39}$$

is continuous for $x \in [0, 1]$.

Finally, if

$$Y_i(y, \lambda) = \sum_{j=0}^{\infty} Y_{ij}(y)\lambda^j, \tag{5.40}$$

then we will show that

$$\sum_{j=0}^{\infty} |Y_{ij}(y)|\,|\lambda^j| \tag{5.41}$$

is continuous in the variables (y, λ) and uniformly convergent in (y, λ) near $\lambda = 0$. If necessary, we agree that

$$\left[\frac{\partial^{(m-1)}}{\partial x^{(m-1)}} S\right](0, 0) = \lim_{y \to 0^+} \left[\frac{\partial^{(m-1)}}{\partial x^{(m-1)}} S\right](0, y) \tag{5.42}$$

with a similar definition of the derivative at $(x, y) = (1, 1)$ via a limit as $y \to 1^-$ with $x = 1$. The uniform convergence (and continuity) of the series (5.41) in (y, λ) near $\lambda = 0$ follows from the particularly simple form $Y_i(y, \lambda)$ takes upon expansion of this determinant, namely,

$$Y_i(y, \lambda) = \sum_{k=0}^{m-1} \left\{ a_{ik}(\lambda)\left[\frac{\partial^k S}{\partial x^k}\right](0, y) + b_{ik}(\lambda)\left[\frac{\partial^k S}{\partial x^k}\right](1, y)\right\}, \tag{5.43}$$

where the functions $a_{ik}(\lambda)$ and $b_{ik}(\lambda)$ are certain entire functions. Incidentally, the continuity of the functions X_i and Y_i indicates that the functions ${}_pK(x, x)$ are independent of how we define $K(x, x, \lambda)$, provided $p > s$, since (5.36), the only possibly discontinuous part of $K(x, y, \lambda)$, terminates. Indeed, the functions ${}_pK(x, y)$ are continuous on $[0, 1]^2$ if $p > s$. If

$$\sum_{i=0}^{\infty} {}_iK(x, y)\lambda^i \tag{5.44}$$

is the Taylor expansion of $K(x, y, \lambda)$ about $\lambda = 0$, then the series

$$\sum_{i=0}^{\infty} |{}_iK(x, y)|\,|\lambda|^i \tag{5.45}$$

converges for $|\lambda|$ small and is piecewise continuous and uniformly convergent for $(x, y) \in [0, 1]^2$ and small λ as a result of the above.

Next we show that

$$\sum_{j=0}^{\infty} (j+1)|_{j+1}K(x,y)|\,|\lambda|^j \tag{5.46}$$

is piecewise continuous and uniformly convergent in the variables (x, y, λ) for small λ.

Let $\varepsilon > 0$ be small, and let C be the circle $|z| = \varepsilon$. If $|\lambda| < \varepsilon$, then

$$(j+1)|\lambda|^j = \frac{1}{2\pi i}\left|\int_C \frac{z^{j+1}}{(\lambda - z)^2}\,dz\right| \leqq \frac{\varepsilon^{j+2}}{(\varepsilon - |\lambda|)^2}. \tag{5.47}$$

Hence convergence of (5.46) is uniform in the asserted variables provided $|\lambda| \leqq \varepsilon/2$; this follows from the uniform convergence of (5.45).

If y is replaced by x in either (5.45) or (5.46), one can show in similar fashion that the resulting series are continuous in x, provided $H(x, x, \lambda) = H(x, x^+, \lambda)$ in accordance with the remarks after (2.18).

We give some indication of the justification of the interchange of sum and integral and the rearrangement which are needed in order to justify (5.25) and (5.27), respectively.

Let us assume $\lambda > 0$ and ${}_iK(x, y) \geqq 0$. Hence we replace λ and ${}_iK(x, y)$ by their magnitudes. The new function $K(x, y, \lambda)$ so formed is piecewise continuous and integrable. For each fixed $\varepsilon_0 > 0$, there exists a $\delta_0 > 0$ such that

$$\sum_{j=0}^{\infty} \lambda^j K_{j+1}(x, y, \varepsilon_0) \tag{5.48}$$

converges for $0 < \lambda < \delta_0$ to a Lebesgue integrable (indeed piecewise continuous) function (see Tricomi [11, p. 50]), and hence the right side of (5.6) converges to an integrable function for $0 < \lambda < \varepsilon$ if ε is small. We are using here the fact that (5.48) is a monotonic function of λ and ε_0 under the positivity assumptions made. Under the positivity assumptions, we have (see (5.21))

$$K(x, z_1, \lambda)K(z_1, z_2, \lambda) \cdots K(z_j, y, \lambda) \geqq \sum_{h=0}^{p} f_h^{(j)}(x, y, z_1, \cdots, z_j)\lambda^h. \tag{5.49}$$

Hence the term by term integration needed to obtain (5.25)[4] may be justified by the Lebesgue dominated convergence theorem. The rearrangement in (5.27) is valid since the series involved are absolutely convergent. Similarly the manipulations involving (5.29)–(5.30) can be justified; in order to do this, we would use the fact that (5.46) converges to a piecewise continuous function.

We would like to make a comment about the application of Theorem 3. It is best made through an example. Suppose we consider the system

$$\sum_{j=0}^{3} p_{3-j}u^{(j)} = \lambda \sum_{j=0}^{1} q_{1-j}u^{(j)} \tag{5.50}$$

plus suitable boundary conditions involving λ only as a polynomial for simplicity. Assume that $p_0 \in C^3[0, 1]$, $p_1 \in C^2[0, 1]$ and that all the other coefficients p_j, q_j are merely continuous. Theorem 3 cannot be applied directly, and Theorem 1 yields only the convergence of $\sum_{i=1}^{\infty} |\lambda_i|^{-(1+\varepsilon)}$ for $\varepsilon > 0$. Theorem 2 is inapplicable

[4] and (5.26).

because of insufficient regularity of the coefficients q_0 and q_1. We rewrite the system in the form

$$\sum_{j=2}^{3} p_{3-j}u^{(j)} = \sum_{j=0}^{1} q_{1-j}u^{(j)} - \sum_{j=0}^{1} p_{3-j}u^{(j)} \tag{5.51}$$

and now Theorem 3 yields convergence of the sum

$$\sum_{i=1}^{\infty} |\lambda_i|^{-(1/2+\varepsilon)}.$$

The comment we wish to make is as follows. If $K(x, y, \lambda)$ is given by (2.13), where $M(x, y, \lambda)$ is the Green's function of the operator on the left side of (5.50) for $|\lambda|$ small, if one exists, then (5.13) is *still valid.* This is true because of the following reasons. The function $\Delta(\lambda, \lambda)$ for (5.50), with

$$Mu = \sum_{j=0}^{3} p_{3-j}u^{(j)}, \qquad N_\lambda u = \sum_{j=0}^{1} q_{1-j}u^{(j)}$$

and $Mu = \lambda N_\lambda u$, is the same as the function $\Delta(1, \lambda)$ of (5.51), with

$$Mu = \sum_{j=2}^{3} p_{3-j}u^{(j)}, \qquad N_\lambda u = \lambda \sum_{j=0}^{1} q_{1-j}u^{(j)} - \sum_{j=0}^{1} p_{3-j}u^{(j)}$$

and $Mu = N_\lambda u$. The series $\sum_{i=1}^{\infty} \lambda_i^{-p}$ is precisely the same for both interpretations of the original system. However, the function $\Delta(0, \lambda)$ and the series $\sum_{i=1}^{\infty} \mu_i^{-p}$ are different for each of the two interpretations of the original system, but the order of $\Delta(0, \lambda)$ is zero in each interpretation, since the boundary conditions involve polynomials in λ, and $t = 0$ in each case.

Finally we note that the Green's function $M(x, y)$ for the system

$$u' = i\lambda u,$$

$$u(0) = u(1)$$

is discontinuous along the line $y = x$. However, $\Delta(\lambda)$ and $D_K(\lambda)$ are of order one. Carleman [13] has exhibited continuous kernels (which are evidently not Green's functions) such that the order of $D_K(\lambda)$ is precisely two. Evidently, then, Green's functions have very special properties, even when they are discontinuous.

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SUCCESSIVE APPROXIMATIONS OF SOME NONLINEAR INITIAL-BOUNDARY VALUE PROBLEMS*

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Abstract. The method of successive approximations is used to show the existence of a unique solution for a nonlinear initial-boundary value problem for the heat equation. A constructive method for the determination of approximate solutions and their error estimates is given. The existence problem includes both global and local solutions as well as continuation of a local solution. The space domain can either be bounded or unbounded. Some qualitative analysis on the boundedness of the solution and the stability of an equilibrium solution is included. The mathematical system considered in the paper is motivated by a physical problem arising from the theory of tubular chemical reactors. An application to this particular problem is given.

1. Introduction. Let Ω be an open domain in the n-dimensional Euclidean space R^n with boundary $\partial\Omega$ which consists of a bounded part $\partial\Omega_1$ and an unbounded part $\partial\Omega_2$. Consider the following initial-boundary value problem:

$$u_t - Lu \equiv u_t - \left(\sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(t,x)u_{x_i} + c(t,x)u \right) = f(t,x,u), \quad t \in (0,T], \quad x \in \Omega, \tag{1.1}$$

$$\alpha(t,x)\frac{\partial u}{\partial \nu} + \beta(t,x)u = h(t,x), \qquad t \in (0,T], \quad x \in \partial\Omega_1, \tag{1.2}$$
$$\lim_{x \to \partial\Omega_2} u(t,x) = 0, \qquad t \in (0,T],$$

$$u(0,x) = \phi(x), \qquad x \in \Omega, \tag{1.3}$$

where ν is the unit outward normal vector on $\partial\Omega_1$, $\alpha \geqq 0$, $\beta > 0$ are bounded functions on $[0,T] \times \partial\Omega_1$ and h, ϕ are continuous functions on $[0,T] \times \partial\Omega_1$ and $\bar{\Omega}$ (the closure of Ω), respectively. The coefficients a_{ij}, b_i, c of L are continuous on $[0,T] \times \bar{\Omega}$ and there exists a constant $a_0 > 0$ such that for every vector $\xi = (\xi_1, \cdots, \xi_n)$ in R^n, $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geqq a_0|\xi|^2$ for $(t,x) \in [0,T] \times \bar{\Omega}$. The function f is assumed to be continuous with respect to all its arguments. Either the bounded part $\partial\Omega_1$ or the unbounded part $\partial\Omega_2$ of the boundary $\partial\Omega$ is allowed to be empty. Thus Ω is a bounded domain if $\partial\Omega_2$ is empty and Ω is the whole space R^n if $\partial\Omega_1$ is empty. In general, $\partial\Omega$ consists of a bounded part and an unbounded part such as the exterior of a sphere or a half-space, etc. The purpose of this paper is to show that the problem (1.1)–(1.3) has a unique solution (both global and local) for a class of nonlinear functions f and this solution can be constructed by the method of successive approximations as well as error estimates for the approximations. We also discuss the bounded property of the solution and the stability problem of a steady state (or equilibrium) solution.

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The problem (1.1)–(1.3) is a generalization of the following physical problem arising from the theory of tubular chemical reactors (cf. [2], [4]):

$$u_t - (au_{xx} + bu_x) = f_0(u) \equiv A(r - u)\exp\left(-\frac{B}{(1+u)}\right), \qquad t > 0, \quad 0 < x < 1, \tag{1.4}$$

$$\alpha_1 u_x(0) - \beta_1 u(0) = 0, \quad \alpha_2 u_x(1) + \beta_2 u(1) = 0, \tag{1.5}$$

$$u(0, x) = \phi(x), \qquad 0 < x < 1, \tag{1.6}$$

where $\alpha_i \geqq 0, \beta_i \geqq 0$ with $\alpha_i + \beta_i \neq 0\,(i = 1, 2)$, b is a real number and a, r, A, B are positive constants. We shall show that the problem (1.4)–(1.6) has a unique solution $u(t, x)$ satisfying $0 \leqq u(t, x) \leqq r$ provided that $0 \leqq \phi(x) \leqq r$. This implies in particular that every steady state solution $u_e(x)$ of the system satisfies $0 \leqq u_e(x) \leqq r$, a result that was established in [4] (see also [7]) by considering the corresponding boundary value problem directly.

The existence problem of (1.1)–(1.3) by the method of successive approximations has recently been discussed by Chan [3] and by Sattinger [10], and a number of earlier works on the subject have been referenced by the authors. In [3], Chan approximates the solution by solving an integro-differential equation and the construction of a Neumann function of a boundary-valued problem. Sattinger [10] uses the monotone iterate scheme by constructing a monotone increasing and a monotone decreasing sequence and then shows that these sequences converge from below and above, respectively, to the unique solution. Both authors have treated the problem with bounded space domains. In this paper we also use the concept of successive approximations, but we take a different process of approximations and a different approach in the convergence proof. This process requires indirectly some properties of the Green's function of the corresponding linear problem. The advantage of this approach is that it can be applied to a bounded or an unbounded space domain and leads to an error estimate for the approximations.

In § 2 we show the existence of a global solution and a local solution as well as its continuation. In § 3 we study the boundedness of a solution and the existence of a global solution for locally Lipschitz continuous functions. Section 4 deals with some stability problems of an equilibrium solution and § 5 gives an application of the existence and stability theorems to the problem (1.4)–(1.6).

2. The existence problem. Let $D = (0, T] \times \Omega$ with its closure denoted by $\bar{D}$ and let λ be any real number. Then by letting $v = e^{-\lambda t}u$ we transform the problem (1.1)–(1.3) into the form

$$v_t - (L - \lambda)v = e^{-\lambda t} f(t, x, e^{\lambda t} v), \qquad (t, x) \in D, \tag{2.1}$$

$$\begin{aligned} &\alpha(t, x)\frac{\partial v}{\partial \nu} + \beta(t, x)v = e^{-\lambda t} h(t, x), && t \in (0, T], \quad x \in \partial\Omega_1, \\ &\lim_{x \to \partial\Omega_2} v(t, x) = 0, && t \in (0, T], \end{aligned} \tag{2.2}$$

$$v(0, x) = \phi(x), \qquad x \in \Omega. \tag{2.3}$$

Thus the existence and uniqueness problem of (1.1)–(1.3) follows from the same as for (2.1)–(2.3). We denote by $C(\bar{D})$ the Banach space of all bounded continuous functions $v(t, x)$ on $\bar{D}$ with norm

$$\|v\| = \sup \{|v(t, x)| ; (t, x) \in \bar{D}\}.$$

A similar definition holds for $C(\bar{\Omega})$ with its norm denoted by $\|\cdot\|_{\Omega}$. Let

$$\mathscr{L}v \equiv v_t - (L - \lambda)v$$

and consider $\mathscr{L}$ as an operator in $C(\bar{D})$ into itself with domain $D(\mathscr{L})$ given by

$$D(\mathscr{L}) = \{v \in C(\bar{D}) ; v_t \in C(\bar{D}), v \in C^2(\bar{\Omega}) \text{ for each } t, \text{ and } v \text{ satisfying (2.2), (2.3)}\},$$

where $C^2(\bar{\Omega})$ denotes the set of all twice continuously differentiable functions on $\bar{\Omega}$. Consider F defined by $(F(v))(t, x) = e^{-\lambda t} f(t, x, e^{\lambda t} v)$ as a nonlinear operator on $C(\bar{D})$. Then the problem (2.1)–(2.3) is equivalent to the operator equation

$$\mathscr{L}v = F(v), \qquad v \in D(\mathscr{L}), \tag{2.4}$$

in the Banach space $C(\bar{D})$. The requirement of v in $D(\mathscr{L})$ in (2.4) insures that v satisfies the boundary and initial conditions (2.2), (2.3).

Let S be a closed subset of $C(\bar{D})$. Consider the linear equation

$$\mathscr{L}v \equiv v_t - (L - \lambda)v = g(t, x), \qquad (t, x) \in D, \tag{2.5}$$

together with the boundary and initial conditions (2.2), (2.3), where $g(t, x)$ is in $C(\bar{D})$. If to each $g \in S$ the linear problem (2.5), (2.2), (2.3) has a unique solution in S, then for any $v_0 \in S$ the system

$$\left.\begin{aligned}
&(\mathscr{L}v_m)(t, x) = F(v_{m-1})(t, x), && ((t, x) \in D)\\
&\alpha(t, x)\frac{\partial v_m}{\partial \nu} + \beta(t, x)v_m = e^{-\lambda t}h(t, x), && (t \in (0, T], \quad x \in \partial\Omega_1)\\
&\lim_{x \to \partial\Omega_2} v_m(t, x) = 0, && (t \in (0, T])\\
&v_m(0, x) = \phi(x), && (x \in \Omega)
\end{aligned}\right\} m = 1, 2, \cdots, \tag{2.6}$$

determines a sequence $\{v_m\}$ in S. Our aim is to show that if $f(t, x, u)$ satisfies a Lipschitz condition (in u) then the sequence $\{v_m\}$ converges to a unique solution of the problem (2.1)–(2.3). Before proving this we derive some properties for the operator $\mathscr{L}$. For convenience, we set $c_0 \equiv \sup \{c(t, x) ; (t, x) \in \bar{D}\}$.

LEMMA 2.1. *Let $v_1, v_2 \in D(\mathscr{L})$ and let $v = v_1 - v_2$. Then there exists $t_0 \in (0, T]$, $x_0 \in \Omega$ such that $\|v\| = |v(t_0, x_0)|$ and*

$$v(t_0, x_0)(\mathscr{L}v_1 - \mathscr{L}v_2)(t_0, x_0) \geqq (\lambda - c(t_0, x_0))|v(t_0, x_0)|^2. \tag{2.7}$$

Furthermore for any $\lambda > c_0$ the inverse operator $\mathscr{L}^{-1}$ of $\mathscr{L}$ exists on $R(\mathscr{L})$, the range of $\mathscr{L}$, and

$$\|\mathscr{L}^{-1}w_1 - \mathscr{L}^{-1}w_2\| \leqq (\lambda - c_0)^{-1}\|w_1 - w_2\|, \qquad w_1, w_2 \in R(\mathscr{L}). \tag{2.8}$$

Proof. It is obvious that (2.7) holds when $v \equiv 0$. We assume that $v \not\equiv 0$. Then for any (t_0, x_0) in $\bar{D}$ such that $\|v\| = |v(t_0, x_0)|$ we must have $t_0 \in (0, T]$ and $x_0 \in \Omega$. For from $v \not\equiv 0$ we have $x_0 \notin \partial\Omega_2$ and from $v(0, x) = \phi(x) - \phi(x) = 0$,

$t_0 \neq 0$. Now if $x_0 \in \partial\Omega_1$, then from the boundary condition (2.2) we would have $v(t_0, x_0) = 0$ if $\alpha(t_0, x_0) = 0$ and $\partial v(t_0, x_0)/\partial \nu = -(\beta(t_0, x_0)/\alpha(t_0, x_0))v(t_0, x_0)$ if $\alpha(t_0, x_0) \neq 0$. These are not possible since $\beta > 0$ and $v(t_0, x_0)$ is either a positive maximum or a negative minimum in $\bar{\Omega}$ (with t_0 fixed). Knowing $x_0 \in \Omega$ and $t_0 \neq 0$ we obtain $v_{x_i}(t_0, x_0) = 0\,(i = 1, \cdots, n)$, $v_t(t_0, x_0) = 0$ if $t_0 \neq T$, and

$$\sum_{i,j=1}^{n} a_{ij}(t_0, x_0)v_{x_ix_j}(t_0, x_0) \lesseqgtr 0 \quad \text{according to } v(t_0, x_0) \gtrless 0 \tag{2.9}$$

(cf. [6, p. 34]). The above relations imply that

$$v(t_0, x_0) \sum_{i,j=1}^{n} a_{ij}(t_0, x_0)v_{x_ix_j}(t_0, x_0) \leqq 0. \tag{2.10}$$

It follows that

$$v(t_0, x_0)(\mathscr{L}v_1 - \mathscr{L}v_2)(t_0, x_0) \geqq v(t_0, x_0)v_t(t_0, x_0) + (\lambda - c(t_0, x_0))|v(t_0, x_0)|^2 \tag{2.11}$$

and (2.7) follows from (2.11) if $t_0 \neq T$. In case $t_0 = T$ then $v_t(T, x_0) \geqq 0$ or $\leqq 0$ depending on whether $v(T, x_0) > 0$ or < 0, respectively. Thus $v(T, x_0)v_t(T, x_0) \geqq 0$ and (2.7) also holds. Finally if $\lambda > c_0$ then from (2.7),

$$\begin{aligned} \|v_1 - v_2\|\,\|\mathscr{L}v_1 - \mathscr{L}v_2\| &\geqq v(t_0, x_0)(\mathscr{L}v_1 - \mathscr{L}v_2)(t_0, x_0) \\ &\geqq (\lambda - c_0)\|v_1 - v_2\|^2. \end{aligned} \tag{2.12}$$

Hence $\mathscr{L}^{-1}$ exists and (2.8) holds.

THEOREM 2.1. *Assume that $f(t, x, \cdot)$ maps a closed subset S of $C(\bar{D})$ into itself, and for any g in S the linear problem* (2.5), (2.2), (2.3) *has a unique solution in S. If there exists a constant k such that*

$$|f(t, x, \eta_1) - f(t, x, \eta_2)| \leqq k|\eta_1 - \eta_2|, \qquad (t, x) \in \bar{D}, \quad -\infty < \eta_1, \eta_2 < \infty, \tag{2.13}$$

then the nonlinear problem (1.1)–(1.3) *has a unique solution in S. Moreover given any $\lambda > k + c_0$ and any v_0 in S, the sequence $\{v_m\}$ determined from* (2.6) *converges to a unique solution $v(t, x)$ of* (2.1)–(2.3) *in S and*

$$\|v - v_m\| \leqq \frac{k}{\lambda - k - c_0}\left(\frac{k}{\lambda - c_0}\right)^{m-1} \|v_1 - v_0\|, \quad m = 1, 2, \cdots. \tag{2.14}$$

Proof. Let $w \in S$ and let $F(t, x, w(t, x)) \equiv g(t, x)$. Then by hypothesis, $g \in S$ and there exists $v \in D(\mathscr{L}) \cap S$ such that $(\mathscr{L}v)(t, x) = F(t, x, w(t, x))$. By Lemma 2.1 we may write $v = \mathscr{L}^{-1}F(w)$. Since condition (2.13) implies that for any $w_1, w_2 \in S$

$$|f(t, x, w_1) - f(t, x, w_2)| \leqq k\, e^{-\lambda t}|e^{\lambda t}w_1 - e^{\lambda t}w_2| = k|w_1 - w_2|,$$

we obtain $\|F(w_1) - F(w_2)\| \leqq k\|w_1 - w_2\|$. It follows from Lemma 2.1 that

$$\|\mathscr{L}^{-1}F(w_1) - \mathscr{L}^{-1}F(w_2)\| \leqq k(\lambda - c_0)^{-1}\|w_1 - w_2\|. \tag{2.15}$$

By choosing $\lambda > k + c_0$, the operator $(\mathscr{L}^{-1}F)$ is a contraction mapping on S. Therefore the sequence $\{v_m\}$ defined successively by

$$v_m = \mathscr{L}^{-1}F(v_{m-1}), \qquad m = 1, 2, \cdots, \tag{2.16}$$

converges to a unique element $v \in S$ such that $v = \mathscr{L}^{-1}F(v)$. This shows that

$v \in D(\mathscr{L})$ and $\mathscr{L}v = F(v)$. Hence the problem (2.1)–(2.3) (and thus (1.1)–(1.3)) has a unique solution in S. Since the sequence given in (2.16) is the same as that given in (2.6) and since the contraction constant of $(\mathscr{L}^{-1}F)$ is $k(\lambda - c_0)^{-1}$, the convergence of (2.6) and the inequality (2.14) follow from the standard proof of the contraction mapping theorem. This completes the proof of the theorem.

Remarks 2.1. (a) In case S is an arbitrary set not necessarily closed then the sequence v_m in (2.16) converges to an element v in $C(\bar{D})$. If we let $\mathscr{L}_c$ be the closure of $\mathscr{L}$ (in the sense that if $w_n \in D(\mathscr{L})$, $w_n \to w$ and $\mathscr{L}w_n \to v$ in $C(\bar{D})$ then $w \in D(\mathscr{L}_c)$ and $\mathscr{L}_c w = v$. Notice that $D(\mathscr{L})$ is not a linear subspace) then $\mathscr{L}_c^{-1}$ exists and satisfies (2.8) as can be seen from (2.12). Since $\mathscr{L}v_m = F(v_{m-1})$ and $F(v_{m-1}) \to F(v)$ as $m \to \infty$ we have $v \in D(\mathscr{L}_c)$ and $\mathscr{L}_c v = F(v)$. In this case, the solution $v(t, x)$ is in the above extended sense. However, it is easy to show that $v(t, x)$ satisfies the conditions (2.2), (2.3) and by definition $\mathscr{L}_c v = \mathscr{L}v$ for $v \in D(\mathscr{L})$. We see that if $v_t \in C(\bar{D})$ and $v \in C^2(\bar{\Omega})$ for each t the function $v(t, x)$ is the classical solution of (2.1)–(2.3).

(b) The existence of a solution $v(t, x)$ to the linear problem (2.5), (2.2), (2.3) can be insured under sufficiently smooth conditions on the boundary of Ω and some conditions of Hölder continuity on f, ϕ, h, α, β and the coefficients of L, where S may be chosen as the set of Hölder continuous functions (cf. [8, p. 320]). Using the a priori estimate for v, one can show that $v \in D(\mathscr{L})$ so that v is the solution in the classical sense. On the other hand, from Theorem 2.1 if the Green's function of the linear problem is obtained, then the solution of the nonlinear problem (2.1)–(2.3) can be constructed from (2.6) by successive approximations.

In Theorem 2.1 it is assumed that f satisfies a global Lipschitz condition so that the "global solution" $v(t, x)$ can be constructed by successive approximations. However, many classes of nonlinear functions such as polynomials satisfy only local Lipschitz conditions. It is therefore desirable to know whether solutions to the problem (1.1)–(1.3) exist for locally Lipschitz continuous functions. The following theorem shows the existence of a unique "local solution".

THEOREM 2.2. *If in Theorem* 2.1 *the condition* (2.13) *is replaced by the local Lipschitz condition*

$$(2.17) \qquad |f(t, x, \eta_1) - f(t, x, \eta_2)| \leqq k|\eta_1 - \eta_2|, \qquad \eta_1, \eta_2 \in [-r, r],$$

for some $r > 0$ *then for* $|\phi(x)| < r$ *on* $\bar{\Omega}$ *the problem* (1.1)–(1.3) *has a unique "local solution"* $u(t, x)$ *in the sense that for some* $T_0 > 0$, $u(t, x)$ *satisfies* (1.1)–(1.3) *for all* $x \in \bar{\Omega}$ *and* $t \in [0, T_0]$. *The value of* T_0 *is determined by the largest interval* $[0, T_0]$ *on which* $|u(t, x)| \leqq r$ *on* $\bar{\Omega}$. *If, in addition,* f *satisfies* (2.17) *for every finite* r *where* $k = k(r) < \infty$, *then* $u(t, x)$ *can be continued (in* t*) for as long as it remains bounded on* $\bar{\Omega}$.

Proof. Define a modification for f by

$$(2.18) \qquad \bar{f}(t, x, \eta) = \begin{cases} f(t, x, -r) & \text{if } \eta \leqq -r, \\ f(t, x, \eta) & \text{if } |\eta| \leqq r, \\ f(t, x, r) & \text{if } \eta \geqq r. \end{cases}$$

Then $\bar{f}$ is continuous in η and satisfies (2.13). Thus by Theorem 2.1 the problem (1.1)–(1.3) with f replaced by $\bar{f}$ has a unique solution $\bar{u}(t, x)$. Since $|\bar{u}(0, x)| = |\phi(x)| < r$, there exists $T_0 > 0$ such that $|\bar{u}(t, x)| \leqq r$ for $t \in [0, T_0]$. We choose T_0 with the largest possible value such that $|\bar{u}(t, x)| \leqq r$ on $[0, T_0] \times \bar{\Omega}$. But $\bar{f}$ coincides with f when $|\bar{u}(t, x)| \leqq r$. We see that $\bar{u}(t, x)$ is the unique solution of (1.1)–(1.3) for $t \in [0, T_0]$. In case f satisfies (2.17) for every finite r, where k may depend on r, we choose $\bar{u}(T_0, x)$ as the new initial element in (1.3) and define a modification $\bar{f}_1$ as in (2.18) with $r_1 > r$. Then by the same argument there exists a unique solution to (1.1)–(1.3) on $[T_0, T_1] \times \bar{\Omega}$ for some $T_1 > T_0$. Continuing this process and using the uniqueness property we can continue the solution for as long as it remains bounded on $\bar{\Omega}$. This proves the theorem.

An immediate consequence of Theorem 2.2 is the existence of a local solution of the equation

$$u_t - Lu = \sum_{j=0}^{N} d_j(t, x)u^j, \qquad (t, x) \in D, \tag{2.19}$$

under the boundary and initial conditions (1.2), (1.3).

COROLLARY. *Assume that the functions* $d_j(t, x)$, $j = 0, 1, 2, \cdots$, *are bounded continuous on* $\bar{D}$. *Then the problem* (2.19), (1.2), (1.3) *has a unique local solution* $u(t, x)$. *Furthermore* $u(t, x)$ *is either the solution on the whole domain* $\bar{D}$ *or it is unbounded on* $\bar{\Omega}$ *at some* $t \in (0, T]$.

Proof. Since the function $f = \sum_{j=0}^{N} d_j u^j$ satisfies the local Lipschitz condition (2.17) in every finite interval $[-r, r]$ for some $k = k(r) < \infty$ the conclusion in the corollary follows directly from Theorem 2.2.

The results in Theorems 2.1 and 2.2 are analogous to the Cauchy problem of ordinary differential equations. The possibility of the unboundedness of a solution for functions satisfying only local Lipschitz conditions may be demonstrated by the simple example $dy/dt = y^{\mu}$ $(\mu > 1)$ which has an unbounded solution over a finite interval of t for any initial element $y(0) \neq 0$. Thus in order to obtain bounded solutions of (1.1)–(1.3) on $\bar{D}$ for this class of functions, some additional assumptions seem to be imperative. These assumptions will be given in the following section.

3. The boundedness problem. In this section we give some boundedness properties of a solution of (1.1)–(1.3). These properties lead to the existence of a global solution on $\bar{D}$ for a class of functions satisfying local Lipschitz conditions. We recall that any continuous function $u(t, x)$ on $\bar{D}$ may be considered as a vector-valued function $u(t)$ in $C(\bar{\Omega})$ with $\|u(t)\|_{\Omega} = \sup\{|u(t, x)|; x \in \bar{\Omega}\}$ for each t in $[0, T]$. The following lemma plays an essential role in the study of the boundedness problem as well as the stability problem.

LEMMA 3.1. *Let* $u(t, x) \in C(\bar{D})$ *such that* $u(t, x) \to 0$ *as* $x \to \partial\Omega_2$ *and* $u_t(t_0, x) \in C(\bar{\Omega})$ *at* $t_0 \in [0, T)$. *Then there exists* $x_0 \in \bar{\Omega}$ (x_0 *depends on* t_0) *with* $\|u(t_0)\|_{\Omega} = |u(t_0, x_0)|$ *such that the right-derivative* $(d^+/dt)(|u(t, x_0)|)$ *exists at* $t = t_0$ *and*

$$|u(t_0, x_0)|\frac{d^+}{dt}(|u(t_0, x_0)|) \leqq u(t_0, x_0)u_t(t_0, x_0). \tag{3.1}$$

Proof. We first show that for any fixed $x \in \bar{\Omega}$, $(d^+/dt)(|u(t, x)|)$ exists at $t = t_0$ and

$$\frac{d^+}{dt}(|u(t_0, x)|) = \lim_{h \to 0^+} h^{-1}(|u(t_0, x) + hu_t(t_0, x)| - |u(t_0, x)|). \tag{3.2}$$

The limit in (3.2) exists since the function $g(h) \equiv h^{-1}(|u(t_0, x) + hu_t(t_0, x)| - |u(t_0, x)|)$ is monotonically nondecreasing in h and is uniformly bounded by $\pm\, |u_t(t_0, x)|$

(cf. [5, p. 3]). Since for each $h > 0$,

$$h^{-1}|[|u(t_0 + h, x)| - |u(t_0, x)|] - [|u(t_0, x) + hu_t(t_0, x)| - |u(t_0, x)|]|$$
$$\leqq h^{-1}|u(t_0 + h, x) - u(t_0, x) - hu_t(t_0, x)|,$$

we see by taking $h \to 0^+$ that (3.2) holds. Let $S_0 = \{x \in \bar{\Omega};\ \|u(t_0)\|_\Omega = |u(t_0, x)|\}$. Then by taking any $x_0 \in S_0$ in (3.2) we obtain

$$\frac{d^+}{dt}(|u(t_0, x_0)|) \leqq \lim_{h \to 0^+} h^{-1}(\|u(t_0) + hu_t(t_0)\|_\Omega - \|u(t_0)\|_\Omega). \tag{3.3}$$

Notice that the limit in (3.3) also exists. Since (3.1) holds for $|u(t_0, x_0)| = 0$ we need only to show the case where $|u(t_0, x_0)| \neq 0$. For convenience, we suppress the fixed t_0 in $u(t_0, x)$.

For each $h > 0$ we choose $x_h \in \bar{\Omega}$ such that $\|u + hu_t\|_\Omega = |u(x_h) + hu_t(x_h)|$. Then for any $x^* \in S_0$,

$$|u(x^*) + hu_t(x^*)| \leqq \|u + hu_t\|_\Omega = |u(x_h) + hu_t(x_h)|. \tag{3.4}$$

Moreover there exists $h_0 > 0$ such that for $h < h_0$, $x_h \notin \partial\Omega_2$. For if $x_h \in \partial\Omega_2$, then from the relation

$$\|u\|_\Omega \leqq |u(x_h) + hu_t(x_h)| + \|hu_t\|_\Omega \leqq 2h\|u_t\|_\Omega$$

we would have $\|u\|_\Omega < \|u\|_\Omega$ for all $h < h_0 \equiv \|u\|_\Omega(2\|u_t\|_\Omega)^{-1}$. We next show that $\{x_h\}$ contains a convergent subsequence. This is certainly true if $\{x_h\}$ is bounded. Assume $\{x_h\}$ is unbounded and contains no convergent subsequence. Then there exists a subsequence $\{x_{h_k}\}$ with its limit value on $\partial\Omega_2$. By taking x_{h_k} and h_k in (3.4) and letting $h_k \to 0$ we would have $|u(x^*)| \leqq 0$. Thus we obtain a contradiction. Without any loss we may assume $x_h \to x_0$ (say). Then from (3.4) we obtain $|u(x^*)| \leqq |u(x_0)|$ which shows that $x_0 \in S_0$. Notice that $x_0 \in (\Omega + \partial\Omega_1)$ and (3.3) holds with this particular choice of $x_0 \in S_0$.

Now if $u(x_0) \gtrless 0$ then for sufficiently small h, $u(x_h) \gtrless 0$ and $u(x_h) + hu_t(x_h) \gtrless 0$, and thus

$$h^{-1}(|u(x_h) + hu_t(x_h)| - |u(x_h)|) = (\operatorname{sgn} u(x_0))u_t(x_h). \tag{3.5}$$

Since for each $h > 0$

$$h^{-1}(|u(x_h) + hu_t(x_h)| - |u(x^*)|) \leqq h^{-1}(|u(x_h) + hu_t(x_h)| - |u(x_h)|), \tag{3.6}$$

we obtain from (3.5) and (3.6) that

$$\lim_{h \to 0^+} h^{-1}(\|u + hu_t\|_\Omega - \|u\|_\Omega) \leqq (\operatorname{sgn} u(x_0))u_t(x_0) = \frac{u(x_0)}{\|u\|_\Omega}u_t(x_0). \tag{3.7}$$

Therefore (3.1) follows from (3.3) and (3.7) which proves the lemma.

The following theorem gives a maximum property of a solution of (1.1)–(1.3). For simplicity, we assume in this section that $h(t, x) \equiv 0$ in (1.2). (This assumption can be removed by a suitable transformation.)

THEOREM 3.1. *Assume that there exists a constant $r > 0$ such that for all $(t, x) \in \bar{D}$*

$$\eta f(t, x, \eta) < -c(t, x)\eta^2 \quad \textit{when } 0 < |\eta| \leqq r. \tag{3.8}$$

Then for $|\phi(x)| \leqq r$ *every solution* $u(t, x)$ *of* (1.1)–(1.3) (*with* $h(t, x) \equiv 0$) *satisfies*

$$\|u(t)\|_\Omega \leqq \|\phi\|_\Omega \leqq r \quad \text{for all } t \in [0, T]. \tag{3.9}$$

Proof. Let t_0 be an arbitrary point in $[0, T]$. By Lemma 3.1 there exists $x_0 \in \bar{\Omega}$ (x_0 depends on t_0) such that $|u(t_0, x_0)| = \|u(t_0)\|_\Omega$ and

$$|u(t_0, x_0)| \frac{d^+}{dt}(|u(t_0, x_0)|) \leqq u(t_0, x_0)(Lu)(t_0, x_0) + u(t_0, x_0) f(t_0, x_0, u(t_0, x_0)). \tag{3.10}$$

Suppose that $|u(t_0, x_0)| \neq 0$. Then from the proof of Lemmas 2.1 and 3.1, $x_0 \in \Omega$ and (2.10) holds. It follows from (3.10), (3.8) that

$$|u(t_0, x_0)| \frac{d^+}{dt}(|u(t_0, x_0)|) \leqq c(t_0, x_0)|u(t_0, x_0)|^2 + u(t_0, x_0) f(t_0, x_0, u(t_0, x_0)) < 0 \tag{3.11}$$

whenever $0 < |u(t_0, x_0)| \leqq r$. Thus at any time $t_0 \in [0, T]$ where $|u(t_0, x)| \not\equiv 0$ on $\bar{\Omega}$ we have $(d^+/dt)(|u(t_0, x_0)|) < 0$. This shows that at the point x_0

$$|u(t, x_0)| \leqq |u(t_0, x_0)| \quad \text{for } t \in [t_0, t_0 + \delta]$$

for some $\delta > 0$. The above inequality holds for every $t_0 \in [0, T)$ and a corresponding x_0 so long as $0 < |u(t_0, x_0)| \leqq r$. But $|u(0, x)| = |\phi(x)| \leqq r$ on $\bar{\Omega}$. We conclude that $|u(t_0, x_0)| \leqq \|\phi\|_\Omega \leqq r$ at any such point $(t_0, x_0) \in \bar{D}$. The boundedness of $|u(t_0, x_0)|$ when $t_0 = T$ or $u(t_0, x_0) = 0$ is obvious. This proves (3.9) and thus the theorem.

Under a slightly stronger assumption than (3.8) we have the following.

COROLLARY 1. *If there exists* $\varepsilon > 0$ *such that for all* $t \in [0, \infty)$, $x \in \Omega$,

$$\eta f(t, x, \eta) \leqq -[c(t, x) + \varepsilon]\eta^2 \quad \text{when } |\eta| \leqq r, \tag{3.12}$$

then for $|\phi(x)| \leqq r$ *and any* ε_1 *with* $0 < \varepsilon_1 < \varepsilon$ *the solution* $u(t, x)$ *of* (1.1)–(1.3) *satisfies*

$$\|u(t)\|_\Omega \leqq e^{-\varepsilon_1 t}\|\phi\|_\Omega \qquad \text{for all } t \geqq 0. \tag{3.13}$$

Proof. Under the condition (3.12) the inequality (3.11) becomes

$$|u(t_0, x_0)| \frac{d^+}{dt}(|u(t_0, x_0)|) \leqq -\varepsilon |u(t_0, x_0)|^2. \tag{3.14}$$

It is easily seen from the boundedness of u_t that $|u(t, x)|$ is Lipschitz continuous in t for each $x \in \bar{\Omega}$. In fact,

$$\big||u(t, x)| - |u(s, x)|\big| \leqq |u(t, x) - u(s, x)| \leqq M|t - s|, \qquad t, s \in [0, T], \quad x \in \bar{\Omega},$$

where M is a constant independent of t and x. Since the usual rules of differentiation hold for right derivatives of a continuous function we may write (3.14) as

$$\frac{d^+}{dt}(e^{2\varepsilon_1 t_0}|u(t_0, x_0)|^2) \leqq -2(\varepsilon - \varepsilon_1)\, e^{2\varepsilon_1 t_0}|u(t_0, x_0)|^2 < 0. \tag{3.15}$$

It follows from the same argument as in the proof of the theorem that $e^{2\varepsilon_1 t_0}|u(t_0, x_0)|^2 \leqq \|\phi\|_\Omega^2 \leqq r^2$ for all $t_0 \geqq 0$ which is equivalent to (3.13).

THEOREM 3.2. *Assume that the condition* (3.8) *holds for* $|\eta| > r$ *instead of* $|\eta| \leqq r$. *Then for* $|\phi(x)| \leqq r$ *any solution* $u(t, x)$ *of* (1.1)–(1.3) *satisfies* $\|u(t)\|_\Omega \leqq r$. *If* (3.12) *holds for* $|\eta| > r$ *instead of* $|\eta| \leqq r$, *then for* $|\phi(x)| < \infty$, $\|u(t)\|_\Omega$ *is strictly decreasing in* t *so long as* $\|u(t)\|_\Omega > r$. *In this case there exists* $T_1 > 0$ *such that* $\|u(t)\|_\Omega \leqq r$ *for* $t \geqq T_1$.

Proof. Suppose that (3.8) holds for $|\eta| > r$. Then from the proof of Theorem 3.1 we have $\|u(t)\|_\Omega \leqq \|u(t_0)\|_\Omega$ for $t \in [t_0, t_0 + \delta]$ whenever $\|u(t_0)\|_\Omega > r$. Thus if $\|u(0)\|_\Omega \equiv \|\phi\|_\Omega \leqq r$, the value of $\|u(t)\|_\Omega$ cannot be greater than r as can be seen from the same argument as in the proof of Theorem 3.1. Similarly if (3.12) holds for $|\eta| > r$ then the proof of Corollary 1 shows that (3.15) is valid when $|u(t_0, x_0)| > r$. This implies that $\|u(t_0)\|_\Omega$ is strictly decreasing in t_0 when it has value greater than r. Hence for $\|\phi\|_\Omega \leqq r$, we obtain $\|u(t_0)\|_\Omega \leqq r$ for all $t_0 \geqq 0$ and for $\|\phi\|_\Omega > r$, $\|u(t_0)\|_\Omega$ is strictly decreasing in t_0 until it is bounded by r. This proves the theorem.

Remark 3.1. Let S_r denote the sphere with radius $r > 0$ in the space $C(\bar{\Omega})$. Then Theorem 3.2 states that under the condition (3.12) for $|\eta| > r$ any solution of (1.1)–(1.3) remains in S_r for all t if it starts from S_r, and it is "attracted" toward S_r until it reaches to S_r if it starts outside of S_r.

Based on the results in Theorems 2.2 and 3.1 it is possible to show the existence of a bounded solution on $\bar{D}$ for a class of nonlinear functions satisfying only local Lipschitz conditions.

THEOREM 3.3. *Assume that* f *satisfies* (2.17) *and* (3.8) *for some* $r > 0$. *Then for* $|\phi(x)| \leqq r$ *the problem* (1.1)–(1.3) (*with* $h(t, x) \equiv 0$) *has a unique solution* $u(t, x)$ *such that* $|u(t, x)| \leqq r$ *on* $\bar{D}$.

Proof. Define a modification $\bar{f}$ by (2.18). Then by Theorem 2.1 the modified problem (1.1)–(1.3) with f replaced by $\bar{f}$ has a unique solution $u(t, x)$. Since $\bar{f}$ also satisfies (3.8), Theorem 3.1 ensures that $|u(t, x)| \leqq r$ on $\bar{D}$. But $\bar{f}$ coincides with f when $|u(t, x)| \leqq r$. We conclude that $u(t, x)$ is the solution of the original problem (1.1)–(1.3).

COROLLARY. *Let* $d_0(t, x) = 0$, $d_j(t, x), j = 1, 2, \cdots$, *be bounded continuous on* $\bar{D}$. *If for some* $r > 0$,

$$\sum_{j=1}^{N} d_j(t, x)\eta^{j+1} < -c(t, x)\eta^2 \quad \text{for } 0 < |\eta| \leqq r, \tag{3.16}$$

then for $|\phi(x)| \leqq r$ *the problem* (2.19), (1.2), (1.3) *has a unique solution* $u(t, x)$ *such that* $|u(t, x)| \leqq r$ *on* $\bar{D}$.

Proof. This is a direct consequence of Theorem 3.3.

A simple example of the above corollary is the equation

$$u_t - Lu = d_1(t, x)u + d_3(t, x)u^3,$$

where $d_1(t, x) \leqq -c(t, x)$ and $d_3(t, x) < 0$ on $\bar{D}$. In this case the solution $u(t, x)$ exists on $\bar{D}$ and satisfies $|u(t, x)| \leqq \|\phi\|_\Omega$ for any $|\phi(x)| < \infty$ since (2.17), (3.8) holds for $r = \|\phi\|_\Omega$.

4. The stability problem. In this section we discuss the stability problem of an equilibrium (or steady state) solution $u_e(x)$. Definitions of stability and asymptotic stability of u_e are in the sense of Lyapunov (cf. [9]). Notice that an equilibrium solution $u_e(x)$ may be considered as a solution of (1.1)–(1.3) with $\phi(x) = u_e(x)$, $du_e/dt \equiv 0$. In particular, if $f(t, x, 0) = h(t, x) = \phi(x) = 0$, then $u(t, x) \equiv 0$ is an equilibrium solution. In the following theorem we assume the existence of an equilibrium solution u_e to (1.1)–(1.3). Here the function $h(t, x) \equiv h(x)$ in (1.2) is assumed independent of t but not necessarily zero.

THEOREM 4.1. *Let $u_e(x)$ be an equilibrium solution and let $\mathcal{N}_\delta = \{u \in C(\bar{\Omega});\ \|u - u_e\|_\Omega \leqq \delta\}$ for some $\delta > 0$. Assume for some $\varepsilon > 0$,*

$$(4.1)\qquad (u - u_e)[f(t, x, u) - f(t, x, u_e)] \leqq -[c(t, x) + \varepsilon](u - u_e)^2, \qquad u \in \mathcal{N}_\delta,$$

for all $(t, x) \in \bar{D}$. Then the equilibrium solution u_e is (exponentially) asymptotically stable.

Proof. Let $u(t, x)$ be any solution of (1.1)–(1.3) and let $w(t, x) = u(t, x) - u_e(x)$. Then $w(t, x)$ satisfies the equation

$$(4.2)\qquad w_t - Lw = f(t, x, u(t, x)) - f(t, x, u_e(x))$$

and the boundary condition (1.2) with $h(x) \equiv 0$. For each t where $\|w(t)\|_\Omega \neq 0$ the proof of Lemma 2.1 shows that for some $x_0 \in \Omega$ (x_0 may depend on t) $\|w(t)\|_\Omega = |w(t, x_0)|$ and (2.10) holds for $w(t, x_0)$. Hence by Lemma 3.1,

$$|w(t, x_0)| \frac{d^+}{dt}(|w(t, x_0)|) \leqq w(t, x_0)[Lw(t, x_0) + f(t, x_0, u(t, x_0)) - f(t, x_0, u_e(x_0))]$$

$$\leqq c(t, x_0) w^2(t, x_0) + w(t, x_0)[f(t, x_0, u(t, x_0)) - f(t, x_0, u_e(x_0))]. \tag{4.3}$$

In case $|w(t, x_0)| = 0$, (4.3) is trivially satisfied. In view of the hypothesis (4.1) we have

$$(4.4)\qquad |w(t, x_0)| \frac{d^+}{dt}(|w(t, x_0)|) \leqq -\varepsilon |w(t, x_0)|^2$$

for as long as $u(t)$ remains in $\mathcal{N}_\delta$. It follows from (4.4) that

$$(4.5)\qquad \|w(t)\|_\Omega = |w(t, x_0)| \leqq e^{-\varepsilon_1 t} \|w(0)\|_\Omega = e^{-\varepsilon_1 t} \|\phi - u_e\|_\Omega,$$

where $0 < \varepsilon_1 < \varepsilon$. Hence if $\phi \in \mathcal{N}_\delta$ then $u(t)$ remains in $\mathcal{N}_\delta$ and satisfies (4.5) for all $t \geqq 0$. The asymptotic stability of u_e follows immediately from definition.

Remarks 4.1. (a) If $f_u(t, x, u)$ exists and

$$(4.6)\qquad f_u(t, x, u_e(x)) < -c(t, x), \qquad (t, x) \in \bar{D},$$

then (4.1) is satisfied in some neighborhood $\mathcal{N}_\delta$. Hence under the condition (4.6) the equilibrium solution u_e is asymptotically stable. The stability of any unperturbed solution such as periodic solution can similarly be treated.

(b) From the proof of Theorem 4.1 it is seen that any solution $u(t, x)$ of (1.1)–(1.3) starting from $\mathcal{N}_\delta$ at $t = 0$ will remain in $\mathcal{N}_\delta$ for all $t > 0$ and $u(t, x) \to u_e(x)$ as $t \to \infty$. This property means that $\mathcal{N}_\delta$ is a stability region of the equilibrium solution u_e. Hence the condition (4.1) insures not only the stability of u_e but also a stability region for u_e. In addition, (4.5) shows that $u(t, x)$ decays exponentially to $u_e(x)$ with a decay constant ε_1.

In case $f(t, x, 0) = h(t, x) = 0$, that is, if $u_e(x) \equiv 0$ is an equilibrium solution, the requirement (4.1) can be replaced by (3.12) for some $\varepsilon > 0$. In this case the zero solution is asymptotically stable (see Corollary 1 to Theorem 3.1). In the special case of the equation (2.19), if $d_0(t, x) \equiv 0$ and $d_1(t, x) \leqq -(c(t, x) + \varepsilon)$ for some $\varepsilon > 0$

then there exists $\delta > 0$ such that for $(t, x) \in \bar{D}$,

$$\sum_{j=1}^{N} d_j(t, x)\eta^{j+1} \leqq -(c(t, x) + \varepsilon)\eta^2 \quad \text{when } |\eta| \leqq \delta.$$

Hence the zero solution of (2.19), (1.2), (1.3) (with $h(x) \equiv 0$) is asymptotically stable. In conclusion we have the following corollary.

COROLLARY. *Let $f(t, x, 0) = h(t, x) = 0$ and let* (3.12) *hold for some $\varepsilon > 0$. Then the solution $u_e(x) \equiv 0$ is asymptotically stable. In particular, if $d_0(t, x) \equiv 0$ and $d_1(t, x) \leqq -(c(t, x) + \varepsilon)$ for some $\varepsilon > 0$ then the zero solution of* (2.19), (1.2), (1.3) *is asymptotically stable.*

The above corollary may be considered as the stability of the zero solution of the corresponding linear system under simultaneously initial and "forcing" perturbations.

5. An example. As an application of the results obtained in the previous sections we consider the problem (1.4)–(1.6) with $0 \leqq \phi(x) \leqq r$. It is easily seen that the function f_0 possesses the following properties: (i) $f_0(u) > 0$ for $-\infty < u < r$, (ii) $uf_0(u) < 0$ for $u > r$, (iii) $|\partial f_0/\partial u| \leqq k < \infty$ for $0 \leqq u < \infty$. Define $\bar{f}_0(\eta) = f_0(\eta)$ if $\eta \geqq 0$ and $\bar{f}_0(\eta) = f_0(0)$ if $\eta \leqq 0$. Then by the property (iii) of f_0, $\bar{f}_0$ satisfies the global Lipschitz condition (2.13). It is easily seen that the inequality (2.8) remains valid under the boundary condition (1.5) whether β_i is zero or not ($i = 1, 2$) since in this case the proof of Lemma 2.1 shows that (2.10) holds even if $x_0 = 0$ or 1. It follows from Theorem 2.1 that the modified problem (1.4)–(1.6) with f_0 replaced by $\bar{f}_0$ has a unique solution $\bar{u}(t, x)$. In view of the properties (i), (ii) of f_0, we have $\eta \bar{f}_0(\eta) < 0$ for $|\eta| > r$, that is, $\bar{f}_0$ satisfies the condition (3.8) with $c(t, x) = 0$ for $|\eta| > r$. Hence by Theorem 3.2, $|\bar{u}(t, x)| \leqq r$ on $\bar{D}$. Since the property (i) of f_0 implies $\bar{f}_0(\eta) \geqq 0$ for $|u| \leqq r$ the maximum principle insures that $\bar{u}(t, x) \geqq 0$ on $\bar{D}$. But $\bar{f}_0$ coincides with f_0 when $0 \leqq \bar{u}(t, x) \leqq r$. We conclude that $\bar{u}(t, x)$ is the unique solution of (1.4)–(1.6).

As an application of the results in Theorem 4.1 to the stability problem (1.4)–(1.6) we differentiate f_0 with respect to u to obtain

$$\frac{\partial f_0(u)}{\partial u} = -A(1 + u)^{-2} \exp\left(-\frac{B}{1 + u}\right)[u^2 + (B + 2)u + (1 - Br)]. \tag{5.1}$$

Hence by Theorem 4.1 and Remark 4.1 we see that any equilibrium solution $u_e(x)$ satisfying the relation

$$u_e^2 + (B + 2)u_e + (1 - Br) > 0 \quad \text{for } x \in [0, 1] \tag{5.2}$$

is asymptotically stable. For example, when $r = 0.4$, $B = 20$, $A = 2(10^7)$, three equilibrium solutions of the problem (1.4)–(1.6) with $a = 5$, $\alpha_1 = \alpha_2 = -b = 1$, $\beta_1 = 0.2$, $\beta_2 = 0$ have been computed in [1] (see also [4]). One of these equilibrium solutions is found to have values between 0.32 and 0.37 for $x \in [0, 1]$. Since this solution satisfies (5.2), Theorem 4.1 implies that it is asymptotically stable. This fact is known in [4] by a different definition of stability. Notice that in our case condition (5.1) also gives an estimate of a stability region and a decay constant for this particular equilibrium solution.

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A CONSTRUCTIVE EXISTENCE THEOREM FOR A NONLINEAR PARABOLIC EQUATION*

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Abstract. This paper considers the first boundary value problem for the quasilinear parabolic differential equation

$$\text{(A)} \qquad L[u] = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} - u_t = -f(x,t,u)$$

on a cylindrical domain in E_{n+1}. The central theorem is an existence theorem offering an iterative procedure for approximating or constructing a classical solution $u(x, t)$ for this problem. The iteration is done by successively applying a monotone integral operator with an appropriate Green's function as its kernel. The procedure starts with either of a pair of functions $\underline{\sigma}(x, t) \leqq \bar{\sigma}(x, t)$ which satisfy

$$L[\bar{\sigma}] \geqq -f(x,t,\underline{\sigma})$$

and

$$L[\bar{\sigma}] \leqq -f(x,t,\bar{\sigma}).$$

This theorem is then used to prove an existence theorem for equation (A) on E_{n+1} by solving a sequence of problems on nested cylinders in E_{n+1}.

Introduction. In this paper we shall be dealing with the nonlinear parabolic differential equation

$$\text{(A)} \qquad L[u] = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} - u_t = -f(x,t,u).$$

The main theorem is a constructive existence theorem which gives an iterative method for solving the first boundary value problem in noncylindrical domains for equation (A). The method is patterned after a Čaplygin type of iteration used by Schmitt [6] to solve a nonlinear ordinary differential equation. The iteration scheme is a parallel to a similar theorem due to Kusano [5], but the present theorem permits us to drop Kusano's hypothesis that $f(x, t, u)$ be nondecreasing in u. As a corollary, we are also able to prove Theorem 2 of Chabrowski [1] under these weakened hypotheses.

1. Preliminary notation and results. We shall let E_{n+1} denote the $(n + 1)$-dimensional space-time of points $(x, t) = (x_1, x_2, \cdots, x_n, t)$. By a cyclindrical domain Q in E_{n+1}, we mean a set of the form $Q = D \times (T_0, T_1)$ where D is a bounded domain in E_n. We shall let $\bar{Q}$ denote the closure of Q, S denote the lateral surface of Q, and B denote the base of Q; i.e., $B = \bar{Q} \cap \{(x,t) | t = T_0\}$. The surface $\Gamma = S \cup B$ is called the parabolic surface of Q.

For points $P = (x, t)$ and $P' = (x', t')$ in E^{n+1}, we define two metrics

$$\rho(P, P') = (|x - x'|^2 + |t - t'|^2)^{1/2},$$
$$d(P, P') = (|x - x'|^2 + |t - t'|)^{1/2},$$

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where $|x|$ denotes the Euclidean norm of x. The quantity $d(P, P')$ is called the parabolic distance between P and P'.

Following a standard notation, we introduce the following norms on a function $u(x, t)$ defined on a set A in E_{n+1}:

$$|u|_0^A = \sup_{P \in A} |u(P)|,$$

$$|u|_\alpha^A = |u|_0^A + \sup_{\substack{P, P' \in A \\ P \neq P'}} \frac{|u(P) - u(P')|}{d(P, P')^\alpha},$$

$$|u|_{1+\alpha}^A = |u|_\alpha^A + \sum_{i=1}^n |u_{x_i}|_\alpha^A,$$

$$|u|_{2+\alpha}^A = |u|_{1+\alpha}^A + \sum_{i=1}^n |u_{x_i}|_{1+\alpha}^A + |u_t|_{1+\alpha}^A,$$

where $0 < \alpha < 1$. We shall say that $u(x, t)$ is in Hölder class $C^q(A)$ or that $u \in C^q(A)$ in case $|u|_q^A < \infty$, $q = 0, \alpha, 1 + \alpha, 2 + \alpha$.

A function $h(y, z)$ is said to satisfy a *Hölder condition in y with exponent* α on a set $Y \times Z$ in case

(1.1) $$|h(y, z) - h(y', z)| \leqq H|y - y'|^\alpha \quad \text{for all } (y, z), (y', z) \in Y \times Z,$$

for some constant H, which may depend on z. If (1.1) holds for some H, which is independent of $z \in Z$, we say that h satisfies a Hölder condition in y, *uniformly* with respect to z. Finally, if (1.1) holds for $\alpha = 1$, we say that h satisfies a *Lipschitz condition* in y.

It will be convenient to let Λ_Q denote the linear space of functions $v(x, t)$ which are continuous and continuously differentiable with respect to each component of x on the cylinder Q. We define a norm on Λ_Q by

$$\|v\| = \max_{P \in \bar{Q}} |v(P)| + \sum_{i=1}^n \max_{P \in \bar{Q}} |v_{x_i}(P)|.$$

By a solution to (0.1) on Q, we shall mean a function $u(x, t)$ such that u and u_{x_i} are continuous on $\bar{Q}$, $u_{x_i x_j}$ and u_t are continuous on Q, and that these functions satisfy equation (A) on Q.

In order to prove a constructive existence theorem for the first boundary value problem on Q for equation (A) we shall need the following conditions:

(I) At every $(x, t) \in \bar{Q}$ and for all real n-tuples $(\xi_1, \xi_2, \cdots, \xi_n)$,

$$\sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j \geqq a_0 \sum_{i=1}^n \xi_i^2,$$

where a_0 is a fixed positive constant.

(II) The coefficients $a_{ij}(x, t)$ are in class $C^\alpha(\bar{Q})$, $0 < \alpha < 1$. Moreover they satisfy a Lipschitz condition on the lateral surface S.

(III) The coefficients $b_i(x, t)$ are in class $C^\alpha(\bar{Q})$.

(IV) The boundary condition, a function $\varphi(x, t)$ defined on the parabolic boundary Γ, may be extended over $\bar{Q}$ to a function $\Phi(x, t)$ which is in class $C^{2+\alpha}(Q)$. We will define

$$|\varphi|_{2+\alpha}^\Gamma = \inf |\Phi|_{2+\alpha}^Q,$$

where the infimum is taken over all possible extensions.

(V) The right member $f(x, t, u)$ is continuous on $\bar{Q} \times (-\infty, \infty)$ and satisfies a Hölder condition in (x, t) for each fixed u and a Hölder condition in u, uniformly with respect to $(x, t) \in \bar{Q}$ (both with exponent α).

(VI) For every $(x, t) \in \bar{Q}$,

$$-K(u^1 - u^2) \leqq f(x, t, u^1) - f(x, t, u^2) \tag{1.2}$$

for some positive constant K and any $u^1 \geqq u^2$.

(VII) There exist functions $\underline{\sigma}(x, t)$ and $\bar{\sigma}(x, t)$ satisfying the differential inequalities

$$\begin{aligned} L[\underline{\sigma}](x, t) &\geqq -f(x, t, \underline{\sigma}(x, t)), \\ L[\bar{\sigma}](x, t) &\leqq -f(x, t, \bar{\sigma}(x, t)) \end{aligned} \tag{1.3}$$

on $\bar{Q}$ with

$$\underline{\sigma}(x, t) \leqq \bar{\sigma}(x, t) \quad \text{on } \bar{Q}$$

and

$$\underline{\sigma}(x, t) \leqq \varphi(x, t) < \bar{\sigma}(x, t) \quad \text{on } \Gamma.$$

For the rest of this paper, we shall assume that the domain Q is such that for every point P of $\bar{Q}$ there exists an $(n + 1)$-dimensional neighborhood V such that $V \cap \bar{Q}$ can be represented in the form

$$x_k = h(x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n, t)$$

for some k. Moreover, $h, h_{x_i}, h_{x_i x_j}, h_t$ are assumed to be Hölder continuous with exponent α and h_{tx_i} and h_{tt} are continuous on $V \cap \bar{Q}$.

We need the following lemma, due to Friedman [2].

LEMMA 1. *Consider the first boundary problem for the linear parabolic equation.*

$$\sum_{i,j=1}^{n} a_{ij}(x, t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t)u_{x_i} + c(x, t)u - u_t = -f(x, t) \tag{1.4}$$

on Q with boundary condition

$$u(x, t) = \varphi(x, t) \quad on\ \Gamma. \tag{1.5}$$

Suppose that conditions (I) *and* (IV) *are satisfied. Suppose also that each of the functions $a_{ij}(x, t)$, $b_i(x, t)$, $c(x, t)$, and $f(x, t)$ belongs to $C^\alpha(Q)$.*

Then there exists one and only one solution $u(x, t)$ to problem (A), (1.5) *satisfying the* $(2 + \alpha)$*-estimate*

$$|u|^Q_{2+\alpha} \leqq C(|f|^Q_\alpha + |\varphi|^\Gamma_{2+\alpha}).$$

We shall also make use of the following lemma, again due to Friedman [3], which gives a priori estimates for solutions.

LEMMA 2. *Consider the first boundary problem* (1.4), (1.5). *Assume that conditions* (I) *and* (IV) *hold. Also assume that the coefficients and right member of* (1.4) *are continuous in $\bar{Q}$ and satisfy the inequalities*

$$\sum_{i,j=1}^{n} |a_{ij}|^Q_\alpha + \sum_{i=1}^{n} |b_i|^Q_0 + |c|^Q_0 \leqq C_1, \qquad C_1 > 0$$

and

$$\sum_{i,j=1}^{n}\left\{|a_{ij}|_0^S + \sup_{\substack{S\\ P\neq P'}}\frac{|a_{ij}(P)-a_{ij}(P')|}{\rho(p,p')}\right\}\leqq C_2, \qquad C_2>0.$$

Then if $u(x,t)$ is any solution of problem (1.4), (1.5), *for any* $0<\delta<1$ *we have the* $(1+\delta)$*-estimate*

$$|u|_{1+\delta}^{Q}\leqq C(|f|_0^Q+|\varphi|_{2+\alpha}^{\Gamma}),$$

where C is a positive constant depending only on δ, a_0, C_1, C_2, and the cylinder Q; that is, C is independent of the function $f(x,t)$.

2. The constructive existence theorem on a rectangle. We are now in a position to state and prove the central result of this paper.

THEOREM 1. *Suppose that conditions* (I)–(VI) *are satisfied. Then there exists a solution $u(x,t)$ to problem* (A), (1.5) *satisfying*

$$\underline{\sigma}(x,t)\leqq u(x,t)\leqq\bar{\sigma}(x,t)\quad \text{on } \bar{Q},$$

which may be constructed by successive approximations of the čaplygin type. Moreover, u is in $C^{1+\delta}(\bar{Q})$ for any $0<\delta<1$ and is also an element of $C^{2+\varepsilon}(\bar{Q})$ for some $0<\varepsilon<1$.

Proof. We begin by considering the linear problem

$$(1.6)\qquad L_K[u]\equiv\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_ix_j}+\sum_{i=1}^{n}b_i(x,t)u_{x_i}-Ku-u_t=-h(x,t)$$

with boundary conditions (1.5), where $h(x,t)$ is taken to be continuous on $\bar{Q}$ and satisfying a Hölder condition in x on Q, uniformly with respect to t. By Theorem 16 and Corollary 1 in Friedman [4, pp. 82–83], there is a Green's function $G(x,t;\xi,\tau)$ for $L_K[u]=0$ on Q and $G(x,t;\xi,\tau)\geqq 0$ on $\bar{Q}\times\bar{Q}$ for $T_0\leqq\tau<t\leqq T_1$.

If we let $v_K(x,t)$ denote the (unique) solution of $L_K[u]=0$ satisfying (1.5) (Lemma 1 guarantees this), then the function

$$u(x,t)=\int_{T_0}^{t}\int_D G(x,t;\xi,\tau)h(\xi,\tau)\,d\xi\,d\tau+v_K(x,t)$$

is the unique solution of problem (1.6), (1.5).

Consider the function $Ku+f(x,t,u)$. By condition (V), we know that for any $(x,t)\in\bar{Q}$,

$$Ku^1+f(x,t,u^1)\geqq Ku^2+f(x,t,u^2),$$

for $u^1\geqq u^2$, that is, this function is monotonically nondecreasing in u.

We proceed by defining an operator on the space Λ_Q by

$$F[v](x,t)=\int_{T_0}^{t}\int_D G(x,t;\xi,\tau)[Kv(\xi,\tau)+f(\xi,\tau,v(\xi,\tau))]\,d\xi\,d\tau+v_K(\xi,\tau).$$

It is clear that F is a monotonically increasing operator; that is, if $v^1(x,t)\leqq v^2(x,t)$ on $\bar{Q}$ (v^1 and v^2 elements of Λ_Q), then $F[v^1](x,t)\leqq F[v^2](x,t)$ on $\bar{Q}$. Also, observe that in view of the above comments, $F[v]$ is the solution of the linear equation

$$L_K[u]=-Kv(x,t)-f(x,t,v(x,t))\quad\text{on } Q$$

satisfying (1.5). It therefore follows that a fixed point of F will be a solution of problem (A), (1.5). The rest of the proof will be the construction of such a fixed point.

For the first step, we claim that $F[\bar{\sigma}](x,t) \leqq \bar{\sigma}(x,t)$ on $\bar{Q}$. Define a function

$$w(x,t) = F[\bar{\sigma}](x,t) - \bar{\sigma}(x,t) \quad \text{on } \bar{Q}.$$

Clearly on Γ, we have

$$w(x,t) = \varphi(x,t) - \bar{\sigma}(x,t) \leqq 0.$$

For purposes of contradiction, assume that $w(x,t)$ achieves a positive maximum at some point (x',t') in the interior or on the upper surface of Q. We then have

$$w_t(x',t') \geqq 0; \quad w_{x_i}(w',t') = 0, \qquad i = 1,2,\cdots,n,$$

and

$$\frac{\partial^2 w(x',t')}{\partial\lambda_k \partial\lambda_k} \leqq 0 \tag{1.7}$$

in any direction $\lambda_k = \sum_{i=1}^n \alpha_{ki}(x_i - x_i')$, $k = 1,2,\cdots,n$. From (1.7) and hypothesis (I) it follows that

$$\sum_{i,j=1}^n a_{ij}(x',t') w_{x_i x_j}(x',t') \leqq 0. \tag{1.8}$$

Therefore, at the point (x',t'), we obtain the contradiction

$$\begin{aligned}
0 &\leqq w_t - \sum_{i,j} w_{x_i x_j} \\
&= F[\bar{\sigma}]_t - \sum_{i,j} a_{ij} F[\bar{\sigma}]_{x_i x_j} - \bar{\sigma}_t + \sum_{i,j} a_{ij} \bar{\sigma}_{x_i x_j} \\
&\leqq \sum_i b_i F[\bar{\sigma}]_{x_i} - K \cdot F[\bar{\sigma}] + F(x',t',\bar{\sigma}) \\
&\quad + K\bar{\sigma} - \sum_i b_i \bar{\sigma}_{x_i} - f(x',t',\bar{\sigma}) \\
&= \sum_i b_i [F[\bar{\sigma}]_{x_i} - \bar{\sigma}_{x_i}] - K[F[\bar{\sigma}] - \bar{\sigma}] \\
&< 0.
\end{aligned}$$

By an analogous procedure, it follows that

$$\underline{\sigma}(x,t) \leqq F[\underline{\sigma}](x,t) \quad \text{on } \bar{Q}.$$

We inductively define two sequences of functions $\{y_k(x,t)\}$ *and* $\{z_k(x,t)\}$ by

$$y_1(x,t) = \underline{\sigma}(x,t), \qquad y_{k+1}(x,t) = F[y_k](x,t), \qquad k = 1,2,\cdots,$$

and

$$z_1(x,t) = \bar{\sigma}(x,t), \qquad z_{k+1}(x,t) = F[z_k](x,t), \qquad k = 1,2,\cdots.$$

By monotonicity, we have

$$y_1(x,t) \leqq y_2(x,t) \leqq \cdots \leqq y_k(x,t) \leqq z_k(x,t) \leqq \cdots \leqq z_2(x,t) \leqq z_1(x,t) \tag{1.9}$$

on $\bar{Q}$ for all k. Both $\{y_k\}$ and $\{z_k\}$ are bounded monotone sequences and therefore converge pointwise to some function

$$\lim_{k\to\infty} y_k(x,t) = y(x,t); \qquad \lim_{k\to\infty} z_k(x,t) = z(x,t)$$

on $\bar{Q}$ with $y(x,t) \leqq z(x,t)$ on $\bar{Q}$.

Let

$$M \equiv \max_{\bar{Q}} \{|\underline{\sigma}(x,t)|, |\bar{\sigma}(x,t)|\}.$$

By (1.9), we have that

$$|y_k(x,t)|, |z_k(x,t)| \leqq M \quad \text{on } \bar{Q}, \qquad k = 1, 2, \cdots.$$

The function $f(x,t,u)$ is uniformly continuous for $(x,t) \in \bar{Q}$ and $|u| \leqq M$. Therefore if we can obtain a subsequence, for example, $\{z_{k(j)}\}$ of $\{z_k\}$, which is uniformly convergent on $\bar{Q}$, then $\{f(x,t,z_{k(j)}(x,t))\}$ will converge uniformly to $f(x,t,z(x,t))$. By showing the equicontinuity of $\{z_k\}$ we shall be able to assert the existence of such a sequence.

With this goal in mind, we let

$$N = \max_{\substack{(x,t)\in\bar{Q}\\ u \leqq M}} |f(x,t,u)|.$$

Note that $z_{k+1}(x,t)$ is a solution of the linear equation

$$L_K[u] = f(x,t,z_k(x,t)) + Kz_k(x,t) \tag{1.10}$$

satisfying (1.5). By Lemma 2 applied to problem (1.10), (1.5), we know that for any $0 < \delta < 1$,

$$|z_{k+1}|_{1+\delta}^{Q} \leqq C(N + KM + |\varphi|_{2+\alpha}), \qquad k = 1, 2, \cdots,$$

in which the constant C is independent of k. From this it is clear that the norms $|z_k|_\alpha^Q$ are bounded independently of k. We may now use Lemma 1 to conclude that

$$|z_{k+1}|_{2+\alpha}^{Q} \leqq C'(|f(x,t,z_k)|_\alpha^Q + K \cdot |z_k|_\alpha^Q + |\varphi|_{2+\alpha}^{\Gamma}),$$

where C' is independent of k. Then by using condition (V), this last inequality, and the uniform boundedness of $|z_k|_\alpha^Q$, we find that the norms $|z_k|_{2+\alpha}^Q$ are bounded independently of k.

By repeated use of the Ascoli–Arzela theorem we are able to extract a subsequence of $\{z_k\}$, say $\{z_{k(j)}\}$, such that the subsequence converges uniformly on $\bar{Q}$ to some function. Of course, this limit function is $z(x,t)$. Immediately we also have that

$$f(x,t,z_{k(j)}(x,t)) + Kz_{k(j)}(x,t) \to f(x,t,z(x,t)) + Kz(x,t)$$

uniformly on $\bar{Q}$. We therefore obtain that $z(x,t)$ is a fixed point of the operator F and consequently is a solution of (A), (1.5) satisfying

$$\underline{\sigma}(x,t) \leqq z(x,t) \leqq \bar{\sigma}(x,t) \quad \text{on } \bar{Q}.$$

The asserted smoothness properties of z are shown by standard arguments with the $(1 + \delta)$ and $(2 + \alpha)$-estimates on $\{z_k\}$.

Remark 1. By analogous arguments, it may be shown that $y(x, t)$ is also a solution of (A), (1.5).

Remark 2. It is important to realize that although the nonconstructive Ascoli–Arzela theorem has been used in the proof, the solution $z(x, t)$ is determined as the pointwise limit of the iteration. Of course, the actual approximation of a solution to (A), (1.5) depends on finding suitable $\underline{\sigma}$, $\bar{\sigma}$ and our ability to find or approximate the Green's function and the solution $v_k(x, t)$.

As a corollary to Theorem 1, we easily obtain what is essentially Theorem 2.1 of Kusano [5] for a "system" of one equation on a cylindrical domain.

COROLLARY 1. *Suppose that conditions* (I)–(V) *and* (VII) *of Theorem* 1 *are satisfied. Suppose also that for each* $(x, t) \in \bar{Q}, f(x, t, u)$ *is nondecreasing as a function of* u. *Then the conclusion of Theorem* 1 *holds.*

Proof. Observe that inequality (1.2) is satisfied for any positive K under the additional hypothesis.

COROLLARY 2. *Suppose that conditions* (I)–(V) *and* (VII) *of Theorem* 1 *are satisfied. Suppose also that* $f(x, t, u)$ *satisfies a Lipschitz condition in* u, *uniformly with respect to* $(x, t) \in \bar{Q}$. *Then the conclusion of Theorem* 1 *holds.*

Proof. Note that inequality (1.1) is satisfied with K taken to be the Lipschitz constant.

Remark 3. Under either of these additional hypotheses, the solution constructed will be unique and so $y(x, t) = z(x, t)$. (See Friedman [6, pp. 201–202].)

3. Constructive existence on E_{n+1}. We shall show that by using Corollary 2 we are able to offer an improvement of a constructive existence theorem of Chabrowski [1] for equation (A) on all of E_{n+1}.

THEOREM 2. *Suppose that*

(i) *for every* $(x, t) \in E_{n+1}$ *and for all real n-tuples* $(\xi_1, \xi_2, \cdots, \xi_n)$,

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqq a_0 \sum_{i=1}^{n} \xi_i^2, \qquad a_0 \geqq 0;$$

(ii) *the coefficients* $a_{ij}(x, t)$ *are bounded in* E_{n+1} *and satisfy a Lipschitz condition on any subset* A *of* E_{n+1};

(iii) *the coefficient* $b_i(x, t)$ *are bounded in* E_{n+1} *and members of Hölder class* $C^\alpha(A)$, $0 < \alpha < 1$, *for any subset of* E_{n+1};

(iv) *the right member* $f(x, t, u)$, *defined on* $E_{n+1} \times (-\infty, \infty)$, *satisfies a Hölder condition* (*exponent* α) *in* (x, t) *on any subset* A *of* E_{n+1}, *for fixed* u; *and a Lipschitz condition in* u, *uniformly with respect to* (x, t);

(v) *there exist functions* $\underline{\sigma}(x, t)$ *and* $\bar{\sigma}(x, t)$, *bounded in* E_{n+1}, *satisfying the inequalities*

$$L[\underline{\sigma}](x, t) \geqq -f(x, t, \underline{\sigma}(x, t)),$$

$$L[\bar{\sigma}](x, t) \leqq -f(x, t, \bar{\sigma}(x, t))$$

and

$$\underline{\sigma}(x, t) \leqq \bar{\sigma}(x, t)$$

for any $(x, t) \in E_{n+1}$.

Then there exists at least one bounded solution to equation (A) *on* E_{n+1} *which satisfies*

$$\underline{\sigma}(x, t) \leqq u(x, t) \leqq \bar{\sigma}(x, t) \quad \text{on } E_{n+1}.$$

Proof. This theorem has essentially been proven by Chabrowski. In order to make use of Corollary 2 in his proof, replacing the use of the analogous Kusano theorem, it is only necessary to replace his nested sequence $\{D_m\}$ of subsets in E_{n+1} with a sequence of cylinders $\{Q_m\}$ in E_{n+1} such that

(i) $\bar{Q}_m \subset Q_{m+1}$ for each m;

(ii) $\lim_{m\to\infty} Q_m = E_{n+1}$;

(iii) each Q_m satisfies the geometrical requirements of Theorem 1.

Because the details of the proof are in the Chabrowski paper, we shall not give them here. It should, however, be apparent that the proof consists of solving (uniquely) equation (A) on each Q_m (with appropriate boundary conditions) by using Corollary 2, then extracting a subsequence which converges uniformly together with its x and t-derivatives.

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POISSON INTEGRAL FORMULAS IN GENERALIZED BI-AXIALLY SYMMETRIC POTENTIAL THEORY*

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Abstract. Poisson integral formulas are given which solve certain Dirichlet problems for a "quarter-ball" for the equation of generalized bi-axially symmetric potential theory. The formulas are useful in studying properties of solutions of this equation and related equations. The equation considered includes the equation of Weinstein's generalized axially symmetric potential theory for which corresponding Poisson integral formulas were given by A. Huber.

1. Introduction. In this paper we give Poisson integral formulas for the equation of generalized bi-axially symmetric potential theory

$$L[u] \equiv L_{\alpha,\beta}[u] \equiv \sum_{i=1}^{n} u_{x_i x_i} + \frac{\alpha}{x_{n-1}} u_{x_{n-1}} + \frac{\beta}{x_n} u_{x_n} = 0 \tag{1.1}$$

which reduces to Weinstein's [1], [2] generalized axially symmetric potential theory (abbreviated GASPT) for the real constant $\alpha = 0$; β is also a real constant and n is an integer, $n \geqq 2$. These formulas are useful in studying properties of the solutions of (1.1) and related equations. In GASPT corresponding formulas were given by Huber [3] to which our results reduce in the appropriate special cases.

Equation (1.1) has been treated by Gilbert [4], [5] by integral operator methods, and fundamental solutions in the large of (1.1) were given by Weinacht [6]. Kapilevich has given mean value theorems for a class of equations including (1.1) (see e.g. [7]).

The integral formulas solve certain Dirichlet problems for (1.1) for a "quarter-ball" with data prescribed on various portions of the boundary depending on the values of α and β. Section 3 treats the case $\alpha \geqq 1$, $\beta \geqq 1$ while the cases $\alpha < 1$, $\beta < 1$ and $\alpha < 1$, $\beta \geqq 1$ are dealt with in §§ 4 and 5, respectively. The remaining case ($\alpha \geqq 1$, $\beta < 1$) is obtained by merely changing notation in the case $\alpha < 1$, $\beta \geqq 1$.

2. Notations. The usual notations for vectors in Euclidean n-space will be used. At times it will be convenient in considering a vector $x = (x_1, x_2, \cdots, x_n)$ to put $y = x_{n-1}$ and $z = x_n$; in a similar way for the vector ξ we shall write sometimes η for ξ_{n-1} and ζ for ξ_n.

The ball of radius R and center at the origin will be denoted by $B(R)$, Q will denote the open quarter-space $\{x : x_{n-1} > 0, x_n > 0\}$, $B^+(R) \equiv B(R) \cap Q$ is the quarter-ball and $Q(R)$ will denote the quarter-sphere $\{x : |x| = R, x_{n-1} > 0, x_n > 0\}$. The boundary of a set S will be denoted by ∂S. Since coefficients in (1.1) become singular on $x_{n-1} = 0$ or $x_n = 0$ we shall refer to these sets as singular hyperplanes.

Some of the results presented here were given in the thesis [8].

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3. The case $\alpha \geqq 1$, $\beta \geqq 1$. Here data are prescribed only on $Q(R)$. The formula (3.3) below was obtained by considering the Poisson integral formula for a sphere for Laplace's equation in $(n + \alpha + \beta)$-dimensional space for positive integral α, β: $\sum_{i=1}^{n+\alpha+\beta} \partial^2 u/\partial t_i^2 = 0$ and then assuming bi-axial symmetry, i.e., considering solutions u which depend only on $t_i = x_i$ for $i = 1, 2, \cdots, n-2$, $x_{n-1} = (t_{n-1}^2 + \cdots + t_{n-1+\alpha}^2)^{1/2}$ and $x_n = (t_{n+\alpha}^2 + \cdots + t_{n+\alpha+\beta}^2)^{1/2}$. The resulting formula, however, is valid for nonintegral α, β (Weinstein's spaces of fractional dimension).

THEOREM 1. *Let* $\alpha \geqq 1, \beta \geqq 1$. *Suppose* f *is continuous on* $Q(R)$ *and has the following behavior near the singular hyperplanes*:

$$\lim_{\xi \to \xi_0} \rho(\eta; \alpha)\rho(\zeta; \beta) f(\xi) = 0, \tag{3.1}$$

where $\xi \in Q(R)$, $\xi_0 \in \overline{Q(R)} \cap \partial Q$ *and*

$$\rho(\lambda; \gamma) = \begin{cases} \lambda^{\gamma-1}, & \gamma > 1, \\ (\log \lambda)^{-1}, & \gamma = 1, \end{cases} \tag{3.2}$$

for positive real λ. *Then the function* u *defined on* $B(R)$ *by*

$$u(x) = \int_{Q(R)} \eta^\alpha \zeta^\beta K(x, \xi) f(\xi)\, dS, \tag{3.3}$$

where

$$K(x, \xi) = \frac{2(R^2 - |x|^2)\Gamma(p)}{R\pi^{n/2}\Gamma(\alpha/2)\Gamma(\beta/2)} \int_0^\pi \int_0^\pi \sigma^{-2p} \sin^{\alpha-1} s \sin^{\beta-1} t\, dt\, ds,$$

$$\sigma = [|x - \xi|^2 + 2\eta y(1 - \cos s) + 2\zeta z(1 - \cos t)]^{1/2} \tag{3.4}$$

and $2p = n + \alpha + \beta$ *has the following properties*:

(a) u *is even in* y *and even in* z,

(b) u *is analytic in* $B(R)$ *and satisfies* (1.1) *in* $B(R)$ *except for* $y = 0$ *and* $z = 0$,

(c) u *assumes the boundary value* f *on* $Q(R)$:

$$\lim_{x \to \xi_0} u(x) = f(\xi_0), \qquad x \in B^+(R), \quad \xi_0 \in Q(R),$$

(d) u *inherits the behavior of* f *near the singular hyperplanes*:

$$\lim_{x \to x_0} \rho(y; \alpha)\rho(z; \beta) u(x) = 0, \qquad x \in B^+(R), \quad x_0 \in \overline{B(R)} \cap \partial Q, \tag{3.5}$$

the convergence being uniform in x^0.

Proof. (a) An elementary direct computation shows that K and hence u is even in y and even in z.

(b) By extending each x_i, $i = 1, 2, \cdots, n$, into the complex plane it is easy to see that $K(x, \xi)$ is analytic in x in a neighborhood of each $x \in B(R)$ and continuous in (x, ξ) for x in that neighborhood and ξ in $\overline{Q(R)}$. From this it follows that u is analytic in $B(R)$. Then the (permissible) differentiation of (3.3) under the integral sign yields for x in $B(R)$ but $y \neq 0$, $z \neq 0$,

$$L[K] = 0$$

after integration by parts with respect to s, t. Thus (b) is verified.

(c) The proof proceeds as in classical potential theory. The following identity, valid for all x in $B(R)$ and $\alpha \geqq 1, \beta \geqq 1$:

$$\int_{Q(R)} \eta^\alpha \zeta^\beta K(x, \xi)\, dS \equiv 1 \tag{3.6}$$

is established in the Appendix. Then for any positive ε and any ξ_0 on $Q(R)$ the continuity of f insures the existence of a ball $B(\delta; \xi_0)$ with center ξ_0 and radius δ (with δ less than half the distance of ξ_0 to ∂Q) such that for any ξ in $Q_\delta \equiv B(\delta; \xi_0) \cap Q(R)$ we have $|f(\xi) - f(\xi_0)| < \varepsilon/2$. On the remainder $\tilde{Q}$ of $Q(R)$ the non-negative kernel K tends uniformly in ξ to zero as x tends to ξ_0 because

$$K(x, \xi) \leqq C(R^2 - |x|^2)(\delta/2)^{-2p}$$

for $|\xi_0 - x| < \delta/2$ as follows from straightforward estimates on the integrals appearing in K. The number C depends only on R, α, β and n. Hence there exists a positive $\delta_1 < \delta/2$ such that $|x - \xi_0| < \delta_1$ implies that

$$2 \sup_{Q(R)} |\eta^\alpha \zeta^\beta f(\xi)| \int_{\tilde{Q}} K(x, \xi)\, dS < \varepsilon/2.$$

Then

$$\begin{aligned}
|u(x) - f(\xi_0)| &= \int_{Q(R)} \eta^\alpha \zeta^\beta K(x, \xi)[f(\xi) - f(\xi_0)]\, dS \\
&\leqq \frac{\varepsilon}{2} \int_{Q_\delta} \eta^\alpha \zeta^\beta K(x, \xi)\, dS + 2 \sup_{Q(R)} |\eta^\alpha \zeta^\beta f(\xi)| \int_{\tilde{Q}} K(x, \xi)\, dS \\
&< \frac{\varepsilon}{2} \int_{Q(R)} \eta^\alpha \zeta^\beta K(x, \xi)\, dS + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

which establishes (c).

(d) We treat the case $\alpha > 1, \beta > 1$; the remaining cases are similar. Without loss of generality we may assume that z_0, the nth coordinate of x_0 in (3.5) is zero; otherwise, we merely interchange the last two coordinates. By (3.1) the function $\eta^{\alpha-1}\zeta^{\beta-1} f(\xi)$ can be extended continuously to $\overline{Q(R)}$ and so there exists for every positive ε a positive $\delta_1 = \delta_1(\varepsilon)$, $\delta_1 < R/2$, independent of ξ_0, such that for ξ_0 in $\overline{Q(R)} \cap \partial Q$, on $Q(R)$ and $|\xi - \xi_0| < \delta_1$ we have

$$|f(\xi)| < \varepsilon \eta^{1-\alpha} \zeta^{1-\beta}.$$

Moreover there exists a positive $\delta_2 = \delta_2(\varepsilon)$ with $\delta_2 < \delta_1$ such that

$$|f(\xi)| \leqq \varepsilon R^{1-\alpha} \delta_2^{1-\beta}$$

for $\eta \geqq \delta_1$ and simultaneously $\zeta \geqq \delta_1$. Now choose $x^* \in B^+(R)$ such that $z_* < 2\delta_2$ where $z_* = x_n^*$ and for later use $y_* = x_{n-1}^*$. Then we have on $Q(R)$, $|f(\xi)| \leqq \phi(\xi)$, where

$$\phi(\xi) = \begin{cases} \varepsilon \eta^{1-\alpha} \zeta^{1-\beta} & \text{on } Q_1 = Q(R) \cap \{\xi : 2\eta \leqq y_*, 2\zeta \leqq z_*\}, \\ \varepsilon \eta^{1-\alpha} (z^*/2)^{1-\beta} & \text{on } Q_2 = Q(R) \cap \{\xi : 2\eta \leqq y_*, 2\zeta > z_*\}, \\ \varepsilon (y^*/2)^{1-\alpha} \zeta^{1-\beta} & \text{on } Q_3 = Q(R) \cap \{\xi : 2\eta > y_*, 2\zeta \leqq z_*\}, \\ \varepsilon (y^*/2)^{1-\alpha} (z^*/2)^{1-\beta} & \text{on } Q_4 = Q(R) \cap \{\xi : 2\eta > y_*, 2\zeta > z_*\}. \end{cases}$$

It then follows that

$$|u(x^*)| \leqq \int_{Q(R)} \eta^\alpha \zeta^\beta K(x^*, \xi)\phi(\xi)\, dS$$
$$= I_1 + I_2 + I_3 + I_4,$$

where

$$I_j = \int_{Q_j} \eta^\alpha \zeta^\beta K(x^*, \xi)\phi(\xi)\, dS, \qquad j = 1, 2, 3, 4.$$

We now wish to estimate these integrals. Of course, I_j is considered to be zero when Q_j is empty and hence the estimate in such cases is not needed. With the notations

$$A = \frac{2\Gamma(p)}{R\pi^{n/2}\Gamma(\alpha/2)\Gamma(\beta/2)}$$

and

$$k(x, \xi) = \int_0^\pi \int_0^\pi \sigma^{-2p} \sin^{\alpha-1} s \sin^{\beta-1} t\, ds\, dt$$

we see that

$$I_1 = A\varepsilon(R^2 - |x^*|^2) \int_{Q_1} \eta\zeta k(x^*, \xi)\, dS.$$

Since $2\eta \leqq y_*$, $2\zeta \leqq z_*$, $2|x^* - \xi| \geqq y_*$ and $2|x^* - \xi| \geqq z_*$ on Q_1 and

$$k(x^*, \xi) \leqq |x^* - \xi|^{-n-\alpha-\beta} \int_0^\pi \int_0^\pi \sin^{\alpha-1} s \sin^{\beta-1} t\, ds\, dt,$$

we have

$$I_1 \leqq A_1 \varepsilon y_*^{1-\alpha} z_*^{1-\beta} \int_{Q_1} K_0(x^*, \xi)\, dS \leqq A_1 \varepsilon y_*^{1-\alpha} z_*^{1-\beta},$$

where K_0 is the Poisson kernel for the sphere for the n-dimensional Laplacian so that its integral over $\partial B(R)$ is one. The A_1 depends only on R, α, β and n and the same is true for A_2, A_3, A_4 which appear below. We now consider

$$I_2 = A\varepsilon(z_*/2)^{1-\beta}(R^2 - |x^*|^2) \int_{Q_2} \eta\zeta^\beta k(x^*, \xi)\, dS.$$

Since $2\eta \leqq y_*$ and $2|x^* - \xi| \geqq y_*$ on Q_2 and

$$k(x, \xi) \leqq |x^* - \xi|^{-\alpha} \int_0^\pi \sin^{\alpha-1} s\, ds \cdot ds$$
$$\cdot \int_0^\pi [|x^* - \xi|^2 + 2\zeta z_*(1 - \cos t)]^{-(n+\beta)/2} \sin^{\beta-1} t\, dt,$$

we find

$$I_2 \leqq A_2 \varepsilon y_*^{1-\alpha} z_*^{1-\beta} \int_{Q_2} \zeta^\beta K_\beta(x^*, \xi)\, dS \leqq A_2 \varepsilon y_*^{1-\alpha} z_*^{1-\beta},$$

where $\zeta^\beta K_\beta$ is the Poisson kernel for the sphere for the n-dimensional GASPT operator with parameter β; the integral of this kernel over $Q(R)$ is 1/2 (see Huber [3, (24), p. 353]). The estimate $I_3 \leqq A_3 \varepsilon y_*^{1-\alpha} z_*^{1-\beta}$ is very similar to that for I_2 and is omitted. The estimate

$$\begin{aligned} I_4 &= \varepsilon (y_*/2)^{1-\alpha}(z_*/2)^{1-\beta} \int_{Q_4} \eta^\alpha \zeta^\beta K(x^*, \xi)\, dS \\ &\leqq A_4 \varepsilon y_*^{1-\alpha} z_*^{1-\beta} \end{aligned}$$

follows directly from the identity (3.6). With the above estimates, in which the A_j are independent of $(x_1^*, \cdots, x_{n-2}^*)$ the property (d) follows.

Remarks. 1. For $y = 0$ or $z = 0$ the operator L is not defined. For such points in $B(R)$, u in (3.3) satisfies the limiting form of the homogeneous equation, $u_y = 0$ for $y = 0$ and $u_z = 0$ for $z = 0$.

2. The analyticity of u in $B(R)$ but *not* on $y = 0$ or $z = 0$ can be established via the general theory of elliptic equations but this procedure is *not* valid on or across the singular hyperplanes.

3. Putting $x = 0$ yields a mean value theorem for solutions of (1.1) analogous to Weinstein's mean value theorem in GASPT [1] (see also Kapilevich [7]).

4. The case $\alpha < 1$, $\beta < 1$. In this case data are prescribed on the entirety of $\partial B^+(R)$ with zero data on the singular hyperplanes and the problem is solved by reduction to the previous case via the following simple generalization (Weinacht [6]) of Weinstein's [1] correspondence principle of GASPT:

A function u is a solution of $L_{2-\alpha,2-\beta}[u] = 0$ if and only if $v = x_{n-1}^{1-\alpha} x_n^{1-\beta} u$ is a solution of $L_{\alpha,\beta}[v] = 0$.

This principle is an immediate consequence of the identity

$$L_{\alpha,\beta}[x_{n-1}^{1-\alpha} x_n^{1-\beta} w] = x_{n-1}^{1-\alpha} x_n^{1-\beta} L_{2-\alpha,2-\beta}[w]$$

analogous to that in GASPT.

The formula (4.1) can also be obtained by use of a fundamental solution (Weinacht [6]) and a method of images (see the Appendix to this paper).

THEOREM 2. *Let $\alpha < 1$, $\beta < 1$. Suppose g is a function which is continuous on $\partial B^+(R)$ and vanishes on $B(R) \cap \partial Q$. Then the function v defined on $B^+(R)$ by*

$$v(x) = y^{1-\alpha} z^{1-\beta} \int_{Q(R)} n\zeta J(x, \xi) g(\xi)\, dS, \tag{4.1}$$

where

$$J(x,\xi) = \frac{2(R^2 - |x|^2)\Gamma(q)}{R\pi^{n/2}\Gamma((2-\alpha)/2)\Gamma((2-\beta)/2)} \int_0^\pi \int_0^\pi \sigma^{-2q} \sin^{1-\alpha} s \sin^{1-\beta} t\, ds\, dt$$

with σ as defined previously in (3.4) *and $2q = n + 4 - \alpha - \beta$ is the unique solution of $L[v] = 0$ in $B^+(R)$ taking on the boundary values g:*

$$\lim_{x \to \xi_0} v(x) = g(\xi_0), \qquad x \in B^+(R), \quad \xi_0 \in \partial B^+(R).$$

Proof. The proof consists of a reduction to a situation where Theorem 1 applies. Define $f(\xi) = \eta^{\alpha-1}\zeta^{\beta-1} g(\xi)$. Then f is continuous on $Q(R)$ and satisfies the hypothesis (3.1) of Theorem 1 with (α, β) replaced by $(2 - \alpha, 2 - \beta)$; note that $2 - \alpha > 1, 2 - \beta > 1$. Hence we are assured by Theorem 1 that there is a function u given by (3.3) with (α, β) replaced by $(2 - \alpha, 2 - \beta)$ and having the following properties: $L_{2-\alpha,2-\beta}[u] = 0$ in $B^+(R)$, u takes on the boundary values f on $Q(R)$ and u satisfies (3.5). By the correspondence principle the function $v = y^{1-\alpha}z^{1-\beta}u$ satisfies $L_{\alpha,\beta}[v] = 0$ in $B^+(R)$. This is precisely the v in (4.1). Moreover, we see directly from its definition in terms of u that v takes on the boundary values g on $Q(R)$ and on $\overline{B(R)} \cap \partial Q$ tends to zero which is the value of g there. Thus, v is a solution of the boundary value problem as stated. The uniqueness follows from a simple application of the usual maximum principle (Hopf [9]) for second order *nonsingular* elliptic equations in the region $B^+(R)$.

Remark. Putting $\alpha = 0$ or $\beta = 0$ in (4.1) yields after a short computation the Poisson integral formula obtained by Huber [3, p. 357] for GASPT.

5. The case $\alpha < 1$, $\beta \geqq 1$. Here data are prescribed on $Q(R)$ as well as zero data on $y = 0$. The formula (5.2) was obtained via the correspondence principle which also forms the basis of the verification.

THEOREM 3. *Let* $\alpha < 1$, $\beta \geqq 1$. *Suppose h is continuous on* $Q(R)$ *and suppose h has the following behavior near the singular hyperplanes*:

$$\lim_{\xi \to \xi_0} \rho(\zeta; \beta)h(\xi) = 0, \qquad \xi \in Q(R), \quad \xi_0 \in \overline{Q(R)} \cap \partial Q, \tag{5.1}$$

where ρ *is as defined in* (3.2). *Then the function* w *defined in the half-ball* $H(R) \equiv B(R) \cap \{x : x_{n-1} > 0\}$ *by*

$$w(x) = y^{1-\alpha} \int_{Q(R)} \eta \zeta^{\beta} I(x, \xi) h(\xi)\, dS, \tag{5.2}$$

where

$$I(x, \xi) = \frac{2(R^2 - |x|^2)\Gamma(r)}{R\pi^{n/2}\Gamma((2-\alpha)/2)\Gamma(\beta/2)} \int_0^{\pi}\int_0^{\pi} \sigma^{-2r} \sin^{1-\alpha} s \sin^{\beta-1} t\, ds\, dt,$$

$2r = n + \beta + 2 - \alpha$ *and* σ *is as defined in* (3.4) *has the following properties*:

(a) w *is even in* z;

(b) w *is analytic in* $H(R)$ *and satisfies* (1.1) *there except for* $z = 0$;

(c) w *assumes the boundary values* h *on* $Q(R)$:

$$\lim_{x \to \xi_0} w(x) = h(\xi_0), \qquad x \in B^+(R), \quad \xi_0 \in Q(R);$$

(d) w *inherits the behavior of* h *near the singular hyperplanes*:

$$\lim_{x \to x_0} \rho(z; \beta)w(x) = 0, \qquad x \in B^+(R), \quad x_0 \in B(R) \cap \partial Q.$$

Proof. The proof is similar to that of Theorem 2 and so will be sketched briefly. Define $f(\xi) = \eta^{\alpha-1}h(\xi)$. Then because of (5.1), f satisfies (3.1) with (α, β) replaced by $(2 - \alpha, \beta)$. For this f Theorem 1 yields a function u via (3.3) with (α, β) replaced by $(2 - \alpha, \beta)$. The w in (5.2) is precisely $y^{1-\alpha}u$. Using again the correspondence principle and the properties of u one sees that w has all of the properties asserted.

Remarks. Putting $\alpha = 0$ in (5.2) yields, after a short computation, the Poisson integral formula of Huber for GASPT [3, p. 352].

Appendix. We prove here (3.6), i.e.,

$$\int_{Q(R)} \eta^{\alpha}\zeta^{\beta}K(x,\xi)\,dS \equiv 1 \tag{A.1}$$

for $\alpha \geqq 1$, $\beta \geqq 1$ and $x \in B(R)$. It is sufficient to consider x in $B^+(R)$ because K is even in y and z (so that only $y \geqq 0$ and $z \geqq 0$ need be considered) and because, in addition, $K(x,\xi)$ is continuous for x in $B(R)$ and on $\overline{Q(R)}$ (so that only $y > 0$ and $z > 0$ need be considered). Moreover it is sufficient to establish (A.1) for $\alpha > 1$ and $\beta > 1$ and then argue by continuity to obtain the result for $\alpha \geqq 1$ and $\beta \geqq 1$.

In the fundamental solution E for $\tilde{L} \equiv L_{2-\alpha,2-\beta}$ in the quarter space Q with pole x (Weinacht [6, p. 577])

$$E(\xi,x) = \frac{-yz\eta^{\alpha-1}\zeta^{\beta-1}}{\pi^{n/2}\Gamma(\alpha/2)\Gamma(\beta/2)}\int_0^{\pi}\int_0^{\pi}\sigma^{2-n-\alpha-\beta}\sin^{\alpha-1}s\sin^{\beta-1}t\,ds\,dt$$

replace σ by $(|x|/R)\bar{\sigma}$, where $\bar{\sigma}$ is obtained by replacing x_i by $(R^2/|x|^2)x_i$ $(i = 1, 2, \cdots, n)$ in σ (see equation (3.4)) and call the resulting function $g = g(\xi, x)$. The above process is a variant of the method of images. It is easy to see that $G = E - g$ is a Green's function for $\tilde{L}$ in $B^+(R)$ with pole x which vanishes for ξ on $\partial B^+(R)$. Then the introduction of $v(\xi) = G(x,\xi)$ and the particular solution $u(\xi) = \eta^{\alpha-1}\zeta^{\beta-1}$ of $\tilde{L}[u] = 0$ with x replaced by ξ into the Green's second identity

$$\int_{B^+(R)}\eta^{2-\alpha}\zeta^{2-\beta}(u\tilde{L}[v] - v\tilde{L}[u])\,d\xi = \int_{\partial B^+(R)}\eta^{2-\alpha}\zeta^{2-\beta}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$$

yields after approach from compact subregions

$$yz = \int_{Q(R)}\eta\zeta\frac{\partial G}{\partial n}\,dS,$$

which upon simplification yields (A.1).

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JACOBI POLYNOMIALS, I. NEW PROOFS OF KOORNWINDER'S LAPLACE TYPE INTEGRAL REPRESENTATION AND BATEMAN'S BILINEAR SUM*

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Abstract. This is the first of a series of papers which will give simple proofs of a number of recent formulas for Jacobi polynomials. In this paper one of Gegenbauer's proofs for his integral representation of ultraspherical polynomials is given, and then a fractional integration gives Koornwinder's integral representation for Jacobi polynomials. This is then combined with Koornwinder's product formula to give a new proof of a bilinear sum of Bateman.

1. Introduction. Gegenbauer's fundamental work in the last quarter of the nineteenth century led to a deep understanding of ultraspherical (or Gegenbauer) polynomials $C_n^\lambda(x)$. These polynomials are orthogonal with respect to the weight function $(1 - x^2)^{\lambda - 1/2}$. The more general Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, which are orthogonal with respect to $(1 - x)^\alpha(1 + x)^\beta$, also arise in a number of different contexts. A brief glance at the standard list of formulas [8] shows that many of the known formulas for ultraspherical polynomials have not been generalized to Jacobi polynomials. The most important missing formula is the addition formula [8, 10.9 (34)]

$$
\begin{aligned}
&C_n^\lambda(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\varphi)\\
(1.1)\qquad &= \sum_{m=0}^{n} 2^{2m}(2m + 2\lambda - 1)(n - m)!\frac{[(\lambda)_m]^2}{(2\lambda - 1)_{n+m+1}}(\sin\theta)^m C_{n-m}^{\lambda+m}(\cos\theta)\\
&\quad\cdot(\sin\psi)^m C_{n-m}^{\lambda+m}(\cos\psi)C_m^{\lambda-1/2}(\cos\varphi).
\end{aligned}
$$

The Laplace type integral, [8, 10.9 (31)],

$$
(1.2)\qquad C_n^\lambda(x) = \frac{2^{1-2\lambda}\Gamma(n + 2\lambda)}{n![\Gamma(\lambda)]^2}\int_0^\pi [x + (x^2 - 1)^{1/2}\cos\varphi]^n(\sin\varphi)^{2\lambda-1}\,d\varphi
$$

is another such formula. These gaps have been particularly unfortunate, for not only are these formulas very useful, but Jacobi polynomials form a more natural class of polynomials than the subclass of ultraspherical polynomials. There are some operations which are very natural in the class of Jacobi polynomials; for example, the kernel which gives the partial sum of a Jacobi series at $y = 1$,

$$
\begin{aligned}
K_n^{(\alpha,\beta)}(x, 1) &= \sum_{k=0}^{n}\frac{(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)\Gamma(k + 1)}{2^{\alpha+\beta+1}\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(1)\\
(1.3)\qquad &= 2^{-\alpha-\beta-1}\frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(n + \beta + 1)}P_n^{(\alpha+1,\beta)}(x)\qquad [17, (4, 5, 3)]
\end{aligned}
$$

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and Bateman's integral

$$(1.4)\qquad (1-x)^{\alpha+\mu}\frac{P_n^{(\alpha+\mu,\beta-\mu)}(x)}{P_n^{(\alpha+\mu,\beta-\mu)}(1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)}\int_x^1 (1-y)^\alpha \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}(y-x)^{\mu-1}\,dy$$

[1, (3.3)];

which take ultraspherical polynomials out of the class of ultraspherical (or symmetric Jacobi) polynomials.

After almost one hundred years this gap has finally been closed by some deep work of Tom Koornwinder [11], [12]. He has found the addition formula for Jacobi polynomials, and as a consequence has found the Laplace integral representation. His original proof used the unitary group and is a good example of the power of algebraic methods applied to special functions. While this proof is very natural there are many people who could use Koornwinder's results who do not have the background to understand his proof. In this series of papers by Askey, Koornwinder, and others an alternate proof will be given of this work, and applications will be given of some of the special cases. One such application of an old (and almost forgotten) formula of Bateman [3]

$$(1.5)\qquad \left(\frac{x+y}{2}\right)^n \frac{P_n^{(\alpha,\beta)}((1+xy)/(x+y))}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^n c_{k,n}\frac{P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y)}{P_k^{(\alpha,\beta)}(1)P_k^{(\alpha,\beta)}(1)},$$

where $c_{k,n}$ is defined by

$$(1.6)\qquad \left(\frac{1+x}{2}\right)^n = \sum_{k=0}^n c_{k,n}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)},$$

is the variation diminishing property of the de la Vallée Poussin means [2] for Jacobi series. This will be given in a later paper by R. Horton. H. Bavinck pointed out that Bateman's sum (1.5) gives an explicit formula for the kernel of the de la Vallée Poussin means. The variation diminishing property for the case $\alpha = \beta = -\frac{1}{2}$ was previously given by Pólya and Schoenberg [15]. Their proofs were very complicated, because they did not have the explicit formula (1.5). Thus even in the case of Fourier series it sometimes pays to consider generalizations to Jacobi series. As often happens in mathematics, the correct generalization tends to simplify matters.

One application of these results is the positivity of some Cesàro mean for Jacobi series. Without the product formula (4.4), it would be impossible to prove this result. The Laplace integral (3.6) can be used to study biaxially symmetric potentials. Once these results become well known, other applications will arise.

2. Gegenbauer's Laplace integral. Gegenbauer's Laplace type integral is reasonably well known, but the impression exists that it is hard to prove; see [16] where a very unnatural proof is given. Gegenbauer's second proof [10] is simple, natural, and almost unknown, so we start with it.

One of the most natural ways to define ultraspherical polynomials is by the generating function

$$(2.1)\qquad (1-2r\cos\theta+r^2)^{-\lambda} = \sum_{n=0}^\infty C_n^\lambda(\cos\theta)r^n.$$

Use $2\cos\theta = e^{i\theta} + e^{-i\theta}$ in (2.1), factor and expand the left-hand side to obtain

$$C_n^\lambda(\cos\theta) = \sum_{k=0}^{n} \frac{(\lambda)_k(\lambda)_{n-k}}{k!(n-k)!} e^{i(n-2k)\theta}. \tag{2.2}$$

The expression $(\lambda)_k$ is defined by

$$(\lambda)_k = \frac{\Gamma(k+\lambda)}{\Gamma(\lambda)}.$$

Thus (2.2) can be rewritten as

$$C_n^\lambda(\cos\theta) = \frac{1}{[\Gamma(\lambda)]^2} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\Gamma(k+\lambda)\Gamma(n-k+\lambda)}{\Gamma(n+1)} e^{i(n-2k)\theta}. \tag{2.3}$$

The presence of $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ suggests the binomial theorem, and the beta function

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}\,dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

can be used with $a = k+\lambda$, $b = n-k+\lambda$ to obtain

$$\begin{aligned} C_n^\lambda(\cos\theta) &= \frac{\Gamma(n+2\lambda)}{[\Gamma(\lambda)]^2 n!} \int_0^1 y^{\lambda-1}(1-y)^{\lambda-1} \sum_{k=0}^{n} \binom{n}{k} y^k(1-y)^{n-k} e^{i(n-2k)\theta}\,dy \\ &= \frac{\Gamma(n+2\lambda)}{[\Gamma(\lambda)]^2 n!} \int_0^1 y^{\lambda-1}(1-y)^{\lambda-1}(1-y)^n e^{in\theta}\left[1 + \frac{y\,e^{-2i\theta}}{1-y}\right]^n dy \\ &= \frac{\Gamma(n+2\lambda)}{[\Gamma(\lambda)]^2 n!} \int_0^1 y^{\lambda-1}(1-y)^{\lambda-1}[(1-y)\,e^{i\theta} + y\,e^{-i\theta}]^n\,dy. \end{aligned}$$

Letting $y = \sin^2\varphi$ gives

$$C_n^\lambda(\cos\theta) = \frac{2\Gamma(n+2\lambda)}{[\Gamma(\lambda)]^2 n!} \int_0^{\pi/2} [\cos\theta + i\sin\theta(\cos^2\varphi - \sin^2\varphi)]^n(\sin\varphi\cos\varphi)^{2\lambda-1}\,d\varphi.$$

Replacing φ by $\varphi/2$ gives

$$C_n^\lambda(\cos\theta) = \frac{\Gamma(n+2\lambda)}{2^{2\lambda-1}[\Gamma(\lambda)]^2 n!} \int_0^{\pi} [\cos\theta + \sin\theta\cos\varphi]^n(\sin\varphi)^{2\lambda-1}\,d\varphi$$

or

$$\frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^{\pi} [\cos\theta + i\sin\theta\cos\varphi]^n(\sin\varphi)^{2\lambda-1}\,d\varphi, \qquad \lambda > 0, \tag{2.4}$$

where

$$C_n^\lambda(1) = (2\lambda)_n/n!$$

and Legendre's duplication formula $\Gamma(2\lambda) = 2^{2\lambda-1}\Gamma(\lambda)\Gamma(\lambda+\frac{1}{2})/\Gamma(\frac{1}{2})$ has been used to simplify (2.4). This is Gegenbauer's formula.

3. Koornwinder's Laplace type integral. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k!}\left(\frac{1-x}{2}\right)^k = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \tag{3.1}$$

$$P_n^{(\alpha,\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\frac{n!}{(2\lambda)_n}C_n^{\lambda}(x), \qquad \lambda = \alpha + \tfrac{1}{2}, \tag{3.2}$$

gives a connection between the ultraspherical polynomials and the symmetric Jacobi polynomials, so Gegenbauer's integral is

$$\frac{P_n^{(\beta,\beta)}(y)}{P_n^{(\beta,\beta)}(1)} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi}[y + i\sqrt{1-y^2}\cos\theta]^n(\sin\theta)^{2\beta}\,d\theta. \tag{3.3}$$

The integral

$$\frac{(1-x)^{\alpha}}{(1+x)^{n+\beta+1}}\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}\int_x^1\frac{(1-y)^{\beta}}{(1+y)^{n+\alpha+1}}\frac{P_n^{(\beta,\beta)}(y)}{P_n^{(\beta,\beta)}(1)}(y-x)^{\alpha-\beta-1}\,dy, \tag{3.4}$$

$\alpha > \beta$, is a consequence of Bateman's integral

$$F(a;b;c+\mu;x) = \frac{\Gamma(c+\mu)}{\Gamma(c)\Gamma(\mu)}\int_0^1 y^{c-1}(1-y)^{\mu-1}F(a;b;c;xy)\,dy, \qquad \mu > 0,$$

and the Euler transformation formula

$$F(a,b;c;x) = (1-x)^{-a}F(a,c-b;c;x/(x-1)).$$

See [1]. Combining (3.3) and (3.4) gives

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(\frac{1}{2})}\int_x^1\int_0^{\pi}\frac{(1-y)^{\beta}(1+x)^{n+\beta+1}}{(1-x)^{\alpha}(1+y)^{n+\alpha+1}}(y-x)^{\alpha-\beta-1} \cdot [y + i\sqrt{1-y^2}\cos\theta]^n(\sin\theta)^{2\beta}\,d\theta\,dy. \tag{3.5}$$

The change of variables

$$u^2 = \frac{(1-y)(1+x)}{(1+y)(1-x)}$$

reduces (3.5) to

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(\frac{1}{2})}\int_0^1\int_0^{\pi}\left[\frac{1+x-(1-x)u^2}{2} + i\sqrt{1-x^2}\,u\cos\theta\right]^n \cdot (1-u^2)^{\alpha-\beta-1}u^{2\beta+1}(\sin\theta)^{2\beta}\,d\theta\,du, \qquad \alpha > \beta > -\tfrac{1}{2}, \tag{3.6}$$

which is Koornwinder's Laplace type integral for Jacobi polynomials. Somewhat earlier Braaksma and Meulenbeld [5] found a different integral representation for Jacobi polynomials and their work was extended by Dijksma and Koornwinder [7]. However Koornwinder's formula [11] is a more useful formula, as will be shown in the remainder of this paper and in the sequel [13] by Koornwinder.

4. Bateman's bilinear sum. Bateman [3] found the formula

$$\left(\frac{x+y}{2}\right)^n \frac{P_n^{(\alpha,\beta)}((1+xy)/(x+y))}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} c_{k,n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \frac{P_k^{(\alpha,\beta)}(y)}{P_k^{(\alpha,\beta)}(1)}, \tag{4.1}$$

where $c_{k,n}$ is defined by (4.1) when $y = 1$, i.e.,

$$\left(\frac{1+x}{2}\right)^n = \sum_{k=0}^{n} c_{k,n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)}. \tag{4.2}$$

A simple calculation using Rodrigues' formula

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

shows that

$$c_{k,n} = \frac{\Gamma(n+\beta+1)n!(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(k+n+\alpha+\beta+2)\Gamma(k+\beta+1)k!(n-k)!\Gamma(\alpha+1)}. \tag{4.3}$$

However for some applications the explicit formula is not needed, only the defining series (4.2), or in Horton's proof, the positivity of (4.3). Replace the variable x in (4.2) by

$$xy - \tfrac{1}{2}(1-x)(1-y)(1-v^2) + \sqrt{1-x^2}\sqrt{1-y^2}\,v\cos\theta.$$

Then use

$$\begin{aligned}\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} &= \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})\beta(\frac{1}{2})} \\ &\cdot \int_0^1\int_0^\pi \frac{P_n^{(\alpha,\beta)}(xy - \frac{1}{2}(1-x)(1-y)(1-v^2) + \sqrt{1-x^2}\sqrt{1-y^2}\,v\cos\theta)}{P_n^{(\alpha,\beta)}(1)} \\ &\cdot (1-v^2)^{\alpha-\beta-1}v^{2\beta+1}(\sin\theta)^{2\beta}\,d\theta\,dv\end{aligned} \tag{4.4}$$

and (3.6), and a routine calculation to obtain (4.1).

Equation (4.4) is Koornwinder's version of the product formula for Jacobi polynomials. An equivalent expression, which however is a little harder to use as it is given, was found somewhat earlier by Gasper [9]. Gasper's original proof used a very complicated integral involving Bessel functions and Koornwinder's first proof used his addition formula. In the next paper Koornwinder will give a very simple proof of (4.4). His proof consists of finding a simple direct proof of (4.1) and its inverse, and then applying the Laplace type integral as above. After all of this work had been done, Gasper called our attention to Bateman's paper [3] and then Koornwinder found that his proof of (4.1) had been anticipated by Bateman in 1932 [4]. A possible reason that this important work has been overlooked is the nonstandard notation which Bateman used. Two other papers should be mentioned, [6] and [14]. Cowling proved the case $\alpha = 0$ of the following formula and Koschmieder proved the general case.

$$(4.5)\quad (u^2 + v^2 - 1)^{n/2} \frac{P_n^{(\alpha,\alpha)}(uv/(u^2 + v^2 - 1)^{1/2})}{P_n^{(\alpha,\alpha)}(1)} = \sum_{k=0}^{n} d_{k,n} \frac{P_k^{(\alpha,\alpha)}(u)}{P_k^{(\alpha,\alpha)}(1)} \frac{P_k^{(\alpha,\alpha)}(v)}{P_k^{(\alpha,\alpha)}(1)},$$

where $d_{k,n}$ is given by (4.5) when $v = 1$. I knew (4.5) from the review of [14] in Mathematical Reviews and (4.1) was discovered for $\beta = \pm\frac{1}{2}$ by applying the classical quadratic transformation formulas

$$(4.6)\qquad \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-1/2)}(2x^2 - 1)}{P_n^{(\alpha,-1/2)}(1)} \qquad [17,(4,1,5)]$$

and

$$(4.7)\qquad \frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = x\frac{P_n^{(\alpha,1/2)}(2x^2 - 1)}{P_n^{(\alpha,1/2)}(1)} \qquad [17,(4,1,5)]$$

to (4.5).

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JACOBI POLYNOMIALS, II. AN ANALYTIC PROOF OF THE PRODUCT FORMULA*

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Abstract. An analytic proof is given for the author's product formula for Jacobi polynomials and a new integral representation is obtained for the product $J_\alpha(x)J_\beta(y)$ of two Bessel functions. Similarly, a product formula for Jacobi polynomials due to Dijksma and the author is derived in an analytic way. The proofs are based on Bateman's work on special solutions of the biaxially symmetric potential equation. The paper concludes with new proofs for Gasper's evaluation of the convolution kernel for Jacobi series and for Watson's evaluation of the integral

$$\int_0^\infty J_\alpha(\lambda x)J_\beta(\lambda y)J_\beta(\lambda z)\lambda^{1-\alpha}\,d\lambda.$$

1. Introduction. In recent papers [13], [14], [15] the author derived the addition formula for Jacobi polynomials by group theoretic methods. It was pointed out in [13] that the product formula and the Laplace type integral representation for Jacobi polynomials immediately follow from the addition formula. The way of obtaining these results illustrated the power of the group theoretic approach to special functions. However, it was felt unsatisfying that no analytic proofs were available for the addition formula and its corollaries.

Next, an elementary analytic proof of the Laplace type integral representation was given by Askey [1]. Our main result in the present paper is an analytic derivation of the product formula. It is based on important but rather unknown results of Bateman [3], [4] concerning special solutions of the biaxially symmetric potential equation. The present paper is a continuation of Askey's paper [1]. We would like to thank Askey for communicating us the results contained in [1] and Gasper for calling our attention to [3].

Immediately after this work was done both Gasper and the author extended the results to an analytic proof of the addition formula. They used different methods and will publish their proofs separately in subsequent papers.

Section 2 of this paper contains a review of Bateman's work on the biaxially symmetric potential equation [3], [4]. Admitting transformations of the variables, Bateman obtained solutions of this equation by separating the variables in three different ways. We prove that, in a certain sense, these three possibilities are the only ones. Bateman's special solutions involve Bessel functions, Jacobi polynomials and nth powers. They can be expressed in terms of each other by means of a number of identities, one of which is the bilinear sum obtained in [1].

By using these identities the product formula for Jacobi polynomials and a new product formula for Bessel functions can be derived from the Laplace type integral representation for Jacobi polynomials. This is done in § 3. Section 4 discusses the analogous results connected with an integral representation for Jacobi polynomials due to Braaksma and Meulenbeld [5] and a new proof is given of a product formula due to Dijksma and the author [7].

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Gasper [10], [11] settled the positivity of the convolution structure for Jacobi series. His explicit expression for the convolution kernel is derived from our product formula in § 5. Some formulas from Watson [17], which Gasper applied in his proof in [10], here arise in a natural way. Thus, a deeper understanding of Gasper's proof is achieved.

2. The biaxially symmetric potential equation. The partial differential equation

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2\beta+1}{u}\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2} + \frac{2\alpha+1}{v}\frac{\partial}{\partial v}\right)F(u,v) = 0 \tag{2.1}$$

arises naturally from the potential equation in two different ways.

First, if α and β are nonnegative integers and if $(x_1, x_2, x_3, x_4) = (u\cos\phi, u\sin\phi, v\cos\chi, v\sin\chi)$, then the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right)(u^\beta v^\alpha e^{i(\beta\phi+\alpha\chi)}F(u,v)) = 0 \tag{2.2}$$

is equivalent to (2.1) (cf. Bateman [4, p. 389]).

Second, if $2\alpha + 1$ and $2\beta + 1$ are nonnegative integers and if

$$u = \sqrt{x_1^2 + x_2^2 + \cdots + x_{2\beta+2}^2}$$

and

$$v = \sqrt{y_1^2 + y_2^2 + \cdots + y_{2\alpha+2}^2},$$

then the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{2\beta+2}^2} + \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{2\alpha+2}^2}\right)F(u,v) = 0 \tag{2.3}$$

is equivalent to (2.1).

Therefore, (2.1) is called the biaxially symmetric potential equation. Special solutions of this equation were studied by Bateman in [3] and in [4, pp. 389–394]. We will summarize some of Bateman's results in this section.

The differential operator in (2.1) has two singular lines $u = 0$ and $v = 0$. It is natural to consider solutions of (2.1) in the upper right quarter-plane. Equation (2.1) admits solutions by separation of variables. Regular solutions of this type are

$$F(u,v) = u^{-\beta}J_\beta(\lambda u)v^{-\alpha}I_\alpha(\lambda v), \tag{2.4}$$

where the functions J_β and I_α are Bessel functions.

Let $\mathscr{D}_1$ be a simply connected domain in the (s,t)-plane and let $\mathscr{D}_2 = \{(u,v) | u > 0, v > 0\}$. Suppose that the mapping $(s,t) \to (u,v)$ is a conformal mapping of $\mathscr{D}_1$ onto $\mathscr{D}_2$. It means that $u(s,t)$ and $v(s,t)$ satisfy the Cauchy–Riemann equations

$$u_s = v_t \quad \text{and} \quad u_t = -v_s \tag{2.5}$$

and that

$$\Delta(s,t) \equiv u_s v_t - u_t v_s \neq 0 \quad \text{on } \mathscr{D}_1. \tag{2.6}$$

After this transformation, equation (2.1) becomes

$$\frac{1}{\Delta(s,t)}\left[\frac{\partial^2}{\partial s^2} + \left((2\beta+1)\frac{u_s}{u} + (2\alpha+1)\frac{v_s}{v}\right)\frac{\partial}{\partial s} + \frac{\partial^2}{\partial t^2} + \left((2\beta+1)\frac{u_t}{u} + (2\alpha+1)\frac{v_t}{v}\right)\frac{\partial}{\partial t}\right]F(u(s,t), v(s,t)) = 0. \tag{2.7}$$

It is not difficult to prove that for a fixed conformal mapping $(s, t) \to (u, v)$ as introduced above the following three statements are equivalent.

(A) For all values of α and β, equation (2.7) admits separation of variables.

(B) Both the functions $u(s, t)$ and $v(s, t)$ are the products of a function of s and a function of t.

(C) The mapping $(s, t) \to (u, v)$ is given by one of the three complex analytic functions

$$u + iv = s + it, \quad u + iv = e^{s+it} \quad \text{or} \quad u + iv = \cos(s + it),$$

up to translations, dilatations and rotations over an angle $k(\pi/2)$ of the (s, t)-plane and up to dilatations of the (u, v)-plane.

We did not succeed in proving or disproving the equivalence of **(B)** with the following statement (A').

(A') There is a value of α and β $(-\frac{1}{2} \neq \alpha \neq \beta \neq -\frac{1}{2})$ for which equation (2.7) admits separation of variables.

However, the equivalence of the statements (A) and (C) suggests that one should especially consider the three forms of equation (2.1) connected by the transformations

$$u + iv = e^{x+iy} = \cos(\xi + i\eta). \tag{2.8}$$

The pictures in Fig. 1 show the domains which are thus mapped onto each other.

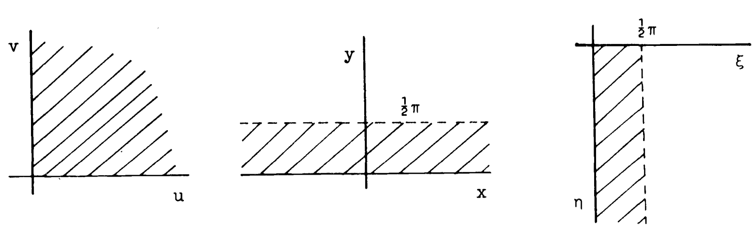

FIG. 1

The first identity in (2.8) is equivalent to

$$u = e^x \cos y, \quad v = e^x \sin y \tag{2.9}$$

and equation (2.7) becomes

$$\left[\frac{\partial^2}{\partial x^2} + 2(\alpha+\beta+1)\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + ((2\alpha+1)\operatorname{cotg} y - (2\beta+1)\operatorname{tg} y)\frac{\partial}{\partial y}\right] \cdot F(e^x \cos y, e^x \sin y) = 0 \tag{2.10}$$

with the special regular solutions

$$F(e^x \cos y, e^x \sin y) = e^{2nx} P_n^{(\alpha,\beta)}(\cos 2y) \tag{2.11}$$

(cf. Bateman [4, p. 389]). Here the function $P_n^{(\alpha,\beta)}$ denotes a Jacobi polynomial.

The mapping $(\xi, \eta) \to (u, v)$ in (2.8) can be written as

$$u = \cos \xi \operatorname{ch} \eta, \quad v = -\sin \xi \operatorname{sh} \eta \tag{2.12}$$

and after this transformation equation (2.1) takes the form

$$\begin{aligned} &\left[\frac{\partial^2}{\partial \xi^2} + ((2\alpha + 1) \operatorname{cotg} \xi - (2\beta + 1) \operatorname{tg} \xi) \frac{\partial}{\partial \xi}\right. \\ &\left. + \frac{\partial^2}{\partial \eta^2} + ((2\alpha + 1) \operatorname{cth} \eta + (2\beta + 1) \operatorname{th} \eta) \frac{\partial}{\partial \eta}\right] \\ &\cdot F(\cos \xi \operatorname{ch} \eta, -\sin \xi \operatorname{sh} \eta) = 0 \end{aligned} \tag{2.13}$$

with the special regular solutions

$$F(\cos \xi \operatorname{ch} \eta, -\sin \xi \operatorname{sh} \eta) = P_n^{(\alpha,\beta)}(\cos 2\xi) P_n^{(\alpha,\beta)}(\operatorname{ch} 2\eta) \tag{2.14}$$

(cf. Bateman [4, pp. 392–393]).

Bateman [3], [4] has derived some identities which relate the special solutions (2.4), (2.11) and (2.14) of equation (2.1) to each other. We need two of these identities.

Solutions of type (2.4) and (2.11) are related to each other by

$$u^{-\beta} J_\beta(u) v^{-\alpha} I_\alpha(v) = \sum_{n=0}^{\infty} a_n (u^2 + v^2)^n \frac{P_n^{(\alpha,\beta)}((u^2 - v^2)/(u^2 + v^2))}{P_n^{(\alpha,\beta)}(1)}, \tag{2.15}$$

where the coefficients a_n are defined by

$$\frac{1}{2^\alpha \Gamma(\alpha + 1)} u^{-\beta} J_\beta(u) = \sum_{n=0}^{\infty} a_n u^{2n} \tag{2.16}$$

(formula (2.15) with $v = 0$). For a detailed proof, see Bateman [3, pp. 113, 114]. Formula (2.15) is a generating function for Jacobi polynomials, which is also mentioned in Erdélyi [8, vol. III, § 19.9(12)].

The substitution

$$s = \cos 2\xi, \quad t = \operatorname{ch} 2\eta \tag{2.17}$$

combined with the substitutions (2.9) and (2.12) gives

$$e^{2x} = s + t, \quad \cos 2y = \frac{1 + st}{s + t}. \tag{2.18}$$

In terms of the variables s and t, the solutions of type (2.11) and (2.14) can be related to each other by the identity

$$\frac{P_n^{(\alpha,\beta)}(s)}{P_n^{(\alpha,\beta)}(1)} \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} b_{k,n} (s + t)^k \frac{P_k^{(\alpha,\beta)}((1 + st)/(s + t))}{P_k^{(\alpha,\beta)}(1)}, \tag{2.19}$$

where $b_{k,n}$ is defined by (2.19) when $t = 1$, i.e.,

$$\frac{P_n^{(\alpha,\beta)}(s)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} b_{k,n}(s+1)^k. \tag{2.20}$$

Formula (2.19) is proved in Bateman [4, pp. 392, 393] by using the fact that both sides of (2.19) are solutions of the same partial differential equation (2.13) (after the transformation (2.17)). The converse identity (formula (4.1) in Askey [1]) was first obtained in [3, pp. 122, 123]. For another result of Bateman, which expresses the solution (2.4) in terms of the solutions (2.14), the reader is referred to [3, p. 115] or [17, p. 370].

The preceding results might be extended by considering other special solutions of (2.1). For instance, one may take n complex in the solutions (2.11) and (2.14). In this way Flensted-Jensen and the author [9] generalized (2.19) for complex values of n. Another possibility is to replace one or both of the factors in (2.4), (2.11), (2.14) by a second solution of the (ordinary) differential equation.

It should be pointed out that Appell's hypergeometric function

$$F_4(\gamma, \delta; 1+\alpha, 1+\beta; -v^2, u^2),$$

defined in [8, vol. I, § 5.7.1], is also a solution of (2.1). This can be verified by termwise differentiating the power series of the function F_4. The methods of this section may be applied in order to prove the generating function for Jacobi polynomials mentioned in [8, vol. III, § 19.10 (26)] and the Poisson kernel for Jacobi polynomials (see Bailey [2, p. 102, example 19]).

It would also be of interest to express the solutions (2.11) and (2.14) in terms of the solutions (2.4) by means of definite integrals over λ.

Finally, we mention the work of Henrici [12], who used equation (2.1) in order to prove the addition formula for Gegenbauer functions.

3. The product formulas for Jacobi polynomials and for Bessel functions. The Laplace type integral representation for Jacobi polynomials is

$$R_n^{(\alpha,\beta)}(x) = \int_{r=0}^{1}\int_{\phi=0}^{\pi}\left(\frac{1+x}{2} - \frac{1-x}{2}r^2 + i\sqrt{1-x^2}\,r\cos\phi\right)^n dm_{\alpha,\beta}(r,\phi), \tag{3.1}$$
$$\alpha > \beta > -\tfrac{1}{2},$$

where

$$dm_{\alpha,\beta}(r,\phi) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi. \tag{3.2}$$

Following Gasper [10] we use the notation

$$R_n^{(\alpha,\beta)}(x) \equiv \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}. \tag{3.3}$$

The measure (3.2) is normalized by

$$\int_0^1\int_0^\pi dm_{\alpha,\beta}(r,\phi) = 1. \tag{3.4}$$

Formula (3.1) was first proved by the author [13] from the addition formula. Next, an elementary analytic proof of (3.1) was obtained by Askey [1, § 3]. The derivations given below were suggested by the way Askey proved the converse of (2.19) (see [1, § 4]).

It follows from (3.1) that

$$(x+y)^n R_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) = \int_0^1 \int_0^\pi [\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + \sqrt{1-x^2}\sqrt{1-y^2}\,r\cos\phi]^n \, dm_{\alpha,\beta}(r,\phi) \tag{3.5}$$

and

$$(x^2+y^2)^n R_n^{(\alpha,\beta)}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \int_0^1 \int_0^\pi (x^2 - y^2r^2 + 2ixyr\cos\phi)^n \, dm_{\alpha,\beta}(r,\phi). \tag{3.6}$$

Combination of formulas (2.19), (2.20) and (3.5) gives the product formula

$$R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y) = \int_0^1 \int_0^\pi R_n^{(\alpha,\beta)}[\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + \sqrt{1-x^2}\sqrt{1-y^2}\,r\cos\phi - 1]\, dm_{\alpha,\beta}(r,\phi), \qquad \alpha > \beta > -\tfrac{1}{2}. \tag{3.7}$$

In his original proof the author [13] derived (3.7) from the addition formula by integration.

In a similar way, it follows from the formulas (2.15), (2.16) and (3.6) that

$$\begin{aligned} x^{-\beta}J_\beta(x) y^{-\alpha} I_\alpha(y) &= \sum_{n=0}^\infty a_n \int_0^1 \int_0^\pi (x^2 - y^2r^2 + 2ixyr\cos\phi)^n \, dm_{\alpha,\beta}(r,\phi) \\ &= \int_0^1 \int_0^\pi \sum_{n=0}^\infty a_n (x^2 - y^2r^2 + 2ixyr\cos\phi)^n \, dm_{\alpha,\beta}(r,\phi) \\ &= \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^1 \int_0^\pi \frac{J_\beta((x^2 - y^2r^2 + 2ixyr\cos\phi)^{1/2})}{(x^2 - y^2r^2 + 2ixyr\cos\phi)^{(1/2)\beta}} \, dm_{\alpha,\beta}(r,\phi). \end{aligned}$$

The interchanging of summation and integration is allowed because the infinite sum converges uniformly in r and ϕ. By using that

$$y^{-\alpha} I_\alpha(y) = (iy)^{-\alpha} J_\alpha(iy)$$

and by analytic continuation it follows that

$$x^{-\beta}J_\beta(x) y^{-\alpha} J_\alpha(y) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \cdot \int_0^1 \int_0^\pi \frac{J_\beta((x^2 + y^2r^2 + 2xyr\cos\phi)^{1/2})}{(x^2 + y^2r^2 + 2xyr\cos\phi)^{(1/2)\beta}} \, dm_{\alpha,\beta}(r,\phi), \qquad \alpha > \beta > -\tfrac{1}{2}. \tag{3.8}$$

This formula seems to be new.

It is surprising that the two product formulas (3.7) and (3.8), which seem to be much deeper results than the integral representation (3.1), can be derived from (3.1) so easily. Another surprising fact is that formula (3.1) implies (3.7) but is also a degenerate case of (3.7). In fact, one obtains (3.1) after dividing both sides of (3.7) by $R_n^{(\alpha,\beta)}(y)$ and then taking the limit for $y \to \infty$.

Formula (3.8) follows from (3.7) by applying the confluence relations

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}(1 - y^2/(2n^2))}{P_n^{(\alpha,\beta)}(1)} = 2^\alpha \Gamma(\alpha + 1) y^{-\alpha} J_\alpha(y) \tag{3.9}$$

and

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}(x^2/(2n^2) - 1)}{P_n^{(\alpha,\beta)}(-1)} = 2^\beta \Gamma(\beta + 1) x^{-\beta} J_\beta(x) \tag{3.10}$$

(cf. Erdélyi [8, vol. II, § 10, 8(41)]).

If $\beta \uparrow \alpha$ then the measure $dm_{\alpha,\beta}(r, \phi)$ defined in (3.2) degenerates to the measure

$$\frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} \delta(1 - r)(\sin \phi)^{2\beta}\, dr\, d\phi.$$

Here $\delta(t)$ represents Dirac's delta function.

The degenerate forms of (3.1) and (3.7) for $\alpha = \beta$ are Gegenbauer's classical formulas for ultraspherical polynomials (cf. [8, vol. I, § 3, 15(22), (20)]). Formula (3.8) degenerates to the product formula

$$\begin{aligned} x^{-\beta} J_\beta(x) y^{-\beta} J_\beta(y) &= \frac{1}{2^\beta \sqrt{\pi}\,\Gamma(\beta + \frac{1}{2})} \\ &\quad \cdot \int_0^\pi \frac{J_\beta((x^2 + y^2 + 2xy\cos\phi)^{1/2})}{(x^2 + y^2 + 2xy\cos\phi)^{(1/2)\beta}} (\sin \phi)^{2\beta}\, d\phi, \qquad \beta > -\tfrac{1}{2}. \end{aligned} \tag{3.11}$$

This is an integrated form of Gegenbauer's addition formula for Bessel functions (cf. Watson [17, § 11.4(2)]). It should be pointed out that new proofs are obtained for these two classical product formulas of Gegenbauer if one applies Bateman's identities (2.15) and (2.19) to (3.1) in the case $\alpha = \beta$.

Askey [1] derived the Laplace type integral representation (3.1) from its degenerate case $\alpha = \beta$ by using a fractional integral for Jacobi polynomials. In a similar way we can derive the product formula (3.8) from its special case (3.11) by applying Sonine's first integral

$$y^{-\alpha} J_\alpha(y) = \frac{1}{2^{\alpha-\beta-1}\Gamma(\alpha - \beta)} \int_0^1 (yr)^{-\beta} J_\beta(yr) r^{2\beta+1} (1 - r^2)^{\alpha-\beta-1}\, dr, \tag{3.12}$$

$\alpha > \beta > -1$ (see Watson [17, § 12.11(1)]). This method of reducing the case (α, β) to the case (β, β) fails for the product formula (3.7).

If $\beta \downarrow -\frac{1}{2}$ then the measure $dm_{\alpha,\beta}(r, \phi)$ degenerates to the measure

$$\frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} (1 - r^2)^{\alpha - 1/2} (\delta(\phi) + \delta(\pi - \phi))\, dr\, d\phi.$$

The degenerate forms of (3.1) and (3.7) which are thus obtained are related to the degenerate forms for $\alpha = \beta$ by the quadratic transformation

$$\frac{P_n^{(\alpha,-1/2)}(2x^2-1)}{P_n^{(\alpha,-1/2)}(1)} = \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} \qquad \text{(see [8, vol. II, § 10.9(21)])}. \tag{3.13}$$

Formula (3.8) degenerates for $\beta = -\frac{1}{2}$ to

$$\cos x \cdot y^{-\alpha} J_\alpha(y) = \frac{1}{2^\alpha \sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} \int_{-1}^{+1} \cos(x + yr)(1 - r^2)^{\alpha - 1/2}\, dr, \qquad \alpha > -\tfrac{1}{2}. \tag{3.14}$$

For $x = 0$, this is Poisson's integral ([17, § 3.3(1)])

$$y^{-\alpha} J_\alpha(y) = \frac{1}{2^\alpha \sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} \int_{-1}^{+1} \cos(yr)(1 - r^2)^{\alpha - 1/2}\, dr \tag{3.15}$$

and, conversely, formula (3.14) immediately follows from (3.15). Thus, the double integral (3.8) connects (3.11) with Poisson's integral in a continuous way.

The remarks at the end of § 2 suggest that other integral formulas can be derived by the methods of § 3. One case, for Jacobi functions, is worked out in [9].

The left-hand sides of formulas (3.1), (3.7) and (3.8) can each be considered as the first term of an orthogonal expansion with respect to the measure $dm_{\alpha,\beta}(r, \phi)$. An orthogonal system of functions with respect to this measure is

$$f_{k,l}(r, \phi) = P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2 - 1) r^{k-l} P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\phi), \qquad k \geqq l \geqq 0. \tag{3.16}$$

The expansion corresponding to formula (3.7) is called the addition formula for Jacobi polynomials (see Koornwinder [13]). The expansions corresponding to (3.1) and (3.8) can be obtained as degenerate cases of this addition formula. Recently, Gasper and the author independently gave analytic proofs of these expansions.

Gasper first derived the expansion corresponding to (3.1) in an elementary way and next applied (2.19) and (2.20) in order to obtain the addition formula. Similarly, one might prove the expansion corresponding to (3.8).

The author obtained the higher terms of the addition formula by doing integration by parts in (3.7). The same method might be applied to (3.1) and (3.8).

These two methods of proof will be published in the near future.

4. The integral representation of Braaksma and Meulenbeld. By interpreting Jacobi polynomials as spherical harmonics Braaksma and Meulenbeld [5] obtained an integral representation for Jacobi polynomials which is different from (3.1). Their formula is

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) = {} & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \frac{(\alpha+1)_n(\beta+1)_n}{(\frac{1}{2})_n n! 2^n} \\ & \cdot \int_0^\pi \int_0^\pi (i\sqrt{1-x}\cos\phi + \sqrt{1+x}\cos\psi)^{2n} \\ & \cdot (\sin\phi)^{2\alpha} (\sin\psi)^{2\beta}\, d\phi\, d\psi, \qquad \alpha > -\tfrac{1}{2}, \quad \beta > -\tfrac{1}{2}. \end{aligned} \tag{4.1}$$

As pointed out in [5], the analytic proof of (4.1) is easy.

By using (2.19) a product formula can be derived from (4.1). The explicit form of the coefficients $b_{k,n}$ in (2.21) follows from

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} = \frac{P_n^{(\beta,\alpha)}(-x)}{P_n^{(\beta,\alpha)}(1)} = {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right) = \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\beta+1)_k k!}\left(\frac{1+x}{2}\right)^k. \tag{4.2}$$

Hence,

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \cdot \int_0^\pi \int_0^\pi \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\frac{1}{2})_k k!\ 2^{2k}} \cdot (\sqrt{1-x}\sqrt{1-y}\cos\phi + \sqrt{1+x}\sqrt{1+y}\cos\psi)^{2k} \cdot (\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\, d\phi\, d\psi.$$

Let $C_{2n}^{\alpha+\beta+1}(t)$ denote a Gegenbauer polynomial. By using

$$\sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\frac{1}{2})_k k!} t^{2k} = {}_2F_1(-n, n+\alpha+\beta+1; \tfrac{1}{2}; t^2) = \frac{P_n^{(-1/2,\alpha+\beta+1/2)}(1-2t^2)}{P_n^{(-1/2,\alpha+\beta+1/2)}(1)} = \frac{C_{2n}^{\alpha+\beta+1}(t)}{C_{2n}^{\alpha+\beta+1}(0)}$$

we conclude that

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})C_{2n}^{\alpha+\beta+1}(0)} \cdot \int_0^\pi \int_0^\pi C_{2n}^{\alpha+\beta+1}(\tfrac{1}{2}\sqrt{1-x}\sqrt{1-y}\cos\phi + \tfrac{1}{2}\sqrt{1+x}\sqrt{1+y}\cos\psi) \cdot (\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\, d\phi\, d\psi, \quad \alpha > -\tfrac{1}{2},\ \beta > -\tfrac{1}{2}. \tag{4.3}$$

Formula (4.3) was first obtained by Dijksma and Koornwinder [7]. They used similar group theoretic methods to those of Braaksma and Meulenbeld [5].

We can also derive from (2.15) and (4.1) that

$$x^{-\alpha}J_\alpha(x)y^{-\beta}J_\beta(y) = \frac{1}{2^{\alpha+\beta}\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \cdot \int_0^\pi \int_0^\pi \cos(x\cos\phi + y\cos\psi)(\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\, d\phi\, d\psi, \quad \alpha > -\tfrac{1}{2},\ \beta > -\tfrac{1}{2}. \tag{4.4}$$

Writing

$$\cos(x\cos\phi + y\cos\psi) = \cos(x\cos\phi)\cos(y\cos\psi) - \sin(x\cos\phi)\sin(y\cos\psi),$$

we can reduce (4.4) to the product of two Poisson integrals (3.15).

5. Gasper's product formula. The right-hand sides of the formulas (3.1), (3.7) and (3.8) all have the form

$$\int_0^1 \int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha,\beta}(r,\phi),$$

where the function f is continuous on $(0, \infty)$, the letters a and b represent positive real numbers and the measure $dm_{\alpha,\beta}(r, \phi)$ is defined by (3.2). By a transformation of the integration variables this integral can be rewritten in the so-called kernel form. We will prove that

$$\int_0^1 \int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha,\beta}(r,\phi) = \int_0^\infty f(t^2) K_{\alpha,\beta}(a, b, t) t^{2\beta+1}\, dt, \tag{5.1}$$

where for $\alpha > \beta > -\frac{1}{2}$ the kernel $K_{\alpha,\beta}$ is defined by

$$K_{\alpha,\beta}(a,b,c) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \cdot a^{-2\alpha} \int_0^\pi (a^2 - b^2 - c^2 + 2bc\cos\psi)_+^{\alpha-\beta-1} (\sin\psi)^{2\beta}\, d\psi. \tag{5.2}$$

In formula (5.2) the notation

$$(x)_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0, \end{cases}$$

is used.

Formula (5.1) can be proved by successively performing the following transformations of variables to the left-hand side of (5.1). First, we put

$$x = r\cos\phi, \quad y = r\sin\phi,$$

next,

$$x' = ax + b, \quad y' = ay,$$

and finally,

$$x' = t\cos\psi, \quad y' = t\sin\psi.$$

Thus we obtain the equalities

$$\int_0^1 \int_0^\pi f(|ar\, e^{i\phi} + b|^2)(1 - r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\phi)^{2\beta}\, dr\, d\phi$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} f((ax + b)^2 + (ay)^2)(1 - x^2 - y^2)_+^{\alpha-\beta-1} y^{2\beta}\, dx\, dy$$

$$= a^{-2\alpha} \int_{-\infty}^{+\infty} \int_0^{\infty} f((x')^2 + (y')^2)(a^2 - b^2 - (x')^2 - (y')^2 + 2bx')_+^{\alpha-\beta-1} \cdot (y')^{2\beta}\, dx'\, dy'$$

$$= a^{-2\alpha} \int_0^{+\infty} \int_0^{\pi} f(t^2)(a^2 - b^2 - t^2 + 2bt\cos\psi)_+^{\alpha-\beta-1} t^{2\beta+1}\, dt\, d\psi.$$

Formula (5.1) follows by substitution of (3.2) and (5.2).

The kernel $K_{\alpha,\beta}$, defined by (5.2), is clearly nonnegative. Putting $f(x) \equiv 1$ in (5.1) we find

$$\int_0^{\infty} K_{\alpha,\beta}(a, b, t) t^{2\beta+1}\, dt = 1. \tag{5.3}$$

The analytic form of the kernel K was studied by Macdonald (see Watson [17, p. 412]) and by Gasper [11]. It turns out that three different cases have to be distinguished. Let

$$B \equiv \frac{b^2 + c^2 - a^2}{2bc}. \tag{5.4}$$

Then (5.2) takes the form

$$K_{\alpha,\beta}(a, b, c) = \frac{2^{\alpha-\beta}\Gamma(\alpha + 1)}{\sqrt{\pi}\,\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})} a^{-2\alpha}(bc)^{\alpha-\beta-1} \cdot \int_{-1}^{+1} (s - B)_+^{\alpha-\beta-1}(1 - s^2)^{\beta-1/2}\, ds. \tag{5.5}$$

Case I. $a < |b - c|$. Here $1 < B$, and $K_{\alpha,\beta}(a, b, c) = 0$.

Case II. $|b - c| < a < b + c$. Here $-1 < B < 1$, and

$$K_{\alpha,\beta}(a, b, c) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} a^{-2\alpha}(bc)^{\alpha-\beta-1}(1 - B^2)^{\alpha-1/2} \cdot {}_2F_1\left(\alpha + \beta, \alpha - \beta; \alpha + \tfrac{1}{2}; \frac{1 - B}{2}\right). \tag{5.6}$$

Case III. $b + c < a$. Here $B < -1$, and

$$K_{\alpha,\beta}(a, b, c) = \frac{2^{\alpha-\beta}\Gamma(\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\beta + 1)} a^{-2\alpha}(bc)^{\alpha-\beta-1} \frac{(1 - B)^{\alpha-1/2}}{(-1 - B)^{\beta+1/2}} \cdot {}_2F_1\left(\alpha + \beta, \beta + \tfrac{1}{2}; 2\beta + 1; \frac{2}{1 + B}\right). \tag{5.7}$$

For these results, cf. [17, p. 412] and [11].

Next, we will rewrite the formulas (3.1), (3.7) and (3.8) in kernel form using formula (5.1). It follows from (3.1) that

$$R_n^{(\alpha,\beta)}(x) = \frac{1}{2}\int_0^\infty y^{n+\beta} K_{\alpha,\beta}\,(\sqrt{(x-1)/2}, \sqrt{(x+1)/2}, \sqrt{y})\,dy, \qquad x > 1. \tag{5.8}$$

A Mehler type integral for Jacobi functions (also for complex n) which follows from (5.8) leads to an explicit expression for the Radon transform for Jacobi function expansions (to be published by the author). The analogous Mehler type integral for Jacobi polynomials was independently obtained by Gasper (yet unpublished). He applied the formulas [8, vol. I, § 2.4(3) and § 2.8(11)]. The kernel form of (3.7) was first obtained by Gasper [10]. It is

$$\begin{aligned} R_n^{(\alpha,\beta)}(\cos 2\theta_1)R_n^{(\alpha,\beta)}(\cos 2\theta_2) = \int_0^{\pi/2} & R_n^{(\alpha,\beta)}(\cos 2\theta_3)K_{\alpha,\beta}(\sin\theta_1\sin\theta_2, \\ & \cos\theta_1\cos\theta_2, \cos\theta_3)(\cos\theta_3)^{2\beta+1}\sin\theta_3\,d\theta_3, \\ & 0 < \theta_2 < \frac{\pi}{2}, \quad 0 < \theta_2 < \frac{\pi}{2}, \quad \alpha > \beta > -\tfrac{1}{2}. \end{aligned} \tag{5.9}$$

Here, the range of integration is restricted, because $a = \sin\theta_1 \sin\theta_2$, $b = \cos\theta_1\cos\theta_2$ and $c > 1$ would imply the condition of Case I.

Formula (3.8) can be rewritten as

$$\frac{J_\alpha(x)}{x^\alpha}\frac{J_\beta(y)}{y^\beta} = \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^\infty \frac{J_\beta(z)}{z^\beta}K_{\alpha,\beta}(x, y, z)z^{2\beta+1}\,dz. \tag{5.10}$$

It follows by the homogeneity of $K_{\alpha,\beta}$ that

$$J_\alpha(\lambda x)J_\beta(\lambda y)\lambda^{-\alpha} = \int_0^\infty \frac{x^\alpha y^\beta z^\beta K_{\alpha,\beta}(x, y, z)}{2^\alpha\Gamma(\alpha+1)}J_\beta(\lambda z)z\,dz. \tag{5.11}$$

By duality it follows from (5.9) that

$$\begin{aligned} &\sum_{n=0}^\infty h_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos 2\theta_1)R_n^{(\alpha,\beta)}(\cos 2\theta_2)R_n^{(\alpha,\beta)}(\cos 2\theta_3) \\ &\qquad = \frac{K_{\alpha,\beta}(\sin\theta_1\sin\theta_2, \cos\theta_1\cos\theta_2, \cos\theta_3)}{2^{\alpha+\beta+2}(\sin\theta_3)^{2\alpha}}, \end{aligned} \tag{5.12}$$

where

$$(h_n^{(\alpha,\beta)})^{-1} = \int_{-1}^{+1} (R_n^{(\alpha,\beta)}(x))^2(1-x)^\alpha(1+x)^\beta\,dx$$

and $\cos\theta_3 \neq |\cos(\theta_1 \pm \theta_2)|$. It follows from (5.11) that

$$\int_0^\infty J_\alpha(\lambda x)J_\beta(\lambda y)J_\beta(\lambda z)\lambda^{1-\alpha}\,d\lambda = \frac{x^\alpha y^\beta z^\beta K_{\alpha,\beta}(x, y, z)}{2^\alpha\Gamma(\alpha+1)} \tag{5.13}$$

for $z \neq |x \pm y|$.

In order to prove (5.12) and (5.13) by duality one has to use that the function $K_{\alpha,\beta}(a, b, t)$ is continuous differentiable on the intervals $(0, |a-b|)$, $(|a-b|, a+b)$

and $(a + b, \infty)$. In the Jacobi case, the equiconvergence theorem for Jacobi series (Szegö [16, Thm. 9.1.2]) and well-known convergence properties of Fourier-cosine series then can be applied. In the Bessel case, the tool is Hankel's inversion theorem [17, p. 456].

Combination of (5.12) and (5.13) gives

$$\begin{aligned}&\sum_{n=0}^{\infty} h_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos 2\theta_1) R_n^{(\alpha,\beta)}(\cos 2\theta_2) R_n^{(\alpha,\beta)}(\cos 2\theta_3)\\ &\quad = 2^{-\beta-2}\Gamma(\alpha+1)(\sin\theta_1 \sin\theta_2 \sin^2\theta_3)^{-\alpha}(\cos\theta_1\cos\theta_2\cos\theta_3)^{-\beta}\\ &\quad\quad \cdot \int_0^{\infty} J_\alpha(\lambda \sin\theta_1 \sin\theta_2) J_\beta(\lambda\cos\theta_1\cos\theta_2) J_\beta(\lambda\cos\theta_3)\lambda^{1-\alpha}\,d\lambda,\\ &\quad\quad\quad \alpha > \beta > -\tfrac{1}{2}, \quad \cos\theta_3 \neq |\cos(\theta_1 \pm \theta_2)|.\end{aligned} \tag{5.14}$$

For (5.13) and (5.14) see Watson [17, pp. 411, 413]. Gasper [10] obtained (5.12) by combining these two formulas of Watson. Formula (5.13) was applied by Copson [6, p. 352] to the Riemann–Green function for the hyperbolic analogue of (2.1).

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SOME ADDITIONAL REMARKS ON THE NONEXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR WAVE EQUATIONS*

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Abstract. Let P and A be symmetric linear operators defined on a dense domain $D \subset H$, a (real) Hilbert space. Let $(x, Ax) \geqq \lambda(x, x)$ for all $x \in D$ and some $\lambda > 0$ and $(x, Px) > 0$ for all $x \in D$, $x \neq 0$. Let D have a Hilbert space structure and let the embedding be continuous. Let $\mathscr{F}: D \to H$ be a nonlinear gradient operator and let $\mathscr{G}$ be a potential associated with $\mathscr{F}$. Suppose that $2(2\alpha + 1)\mathscr{G}(x) \leqq (x, \mathscr{F}(x))$ for all $x \in D$ and some $\alpha > 0$. If $u:[0, T) \to D$ is a solution to $Pu_{tt} = -Au + \mathscr{F}(u)$ with $u(0) = u_0$, $v_t(0) = v_0$, and if $\mathscr{G}(u_0) > \frac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)] - (\lambda\alpha/(2\alpha + 1))(u_0, Pu_0)$, then, for some $T < \infty$, $\lim_{t \to T^-} (u, Pu) = +\infty$. An analogous result holds for weak solutions to this equation and for the damped equation $Pu_{tt} + \hat{A}u_t + Au = \mathscr{F}(u)$, where $\hat{A}$ is a positive semidefinite linear operator.

Introduction. In this paper, we shall extend the results of [1] to weak solutions and examine the consequences of imposing an additional assumption on the operator A, namely, that $(x, Ax) \geqq \lambda(x, Px)$ for all $x \in D_A$ and some $\lambda > 0$.

The notation and definitions used in [1] *are assumed to be in force here.* Here we assume that the operator A does not depend upon time.

We shall also extend the result of [1, Theorem VI], for the damped equation $u_{tt} + au_t + Au = \mathscr{F}(u)$, with A a positive operator and a a constant, $a > 0$, to $Pu_{tt} + \hat{A}u_t + Au = \mathscr{F}(u)$, with P and $\hat{A}$ positive, symmetric, linear operators so that our instability and nonexistence results hold even when we "damp" the motion with an (unbounded) operator.

1. The case $(x, Ax) \geqq \lambda(x, x)$**.** In [1], the following theorem was proved.

THEOREM 1. *Consider the abstract Cauchy problem in the* (*real*) *Hilbert space* H:

$$Pu_{tt} = -Au + \mathscr{F}(u) \quad in \quad [0, T), \qquad u(0) = u_0, \qquad u_t(0) = v_0. \tag{1.1}$$

Suppose that P and A are symmetric linear operators with $P > 0$ and $A \geqq 0$ in the usual sense of quadratic forms (*P and A being defined on a dense subdomain*[1] $D \subset H$). *Assume that the nonlinearity $\mathscr{F}$* (*defined on D*) *has a symmetric, continuous*[2] *Fréchet derivative at each $x \in D$ and that there is a constant $\alpha > 0$ such that the potential*

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[1] D is presumed to be equipped with a scalar product $(\ ,\ )_D$ such that it in turn is a Hilbert space in this scalar product and such that the injection from D into H is a continuous mapping of Hilbert spaces.

[2] In the D norm.

$\mathscr{G}(x) \equiv \int_0^1 (\mathscr{F}(\rho x), x)\, d\rho$ *satisfies, for all* $x \in D$,

$$(x, \mathscr{F}(x)) \geqq 2(2\alpha + 1)\mathscr{G}(x). \tag{*}$$

Suppose that u_0 *satisfies*

$$d_0: \quad \mathscr{G}(u_0) > \tfrac{1}{2}(u_0, Au_0)$$

and let

$$r(u_0) = \sqrt{2}\{\mathscr{G}(u_0) - \tfrac{1}{2}(u_0, Au_0)\}^{1/2}.$$

Define $\cos\theta = (u_0, Pv_0)/(u_0, Pu_0)^{1/2}(u_0, Pv_0)^{1/2}$ *and let*

$$S_{u_0} = \{v_0 \in D \mid (v_0, Pv_0) < r^2(u_0)\},$$
$$B_{u_0} = \{v_0 \in D \mid (v_0, Pv_0) = r^2(u_0)\},$$
$$H_+(u_0) = \{v_0 \in D \mid \cos\theta > 0\},$$
$$E_{u_0} = \{v_0 \in D \mid (1 - \tfrac{1}{2}\alpha(2\alpha + 1)^{-1}\cos^2\theta)(v_0, Pv_0) < r^2(u_0)\},$$
$$C_{u_0} = \{v_0 \in D \mid \sin^2\theta(v_0, Pv_0) < r^2(u_0)\}.$$

Then if $u: [0, T) \to H$ *is a solution to* (1.1) *in the sense of* [1] *we have*:

(i) *If* $v_0 \in S_{u_0} \cup H_+(u_0) \cap B_{u_0}$, *there exists* T, $0 < T < +\infty$, *such that*

$$\lim_{t \to T^-} (u(t), Pu(t)) = +\infty.$$

(ii) *If* $v_0 \in (E_{u_0} - S_{u_0}) \cap H_+(u_0)$ *and the solution exists on* $[0, \infty)$, *then there exists* $\gamma > 0$ *such that*

$$\liminf_{t \to +\infty} e^{-\gamma t}(u(t), Pu(t)) > 0.$$

(iii) *If* $v_0 \in (C_{u_0} - E_{u_0}) \cap H_+(u_0)$ *and* u *exists on* $[0, \infty)$, *then*

$$\liminf_{t \to +\infty} t^{-2}(u(t), Pu(t)) > 0.$$

That is to say, we have the situation indicated in Fig. 1. If v_0 is in the interior of the "disk" S_{u_0} or on the open "semicircle" $\overline{P_0P_1}$, u will have finite escape time. If v_0 is in the region Γ_1 excluding the "arc" $\overline{P_0P_2P_1}$, we have at least exponential growth of (u, Pu). If v_0 is in the region Γ_2, we have at least quadratic growth of (u, Pu).

Suppose we assume that there is a constant $\lambda > 0$ such that $(x, Ax) \geqq \lambda(x, Px)$ for all $x \in D$ and $t \in [0, \infty)$, the other hypotheses on $\mathscr{F}$ and A being unchanged except that u_0 satisfies

$$\mathrm{d}_1: \quad \alpha\lambda(u_0, Pu_0)/(2\alpha + 1) + \mathscr{G}(u_0) > \tfrac{1}{2}(u_0, Au_0).$$

In this case we have the following theorem.

THEOREM 2. *Let* u_0 *satisfy* d_1. *Then Theorem* 1, *statements* (ii) *and* (iii) *hold with* $r(u_0)$ *replaced by*

$$R(u_0) = \sqrt{2}[\alpha\lambda(2\alpha + 1)^{-1}(u_0, Pu_0) + \mathscr{G}(u_0) - \tfrac{1}{2}(u_0, Au_0)]^{1/2}.$$

(These statements will be referred to as (ii)′ and (iii)′ and the corresponding subsets referred to as $\tilde{S}_{u_0}$, $\tilde{B}_{u_0}$, $\tilde{C}_{u_0}$, and $\tilde{E}_{u_0}$ respectively.) *Let* $H_+ \equiv H_+(u_0)$. *Instead of statement* (i) *we have*:

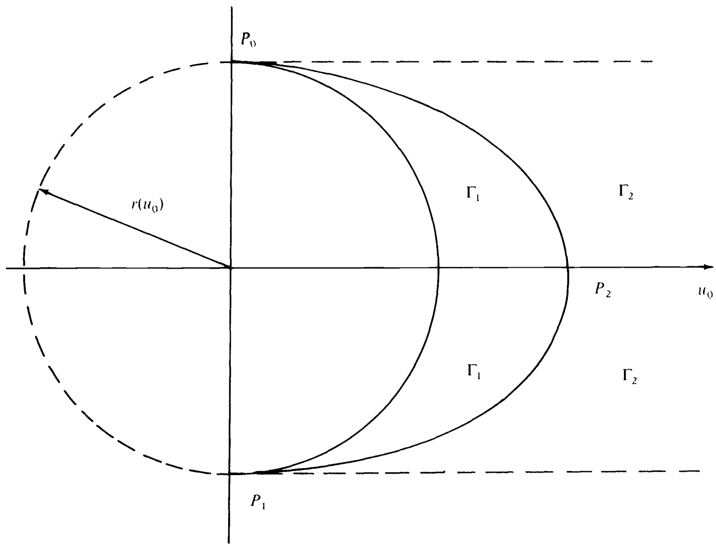

FIG. 1

(i)′ *If* $v_0 \in (\tilde{S}_{u_0} \cap H_+) \cup (\tilde{B}_{u_0} \cap H_+)$, *then there is a number* $T > 0$, $T < \infty$ *such that*

$$\lim_{t \to T^-} (u(t), Pu(t)) = +\infty.$$

If, in addition, the stronger inequality

$$\mathrm{d}_0: \quad \mathscr{G}(u_0) > \tfrac{1}{2}(u_0, Au_0)$$

holds, then statement (i) *holds as well. That is, if* $v_0 \in S_{u_0} \cap (D - H_+)$, *then* $\lim_{t \to T^-} (u, Pu) = +\infty$ *for some finite* $T > 0$.

Proof. The proof is an immediate consequence of the results of [1]. If we set $F(t) = (u, Pu) + Q^2$, then we find from (II-3), (II-5) and (II-6) of [1] with $\beta = 0$ that

$$FF'' - (\alpha + 1)(F')^2 \geqq 2F\{2\alpha(u, Au) + 2(2\alpha + 1)\{\mathscr{G}(u_0) - \tfrac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)]\}\}.$$

Since $F'(t) = 2(u, Pu_t)$ and $F'(0) = 2(u_0, Pv_0)$, $v_0 \in H_+$ if and only if $F'(0) > 0$ so that *near* $t = 0$, $F'(t) \geqq 0$ and $F(t)$ is increasing. Thus $(u, Au) \geqq \lambda(u, Pu) \geqq \lambda(u_0, Pu_0)$ so that

$$\begin{aligned} FF'' - (\alpha + 1)(F')^2 \geqq 2F\{2\alpha\lambda(u_0, Pu_0) + 2(2\alpha + 1)[\mathscr{G}(u_0) \\ - \tfrac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)]]\}. \end{aligned} \tag{1.2}$$

It then follows from (1.2) that the analogues of statements (ii) and (iii) follow by simply modifying the proofs of Theorems III and IV of [1] to the extent that $\mathscr{G}(u_0)$ is replaced by $\alpha\lambda(u_0, Pu_0)(2\alpha + 1)^{-1} + \mathscr{G}(u_0)$ in these proofs.

In order to show that (1.2) holds on the *entire* existence interval, one need only establish that F' does not change sign if $F'(0) > 0$. The arguments of [1] are easily modified to show that near $t = 0$, $(F^{-\alpha})' < \sqrt{\alpha\mu}(F^{-\alpha})$ in the case of (ii)′ and $F' > 2\nu F^{1/2}$ in the case of (iii)′ (μ and ν are chosen as in [1] with the appropriate replacement of $\mathscr{G}(u_0)$.) From these latter it follows that F' does not change sign.

If $v_0 \in \tilde{S}_{u_0} \cap H_+ \cup \tilde{B}_{u_0} \cap H_+$, that is, if the coefficient of $2F$ on the right of (1.2) is nonnegative, then $(F^{-\alpha})'' \leqq 0$ near $t = 0$. This inequality persists in an interval $[0, t_0)$, say, as long as $F'(t) > 0$ there. In this interval,

$$(F^{-\alpha})'(F^{-\alpha})'' \geqq 0,$$

so that

$$\left[\frac{dF^{-\alpha}(t)}{dt}\right]^2 \geqq \left[\frac{dF^{-\alpha}(0)}{dt}\right]^2 > 0.$$

Therefore, $(F^{-\alpha})'$ cannot change sign and hence, since it is initially negative, $F'(t) > 0$ *for all* t for which u is defined so that $(F^{-\alpha})'' \leqq 0$ wherever the solution exists. Thus statement (i)′ holds as we see from (II-4) of [1]. The last conclusion of Theorem 2 is simply part of statement (i) of Theorem 1.

Remark 1. The extension is important because, as is the case in many initial boundary value problems for bounded space domains, we often have $(x, Ax) \geqq \lambda(x, Px)$ for some $\lambda > 0$ and all $x \in D$. We are thus able to locate more precisely the candidates for u_0 and v_0 which give rise to unstable solutions to (1.1). (Compare Figs. 2 and 3.)

Remark 2. Figure 2 illustrates the geometric content of Theorem 2 if d_1 holds and $\mathscr{G}(u_0) \leqq \frac{1}{2}(u_0, Au_0)$, while Fig. 3 illustrates the geometric content of Theorem 2 if $\mathscr{G}(u_0) > \frac{1}{2}(u_0, Au_0)$. The labeling in both figures has the same significance as that used in Fig. 1. In Fig. 2 $\mathscr{G}(u_0) \leqq \frac{1}{2}(u_0, Au_0)$ and $u_0 \neq 0$. We have unbounded growth of (u, Pu) in finite time if v_0 is in the right half "disk" including the (open) "arc" $\overline{P_0P_1}$ (see Remark 2). In Fig. 3 $\mathscr{G}(u_0) > \frac{1}{2}(u_0, Au_0)$ (see Remark 2).

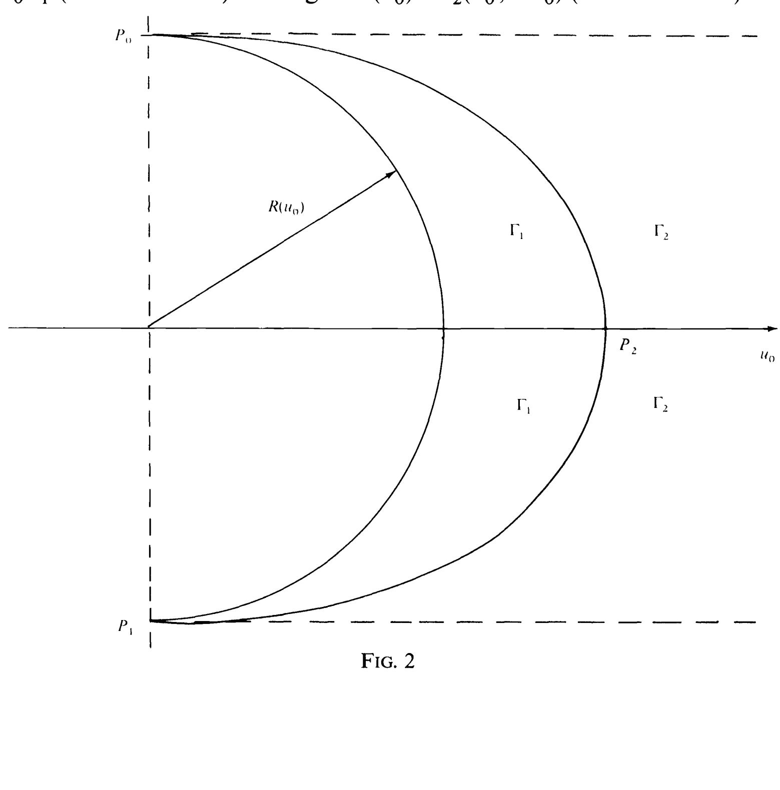

FIG. 2

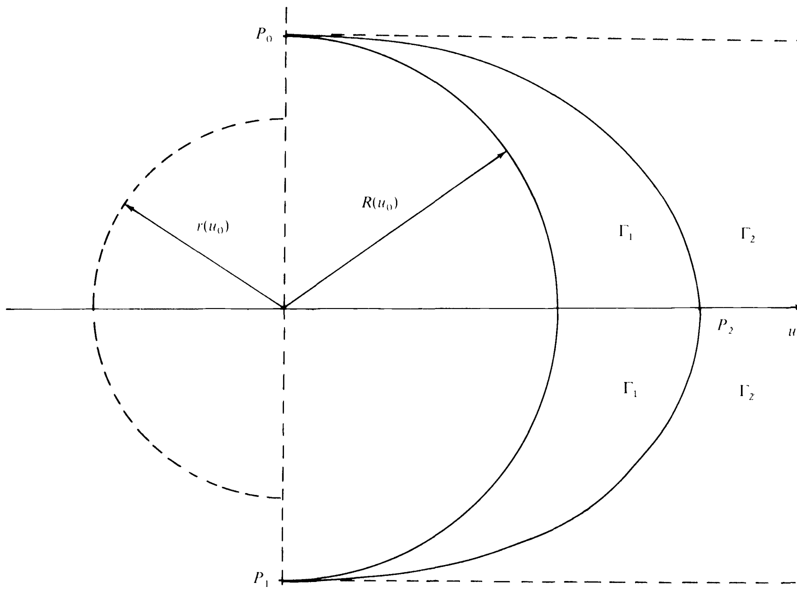

FIG. 3

Remark 3. Note that d_1 cannot hold unless $\mathscr{G}(u_0) > 0$. Otherwise, from d_1 and the lower bound on (x, Ax), we have $2\alpha(u_0, Au_0) > (2\alpha + 1)(u_0, Au_0)$, which is a contradiction.

Remark 4. Applying "energy" arguments, we find that

$$\begin{aligned} E_N(t) &= \tfrac{1}{2}[(u, Au) + (u_t, Pu_t)] - \mathscr{G}(u(t)) \\ &= \tfrac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)] - \mathscr{G}(u_0) = E_N(0). \end{aligned}$$

(That is, we take the scalar product of both sides of (1.1) with u_t, integrate, use (**) of [1] $(\int_0^t (\mathscr{F}(u), u_\eta)\, d\eta = \mathscr{G}(u) - \mathscr{G}(u_0))$ and rearrange to arrive at the above identity.) We saw in [1] that if $E_N(0) \leqq 0$, then we have nonexistence of global solutions to (1.1). Theorem 2 says that even if $E_N(0) > 0$ the solution to (1.1) will not be global if $E_N(0)$ is not too positive, that is, if $E_N(0) \leqq \alpha\lambda(u_0, Pu_0)(2\alpha + 1)^{-1}$.

Note that, in view of Remark 3, $E_N(t) \equiv 0$ does not imply that $u \equiv 0$ if d_1 holds. ($E_N(t)$ is called the "energy" of the nonlinear problem at time t.)

Remark 5. If, as in [1], $A = A(t)$, $(x, \dot{A}(t)x) \leqq 0$ and $(x, A(t)x) \geqq \lambda(x, Px)$ for some constant $\lambda > 0$, all $t \in [0, \infty)$ and all $x \in D$, Theorem 2 remains valid. ($\dot{A}(t)$ denotes the strong limit of $[A(t + h) - A(t)] \cdot h^{-1}$ as $h \to 0$.)

Remark 6. It is interesting to note that the solutions to (1.1) were not required to be uniquely determined by u_0 and v_0 although this is the case if $\mathscr{F}_x$ is continuous.

Remark 7. If d_0 or d_1 fail, then (1.1) can exhibit bounded solutions. To see this, let f be a nontrivial solution of

$$\begin{gathered} f'' + f^2 = 0 \quad \text{on} \quad [0, \pi], \\ f(0) = f(\pi) = 0. \end{gathered}$$

Here $\mathscr{F}(g) = g^2, \alpha = 1/4, \lambda = 1$ and $\mathscr{G}(g) = \frac{1}{3}\int_0^\pi g^3\,dx\,(g \in \mathscr{C}^2[0,\pi], g(0) = g(\pi) = 0)$. Then $u(x,t) = f(x)$ satisfies

$$\begin{aligned} &u_{tt} = u_{xx} + u^2, && (x,t) \in (0,\pi) \times [0,\infty), \\ &u(0,t) = u(\pi,t) = 0, && t \geqq 0, \\ &u(x,0) = f(x), u_t(x,0) = 0, && 0 \leqq x \leqq \pi, \end{aligned}$$

and is bounded and global. (d_0 fails because $\mathscr{G}(f) = \frac{1}{3}\int_0^\pi f^3\,dx = -\frac{1}{3}\int_0^\pi ff''\,dx = \frac{1}{3}\int_0^\pi (f')^2\,dx < \frac{1}{2}\int_0^\pi (f')^2\,dx$, while d_1 fails because if it held it would imply that

$$\tfrac{1}{6}\int_0^\pi f^2\,dx + \tfrac{1}{3}\int_0^\pi f^3\,dx > \tfrac{1}{2}\int_0^\pi (f')^2\,dx \quad \text{or} \quad \tfrac{1}{6}\int_0^\pi f^2\,dx > \tfrac{1}{6}\int_0^\pi (f')^2\,dx,$$

which contradicts the well-known inequality of Poincaré, namely, if $g \in C^1$, $g(0) = g(\pi) = 0$, then $1 \leqq \int_0^\pi (g')^2\,dx / \int_0^\pi g^2\,dx$.)

2. Weak solutions. We observe that we can also prove results analogous to those of [1] and the present note for weak solutions to (1.1), provided we properly formulate the definitions of a weak solution to (1.1). To do this, *at least formally*, let u exist on $[0,T)$ and let $P^* = P^{1/2}$, $A^* = A^{1/2}$. Let D_* ($\supseteq D$) denote a common domain of definition for P^*, $\mathscr{F}$ and A^*[3]. (It may happen that $D_* = D$.) *We say that u is a weak solution to* (1.1) *if $u_0, v_0 \in D_*$ and, for all "smooth enough" $\phi : [0,T) \to D_*$ we have, for all $t \in [0,T)$, that*

$$\begin{aligned} (P^*\phi, P^*u_t) + \int_0^t (A^*\phi, A^*u)\,d\eta &= (P^*\phi(0), P^*v_0) + \int_0^t (P^*\phi_\eta, P^*u_\eta)\,d\eta \\ &\quad + \int_0^t (\phi, \mathscr{F}(u))\,d\eta, \end{aligned} \tag{2.1}$$

$$u(0) = u_0, \qquad u_t(0) = v_0,$$

and that u is an admissible ϕ. (This clearly requires that both u and u_t take values in D_*.) We now prove Theorem 1, statement (i) assuming only that (a) $\mathscr{G}(u_0) > \frac{1}{2}(u_0, Au_0)$, (b) all such weak solutions u satisfy the following "energy" inequality for $\alpha > 0$:

$$\begin{aligned} E_N(t) &\equiv \tfrac{1}{2}[(A^*u, A^*u) + (P^*u_t, P^*u_t)] - \mathscr{G}(u(t)) \leqq E_N(0), \\ E_N(0) &\equiv \tfrac{1}{2}[(A^*u_0, A^*u_0) + (P^*v_0, P^*v_0)] - \mathscr{G}(u_0), \end{aligned} \tag{2.2}$$

and (c) $\mathscr{F}$ and $\mathscr{G}$ satisfy F-I, F-II of [1] and (*). Note that (2.2) is weaker than the statement $E_N(t) = E_N(0)$ of Remark 4.

We now prove the following theorem.

THEOREM 3. *If* (2.1) *and* (2.2) *hold and if* $(P^*v_0, P^*v_0) < 2[\mathscr{G}(u_0) - \frac{1}{2}(A^*u_0, A^*u_0)]$, *then*

$$\lim_{t \to T^-} (P^*u(t), P^*u(t)) = +\infty \quad \textit{for some} \quad T, \qquad 0 < T < \infty.$$

[3] Footnote 1 about the embedding of D in H is assumed to hold also for D^* with scalar product $(\ ,\ )_{D^*}$.

Proof. Let

$$F(t) = (P^*u, P^*u) + \beta(t + \tau)^2,$$

where β and τ are positive constants to be determined later. Then

$$F'(t) = 2(P^*u, P^*u_t) + 2\beta(t + \tau).$$

Putting $\phi = u$ in (2.1) and solving the resulting equation for (P^*u, P^*u_t) we find that

$$F''(t) = 4(\alpha + 1)[(P^*u_t, P^*u_t) + \beta] + 2\{(u, \mathscr{F}(u)) - (A^*u, A^*u) - 2(2\alpha + 1)\mathscr{G}(u) + 2(2\alpha + 1)[\mathscr{G}(u) - \tfrac{1}{2}(P^*u_t, P^*u_t) - \beta]\}.$$

Using (2.2) and (*) of [1], we see that

$$\begin{aligned} F''(t) &\geqq 4(\alpha + 1)[(P^*u_t, P^*u_t) + \beta] + 4\alpha(A^*u, A^*u) + 2(2\alpha + 1)\{\mathscr{G}(u_0) \\ &\qquad - \tfrac{1}{2}[(A^*u_0, A^*u_0) + (P^*v_0, P^*v_0)] - \tfrac{1}{2}\beta\} \\ &\geqq 4(\alpha + 1)[(P^*u_t, P^*u_t) + \beta], \end{aligned}$$

which follows after noting that we have assumed $E_N(0) < 0$ and subsequently put $\beta = -2E_N(0)$. Thus here too, $(F^{-\alpha})'' \leqq 0$. If τ is then chosen so large that $F'(0) = 2(P^*u_0, P^*v_0) + 2\beta\tau > 0$, we see from (II-4) in [1] that $\lim_{t \to T^-} (P^*u, P^*u) = +\infty$, where $0 < T \leqq F(0)/\alpha F'(0)$.

Analogues of (ii) and (iii) in Theorem 1 and an analogue of Theorem 2 can be similarly proved.

Following [3], it is possible to relax the definition of a weak solution.

3. The damped equation. Here we let P, $\mathscr{F}$, $\mathscr{G}$, A, α be as in [1], A being time independent and the conditions on $\mathscr{F}$ and $\mathscr{G}$ being in force. Let $\hat{A}: D \to H$ be another symmetric linear operator such that $(x, \hat{A}x) \geqq 0$ for all $x \in D$. We prove the following theorem.

THEOREM 4. *Let* $u:[0, T) \to H$ *be a solution in the sense of* [1] *to*

$$P\frac{d^2u}{dt^2} + \hat{A}\frac{du}{dt} + Au = \mathscr{F}(u) \quad in \quad [0, T), \tag{3.1}$$

$$u(0) = u_0,$$

$$u_t(0) = v_0, \qquad u_0, v_0 \in D.$$

If

$$\mathrm{d}_0': \quad \mathscr{G}(u_0) > \tfrac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)],$$

then there exists T *with*

$$0 < T \leqq \alpha^{-2}\beta_0^{-1}\{[(\tfrac{1}{2}(u_0, \hat{A}u_0) - \alpha(v_0, Pu_0))^2 + (u_0, Pu_0)\alpha^2\beta_0]^{1/2} + (\tfrac{1}{2}(u_0, \hat{A}u_0) - \alpha(v_0, Pu_0))\},$$

where

$$\beta_0 \equiv 2\{\mathscr{G}(u_0) - \tfrac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)]\}$$

such that

$$\lim_{t\to T^-}\left[(u(t), Pu(t)) + \int_0^t (u(\eta), \hat{A}u(\eta))\, d\eta\right] = +\infty.$$

(We are assuming local existence here.)

Proof. Suppose the theorem fails so that u exists on $[0, \infty)$. For any T_0, β, τ, arbitrary but positive, let

$$F(t) = (u, Pu) + \int_0^t (u, \hat{A}u)\, d\eta + (T_0 - t)(u_0, \hat{A}u_0) + \beta(t + \tau)^2 \tag{3.2}$$

for $t \in [0, T_0]$. Then

$$\begin{aligned} F'(t) &= 2(u_t, Pu) + (u, \hat{A}u) - (u_0, \hat{A}u_0) + 2\beta(t + \tau) \\ &= 2(u_t, Pu) + 2\int_0^t (u_\eta, \hat{A}u)\, d\eta + 2\beta(t + \tau) \end{aligned} \tag{3.3}$$

and

$$F''(t) = 2(u_{tt}, Pu) + 2(u_t, Pu_t) + 2(u_t, \hat{A}u) + 2\beta \tag{3.4}$$

so that

$$\begin{aligned} FF'' - (\alpha + 1)(F')^2 \geqq{}& 4(\alpha + 1)S^2 + 2F\{(u, \mathscr{F}(u)) - (u, Au) \\ & - 2(\alpha + 1)\int_0^t (u_\eta, \hat{A}u_\eta)\, d\eta - (2\alpha + 1)[(u_t, Pu_t) + \beta]\}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} S^2 ={}& \left[(u, Pu) + \int_0^t (u, \hat{A}u)\, d\eta + \beta(t + \tau)^2\right]\left[(u_t, Pu_t) + \int_0^t (u_\eta, \hat{A}u_\eta)\, d\eta + \beta\right] \\ & - \left[(u_t, Pu) + \int_0^t (u_\eta, \hat{A}u)\, d\eta + \beta(t + \tau)\right]^2. \end{aligned}$$

Letting $H(t)$ denote the expression in braces on the right of (3.5), we see that

$$\begin{aligned} H'(t) &= \frac{d}{dt}(u, \mathscr{F}(u)) - (4\alpha + 2)(u_t, Pu_{tt}) - 2(\alpha + 1)(u_t, \hat{A}u_t) - 2(u_t, Au) \\ &= \frac{d}{dt}(u, \mathscr{F}(u)) - (4\alpha + 2)(u_t, \mathscr{F}(u)) + 2\alpha(u_t, \hat{A}u_t) + 4\alpha(u_t, Au). \end{aligned}$$

Therefore using (*) in Theorem 1 and the formula $d\mathscr{G}(u(t))/dt = (u_t, \mathscr{F}(u))$ we find that

$$\begin{aligned} H(t) &\geqq H(0) - (u_0, \mathscr{F}(u_0)) + (4\alpha + 2)\mathscr{G}(u_0) + 2\alpha(u, Au) - 2\alpha(u_0, Au_0) \\ &\geqq (4\alpha + 2)\{\mathscr{G}(u_0) - \tfrac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)] - \beta/2\}. \end{aligned} \tag{3.6}$$

Thus with $\beta \equiv \beta_0 \equiv 2\{\mathscr{G}(u_0) - \frac{1}{2}[(u_0, Au_0) + (v_0, Pv_0)]\}$, we see that $[F^{-\alpha}(t)]'' \leqq 0$ in $[0, T_0]$. Therefore if we choose T_0 and τ such that $F'(0) > 0$ ($(F^{-\alpha})'(0) < 0$) and $F(0)/\alpha F'(0) \leqq T_0$, then $F^{-\alpha}$ will have a zero in $[0, T_0)$ at some point

$T \leqq F(0)/\alpha F'(0)$ and the second statement of the theorem will follow. Now $F'(0) = 2(u_0, Pv_0) + 2\beta_0\tau > 0$ if τ is sufficiently large. Moreover, $F(0) \leqq \alpha F'(0)T_0$ if and only if

$$(3.7) \qquad (u_0, Pu_0) + \beta_0\tau^2 \leqq 2T_0\{\alpha[(u_0, Pv_0) + \beta_0\tau] - \tfrac{1}{2}(u_0, \hat{A}u_0)\}.$$

Now choose τ so large that the quantity in braces on the right of (3.7) is positive, and let

$$T_0 \equiv \tfrac{1}{2}[(u_0, Pu_0) + \beta_0\tau^2]/\{\alpha[(u_0, Pv_0) + \beta_0\tau] - \tfrac{1}{2}(u_0, \hat{A}u_0)\}.$$

Then minimizing over τ, we obtain the estimate on T given in the statement of the theorem.

Remark 8. By specializing Theorem 4 to the case $P = I$ and $\hat{A} = aI, a > 0$, we conclude that if $\|v_0\| < r(u_0) = \sqrt{2}\{\mathcal{G}(u_0) - \tfrac{1}{2}(u_0, Au_0)\}$ then $\limsup_{t\to T^-}\|u(t)\| = +\infty$ for some $T, 0 < T < \infty$. In this case, however, Theorem VI of [1] tells us that if $\|v_0\| < r(u_0)$ and $\|v_0\| \cos\theta > a\|u_0\|/2\alpha$, where $\cos\theta \equiv (u_0, v_0)/\|u_0\|\,\|v_0\|$, then $\lim_{t\to T^-}\|u(t)\| = +\infty$ for some $T, 0 < T < \infty$. Thus, *in this case, with the additional restriction on* v_0, the result of [1] gives us somewhat more information about the nature of the approach to infinity of $\|u\|$. Theorem 4 has wider application of course.

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SOME PROPERTIES OF GENERALIZED BIAXIALLY SYMMETRIC HELMHOLTZ POTENTIALS*

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Abstract. For an equation which includes the equation of generalized biaxially symmetric potential theory an investigation is made of the behavior of solutions near the intersection of the two singular hyperplanes. The main results consist of a continuation theorem and uniqueness theorems analogous to those of Huber [3] in generalized axially symmetric potential theory.

1. Introduction. In this note we investigate solutions of the generalized biaxially symmetric Helmholtz equation

$$L_{\alpha,\beta}[u] + \lambda u = 0, \tag{1.1}$$

where

$$L_{\alpha,\beta}[u] \equiv \sum_{i=1}^{n} u_{x_i x_i} + \frac{\alpha}{x_{n-1}} u_{x_{n-1}} + \frac{\beta}{x_n} u_{x_n}$$

with α, β, λ real constants and $n \geqq 2$. Of course, for $x_{n-1} \neq 0$, $x_n \neq 0$ solutions of this elliptic equation are analytic. Here we examine the behavior of solutions at the intersection of the singular hyperplanes, i.e., near $x_{n-1} = 0$, $x_n = 0$. It is hoped that these results will give insight into the behavior of solutions of singular and degenerate elliptic equations at the intersection of more general singular and degenerate hypersurfaces. As a sample result we mention (see Theorem 1) that for $\alpha \geqq 1$ and $\beta \geqq 1$, if a solution does not grow too rapidly near the intersection, then it can be continued analytically throughout an entire neighborhood of the intersection, i.e., the singularity is removable.

For $\alpha = \lambda = 0$, (1.1) reduces to Weinstein's [1], [2] generalized axially symmetric potential theory (abbreviated GASPT). Our results are analogous to those of Huber [3] for GASPT. For further references for GASPT as well as the present case see the book of Gilbert [4]. See also Kiprijanov [8] and references therein for more general equations involving one singular variable, including equations of higher order.

2. Preliminaries. The usual notations for vectors in Euclidean n-space will be used. With $x = (x_1, x_2, \cdots, x_n)$ we denote by Q the open quarter space $\{x : x_{n-1} > 0, x_n > 0\}$, $B(R)$ the ball of radius R with center at the origin, $B^+(R) \equiv B(R) \cap Q$ is the open quarter-ball and $Q(R)$ the quarter sphere $\{x : |x| = R, x_{n-1} > 0, x_n > 0\}$. The boundary of a set S will be denoted by ∂S.

Our results rest heavily on the following Poisson integral formula [5].

THEOREM. *Let $\alpha \geqq 1, \beta \geqq 1$. Suppose f is continuous on $Q(R)$ and has the following behavior near the singular hyperplanes*:

$$\lim_{\xi \to \xi_0} \rho(\xi_{n-1}; \alpha)\rho(\xi_n; \beta) f(\xi) = 0,$$

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where $\xi \in Q(R)$, $\xi_0 \in \overline{Q(R)} \cap \partial Q$ *and*

$$\rho(t;\gamma) = \begin{cases} t^{\gamma-1}, & \gamma > 1, \\ (\log t)^{-1}, & \gamma = 1, \end{cases}$$

for positive real t. *Then the function* u *defined on* $B(R)$ *by*

$$u(x) = \int_{Q(R)} \xi_{n-1}^{\alpha} \xi_n^{\beta} K(x, \xi) f(\xi)\, dS,$$

where

$$K(x, \xi) = \frac{2(R^2 - |x|^2)\Gamma(p)}{R\pi^{n/2}\Gamma(\alpha/2)\Gamma(\beta/2)} \int_0^{\pi} \int_0^{\pi} \sigma^{-2p} \sin^{\alpha-1}\theta \sin^{\beta-1}\phi \, d\theta \, d\phi,$$

$$\sigma = [|x - \xi|^2 + 2\xi_{n-1}x_{n-1}(1 - \cos\theta) + 2\xi_n x_n(1 - \cos\phi)]^{1/2}$$

and $2p = n + \alpha + \beta$, *has the following properties*:

(a) u *is even in* x_{n-1} *and even in* x_n;

(b) u *is analytic in* $B(R)$ *and satisfies* $L_{\alpha,\beta}[u] = 0$ *in* $B(R)$ *except on* $x_{n-1} = 0$ *and* $x_n = 0$;

(c) u *assumes the boundary value* f *on* $Q(R)$:

$$\lim_{x \to \xi_0} u(x) = f(\xi_0); \qquad x \in B^+(R), \quad \xi_0 \in Q(R);$$

(d) u *inherits the behavior of* f *near the singular hyperplanes*:

$$\lim_{x \to x_0} \rho(x_{n-1}; \alpha)\rho(x_n; \beta)u(x) = 0; \qquad x \in B^+(R), \quad x_0 \in \overline{B(R)} \cap \partial Q,$$

the convergence being uniform in x^0.

In addition we will use the following variant of Weinstein's correspondence principle [1] of GASPT:

A function u *is a solution of* $L_{2-\alpha,2-\beta}[u] + \lambda u = 0$ *in a region* G *of the quarter-space* Q *if and only if* $v = x_{n-1}^{1-\alpha} x_n^{1-\beta} u$ *is a solution of* $L_{\alpha,\beta}[v] + \lambda v = 0$ *in* G.

3. Removable singularities. Our first result concerns singularities of solutions (1.1) near the intersection of the singular hyperplanes. Because of the translation invariance of (1.1) with respect to $x_1, x_2, \cdots, x_{n-2}$ there is no loss of generality in considering the intersection at the origin.

THEOREM 1. *Suppose* $\alpha \geqq 1$, $\beta \geqq 1$. *If* u *is a solution of* (1.1) *in* $B^+(R)$ *such that for* x_0 *in* $B(R) \cap \partial Q$

$$\lim_{x \to x_0} \rho(x_{n-1}; \alpha)\rho(x_n; \beta)u(x) = 0, \qquad x \in B^+(R), \tag{3.1}$$

then u *can be continued analytically into all of* $B(R)$ *as an even function of* x_{n-1} *and* x_n *and satisfying* (1.1) *in* $B(R)$, *except on* $x_{n-1} = 0$ *and* $x_n = 0$.

Remarks. (i) On the singular hyperplanes the operator (1.1) is not defined except for $\alpha = 0$, $\beta = 0$ and the continuation of u satisfies limiting forms of (1.1) there; in particular: $u_{x_{n-1}} = 0$ on $x_{n-1} = 0$ and $u_{x_n} = 0$ on $x_n = 0$.

(ii) The function $[\sum_{i=1}^{n} x_i^2]^{\gamma}$, $\gamma = \frac{1}{2}(2 - n - \alpha - \beta)$, which is a solution of (1.1) in $B^+(R)$ for $\lambda = 0$, shows that the condition (3.1) is not superfluous.

Proof. We first make a reduction to the case $\lambda = 0$ by a method of descent in the manner of Diaz and Ludford [6] as follows. The function u satisfies the hypotheses of the theorem if and only if the function

$$U(x_1, \cdots, x_n, x_{n+1}) \equiv \begin{cases} e^{-\sqrt{\lambda}x_{n+1}}u(x_1, \cdots, x_n), & \lambda > 0, \\ \cos\sqrt{-\lambda}x_{n+1}u(x_1, \cdots, x_n), & \lambda < 0, \end{cases}$$

satisfies the hypotheses of the theorem in a space of one higher dimension for $\lambda = 0$. In particular, U satisfies the equation

$$\sum_{i=1}^{n+1} U_{x_i x_i} + \frac{\alpha}{x_{n-1}} U_{x_{n-1}} + \frac{\beta}{x_n} U_{x_n} = 0 \tag{3.2}$$

if and only if u satisfies (1.1). Consequently, if the theorem is proved for $\lambda = 0$, U has an analytic extension $\hat{U}$ to an $(n + 1)$-dimensional ball of radius R about the origin of the $(n + 1)$-dimensional space such that $\hat{U}$ is even in x_{n-1} and x_n and such that $\hat{U}$ satisfies (3.2) there, except on $x_{n-1} = 0$ or $x_n = 0$; a fortiori, $\hat{U}$ is analytic in $x_1, \cdots, x_n$ on the intersection of the $(n + 1)$-dimensional ball and the hyperplane $x_{n+1} = 0$, i.e., on $B(R)$ in the n-dimensional space. Now put $\hat{u}(x_1, \cdots, x_n) = \hat{U}(x_1, \cdots, x_n, 0)$ to obtain the extension $\hat{u}$ with the properties listed in the theorem. Hence, it remains only to give the proof for the case $\lambda = 0$.

Let R_1 be any positive number with $R_1 < R$. Thanks to (3.1), one may use the values of u on $\partial B(R_1) \cap Q$ as boundary values in the Poisson integral formula (see § 2) to obtain a function v which is analytic in $B(R_1)$, is a solution of (1.1) in $B(R_1)$ for $x_{n-1} \neq 0$, $x_n \neq 0$, equals u on $\partial B(R_1) \cap Q$, is even in x_{n-1} and x_n and satisfies the condition (3.1) for all x_0 in $B(R_1) \cap \partial Q$. It remains then only to show that $v = u$ in $B^+(R_1)$. This is easily accomplished by the maximum principle as follows.

Since u and v satisfy (3.1) there exists for each positive ε a positive δ such that for $x_{n-1} \leqq \delta$ or $x_n \leqq \delta$ one has

$$|u(x) - v(x)| < \varepsilon h(x_{n-1}; \alpha)h(x_n; \beta), \tag{3.3}$$

where the nonnegative function h is defined via

$$h(t; \gamma) = \begin{cases} t^{1-\gamma}, & \gamma > 1, \\ \log(R_1/t), & \gamma = 1. \end{cases}$$

Then the function ψ defined by

$$\psi(x) = u(x) - v(x) - \varepsilon h(x_{n-1}; \alpha)h(x_n; \beta)$$

is a solution of (1.1) for $\lambda = 0$ in $B_\delta = B^+(R_1) \cap \{x : x_{n-1} > \delta,\ x_n > \delta\}$ which may be chosen to contain any given x^* in $B^+(R_1)$. On ∂B_δ one has $\psi \leqq 0$ because of (3.3) and the fact that $u = v$ on $\partial B(R_1) \cap Q$. Hence, by the maximum principle for (nonsingular) elliptic equations $\psi(x^*) \leqq 0$, i.e.,

$$u(x^*) - v(x^*) \leqq \varepsilon h(x^*_{n-1}; \alpha)h(x^*_n; \beta)$$

so that by letting ε tend to zero for fixed x^* one obtains $u(x^*) \leqq v(x^*)$ for arbitrary x^* in $B^+(R_1)$. Similarly $v \leqq u$ in $B^+(R_1)$, completing the proof.

COROLLARY 1. *Suppose* $\alpha < 1$, $\beta < 1$. *If* u *is a solution of* (1.1) *in* $B^+(R)$ *which* *takes on the boundary value zero on* $B(R) \cap \partial Q$:

$$\lim_{x \to x_0} u(x) = 0, \quad x \in B^+(R), \quad x_0 \in B(R) \cap \partial Q,$$

then there exists a function v *such that* $u = x_{n-1}^{1-\alpha} x_n^{1-\beta} v$ *in* $B^+(R)$ *where* v *is analytic in* $B(R)$, *is even in* x_{n-1} *and* x_n *and satisfies* (1.1) *with* (α, β) *replaced by* $(2 - \alpha, 2 - \beta)$ *in* $B(R)$ *for* $x_{n-1} \neq 0$, $x_n \neq 0$.

Proof. Owing to the correspondence principle (see § 2) the function $x_{n-1}^{\alpha-1} x_{n-1}^{\beta-1} u$ satisfies all the hypotheses of the function u in Theorem 1 except (α, β) is replaced by $(2 - \alpha, 2 - \beta)$, where $2 - \alpha > 1$, $2 - \beta > 1$. Hence $x_{n-1}^{\alpha-1} x_n^{\beta-1} u$ can be continued analytically into all of $B(R)$ to a function v which satisfies all of the conditions stated and $u = x_{n-1}^{1-\alpha} x_n^{1-\beta} v$ in $B^+(R)$, completing the proof.

Using the correspondence principle one can also easily establish the following corollary.

COROLLARY 2. *Suppose* $\alpha < 1, \beta \geqq 1$. *If* u *is a solution of* (1.1) *in* $B^+(R)$ *such that for each* x_0 *in* $B(R) \cap \partial Q$

$$\lim_{x \to x_0} \rho(x_n; \beta) u(x) = 0, \qquad x \in B^+(R), \tag{3.4}$$

then there exists a function w *such that* $u = x_{n-1}^{1-\alpha} w$ *in* $B^+(R)$, *where* w *is analytic in* $B(R)$, *is even in* x_{n-1} *and* x_n *and satisfies* (1.1) *with* α *replaced by* $2 - \alpha$ *in* $B(R)$ *for* $x_{n-1} \neq 0$, $x_n \neq 0$.

Remarks. (i) Note that the hypothesis (3.4) requires that u vanish on $B(R) \cap \partial Q$ if $x_{n-1} = 0$.

(ii) It follows immediately from Corollary 2 that the u considered can be continued analytically into the hemi-ball $B(R) \cap \{x : x_{n-1} > 0\}$ as an even function of x_n.

4. Uniqueness theorems. Theorem 1 allows one also to establish the following uniqueness theorems which indicate for (1.1) to what extent the values of a solution on the singular hyperplanes determine a solution. The results depend critically on the values of the parameters α, β.

THEOREM 2. *Suppose* $\alpha \geqq 1$, $\beta \geqq 1$. *Let* G *be a region contained in the quarter-space* Q *such that there exists a ball* $B(R)$ *so that* $B(R) \cap Q$ *is contained in* G. *Let* u *be a solution of* (1.1) *in* G *such that for any* x_0 *in* $B(R) \cap \partial Q$

$$\lim_{x \to x_0} \rho(x_{n-1}; \alpha) \rho(x_n; \beta) u(x) = 0, \qquad x \in B^+(R).$$

Then, if u *takes on the boundary value zero on an open subset* O *of* $B(R) \cap \partial Q$:

$$\lim_{x \to x^*} u(x) = 0, \qquad x \in B^+(R), \quad x^* \in O, \tag{4.1}$$

it follows that $u \equiv 0$ *in* G.

Remark. Observe that Theorem 2 is sharp in the following sense: If the rate of growth allowed by (4.1) is increased, then the conclusion is false. For, if such were the case, the functions $x_{n-1}^{1-\alpha} x_n^{1-\beta}$ (for $\alpha > 1$, $\beta > 1$), $x_n^{1-\beta} \log x_{n-1}$ (for $\alpha = 1$, $\beta > 1$), $x_{n-1}^{1-\alpha} \log x_n$ (for $\alpha > 1$, $\beta = 1$) and $\log x_n \log x_{n-1}$ (for $\alpha = \beta = 1$) are counterexamples.

Proof. Again because of the method of descent it is sufficient to consider the case $\lambda = 0$.

By means of Theorem 1, u can be continued analytically into all of $B(R)$. The result now follows directly from the following lemma, by realizing that solutions of elliptic equations with analytic coefficients are analytic.

LEMMA. *Let P be a linear differential operator*

$$P(x;D) = \sum_{|\gamma| \leqq m} a_\gamma(x) D^\gamma$$

(*where the usual multi-index notation has been used*) *with coefficients a_γ bounded in some region Ω contained in E_n. Suppose for scalar t, $u = u(x, t)$ is a solution of*

$$P(x;D)u = u_{tt} + (k/t)u_t \tag{4.2}$$

on $\Omega \times (0, T)$ and further suppose u is analytic in (x, t) on $\Omega \times [0, T)$. If $u(x, 0)$ vanishes on some open subset of Ω, then for $k \neq 0, -1, -2, \cdots$, we have $u \equiv 0$ in $\Omega \times [0, T)$.

Proof. The proof is an elementary power series argument as given in the case of GASPT by Hyman [7]. We can write

$$u(x, t) = \sum_{l=1}^{\infty} U_l(x) t^l$$

for x in a neighborhood of x_0 with small nonnegative t and so (4.2) implies $U_1(x) \equiv 0$, $U_2(x) \equiv 0$ and $(l + 2)(k + l + 1)U_{l+2}(x) = P(x;D)U_l(x)$, $l = 1, 2, \cdots$, completing the proof.

A proof analogous to that of Theorem 2 establishes the following theorem.

THEOREM 3. *Suppose $\alpha < 1, \beta \geqq 1$. Let G be a region in the quarter-space Q such that there exists a ball $B(R)$ so that $B(R) \cap Q$ is contained in G. Let u be a solution of* (1.1) *such that for any x_0 in $B(R) \cap \partial Q$*

$$\lim_{x \to x_0} \rho(x_n; \beta) u(x) = 0, \qquad x \in B^+(R).$$

Then, if u takes on the boundary value zero on an open subset O of $B^+(R) \cap \{x : x_n = 0\}$:

$$\lim_{x \to x^*} u(x) = 0, \qquad x \in B^+(R), \quad x^* \in O,$$

it follows that $u \equiv 0$ in G.

Remark. Again the result is sharp as shown by the examples $x_{n-1}^{1-\alpha} x_n^{1-\beta}$ (for $\beta > 1$) and $x_{n-1}^{1-\alpha} \log x_n$ (for $\beta = 1$).

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AN EXISTENCE-UNIQUENESS THEOREM FOR TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract. A sufficient condition is given for the existence of a unique solution to the two-point boundary value problem

$$x'' = f(x, x', t), \quad x(0) = x(1) = 0$$

under the assumption that the partial derivative f_2 is bounded and the partial derivative f_1 is bounded from below.

1. Introduction. In [3] Lees shows that if f is a continuous real-valued function on the set $T = R \times [0, 1]$, the two-point boundary value problem

$$x'' = f(x, t),$$
$$x(0) = x(1) = 0$$

has a unique solution whenever the partial derivative f_1 is continuous on T and

$$\inf_T f_1 > -\pi^2.$$

The purpose of this paper is to extend this result to the more general problem

$$\begin{aligned} x'' &= f(x, x', t), \\ x(0) &= x(1) = 0, \end{aligned} \tag{1}$$

where f, f_1, and f_2 are continuous on the set $S = R \times R \times [0, 1]$. In what follows, if h is a real-valued function which is bounded on a set E, we put $\|h\|_E = \sup_{\xi \in E} |f(\xi)|$. The principal result is the following.

THEOREM. *Let there be a constant η for which*

$$f_1(x, y, t) \geqq -\eta > -\pi^2$$

on S. Further, suppose that f_2 is bounded on S and satisfies

$$\|f_2\|_S \leqq \frac{\pi^2 - \eta}{\pi}.$$

Then the boundary value problem (1) *has a unique solution.*

We note that this theorem gives information on the existence of unique solutions for problems of the form

$$x'' = f(x, x', t),$$
$$x(a) = A, \quad x(b) = B,$$

since such a problem may be reduced to (1) by appropriate transformation of the variables.

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2. Preliminary lemmas. The proof of the Theorem will be prefaced by two lemmas. We denote $[0, 1]$ by I and $f(0, 0, t)$ by $f_0(t)$.

LEMMA 1. *If x is a solution of the boundary value problem* (1) *and f satisfies the hypothesis of the Theorem, then there is a constant K such that*

$$\left(\int_0^1 (x')^2\, dt\right)^{1/2} \leqq 2K\|f_0\|_I.$$

Also

$$\|x\|_I \leqq K\|f_0\|_I.$$

Proof. We first show that a solution x of (1) satisfies a differential equation in which f_1 and f_2 appear explicitly. Define a function $G(\xi, t)$ by

$$G(\xi, t) = f(\xi x, \xi x', t).$$

Differentiating G with respect to ξ and integrating the result over I, it is found that

$$\begin{aligned} f(x, x', t) &= \int_0^1 G_1(\xi, t)\, d\xi + f_0(t) \\ &= x\int_0^1 f_1(\xi x, \xi x', t)\, d\xi + x'\int_0^1 f_2(\xi x, \xi x', t)\, d\xi + f_0(t), \end{aligned}$$

and we see that the equation in (1) is equivalent to

$$x'' = xp(x, x', t) + x'q(x, x', t) + f_0(t), \tag{2}$$

where p and q are defined by the integrals above. Multiplying (2) by x and integrating the left-hand side by parts we find that

$$\int_0^1 (x')^2\, dt = -\int_0^1 x^2 p\, dt - \int_0^1 xx'q\, dt - \int_0^1 xf_0\, dt.$$

Noting the definitions of p and q, we have

$$-\int_0^1 x^2 p\, dt \leqq \eta\int_0^1 x^2\, dt \quad \text{and} \quad \left|\int_0^1 xx^1 q\, dt\right| \leqq \|f_2\|_S\int_0^1 |x|\,|x'|\, dt.$$

Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_0^1 (x')^2\, dt \leqq \eta\int_0^1 x^2\, dt &+ \|f_2\|_S\left(\int_0^1 x^2\, dt\right)^{1/2}\left(\int_0^1 (x')^2\, dt\right)^{1/2} \\ &+ \left(\int_0^1 f_0^2\, dt\right)^{1/2}\left(\int_0^1 x^2\, dt\right)^{1/2}. \end{aligned}$$

All occurrences of $(\int_0^1 x^2\, dt)^{1/2}$ may be removed from this expression by applying the well-known inequality [2, p. 182]

$$\pi\left(\int_0^1 x^2\, dt\right)^{1/2} \leqq \left(\int_0^1 (x')^2\, dt\right)^{1/2}\cdot \qquad (x(0) = 0).$$

The result is the inequality

$$(\pi^2 - \eta - \pi\|f_2\|_S)\int_0^1 (x')^2\,dt \leqq \pi\left(\int_0^1 (x')^2\,dt\right)^{1/2}\left(\int_0^1 f_0^2\,dt\right)^{1/2}.$$

Since $(\pi^2 - \eta - \pi\|f_2\|_S) > 0$ by hypothesis, let K be $\pi/(2(\pi^2 - \eta - \pi\|f_2\|_S))$ to obtain

$$\left(\int_0^1 (x')^2\,dt\right)^{1/2} \leqq 2K\left(\int_0^1 f_0^2\,dt\right)^{1/2} \leqq 2K\|f_0\|_I.$$

The second conclusion follows from

$$2\|x\|_I \leqq \left(\int_0^1 (x')^2\,dt\right)^{1/2}.$$

In the second lemma, we establish some inequalities concerning the behavior of the derivative of a solution to problem (1). Let the set $[-K\|f_0\|_I, K\|f_0\|_I] \times \{0\} \times [0,1]$ be denoted by H. Since this set is compact, $\|f_1\|_H$ exists.

LEMMA 2. *If x is a solution of problem* (1) *and f satisfies the hypothesis of the Theorem, then there is a constant L such that for any t_1, t_2 in I,*

$$|x'(t_1) - x'(t_2)| \leqq |t_1 - t_2|^{1/2} L\|f_0\|_I.$$

Moreover,

$$\|x'\|_I \leqq 2(K + L)\|f_0\|_I,$$

where K is the constant from Lemma 1.

Proof. Here we employ another equivalent form of the equation in (1). Successive applications of the fundamental theorem of calculus show that

$$f(x, x', t) = x'\int_0^1 f_2(x, \xi x', t)\,d\xi + x\int_0^1 f_1(\xi x, 0, t)\,d\xi + f_0(t).$$

Hence, the equation in (1) is equivalent to

$$x'' = x'h(x, x', t) + xk(x, t) + f_0(t),$$

where h and k are defined by the integrals above. Noting that $\|h\|_S \leqq \|f_2\|_S$ and $|k(x,t)| \leqq \|f_1\|_H$, we have by the Cauchy–Schwarz inequality

$$\begin{aligned}|x'(t_1) - x'(t_2)| \leqq \int_{t_1}^{t_2} |x''(t)|\,dt &\leqq \|f_2\|_S|t_1 - t_2|^{1/2}\left(\int_0^1 (x')^2\,dt\right)^{1/2}\\ &\quad + \|f_1\|_H|t_1 - t_2|^{1/2}\left(\int_0^1 x^2\,dt\right)^{1/2}\\ &\quad + |t_1 - t_2|^{1/2}\left(\int_0^1 f_0^2\,dt\right)^{1/2}.\end{aligned}$$

Now, recalling the two conclusions of Lemma 1, we see that

$$\begin{aligned}|x'(t_1) - x'(t_2)| &\leqq \|f_2\|_S|t_1 - t_2|^{1/2}2K\|f_0\|_I\\ &\quad + \|f_1\|_H|t_1 - t_2|^{1/2}K\|f_0\|_I + |t_1 - t_2|^{1/2}\|f_0\|_I,\end{aligned}$$

from which the first conclusion of the lemma,

$$|x'(t_1) - x'(t_2)| \leqq |t_1 - t_2|^{1/2} L \| f_0 \|_I,$$

is obtained. From this it is immediate that

$$|x'(t)| \leqq L \| f_0 \|_I + |x'(0)| \tag{3}$$

and

$$|x'(t)| \geqq |x'(0)| - L \| f_0 \|_I.$$

If the right-hand side of this last inequality is negative then $|x'(0)| < L\| f_0 \|_I$; if it is positive, it follows that

$$\left(\int_0^1 (x')^2 \, dt \right)^{1/2} \geqq |x'(0)| - L \| f_0 \|_I.$$

Recalling Lemma 1, we see that in either case

$$|x'(0)| \leqq (2K + L) \| f_0 \|_I.$$

This, together with (3), supplies the second conclusion of the lemma.

3. Proof of the Theorem. First we show that whenever f satisfies the hypothesis of the theorem, problem (1) has at most one solution. Suppose x_1 and x_2 are solutions of (1) for such a function, and let $y = x_1 - x_2$. The procedure used in the proof of Lemma 1 may be applied to show that y satisfies the equation

$$y'' = F(y, y_1', t) = P(t)y + Q(t)y',$$

where

$$P(t) = \int_0^1 f_1(\xi[x_1 - x_2] + x_2, \xi[x_1' - x_2'] + x_2', t) \, d\xi,$$

and

$$Q(t) = \int_0^1 f_2(\xi[x_1 - x_2] + x_2, \xi[x_1' - x_2'] + x_2', t) \, d\xi.$$

Clearly, $P(t) > -\eta$ and $|Q(t)| \leqq \| f_2 \|_S$. Thus $F_1 > -\eta$ and

$$0 < \pi^2 - \eta - \pi \| f_2 \|_S \leqq \pi^2 - \eta - \pi \| Q \|_I = \pi^2 - \eta - \pi \| F_2 \|_S.$$

Since $y(0) = y(1) = 0$, we conclude from Lemma 1 that

$$\| y \|_I \leqq K \| F(0, 0, t) \|_I = 0,$$

and consequently (1) has at most one solution. For the remainder of the proof, let $C^1(I)$ denote the Banach space of real-valued continuously differentiable functions on I, normed by

$$\| x \| = \| x \|_I + \| x' \|_I.$$

To show existence of solutions, we define a map T from $C^1(I)$ to $C^1(I)$ in such a way that any fixed point of T solves (1). If $x \in C^1(I)$, let $T(x)$ be the solution of problem (1) for the linear equation

$$y'' = y'h(x, x', t) + yk(x, t) + f_0(t), \tag{4}$$

where h and k are the functions defined in the proof of Lemma 2. T is well-defined since a solution to (4) may be constructed exactly as in [3] and the preceding uniqueness result applies. Let Ω be the set

$$\{x \in C^1(I) : \|x\| \leqq (3K + 2L)\|f_0\|_I\}.$$

Lemma 1 and Lemma 2 applied to (4) show that T maps Ω into itself. Next we show that T is continuous on Ω. Let $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Then $y = y_1 - y_2$ satisfies the equation

$$\begin{aligned} Y'' = y'h(x_1, x_1', t) + yk(x_1, t) + y_2'[h(x_1, x_1', t) - h(x_2, x_2', t)] \\ + y_2[k(x_1, t) - k(x_2, t)]. \end{aligned}$$

Since $y(0) = y(1) = 0$, we may apply the lemmas to conclude that

$$\begin{aligned} \|y\| &\leqq (3K + 2L)[\|y_2'\|_I \|h(x_1, x_1', t) - h(x_2, x_2', t)\|_I \\ &\quad + \|y_2\|_I \|k(x_1, t) - k(x_2, t)\|_I] \\ &\leqq 2(K + L)(3K + 2L)\|f_0\|_I \|h(x_1, x_1', t) - h(x_2, x_2', t)\|_I \\ &\quad + K(3K + 2L)\|f_0\|_I \|k(x_1, t) - k(x_2, t)\|_I. \end{aligned}$$

Because the set

$$\begin{aligned} [-(3K + 2L)\|f_0\|_I, (3K + 2L)\|f_0\|_I] \times [-(3K + 2L)\|f_0\|_I, \\ (3K + 2L)\|f_0\|_I] \times [0, 1] \end{aligned}$$

is compact, h and k are uniformly continuous there, and it follows from the inequality above that T is in fact uniformly continuous on Ω. Finally we observe that the closure of $T(\Omega)$ is compact. Indeed, given a sequence $\{x_n\}$ in $T(\Omega)$, Lemma 2 shows that the sequence of derivatives $\{x_n'\}$ is uniformly bounded and equicontinuous, so the Ascoli theorem applies to show that the original sequence has a subsequence which converges in $C^1(I)$. The Schauder fixed-point theorem [1, p. 415] implies that there is a function x in Ω such that $T(x) = x$. As was noted in the proof of Lemma 2, any solution of

$$x'' = x'h(x, x', t) + xh(x, t) + f_0(t)$$

is also a solution of the equation in (1), so the proof is complete.

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ASYMPTOTIC EVALUATION OF INTEGRALS INVOLVING A FRACTIONAL DERIVATIVE*

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Abstract. For the integral

$$\int_a^\infty e^{-z(t-a)} I^{\lambda-1} f(t)\, dt$$

an asymptotic expansion is obtained as $z \to \infty$. Here λ is fixed, $0 < \lambda < 1$, $I^{\lambda-1}$ is the operator of fractional integration, and the expansion holds uniformly for $a \geqq 0$. A similar expansion is obtained for the integral from 0 to a and is applied to the solution of an integral equation.

1. In this paper, an asymptotic expansion of the integral

$$F(z, a) = \int_a^\infty e^{-z(t-a)} t^{\lambda-1} g(t)\, dt \tag{1.1}$$

will be obtained as $z \to \infty$, valid uniformly for $a \geqq 0$.

For fixed a, the asymptotic expansion of the integral is well known. If $a > 0$, the expansion is

$$\sum h^{(k)}(a) z^{-k-1},$$

where $h(t) = t^{\lambda-1} g(t)$; if $a = 0$ the expansion is

$$\sum \frac{\Gamma(k+\lambda)}{k!} g^{(k)}(0) z^{-k-\lambda}.$$

Except when λ is a positive integer, the expansion based on a direct application of Watson's lemma clearly cannot hold uniformly in a for $a \geqq 0$.

The lack of uniformity is due to the presence and possible coalescence in (1.1) of two critical points: the end point of integration, $t = a$, and the singularity of the integrand, $t = 0$. There are known methods for obtaining uniform asymptotic expansions of integrals involving two critical points. The two best known methods are based respectively on two-point expansions of the integrand, used in the case of two saddle points by Chester, Friedman and Ursell [2], and on a skillfully arranged procedure of integration by parts, used in the case of a stationary point and an algebraic singularity by Bleistein [1].

Let us indicate the first step of an integration by parts procedure.[1] Writing

$$F(z, a) = \int_a^\infty e^{-z(t-a)} t^{\lambda-1} g(0)\, dt + \int_a^\infty e^{-z(t-a)} t^{\lambda-1} [g(t) - g(0)]\, dt,$$

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[1] A different procedure is outlined in a letter from N. Bleistein to me. A.E.

and integrating by parts in the second integral, we obtain, with the notation

$$\frac{d}{dt}\{t^{\lambda-1}[g(t) - g(0)]\} = t^{\lambda-1}g_1(t),$$

$$\begin{aligned} F(z, a) = [g(0) + z^{-1}g_1(0)] \int_a^\infty e^{-z(t-a)}t^{\lambda-1}\,dt \\ + a^{\lambda-1}[g(a) - g(0)]z^{-1} \\ + z^{-1}\int_a^\infty e^{-z(t-a)}t^{\lambda-1}[g_1(t) - g_1(0)]\,dt. \end{aligned}$$

This can be shown to be the beginning of a uniform asymptotic expansion, and it is not difficult to repeat the process, but it does not appear to be easy to write down explicit expressions for the successive terms, to estimate the remainder term, or determine conditions of validity for the resulting expansion.

If $\lambda = n + 1$ happens to be a positive integer, this difficulty can be overcome by representing $t^n g(t) = h(t)$ as the n times repeated integral of $h^{(n)}(t)$, and then using straightforward integrations by parts. It will be shown in this paper that in case of a general λ, a representation of $t^{\lambda-1}g(t)$ as an integral of fractional order, followed by fairly straightforward integration by parts leads to the desired expansion. Explicit expressions for the generic term of this expansion, and an estimate of the remainder term will be obtained.

For the sake of simplicity it will be assumed that λ, a, z are real. The extension to complex λ and z is almost automatic, and the extension to complex a and analytic functions f does not seem to present any difficulties. Also for the sake of simplicity, f will be assumed to be sufficiently often differentiable. It is well known that in case a is restricted to an interval $[0, A]$, it is sufficient for f to be differentiable in $[0, A_1[$ for some $A_1 > A$.

It seems likely that fractional integration can be used for the asymptotic evaluation of other types of integrals but this has not so far been tested.

2. For $f \in C[0, b[$ and $\alpha > 0$, the operator I^α of integration of order α is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}f(s)\,ds. \tag{2.1}$$

For $\alpha > 0$, $\beta > 0$,

$$I^\alpha I^\beta f = I^{\alpha+\beta}f \tag{2.2}$$

can be proved by interchanging the order of integrations in the repeated integral $I^\alpha(I^\beta f)$, and using Euler's integral for the beta function.

If $f \in C^1[a, b[$ and $\alpha > 0$,

$$\frac{d}{dt}I^{\alpha+1}f(t) = I^\alpha f(t) = \frac{f(0)}{\Gamma(\alpha + 1)}t^\alpha + I^{\alpha+1}f'(t) \tag{2.3}$$

since

$$f(t) = f(0) + I^1 f'(t)$$

and

$$I^{\alpha}1(t) = \frac{t^{\alpha}}{\Gamma(\alpha + 1)}.$$

Equation (2.3) can be used to extend the definition of $I^{\alpha}f$ to $\alpha > -1$ if $f \in C^1[0, b[$ and, by repeated application, to $\alpha > -n$ if $f \in C^n[0, b[$. The addition theorem (2.2) remains valid for this extension. In particular,

$$I^0 f(t) = f(0) + I^1 f'(t) = f(t). \tag{2.4}$$

We can now represent $t^{\lambda - 1}g(t)$ as an integral of order $\lambda - 1$.

LEMMA 1. *Let* $0 < \lambda \leqq 1$, *let* $g \in C^n[0, b[$, *and let* $k = 0, 1, \cdots, n$. *Then* f, *given by*

$$f(t) = \begin{cases} g(t) & \text{if } \lambda = 1, \\ \dfrac{1}{\Gamma(1 - \lambda)} \displaystyle\int_0^t (t - s)^{-\lambda} s^{\lambda - 1} g(s)\, ds & \text{if } 0 < \lambda < 1, \end{cases}$$

has the following properties:

(i) $f \in C^n[0, b[$ *and, if* $0 < \lambda < 1$, *then*

$$f^{(k)}(t) = \frac{t^{-k}}{\Gamma(1 - \lambda)} \int_0^t (t - s)^{-\lambda} s^{k + \lambda - 1} g^{(k)}(s)\, ds;$$

(ii) $f^{(k)}(0) = [\Gamma(k + \lambda)/k!]g^{(k)}(0)$ *and*

$$|f^{(k)}(t)| \leqq \frac{\Gamma(k + \lambda)}{k!} \max \{|g^{(k)}(s)| : 0 \leqq s \leqq t\};$$

(iii)

$$I^{\lambda} f^{(k)}(t) = t^{\lambda} \int_0^1 F(1 - k, 1 - \lambda; 1; 1 - u) g^{(k)}(ut)\, du;$$

(iv)

$$t^{\lambda - 1} g(t) = I^{\lambda - 1} f(t).$$

Proof. For $\lambda = 1$, the lemma is trivially true. Let $0 < \lambda < 1$. Then

$$\Gamma(1 - \lambda) f(t) = \int_0^1 (1 - u)^{-\lambda} u^{\lambda - 1} g(ut)\, dt$$

clearly shows that $f \in C^n[0, b[$, and (i) follows by differentiation under the integral sign followed by a change of variable, $s = ut$. (ii) follows from (i).

To prove (iii), interchange the order of integrations in

$$I^{\lambda} f^{(k)}(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t - r)^{\lambda - 1} \left(\frac{r^{-k}}{\Gamma(1 - \lambda)} \int_0^r (r - s)^{-\lambda} s^{k + \lambda - 1} g^{(k)}(s)\, ds \right) dr$$

to obtain

$$I^{\lambda} f^{(k)}(t) = \frac{1}{\Gamma(\lambda)\Gamma(1 - \lambda)} \int_0^t s^{k + \lambda - 1} g^{(k)}(s) \left(\int_s^t r^{-k} (t - r)^{\lambda - 1} (r - s)^{-\lambda}\, dr \right) ds.$$

With $r = s + (t - s)x$,

$$\frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_s^t (r-s)^{-\lambda}(t-r)^{\lambda-1} r^{-k}\, dr$$

$$= \frac{t^{-k}}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 x^{-\lambda}(1-x)^{\lambda-1}\left[1 - (1-x)\left(1 - \frac{s}{t}\right)\right]^{-k} dx.$$

By Euler's integral representation of the hypergeometric function [4, (2.1.10)] this is

$$t^{-k}F(\lambda, k; 1; 1 - s/t),$$

and by Euler's transformation [4, (2.1.23)] this can also be written as

$$s^{1-k-\lambda}t^{\lambda-1}F(1-k, 1-\lambda; 1; 1 - s/t).$$

Thus,

$$I^\lambda f^{(k)}(t) = t^{\lambda-1}\int_0^t F(1-k, 1-\lambda; 1; 1 - s/t) g^{(k)}(s)\, ds,$$

and this proves (iii).

Lastly, we have from (2.3),

$$I^{\lambda-1}f(t) = \frac{f(0)}{\Gamma(\lambda)} t^{\lambda-1} + I^\lambda f'(t),$$

and since

$$f(0) = g(0)\Gamma(\lambda)$$

from (ii), and $F(0, 1-\lambda; 1; 1 - s/t) = 1$ in (iii) with $k = 1$,

$$I^{\lambda-1}f(t) = g(0)t^{\lambda-1} + t^{\lambda-1}\int_0^t g'(s)\, ds = t^{\lambda-1}g(t),$$

thus proving (iv).

3. For the sake of simplicity we assume in (1.1) that a, λ, z are real, and $g \in C^n[0, \infty[$. We further assume that λ is fixed, and without loss of generality take $0 < \lambda < 1$ ($\lambda = 1$ is covered by existing results); and also assume that $a \geqq 0$, and $z \to \infty$. It is reasonable to take g and its derivatives exponentially bounded, and without further loss of generality we may assume that g and its derivatives are bounded (by a constant). In this case Lemma 1 shows that $t^{\lambda-1}g(t)$ can be represented in the form $I^{\lambda-1}f(t)$ with $f \in C^n[0, \infty[$ and f and its derivatives bounded.

Thus, in place of (1.1) we consider

$$F(z, a) = \int_a^\infty e^{-z(t-a)} I^{\lambda-1}f(t)\, dt \tag{3.1}$$

with λ fixed and $0 < \lambda < 1$, $a \geqq 0$, $z \to +\infty$, $f \in C^n[a, \infty[$ and f and its derivatives bounded.

The simplest integral of the form (3.1), corresponding to $f(t) = 1$, is

$$Q = \frac{1}{\Gamma(\lambda)} \int_a^\infty e^{-z(t-a)} t^{\lambda-1} \, dt.$$

This can be expressed in terms of the incomplete gamma function [4, (9.1.2)] in the form

$$Q = \frac{e^{az}}{\Gamma(\lambda) z^\lambda} \int_{az}^\infty e^{-u} u^{\lambda-1} \, du = \frac{e^{az}\Gamma(\lambda, az)}{\Gamma(\lambda) z^\lambda}. \tag{3.2}$$

Q is used as a comparison integral as it were.

In (3.1) we set $f(t) = f(0) + I^1 f'(t)$ and note that $I^{\lambda-1} I^1 = I^\lambda$ to obtain

$$F(z, a) = f(0)Q + \int_a^\infty e^{-z(t-a)} I^\lambda f'(t) \, dt.$$

Here we integrate by parts noting that, by (2.3), $(d/dt) I^\lambda f'(t) = I^{\lambda-1} f'(t)$. Since f' is bounded and $\lambda > 0$, we see from (2.1) that $|I^\lambda f'(t)| \leqq ct^\lambda$, so that, for $z > 0$, the integrated parts vanish at infinity, and we have

$$F(z, a) = f(0)Q + z^{-1} I^\lambda f'(a) + z^{-1} \int_a^\infty e^{-z(t-a)} I^{\lambda-1} f'(t) \, dt.$$

The integral on the right is of the same form as (3.1) and with $f'(t) = f'(0) + I^1 f''(t)$ we obtain

$$\begin{aligned} F(z, a) &= z^{-1} I^\lambda f'(a) + [f(0) + f'(0) z^{-1}] Q \\ &\quad + z^{-1} \int_a^\infty e^{-z(t-a)} I^\lambda f''(t) \, dt. \end{aligned}$$

This process can be repeated and finally leads to

$$F(z, a) = \sum_{k=1}^{n-1} z^{-k} I^\lambda f^{(k)}(a) + \sum_{k=0}^{n-1} z^{-k} f^{(k)}(0) Q + R_n. \tag{3.3}$$

The remainder is given by

$$R_n = z^{1-n} \int_a^\infty e^{-z(t-a)} I^\lambda f^{(n)}(t) \, dt. \tag{3.4}$$

We shall show that (3.3) is a finite "general" asymptotic expansion in the sense of [3].

Since f and its derivatives are bounded, there exist c_k, dependent on f and λ but independent of a, such that

$$|I^\lambda f^{(k)}(t)| \leqq c_k t^\lambda, \qquad k = 0, 1, \cdots, n, \quad t \geqq 0. \tag{3.5}$$

Then

$$\begin{aligned} |R_n| &\leqq c_n z^{1-n} \int_a^\infty e^{-z(t-a)} t^\lambda \, dt \\ &= c_n z^{-n-\lambda} e^{az} \Gamma(\lambda + 1, az). \end{aligned} \tag{3.6}$$

Now, clearly $\{z^{-n-\lambda} e^{az}\Gamma(\lambda + 1, az) : n = 0, 1, 2, \cdots\}$ is an asymptotic sequence as $z \to \infty$, and (3.6) shows that (3.3) is a finite asymptotic expansion holding uniformly in a for $a \geqq 0$ with respect to this asymptotic sequence (which itself depends on a). Division of both sides of (3.3) by $e^{az}\Gamma(\lambda + 1, az)$ results in an asymptotic expansion holding uniformly in a with respect to the asymptotic sequence $\{z^{-n-\lambda} : n = 0, 1, 2, \cdots\}$ which is independent of a.

4. Returning to (1.1) we see that if λ is fixed, $0 < \lambda < 1$, $a \geqq 0$, $z \to +\infty$, $g \in C^n[0, \infty[$, and g and its derivatives are bounded, then (3.3) represents a uniform asymptotic expansion of (1.1). The coefficients in this expansion can be expressed in terms of g by means of (ii) and (iii) of Lemma 1. At first it would seem that the coefficients in the first sum in (3.3) involve all values of g on the interval $[0, a]$. This is not so. For $k \geqq 1$, the hypergeometric function appearing in (iii) of Lemma 1 is in fact a polynomial of degree $k - 1$, and the integral can be evaluated by successive integrations by parts in terms of the values of g and its derivatives at $t = 0$ and $t = a$, thus confirming the existence of two critical points for this integral. The actual expression is

$$\begin{aligned} I^\lambda f^{(k)}(a) = \sum_{m=1}^{k} \frac{a^{\lambda - m}}{(k-m)!}\Bigg[(-1)^{m-1}\frac{(k-1)!\Gamma(m-\lambda)}{(m-1)!\Gamma(1-\lambda)}g^{(k-m)}(a) \\ - \frac{\Gamma(k+\lambda-m)}{\Gamma(\lambda-m+1)}g^{(k-m)}(0)\Bigg], \qquad k = 1, 2, \cdots, n. \end{aligned}$$

This formula, together with $k!f^{(k)}(0) = \Gamma(k + \lambda)g^{(k)}(0)$, gives the explicit form of the expansion of (1.1) which is not easily obtained by a direct integration by parts of (1.1).

5. Superficially it might seem that the first sum in (3.3) represents the contribution of the critical point $t = a$ and corresponds to the "outer expansion" of singular perturbations, while the second sum represents the contribution of the critical point $t = 0$, is significant only if a is small, and represents a "boundary layer" effect. This is not so. The explicit form in terms of g given in the last section shows that the behavior of g at 0 enters also in the first sum. Moreover, it will be seen from certain estimates to be developed presently that the behavior of $F(z, a)$ as a function of a is somewhat different from that encountered in singular perturbations. The same estimates will also enable us to replace the asymptotic sequence $\{z^{-n-\lambda} e^{az}\Gamma(\lambda + 1, az)\}$ by an asymptotic sequence of elementary functions and to show that the estimate (3.6) is an effective estimate in the sense that in general the first neglected term of the expansion will be of the order of the right-hand side of (3.6).

LEMMA 2. *There are positive numbers* $a_1, \cdots, a_4$ *independent of* x *but dependent on* α *such that*

$$a_1(1 + x^{1-\alpha})^{-1} \leqq e^x\Gamma(\alpha, x) \leqq a_2(1 + x^{1-\alpha})^{-1}, \qquad 0 < \alpha \leqq 1, \quad x \geqq 0,$$

$$a_3(1 + x^{\alpha-1}) \leqq e^x\Gamma(\alpha, x) \leqq a_4(1 + x^{\alpha-1}), \qquad \alpha \geqq 1, \quad x \geqq 0.$$

Proof. It is known [4, § 9.2] that $\Gamma(\alpha, 0) = \Gamma(\alpha)$ if $\alpha > 0$, $\Gamma(\alpha, x) > 0$ for $x > 0$, and

$$x^{1-\alpha} e^x\Gamma(\alpha, x) \to 1 \quad \text{as } x \to \infty.$$

Thus,

$$(1 + x^{1-\alpha})\, e^x\Gamma(\alpha, x) \quad \text{if } 0 < \alpha \leqq 1,$$
$$(1 + x^{\alpha-1})^{-1}\, e^x\Gamma(\alpha, x) \quad \text{if } \alpha \geqq 1$$

approach positive limits as $x \to 0+$ or $x \to +\infty$, are positive and continuous on $]0, \infty[$ and are thus bounded and bounded away from zero. This proves the lemma.

This lemma will enable us to estimate the terms and the remainder in the expansion (3.3). The constants implied by the O symbols are independent of a but may depend on λ as well as f.

For the terms of the first sum in (3.3) we have

$$A_k = z^{-k}I^\lambda f^{(k)}(a) = O(z^{-k}a^\lambda)$$

from (3.5), and for those of the second sum,

$$B_k = z^{-k}f^{(k)}(0)Q = O(z^{-k-\lambda}[1 + (az)^{1-\lambda}]^{-1})$$

from (3.2) and Lemma 2. If $f^{(k)}(0) \neq 0$ we also have

$$B_k^{-1} = O(z^{k+\lambda}[1 + (az)^{1-\lambda}]).$$

Thus,

$$A_k/B_k = O((az)^\lambda + az) \quad \text{if } f^{(k)}(0) \neq 0$$

showing that A_k is small in comparison with B_k if az is small, i.e., $a = o(z^{-1})$, while A_k predominates over B_k if az is large. However, in order for the first sum in (3.3) to be small in comparison with the second sum (to any number of terms) we must have

$$A_0/B_k = O(z^k[(az)^\lambda + az]) \quad \text{if} \quad f^{(k)}(0) \neq 0,$$

small for all k, and this demands $a = O(z^{-m})$ for all m. Likewise, for the first sum to predominate over the second, we must have

$$A_k/B_{k'} = O(z^{k'-k}[(az)^\lambda + az])$$

small for all k' for which $f^{(k')}(0) \neq 0$ and for all k, and this demands that az^{-m} be large for all m. It then seems that while there are comparatively narrow regions of a in which one or the other of the two sums in (3.3) predominates, yet over a much wider range both sums contribute significantly, although the number of terms which must be retained in the two sums in order to achieve a given accuracy may differ.

For the remainder term we have, from (3.6) and Lemma 2,

$$R_n = O(z^{-n-\lambda}[1 + (az)^\lambda]), \tag{5.1}$$

and this appears to be a best possible result in the sense that it cannot be improved for all bounded f with n continuous and bounded derivatives.

The first "neglected" term in the expansion (3.3) is $A_n + B_n$ and this is bounded by a multiple of

$$z^{-n}a^\lambda + z^{-n-\lambda}[1 + (az)^{1-\lambda}]^{-1} = z^{-n-\lambda}[1 + (az)^\lambda + az][1 + (az)^{1-\lambda}]^{-1}$$

so that in the "general case" when the estimate (5.1) is realistic in the sense that

$$z^{n+\lambda}[1 + (az)^{\lambda}]^{-1} R_n$$

is not only bounded but also bounded away from zero, we have

$$\frac{A_n + B_n}{R_n} = O\left(\frac{1 + (az)^{\lambda} + az}{[1 + (az)^{\lambda}][1 + (az)^{1-\lambda}]}\right).$$

Since $(1 + x^{\lambda})^{-1}(1 + x^{1-\lambda})^{-1}(1 + x^{\lambda} + x)$ is bounded and bounded away from zero for $x \geqq 0$, we have that, in the general case, $A_n + B_n$ and R_n are precisely of the same order, uniformly in a, as $z \to \infty$.

Let us now envisage a dependent on z. Under all circumstances, the two sums in (3.3) represent $F(z, a)$ to the accuracy stated in (5.1). Under certain circumstances it may happen that some terms of one or the other of the sums are themselves of the order stated in (5.1) and can be omitted, that is, incorporated in the remainder term without changing the estimate (5.1).

First consider the first sum, $\sum A_k$.

$$z^{n+\lambda}[1 + (az)^{\lambda}]^{-1} A_k = O(z^{n-k}(az)^{\lambda}[1 + (az)^{\lambda}]^{-1})$$

shows that if $(az)^{-1} = O(1)$, so that $(az)^{\lambda}[1 + (az)^{\lambda}]^{-1}$ is bounded and bounded away from zero, all terms of the sum must be taken since z^{n-k} is unbounded if $k \leqq n - 1$. On the other hand, if $(az)^{\lambda} = O(z^{-\rho})$ with $\rho \geqq 0$, only those terms with $k < n - \rho$ of $\sum A_k$ need be considered since

$$z^{n-k}(az)^{\lambda}[1 + (az)^{\lambda}]^{-1} = O(z^{n-k-\rho})$$

is bounded if $k \geqq n - \rho$. In particular, the first sum may be omitted altogether if $(az)^{\lambda} = O(z^{1-n})$.

As to the second sum, $\sum B_k$, here

$$z^{n+\lambda}[1 + (az)^{\lambda}]^{-1} B_k = O(z^{n-k}[1 + (az)^{\lambda} + (az)^{1-\lambda} + az]^{-1}),$$

and considerations similar to those carried out for $\sum A_k$ show that all n terms of $\sum B_k$ must be taken if $az = O(1)$, while only those terms with $k < n - \rho$ need be retained if $(az)^{-1} = O(z^{-\rho})$ with $\rho \geqq 0$. In particular, the second sum may be omitted altogether if $(az)^{-1} = O(z^{-n})$.

In § 3 it was shown that

$$f(0)Q + \sum_{k=1}^{\infty} [I^{\lambda} f^{(k)}(a) + f^{(k)}(0)Q] z^{-k}$$

is an asymptotic expansion of $F(z, a)$ with respect to the scale

$$\{z^{-n-\lambda} e^{az} \Gamma(\lambda + 1, az)\}.$$

It follows from Lemma 2 or (5.1) that this scale may be replaced by the scale

$$\{z^{-n-\lambda}[1 + (az)^{\lambda}]\}$$

of elementary functions. The asymptotic expansion is valid uniformly in a for $a \geqq 0$ in this simplified scale.

The identity (3.3), with R_n given by (3.4), holds with $z > b > 0$ for all f in $C^n[0, \infty[$ (with F defined by (1.1) for all g in $C^n[0, \infty[$) for which $e^{-bt} f^{(k)}(t)$ $[e^{-bt} g^{(k)}(t)$

in the form (1.1)], $k = 0, 1, \cdots, n$, $t \geqq 0$ is bounded. The analysis of A_k and B_k depends on (3.5) and will not hold if $b > 0$, nor will (3.6). If one is interested only in the validity of (3.3) as a general asymptotic expansion, then it is not necessary to make further assumptions on $f, f', \cdots, f^{(n-1)}$. Further conditions imposed on $f^{(n)}$ will determine the asymptotic scale. For instance, if $f^{(n)}$ is bounded, then (3.6) will hold and all statements on asymptotic scales made in § 3 and the present section continue to hold.

If it is assumed that $t^{\beta}f^{(n)}(t)$, $\beta < 1$, is bounded, then (3.5) is replaced by

$$|I^{\lambda}f^{(n)}(t)| \leqq c_n t^{\lambda-\beta}, \tag{5.2}$$

and (3.6), by

$$|R_n| \leqq c_n z^{\beta-n-\lambda} e^{az}\Gamma(\lambda - \beta + 1, az). \tag{5.3}$$

This gives the asymptotic scale in the present case. In particular,

$$e^{az}\Gamma(1, az) = 1,$$

and if $t^{\lambda}f^{(n)}(t)$ is bounded, then (5.3) simplifies to $|R_n| \leqq c_n z^{-n}$, and (3.3) holds uniformly with respect to the asymptotic scale $\{z^{-n}\}$.

6. The integral

$$F_1(z, a) = \int_0^a e^{-zt}I^{\lambda-1}f(t)\,dt \tag{6.1}$$

can be treated similarly. Alternatively, the expansion of F_1 can be derived from that of F since

$$F_1(z, a) = F(z, 0) - e^{-az}F(z, a).$$

In the expansion of $F(z, 0)$ we have $I^{\lambda}f^{(k)}(0) = 0$. If we set

$$P = \frac{1}{\Gamma(\lambda)}\int_0^{\infty} e^{-zt}t^{\lambda-1}\,dt - e^{-az}Q$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^{a} e^{-zt}t^{\lambda-1}\,dt$$

or [4, (9.1.1)]

$$P = \frac{\gamma(\lambda, az)}{\Gamma(\lambda)z^{\lambda}} \tag{6.2}$$

and use (3.3) both for $F(z, 0)$ and $F(z, a)$, then the asymptotic expansion of F_1 emerges in the form

$$F_1(z, a) = -\sum_{k=1}^{n-1} z^{-k}e^{-az}I^{\lambda}f^{(k)}(a) + \sum_{k=0}^{n-1} z^{-k}f^{(k)}(0)P + S_n, \tag{6.3}$$

where

$$S_n = z^{1-n}\int_0^a e^{-zt}I^{\lambda}f^{(n)}(t)\,dt. \tag{6.4}$$

For bounded $f^{(n)}$ the remainder S_n can be estimated by means of (3.5) in the form

$$
(6.5)\qquad \begin{aligned} |S_n| &\leqq c_n z^{1-n} \int_0^a e^{-zt} t^\lambda \, dt \\ &= c_n z^{-n-\lambda} \gamma(\lambda + 1, az). \end{aligned}
$$

This estimate shows that (6.3) is a generalized asymptotic expansion with respect to the asymptotic sequence

$$\{z^{-n-\lambda}\gamma(\lambda + 1, az) : n = 0, 1, 2, \cdots\}.$$

A detailed analysis of the order of the terms in the expansion (6.3) may be based on the observation that for fixed $\alpha > 0$,

$$\gamma(\alpha, x)(1 + x^{-\alpha})$$

is bounded and bounded away from zero on $[0, \infty[$. The work is similar to that carried out in § 5 but the results are somewhat different. It turns out that for small az the contributions of the two sums in (6.3) are of the same order of magnitude, while the first sum is exponentially small in comparison with the second one and may be neglected if az is large.

7. The results obtained in the preceding section will now be applied to the solution of an integral equation investigated by Olmstead and Handelsman [5]. The integral equation in question may be written as

$$
(7.1)\qquad u(t) + z^{1/2} I^{1/2} u(t) = z^{1/2} I^{1/2} f(t)
$$

(see [5, eq. (1.7) with $n = 1$]), where z ($= \varepsilon^{-2}$ of [5]) is a positive parameter, and f is a given function which we assume to be in $C^n[0, \infty[$. We shall first find the explicit solution of (7.1).

Using (2.2) we have

$$I^{1/2} u + z^{1/2} I^1 u = z^{1/2} I^1 f,$$

and from this and (7.1),

$$u - z I^1 u = z^{1/2} I^{1/2} f - z I^1 f.$$

This relation shows that $u(0) = 0$ and also that every continuous solution u of (7.1) is continuously differentiable and satisfies the differential equation

$$u' - zu = z^{1/2} I^{-1/2} f - zf.$$

Hence the explicit solution of (7.1) in the form

$$
(7.2)\qquad u(t) = \int_0^t e^{z(t-s)} [z^{1/2} I^{-1/2} f(s) - z f(s)] \, ds.
$$

u is the difference of two integrals. For the first of these we have from (6.3) with $\lambda = \frac{1}{2}$,

$$\int_0^t e^{z(t-s)} I^{-1/2} f(s)\, ds = -\sum_{k=1}^{n-1} z^{-k} I^{1/2} f^{(k)}(t) + \sum_{k=0}^{n-1} z^{-k} f^{(k)}(0) \pi^{-1/2} \int_0^t e^{z(t-s)} s^{-1/2}\, ds + z^{1-n} \int_0^t e^{z(t-s)} I^{1/2} f^{(n)}(s)\, ds,$$

while for the second one, straightforward integrations by parts yield

$$\int_0^t e^{z(t-s)} f(s)\, ds = -\sum_{k=0}^{n-1} z^{-k-1} [f^{(k)}(t) - f^{(k)}(0)\, e^{zt}] + z^{-n} \int_0^t e^{z(t-s)} f^{(n)}(s)\, ds.$$

We now substitute this in (7.2), note that

$$\pi^{-1/2} \int_0^t e^{z(t-s)} s^{-1/2}\, ds = z^{-1/2}\, e^{zt} - \pi^{-1/2} \int_t^\infty e^{z(t-s)} s^{-1/2}\, ds,$$

and obtain

$$\begin{aligned} u(t) = {} & \sum_{k=0}^{n-1} \left[f^{(k)}(t) - f^{(k)}(0) z^{1/2} \pi^{-1/2} \int_t^\infty e^{-z(s-t)} s^{-1/2}\, ds \right] z^{-k} \\ & - \sum_{k=1}^{n-1} z^{1/2-k} I^{1/2} f^{(k)}(t) + R_n \end{aligned} \tag{7.3}$$

with

$$R_n = z^{1-n} \int_0^t e^{z(t-s)} [z^{1/2} I^{1/2} f^{(n)}(s) - f^{(n)}(s)]\, ds. \tag{7.4}$$

This is in effect (3.14) of [5] except that Olmstead and Handelsman carry the second sum to $n - 1$ or n terms according as their M is even (and $n - 1 = M/2 = [(M+1)/2]$) or odd (and $n - 1 = [M/2] = (M+1)/2 - 1$).

We shall now show that an alternative expression of the remainder term, valid under the additional assumption that $f^{(n)}$ is exponentially bounded, is

$$R_n = z^{1-n} \int_t^\infty e^{-z(s-t)} [f^{(n)}(s) - z^{1/2} I^{1/2} f^{(n)}(s)]\, ds. \tag{7.5}$$

This will follow from what is in effect a simple case of fractional integration by parts.

LEMMA 3. *Assume that* $\alpha > 0$, $h \in C[0, \infty[$, $e^{-bt}h(t)$ *is bounded on* $[0, \infty[$, *and* $z > b$. *Then*

$$\int_0^\infty e^{-zs} I^\alpha h(s)\, ds = z^{-\alpha} \int_0^\infty e^{-zs} h(s)\, ds.$$

Proof. The left-hand side is

$$\int_0^\infty e^{-zs}\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-u)^{\alpha-1}h(u)\,du\right)ds.$$

Here it is permissible to interchange the order of integrations and as

$$\frac{1}{\Gamma(\alpha)}\int_u^\infty e^{-zs}(s-u)^{\alpha-1}\,ds = z^{-\alpha}e^{-zu},$$

one obtains

$$z^{-\alpha}\int_0^\infty e^{-zu}h(u)\,du,$$

thus proving Lemma 3.

It follows from Lemma 3 that

$$\int_0^\infty e^{-zs}[z^{1/2}I^{1/2}f^{(n)}(s) - f^{(n)}(s)]\,ds = 0,$$

and so (7.4) and (7.5) are equivalent. The latter expression of the remainder leads to better estimates. If $f^{(n)}$ is bounded, then much as in § 3 one can prove

$$R_n = O(z^{-n}[1 + (zt)^{1/2}])$$

uniformly for $t \geqq 0$, and if $t^{1/2}f^{(n)}(t)$ is bounded, then $f^{(n)}$, being continuous at 0, is bounded and (5.2) shows that $I^{1/2}f^{(n)}$ is also bounded. This being so,

$$\int_t^\infty e^{-z(s-t)}f^{(n)}(s)\,ds = O\left(\int_t^\infty e^{-z(s-t)}\,ds\right) = O(z^{-1}),$$

$$\int_t^\infty e^{-z(s-t)}I^{1/2}f^{(n)}(s)\,ds = O\left(\int_t^\infty e^{-z(s-t)}\,ds\right) = O(z^{-1}),$$

and it follows from (7.5) that

$$R_n = O(z^{1/2-n}) \tag{7.6}$$

uniformly for $t \geqq 0$ in this case.

Olmstead and Handelsman analyze the asymptotic behavior for $z \to \infty$ of a nonlinear integral equation which includes (7.1) as a special case. Even in the linear case [5, § 3] they derive their results, under conditions on f which differ somewhat from ours, directly from the integral equation rather than using the explicit solution (which is available for (7.1) but is not available for the nonlinear equation). Their conclusion agrees with (7.6) if M of [5] is even; if M is odd, then they take n rather than $n - 1$ terms in the second sum in (7.3) and thereby improve (7.6) to $R_n = O(z^{-n})$.

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ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER SYSTEMS*

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Abstract. This note deals with a priori bounds for solutions (and their derivatives) and applications to existence theorems for two-point boundary value problems for systems of nonlinear, second order, ordinary differential equations. The methods are based on those of the author and McLeod employed in the particular case of the swirling flow problem in fluid mechanics. It depends on the construction of general "Lyapunov-type" functions and comparison theorems for (scalar) second order equations.

1. Introduction. In [2] (cf. [3, pp. 428–434]), we considered the boundary value problem

$$x'' = f(t, x, x') \quad \text{and} \quad x(a) = x_a, \quad x(b) = x_b \tag{1.1}$$

for a system of equations, where $x = (x^1, \cdots, x^d) \in \mathbb{R}^d$. Existence was proved by first obtaining a priori bounds for $|x(t)|$, $|x'(t)|$ of a solution of (1.1) under assumptions involving inequalities for $x \cdot f$ and $|f|$. For papers giving related results and analogues of [2], see, for example, [1], [5], [6] and [7]. In Lemmas 2.1–2.2 of [4], we combined methods of [2] with an argument of McLeod [8] to prove an existence theorem for a particular second order functional-differential equation arising in fluid mechanics. In view of the importance of the problem (1.1), it seems worthwhile to extract general results from the methods of [4].

2. Two lemmas. An a priori bound for $|x(t)|$ will be obtained from the following comparison theorem.

LEMMA 2.1. *Assume that* (a) $H(t, u, v) \in C^0([a, b] \times \{u > 0\} \times \mathbb{R}^1)$ *has the property that there exist a constant* $c^0 > 0$ *and functions* $\eta(t_0, \sigma_0, \tau_0, v_0) > 0$ *and* $H_0(t_0, \sigma_0, \tau_0, v_0) \geqq 0$, *where* $a < t_0 < b, \tau_0 > c^0\sigma_0 > 0$ *and* $-\infty < v_0 < \infty$, *such that*

$$\frac{H(t, \tau, \tau v_2)}{\tau} - \frac{H(t, \sigma, \sigma v_1)}{\sigma} \geqq -H_0|v_1 - v_2| \tag{2.1}$$

when $|t - t_0| < \eta$, $|\sigma - \sigma_0| < \eta$, $|\tau - \tau_0| < \eta$, $|v_1 - v_0| < \eta$, $|v_2 - v_0| < \eta$ *(cf. the remark below)*;

(b) *there exists* $w(t)$ *such that* $0 < w(t) \in C^0[a, b] \cap C^2(a, b)$ *and*

$$w'' \leqq H(t, w, w') \quad \textit{for } a < t < b. \tag{2.2}$$

Let $u(t) \in C^0[a, b] \cap C^2(a, b)$,

$$c_0 = \max\left(\frac{u(a)}{w(a)}, \frac{u(b)}{w(b)}, c^0\right) > 0, \tag{2.3}$$

$$u'' \geqq H(t, u, u') \quad \textit{whenever } u(t) > c_0 w(t). \tag{2.4}$$

Then $u(t) \leqq c_0 w(t)$ *for* $a \leqq t \leqq b$.

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Remark. The condition concerning the existence of η, H_0 and (2.1) holds if

$$\frac{H(t,\tau,\tau v)}{\tau} - \frac{H(t,\sigma,\sigma v)}{\sigma} > 0 \quad \text{for } a < t < b, \quad \tau > c^0\sigma, \quad -\infty < v < \infty. \tag{2.5}$$

It holds, for example, with $c^0 = 1$ if either $H(t, \tau, \tau v)/\tau$ is a strictly increasing function of $\tau > 0$ for fixed $(t, v) \in (a, b) \times \mathbb{R}^1$ or if $H(t, \tau, \tau v)/\tau$ is a nondecreasing function of $\tau > 0$ and $H(t, u, v)$ is locally uniformly Lipschitz continuous in v. The function $H(t, u, v) = -[1 + (2u)^{1/2} + |v|]$ occurring in [7] satisfies these conditions. The use of such functions H in [1], [7] is quite different from that below.

A simpler result below could be used in place of Lemma 2.1.

PROPOSITION 2.1. *Assume that* (a) $H(t, u, v) \in C^0([a, b] \times \{u \geqq 0\} \times \mathbb{R}^1)$ *has the property that there exist functions* $\eta(t_0, \sigma_0, \tau_0, v_0) > 0$ *and* $H_0(t_0, \sigma_0, \tau_0, v_0) \geqq 0$, *where* $a < t_0 < b, \tau_0 > \sigma_0$ *and* $-\infty < v_0 < \infty$, *such that*

$$H(t,\tau,v_2) - H(t,\sigma,v_1) \geqq -H_0|v_2 - v_1| \tag{2.1*}$$

when $|t - t_0| < \eta, |\sigma - \sigma_0| < \eta, |\tau - \tau_0| < \eta, |v_1 - v_0| < \eta$ *and* $|v_2 - v_0| < \eta$;

(b) *there exists* $w(t), 0 \leqq w(t) \in C^0[a, b] \cap C^2(a, b)$, *satisfying* (2.2).

Let $u(t) \in C^0[a, b] \cap C^2(a, b)$ *satisfy*

$$u(a) \leqq w(a) \quad \text{and} \quad u(b) \leqq w(b), \tag{2.3*}$$

$$u'' \geqq H(t,u,u') \quad \text{whenever } u(t) > w(t). \tag{2.4*}$$

Then $u(t) \leqq w(t)$ *for* $a \leqq t \leqq b$.

The proof is similar to that of Lemma 2.1 with $r(t) = u(t) - w(t)$, and will be omitted.

Proof of Lemma 2.1. Let $r(t) = u(t)/w(t)$. If the conclusion $r(t) \leqq c_0$ on $[a, b]$ does not hold, then $r(t)$ has a maximum $r(t_0) > c_0 \geqq c^0$ at some point $t_0, a < t_0 < b$. It can be supposed that $r(t) < r(t_0)$ on $[a, t_0)$. In a vicinity of t_0, (2.2), (2.4) and $r' = (wu' - w'u)/w^2$ give

$$w^2r'' + 2ww'r' = (wu' - w'u)' \geqq uw[H(t,u,u')/u - H(t,w,w')/w].$$

If $v_2 = u'/u$ and $v_1 = w'/w$, then $r'(t_0) = 0$ implies $v_2(t_0) = v_1(t_0)$, so that

$$w^2r'' + 2ww'r' \geqq \gamma(t)r' \tag{2.6}$$

near $t = t_0$, by (2.1), where $\gamma(t) = -w^2H_0 \operatorname{sgn}(u'/u - w'/w)$ and $H_0 = H_0(t_0, w(t_0), u(t_0), w'(t_0)/w(t_0), u'(t_0)/u(t_0))$. The function $\gamma(t)$ is bounded and measurable. Hence, the inequality (2.6) and the fact that $r(t)$ has a maximum at $t = t_0$ implies that $r(t)$ is a constant in a vicinity of $t = t_0$. But this contradicts $r(t) < r(t_0)$ on $[a, t_0)$ and completes the proof.

After obtaining an a priori bound for $|x(t)|$, we can obtain bounds for $|x'(t)|$, for example, by the methods of [2] or of [1], [7]. The former gives the following which involves a Nagumo type of condition

$$|x| \leqq R, \quad |x''| \leqq \phi(|x'|) \quad \text{and} \quad 0 < \phi(s) \in C^0[0,\infty), \quad \int^\infty s\,ds/\phi = \infty; \tag{2.7}$$

cf. [3, Exercise 5.1, p. 431].

LEMMA 2.2. *Let* $p > 0$, $R > 0$, $K_1 > 0$ *be constants*, $b - a \geqq p$, *and* $\rho(t) \in C^2[a, b]$ *satisfy* $|\rho(t)| \leqq K_1$. *Then there exists a number* $M = M[p, R, K_1, \phi]$ *with the property that* $x(t) \in C^2[a, b]$, (2.7) *and*

$$|x| \leqq R \quad and \quad |x''| \leqq \rho'', \tag{2.8}$$

imply $|x'(t)| \leqq M$ *for* $a \leqq t \leqq b$.

In applications (cf. [2] or [3, pp. 434–435] or [4]), it is useful for M to be independent of $b - a$, hence the introduction of the parameter p.

When $d = \dim x = 1$, condition (2.8) can be omitted; Nagumo, cf. [3, p. 428]. As observed in [1] and [7], one obtains an easy, but still useful, result if one assumes a bound for $|\rho'(t)|$ rather than for $|\rho(t)|$. We state this assertion which does not involve the Nagumo condition (2.7) (and we replace ρ by ρ') as follows.

PROPOSITION 2.2. *Let* $R > 0$, $K_0 > 0$ *be constants and* $\rho(t) \in C^1[a, b]$ *satisfy* $|\rho(t) \leqq K_0$. *Then the number* $M = 2K_0 + 2R/(b - a)$ *has the property that* $x(t) \in C^2[a, b]$ *and*

$$|x| \leqq R \quad and \quad |x''| \leqq \rho' \tag{2.9}$$

imply $|x'(t)| \leqq M$ *for* $a \leqq t \leqq b$.

3. Existence theorems. We shall use a Lyapunov type function $E(t, x)$, more general than $|x|^2$, as in [1], [4] and [7].

THEOREM 3.1. *Assume that* (A) $f(t, x, x') \in C^0([a, b] \times \mathbb{R}^{2d})$;

(B) $H(t, u, v)$ *satisfies conditions* (a), (b) *of Lemma* 2.1;

(C) $E(t, x) \in C^2([a, b] \times \mathbb{R}^d)$ *satisfies*

$$E(t, x) \to \infty \quad as\, |x| \to \infty \quad uniformly\ in\ t \in [a, b], \tag{3.1}$$

$$c_0 = \max\left(\frac{E(a, x_a)}{w(a)}, \frac{E(b, x_b)}{w(b)}, c^0\right) > 0, \tag{3.2}$$

$$E''_\delta \geqq H(t, E, E') \quad for\ 0 \leqq \delta \leqq 1, \quad whenever\ E(t, x) > c_0 w(t), \tag{3.3}$$

where $E' = E'(t, x, x')$ *and* $E''_\delta = E''_\delta(t, x, x', \delta)$ *are given by* $E' = E_t + E_x \cdot x'$ *and*

$$E''_\delta = E_{tt} + 2E_{tx} \cdot x' + E_{xx} x' \cdot x' + \delta E_x \cdot f;$$

(D) *finally*, f *satisfies a Nagumo condition*

$$|f(t, x, x')| \leqq \phi(|x'|) \quad for\ a \leqq t \leqq b, \quad |x| \leqq R, \quad x' \in \mathbb{R}^d,$$

where $0 < \phi \in C^0[0, \infty]$, $\int^\infty s\, ds/\phi = \infty$, *and for some constants* $K \geqq 0$ *and* $\alpha > 0$,

$$E''_\delta \geqq -K + \alpha\delta |f| \quad for\ a \leqq t \leqq b, \quad |x| \leqq R, \quad x' \in \mathbb{R}^d, \quad 0 \leqq \delta \leqq 1, \tag{3.4}$$

where R *is the constant given by Lemma* 3.1 *below*.

Then (1.1) *has at least one solution*.

Remark 1. When $H \leqq 0$ and

$$U(t, x) \equiv E_{tt} + 2E_{tx} \cdot x' + E_{xx} x' \cdot x' \geqq 0, \tag{3.5}$$

it is sufficient to assume (3.3) with $\delta = 1$ (cf., [2], [3, p. 432] or [1]), for then $E''_\delta = \delta E''_1 + (1 - \delta)U \geqq \delta E''_1 \geqq \delta H \geqq H$. Also, when (3.5) holds, it suffices to assume (3.4) for $\delta = 1$.

Remark 2. Condition (3.4) can be omitted when $d = \dim x = 1$ or when every component f^k of $f = (f^1, \cdots, f^d)$ satisfies $|f^k| \leqq \phi(|x^{k'}|)$ for $a \leqq t \leqq b$, $|x| \leqq R$, $x' \in \mathbb{R}^d$. In general, E in (3.4) can be replaced by another function $E^0 = E^0(t, x) \in C^2([a, b] \times \mathbb{R}^d)$. In some theorems of [1] and [7], conditions on (3.3) are strengthened to permit an estimate for $|E'|$ (as well as E); in which case, the Nagumo condition can be omitted and an appeal made to Proposition 2.2 (instead of Lemma 2.2). For example, if $H(t, u, v)$ is nonincreasing in u, $E(t, x(t)) \leqq R_0$ (as in Lemma 3.1), and (3.3) holds for all (t, E, E'), then we obtain a first order differential inequality $E''_\delta \geqq H(t, R_0, E')$ for E' which might be useful in estimating $|E'|$; cf., [1, Lemma 3].

LEMMA 3.1. *Assume conditions* (A), (B), (C) *of Theorem* 3.1. *Let* $\delta(t, x, x') \in C^0([a, b] \times \mathbb{R}^{2d})$, $0 \leqq \delta \leqq 1$, *and let* $x(t)$ *be a solution of*

$$x'' = \delta(t, x, x')f(t, x, x') \quad \text{and} \quad x(a) = x_a, \quad x(b) = x_b. \tag{3.6}$$

Then $E(t, x(t)) \leqq c_0 w(t)$ *for* $a \leqq t \leqq b$. *If* $R_0 = c_0 \max w(t)$, *then there exists an* $R = R(R_0)$ *such that*

$$E(t, x) \leqq R_0 \Rightarrow |x| \leqq R, \tag{3.7}$$

by (3.1); *hence* $|x(t)| \leqq R$ *for* $a \leqq t \leqq b$.

Proof. If $u(t) = E(t, x(t))$, then (2.4) follows from (3.3) and (3.6), and the result is a consequence of Lemma 2.1.

LEMMA 3.2. *Assume the conditions of Theorem* 3.1 (*including* (D)) *and of Lemma* 3.1, *and* $b - a \geqq p > 0$. *Then there exists a constant* $M = M(p, \alpha, K, R_0, \phi)$ *such that* $|x'(t)| \leqq M$ *for* $a \leqq t \leqq b$.

Proof. This follows from Lemma 2.2 with $\rho(t) = [E(t, x(t)) + K(t - s)^2/2]/\alpha$; cf., (2.8) and (3.4). Note, that for suitable choices of s, $|\rho(t)| \leqq (R_0 + Kp^2/8)/\alpha$ on subintervals of $[a, b]$ of length p.

Proof of Theorem 3.1. We follow the procedure of [2]. Let $\delta(t, x, x') \in C^0([a, b] \times R^{2d})$ satisfy $0 \leqq \delta \leqq 1$, $\delta \equiv 1$ if $|x| \leqq R$ and $|x'| \leqq M$, while $\delta \equiv 0$ if $|x| \geqq R + 1$ or $|x'| \geqq M + 1$. Then $\delta(t, x, x')f(t, x, x')$ is bounded. Hence (3.6) has a solution $x(t)$ by a theorem of Scorza–Dragoni; cf., [3, Theorem 4.2, p. 424]. But, in view of Lemmas 3.1 and 3.2, this solution satisfies $|x| \leqq R$, $|x'| \leqq M$. Hence $\delta(t, x(t), x'(t)) \equiv 1$ and so, $x(t)$ is a solution of (1.1). This completes the proof.

Because of condition (b) in Lemma 2.1 and its analogues, Theorem 3.1 and the results of [1], [7] may not be as useful as those of [2] in dealing with singular boundary value problems on $0 \leqq t < \infty$. No condition of the type (b) appears in [4]. We give an analogous result here. We replace x by a $(d + 1)$-vector (x, y), where $x \subset \mathbb{R}^d$, $y \in \mathbb{R}^1$, and consider a boundary value problem

$$x'' = f(t, x, y, x', y'), \qquad y'' = g(t, x, y, x', y'), \tag{3.8}$$

$$x(a) = x_a, \quad y(a) = y_a > 0 \quad \text{and} \quad x(b) = x_b, \quad y(b) = y_b > 0. \tag{3.9}$$

THEOREM 3.2. *Let* $d = \dim x \geqq 1$ *and* $\dim y = 1$. *Assume that*

(A) $f, g \in C^0([a, b] \times \mathbb{R}^{2d+2})$;

(B) *if* $x(t) \in C^1[a, b]$ *and* $\delta(t) \in C^0[a, b]$, $0 \leqq \delta \leqq 1$, *are arbitrary, then a solution of*

$$y'' = \delta(t)g(t, x(t), \max(0, y), x'(t), y') \tag{3.10}$$

on $[a, b]$ *satisfying* $y(a) > 0$, $y(b) > 0$ *cannot vanish on* $[a, b]$ (*e.g., suppose that* $g(t, x, 0, x', y') \leqq g_0(t, x, x', y')y'$ *for some continuous* g_0 *for small* $|y'|$);

(C) $E(t, x, y)$, $D(t, x, y) \in C^2([a, b] \times \mathbb{R}^d \times \{y > 0\})$ *are such that* $D > 0$, *there exist constants* $c_0 > 0$ *and* $R > 0$, *satisfying*

$$c_0 \geqq \max\left(\frac{E(a, x_a, y_a)}{D(a, x_a, y_a)}, \frac{E(b, x_b, y_b)}{D(b, x_b, y_b)}\right), \tag{3.11}$$

$$E/D \leqq c_0 \Rightarrow |x| \leqq R, |y| \leqq R, \tag{3.12}$$

and there exist continuous functions $\eta(\delta, t, \sigma, \tau, v) > 0$, $H_0(\delta, t, \sigma, \tau, v_1, v_2) \geqq 0$ *for* $0 \leqq \delta \leqq 1$, $a < t < b, \tau > c_0\sigma > 0$, $-\infty < v < \infty$, $|v_1 - v_2| \leqq \eta$ *such that*

$$DE''_\delta - ED''_\delta \geqq -H_0(\delta, t, D, E, D'/D, E'/E)|D'/D - E'/E| \tag{3.13}$$

whenever $E > c_0 D > 0$ *and* $|D'/D - E'/E| < \eta(\delta, t, D, E, D'/D)$ *where* $E' = E_t + E_x \cdot x' + E_y y'$,

$$E''_\delta = E_{tt} + 2E_{tx} \cdot x' + 2E_{ty}y' + E_{xx}x' \cdot x' + 2E_{xy} \cdot x'y + E_{yy}y'^2 + \delta(E_x \cdot f + E_y g),$$

and D', D''_δ *are similarly defined*;

(D) *finally*, f, g *satisfy a Nagumo condition*

$$|f|, |g| \leqq \phi(|x'| + |y'|) \quad \text{for } a \leqq t \leqq b, \quad |x| \leqq R, \quad |y| \leqq R, \tag{3.14}$$

where $0 < \phi \in C^0[0, \infty)$, $\int^\infty s\, ds/\phi = \infty$, *and for some constants* $K \geqq 0$ *and* $\alpha > 0$,

$$E''_\delta \geqq -K + \alpha\delta(|f| + |g|) \quad \text{for } a \leqq t \leqq b, \quad |x| \leqq R, \quad |y| \leqq R, \tag{3.15}$$

$0 \leqq \delta \leqq 1$, *and arbitrary* (x', y').

Then (3.8)–(3.9) *has at least one solution* $(x(t), y(t))$ [*satisfying* $y(t) > 0$ *and* $E(t, x(t), y(t)) \leqq c_0 D(t, x(t), y(t))$].

Remarks analogous to those following Theorem 3.1 are applicable here. The proof is similar to that of Lemmas 2.1–2.2 of [4] and of Theorem 3.1, and we only indicate it.

Proof. Let $\delta(t, x, y, x', y') \in C^0([a, b] \times R^{2d+2})$, $0 \leqq \delta \leqq 1$, be arbitrary and let $x(t)$, $y(t)$ be a solution of

$$\begin{aligned} x'' &= \delta(t, x, y, x', y')f(t, x, y, x', y'), \\ y'' &= \delta(t, x, y, x', y')g(t, x, \max(0, y), x', y') \end{aligned} \tag{3.16}$$

satisfying the boundary conditions (3.9). Then $y(t) > 0$ by condition (B). Hence $(x(t), y(t))$ is a solution of (3.16) even if $\max(0, y)$ is replaced by y. By condition (C), $E(t, x(t), y(t)) \leqq c_0 D(t, x(t), y(t))$ for $a \leqq t \leqq b$, and so $|x(t)| \leqq R$, $|y(t)| \leqq R$; cf. the proofs of Lemmas 2.1 and 3.1. By condition (D), there exists a constant M, independent of the function δ, such that $|x'(t)| \leqq M$, $|y'(t)| \leqq M$; cf. the proofs of Lemmas 2.2 and 3.2. The proof can now be completed as above.

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THE SOLUTION OF LINEAR EQUATIONS IN NORMED SPACES BY AVERAGING ITERATION*

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Abstract. Averaging iterations are applied to the approximate solution of the linear equation $(I - T)x = f$ in a normed space. The theory obtained by Kwon and Redheffer for the Picard iteration is extended and generalized. Three cases of averaging iteration suitable for numerical computation are introduced, and the convergence subspace associated with an averaging iteration is discussed.

1. Averaging iterations in normed spaces. Suppose T is a linear operator on a normed linear space X. Kwon and Redheffer [9] investigated the Picard iteration $x_{n+1} = Tx_n + f$ for the approximate solution of the equation

$$(I - T)x = f \tag{1}$$

assuming nothing more than the continuity of T on X. Our objective is to extend these results to the general averaging iteration introduced by Dotson [2], [4] without imposing severe restrictions on the operator T (such as asymptotic A-boundedness or asymptotic A-regularity) adopted in [2], [4] and [7].

1.1. An infinite real matrix $A = [a_{nj}]$, where $n, j \geqq 0$, will be called *admissible* [7] if A is nonnegative, lower triangular with each row summing to 1. Following Dotson [2], [4] we define the polynomials

$$a_n(t) = \sum_{j=0}^{n} a_{nj}t^j, \qquad b_n(t) = \frac{1 - a_n(t)}{1 - t}, \qquad n = 0, 1, \cdots,$$

whose coefficients are based on the entries of an admissible matrix A. Suppose T is a continuous linear operator on X. Define linear operators $A_n = a_n(T)$ and $B_n = b_n(T)$ for each $n \geqq 0$. Clearly, $A_n \in \operatorname{co}\{T^j : j = 0, \cdots, n\}$ and $B_n \in \operatorname{sp}\{T^j : j = 0, \cdots, n - 1\}$, where co and sp denote the convex and linear hull respectively. Furthermore, A_n and B_n commute with T, and

$$(I - T)B_n = B_n(I - T) = I - A_n, \qquad n = 0, 1, \cdots. \tag{2}$$

We consider the approximate solution of (1) by means of the averaging iteration [2], [4]

$$x_n = A_n x_0 + B_n f, \qquad x_0 \text{ given}. \tag{3}$$

If A is the infinite unit matrix I, (3) reduces to the Picard iteration $x_n = T^n x_0 + (\sum_{j=0}^{n-1} T^j)f$. Note that any solution x of (1) is a stationary point for the iteration (3) in the sense that $x_n = x$ for each $n \geqq 0$ when $x_0 = x$. Indeed, if $(I - T)x = f$ for some $x \in X$, then $x_n = A_n x + B_n(I - T)x = A_n x + (I - A_n)x = x$. Let Q be the operator defined by

$$\lim_{n \to \infty} A_n x = Qx, \tag{4}$$

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whenever the limit exists in the strong topology of X. In the sequel, the symbols $D(U)$, $N(U)$ and $R(U)$ will be used to denote the domain, null space and range of an operator U in X respectively. Since $A_nT = TA_n$ for each $n \geqq 0$, $\{A_nTx\}$ converges strongly whenever $\{A_nx\}$ converges strongly, i.e. $D(Q)$ is invariant under T, and $TQx = QTx$ for all $x \in D(Q)$. We shall require that the matrix A and the operator T satisfy

$$(5) \qquad TQx = QTx = Qx \quad \text{for each } x \in D(Q),$$

or the more restrictive condition

$$(6) \qquad A_nx_0 + B_nf \to x \quad \text{implies} \quad x = Tx + f \quad \text{for any } x_0 \text{ and } f \in X,$$

where "$\to$" is used to denote strong convergence in X. It is easily verified that (6) implies (5). If A is the infinite unit matrix I or the Cesàro matrix C, (6) holds for each continuous linear operator T on X. Indeed, the sequence $\{x_n\} = \{A_nx_0 + B_nf\}$ satisfies the inductive formula $x_n = Tx_{n-1} + f$ if $A = I$, and the formula $x_n = n(n+1)^{-1}Tx_{n-1} + (n+1)^{-1}x_0 + n(n+1)^{-1}f$ if $A = C$. The result follows on letting $n \to \infty$ in these formulas.

LEMMA 1.2. *Let A be an admissible matrix and T a continuous linear operator on X satisfying* (5). *Then*: (a) *$QA_nx = Qx$ for each $x \in D(Q)$ and each $n \geqq 0$*, (b) *$Q^2x = Qx$ for each $x \in D(Q)$*, (c) *for each $x \in D(Q)$ and each $n \geqq 0$, $QB_nx = p_nQx$, where $\{p_n\}$ is a real sequence with the property that $\lim_n p_n = +\infty$ whenever $\lim_n a_{nj} = 0$ for each $j \geqq 0$.*

Proof. (a) We deduce from (5) that $QT^jx = Qx$ for each $x \in D(Q)$ and each $j \geqq 0$. Hence $QA_nx = \sum_{j=0}^n a_{nj}QT^jx = \sum_{j=0}^n a_{nj}Qx = Qx$ for each $n \geqq 0$ as $\sum_{j=0}^n a_{nj} = 1$. (b) follows immediately from (a). To prove (c), we observe that

$$B_n = b_n(T) = \sum_{j=1}^{n} a_{nj} \sum_{i=0}^{j-1} T^i.$$

For each $x \in D(Q)$ and each $n \geqq 0$, $QB_nx = (\sum_{j=0}^n ja_{nj})Qx$. Setting $p_n = \sum_{j=0}^n ja_{nj}$, we obtain $QB_nx = p_nQx$. Suppose $\lim_n a_{nj} = 0$ for each $j \geqq 0$. Given a sequence $\{s_n\}$ of real numbers, define $\{t_n\}$ by $t_n = \sum_{j=0}^n a_{nj}s_j$. Since the matrix A satisfies the Toeplitz conditions, $t_n \to s$ whenever $s_n \to s$ [5, p. 75]. Let K be an arbitrary real number. For any integer N, $N > 2K$, the sequence $\{t_n\}$ associated with $\{s_n\} = \{1, 2, \cdots, N-1, N, N, N, \cdots\}$ converges to N. Hence there exists a positive integer n_0, $n_0 > N$, such that $p_n \geqq t_n > K$ for all $n > n_0$. This proves $\lim_n p_n = +\infty$.

The preceding lemma contains Theorem 2 of [2] as a special case when X is a Banach space, and $D(Q) = X$.

LEMMA 1.3. *Suppose the condition* (5) *is satisfied. Then*

$$D(Q) = R(Q) \oplus N(Q), \quad R(Q) = N(I - T), \quad N(Q) \subset R(I - T)^-,$$

where bar indicates closure in X.

Proof. According to Lemma 1.2(b), $Q: D(Q) \to D(Q)$ and $Q^2 = Q$. Hence $D(Q) = R(Q) \oplus N(Q)$. If $u \in N(I - T)$, $Tu = u$, and $A_nu = u$ for all $n \geqq 0$. Thus $Qu = u$, and $u \in R(Q)$. If, conversely, $u \in R(Q)$, $u = Qx$ for some $x \in D(Q)$, and $(I - T)u = (I - T)Qx = 0$ in view of (5). To establish $N(Q) \subset R(I - T)^-$, suppose $Qx = 0$. Then $x = x - Qx = \lim_n (I - A_n)x = \lim_n (I - T)B_n x$ by (2), and x lies in the closure of $R(I - T)$.

In the case when X is a Banach space, Koliha [7] has proved that $N(Q) = R(I - T)^-$ under the additional hypotheses that the sequence $\{\|A_n\|\}$ is bounded and that $(I - T)A_n x \rightharpoonup 0$ for each $x \in X$ ("$\rightharpoonup$" denotes weak convergence in X).

PROPOSITION 1.4. *Suppose* $(I - T)y = f$. *Then the sequence* $\{x_n\}$ *in* (3) *converges strongly if and only if* $x_0 - y \in D(Q)$. *If, in addition,* (5) *is fulfilled, the strong limit* x *of* $\{x_n\}$ *is a solution of* (1).

Proof. If $(I - T)y = f$, y is a stationary point for the iteration (3), i.e., $y = A_n y + B_n f$ for all $n \geqq 0$, and $x_n - y = A_n(x_0 - y)$. Hence $\{x_n\}$ converges strongly to $x = Q(x_0 - y) + y$ if and only if $x_0 - y \in D(Q)$. If this is the case and if (5) holds, $(I - T)x = (I - T)Q(x_0 - y) + (I - T)y = f$.

Remarks 1.5. (a) Setting $A = I$ in the preceding proposition, we obtain Remark 1 of [9].

(b) Suppose $\{\|A_n\|\}$ is bounded and suppose $(I - T)A_n x \to 0$ for each $x \in X$. Then it follows from Eberlein's mean ergodic theorem [5] that $x \in D(Q)$ if and only if $\{A_n x\}$ has a weak cluster point. The condition (5) is clearly satisfied. If $f = (I - T)y$ and if $\{x_n\}$ has a weak cluster point, also $\{A_n(x_0 - y)\} = \{x_n - y\}$ has a weak cluster point, and $x_0 - y \in D(Q)$. Then $\{x_n\}$ converges strongly to a solution x of (1) in accordance with 1.4. Thus we have generalized Theorem 2 of [4].

(c) Proposition 1.4 has been proved in [7] under the hypotheses that X is a Banach space, $\{\|A_n\|\}$ is bounded, and $(I - T)A_n x \rightharpoonup 0$ for each $x \in X$.

PROPOSITION 1.6. *Suppose the condition* (6) *is satisfied. If* $\{x_n\}$ *converges strongly*, $(I - T)x_0 - f \in N(Q)$.

Proof. If $x_n \to x$, $(I - T)x = f$ by (6). Moreover, $(I - T)x_n = A_n((I - T)x_0 - f) + f$ as follows from (2), and $A_n((I - T)x_0 - f) \to 0$.

The following proposition generalizes Remark 2 of [9]; apart from considering a general admissible matrix A in place of I, the proposition replaces the strong cluster point considered in [9] by a weak cluster point.

PROPOSITION 1.7. *Let* $(I - T)x_0 - f \in N(Q)$. *If* $\{x_n\}$ *has a weak cluster point* x, $f \in R(I - T)$, *and* x *is a solution of* (1).

Proof. Define a sequence $\{y_n\}$ by $y_n = A_n x_1 + B_n f$, where $x_1 = A_1 x_0 + B_1 f = a_{10}x_0 + a_{11}Tx_0 + a_{11}f$. Then $x_n - y_n = A_n(x_0 - x_1) = a_{11}A_n((I - T)x_0 - f)$. Suppose $x_n \rightharpoonup x$ as $n = n_j \to \infty$. Then also $y_n \rightharpoonup x$ as $n = n_j \to \infty$ in view of the hypothesis $(I - T)x_0 - f \in N(Q)$. After a short calculation based on (2), we obtain $y_n - Tx_n = a_{10}A_n((I - T)x_0 - f) + f$. Passing to the weak limit as $n = n_j \to \infty$, we obtain $x - Tx = f$ since T is strongly (and also weakly) continuous and since the weak topology of X is Hausdorff.

As an application of Proposition 1.7, consider the case when $A_n x \to 0$ for each $x \in X$. Then $D(Q) = N(Q) = X$, $N(I - T) = \{0\}$ and $R(I - T)^- = X$ in view of Lemma 1.3. An element $f \in X$ belongs to $R(I - T)$ if and only if $\{x_n\}$ has a weak cluster point x for some $x_0 \in X$; x is then a solution of (1).

An infinite matrix $A = [a_{nj}]$ will be called *Toeplitz* if A is admissible and if $\lim_n a_{nj} = 0$ for each $j \geqq 0$.

PROPOSITION 1.8. *Suppose* A *is a Toeplitz matrix satisfying the condition* (5). *Let* f *and* $x_0 \in D(Q)$, *and suppose* 0 *is a weak cluster point of* $\{Qx_n/p_n\}$, *where* $\{p_n\}$ *is the sequence described in Lemma* 1.2(c). *Then* $f \in N(Q)$, *and* $(I - T)x_0 - f \in N(Q)$.

Proof. Since $D(Q)$ is invariant under each A_n and B_n, $x_n \in D(Q)$ for each $n \geqq 0$, whenever x_0 and $f \in D(Q)$. Hence $Qx_n = QA_n x_0 + QB_n f = Qx_0 + p_n Qf$. Suppose $Qx_n/p_n \rightharpoonup 0$ as $n = n_j \to \infty$. Then $Qx_n/p_n = Qx_0/p_n + Qf \to Qf$ as $n \to \infty$ in virtue of Lemma 1.2(c), and also $Qx_n/p_n \rightharpoonup Qf$ as $n = n_j \to \infty$. Thus $Qf = 0$ as the weak topology of X is Hausdorff. Finally, $Q((I - T)x_0 - f) = Q(I - T)x_0 - Qf = 0$, as follows from (5).

Remarks 1.9. (a) The preceding proposition has been proved in Remark 3 of [9] for $A = I$ under the hypothesis that 0 is a strong cluster point of $\{Qx_n/n\}$.

(b) Suppose A is a Toeplitz matrix satisfying (5), and suppose Q is continuous on $D(Q) = X$. If x is a weak cluster point of $\{x_n\}$, 0 is a weak cluster point of $\{Qx_n/p_n\}$, and $(I - T)x_0 - f \in N(Q)$ by 1.8; Proposition 1.7 shows that $(I - T)x = f$, and Proposition 1.4 implies that $x_n \to x$. Hence we have generalized Theorem 4(b) of [2].

PROPOSITION 1.10. *Suppose* (6) *is satisfied. Then the sequence* $\{B_n f\}$ *converges strongly if and only if* $f \in (I - T)D(Q) = (I - T)N(Q)$.

Proof. Let $(I - T)x = f$ for some $x \in D(Q)$. Then $B_n f = B_n(I - T)x = (I - A_n)x \to (I - Q)x$. If, conversely, $\{B_n f\}$ converges strongly to x, $(I - A_n)f = (I - T)B_n f \to (I - T)x$, and $(I - T)x = f$ in view of (6). Setting $x_0 = 0$ in Proposition 1.4, we obtain $x \in D(Q)$. The equality $(I - T)D(Q) = (I - T)N(Q)$ follows from Lemma 1.3. Note that the sequence $\{A_x x_0 + B_n f\}$ with $f \in (I - T)D(Q)$ converges strongly if and only if $x_0 \in D(Q)$.

Remark 5 (and consequently also Remark 4) of [9] are obtained by setting $A = I$ in Proposition 1.10. Proposition 2 of [7] establishes the equivalence "$\{B_n f\}$ converges strongly if and only if $f \in (I - T)N(Q)$" under more restrictive conditions on X, A and T, but without the hypothesis (6).

PROPOSITION 1.11. *Suppose* $I - T$ *has a continuous inverse* $(I - T)^{-1}$ *on the subspace* $R(I - T)$, *which is assumed to be closed. If* $f \in N(Q)$, $B_n f \to y$, *where* y *is the unique solution of* (1) *in* $N(Q)$.

Proof. The existence of $(I - T)^{-1}$ on $R(I - T)$ implies that $N(I - T) = \{0\}$, so that $D(Q) = N(Q) \oplus N(I - T) = N(Q)$, and $N(Q) \subset R(I - T)$ in view of Lemma 1.3 and the hypothesis $R(I - T)^- = R(I - T)$. Suppose $f \in N(Q)$. Then $(I - T)B_n f = (I - A_n)f \to f$, and the sequence $\{B_n f\} = \{(I - T)^{-1}(I - A_n)f\}$ converges strongly to $y = (I - T)^{-1}f$. Proposition 1.4 with $x_0 = 0$ yields $y \in N(Q)$.

Suppose X is a Banach space and suppose $I - T$ admits a left inverse $(I - T)^{-1}$ which is bounded on $R(I - T)$. Then $R(I - T)$ is closed and (1) with $f \in N(Q)$ has a unique solution $y \in N(Q)$. Thus we have generalized Remark 6 of [9].

2. Special cases. As in the first part of the paper, X is a linear normed space, and T a continuous linear operator on X. The general averaging iteration (3) is rather impractical for the actual computation of the successive approximations x_n. In fact, each x_n is completely independent of the preceding approximations x_j, where $1 \leqq j \leqq n - 1$, and all the elements $x_0, Tx_0, \cdots, T^n x_0, f, Tf, \cdots, T^{n-1}f$ are needed for its construction. In the case when T is an $m \times m$ matrix with large m, the iteration (3) in its general form can be hardly used for the approximate solution of the equation $(I - T)x = f$ by a computer as the requirements on the memory are enormous. We suggest three special cases of the averaging iteration that seem to be particularly suitable for numerical computation.

PROPOSITION 2.1. *Let* $\{c_n\}_0^\infty$ *be a sequence of real numbers satisfying* $0 \leqq c_n \leqq 1$. *Each of the iterations*

$$x_{n+1} = (1 - c_n)Tx_n + c_n x_0 + (1 - c_n)f, \qquad x_0 \text{ given}, \tag{7}$$

$$y_{n+1} = (1 - c_n)Ty_n + c_n Ty_0 + f, \qquad y_0 \text{ given}, \tag{8}$$

$$z_{n+1} = (1 - c_n)Tz_n + c_n z_n + (1 - c_n)f, \qquad z_0 \text{ given}, \tag{9}$$

is an averaging process as defined in (3). *If* $\lim_n c_n = 0$, *each of the processes* (7), (8), (9) *is based on a Toeplitz matrix and satisfies the condition* (6).

Let us remark that if $c_n = 0$ for all $n \geqq 0$, (7), (8) and (9) reduce to the Picard iteration $x_{n+1} = Tx_n + f$. Setting $c_n = (n + 1)^{-1}$ in (8), we obtain the iteration $y_{n+1} = n(n + 1)^{-1}Ty_n + (n + 1)^{-1}Ty_0 + f$ introduced and investigated by De Figueiredo and Karlovitz in [1] in connection with a modified version of the Kakutani–Yosida mean ergodic theorem. When $c_n = c$ for all $n \geqq 0$, $0 < c < 1$, (9) becomes $z_{n+1} = (1 - c)Tz_n + cz_n + (1 - c)f$, which is an iteration investigated by many authors, particularly the nonlinear case (cf. [1]). Dotson [3] studied the iteration (9) for a nonlinear operator T under the assumption that $0 < c_n \leqq 1$ for all n and that $\sum_0^\infty (1 - c_n) = +\infty$ (the so-called normal Mann process).

Proof of 2.1. Put $G_n = \operatorname{co}\{T^j : j = 0, \cdots, n\}$, $H_n = \operatorname{sp}\{T^j : j = 0, \cdots, n - 1\}$ for each $n \geqq 1$, $G_0 = \{I\}$ and $H_0 = \{0\}$.

(a) Consider the iteration (7). Put $A_0 = I$ and $B_0 = 0$. For induction assume that, for a certain $n \geqq 0$, $x_n = A_n x_0 + B_n f$ with $A_n \in G_n$ and $B_n \in H_n$ satisfying $I - A_n = (I - T)B_n$. The relations are obviously true for $n = 0$. Since

$$\begin{aligned} x_{n+1} &= (1 - c_n)T(A_n x_0 + B_n f) + c_n x_0 + (1 - c_n)f \\ &= ((1 - c_n)TA_n + c_n I)x_0 + (1 - c_n)(TB_n + I)f, \end{aligned}$$

we have $x_{n+1} = A_{n+1}x_0 + B_{n+1}f$, where

$$A_{n+1} = (1 - c_n)TA_n + c_n I, \qquad B_{n+1} = (1 - c_n)(TB_n + I). \tag{10}$$

A_{n+1} lies in G_{n+1} as a convex combination of TA_n and $I \in G_{n+1}$, and B_{n+1} clearly lies in H_{n+1}. Furthermore, $(I - T)B_{n+1} = (1 - c_n)(I - T)(TB_n + I) = (1 - c_n) \cdot (I - TA_n) = I - A_{n+1}$. This proves that $A_n = a_n(t)$ and $B_n = b_n(t)$ for each $n \geqq 0$, where the polynomials $a_n(t) = \sum_{j=0}^n a_{nj}t^j$ are based on the entries of an admissible matrix A, and where $b_n(t)$ are the polynomials $(1 - a_n(t))/(1 - t)$. From (10) we deduce that the entries a_{nj} of the matrix A satisfy the relations

$$a_{00} = 1; \qquad a_{n+1,0} = c_n, \qquad a_{n+1,j} = (1 - c_n)a_{n,j-1} \quad (1 \leqq j \leqq n + 1),$$

for each $n \geqq 0$. If $\lim_n c_n = 0$, also $\lim_n a_{n0} = 0$. By induction, $\lim_n a_{nj} = 0$ for each $j \geqq 1$. Hence A is Toeplitz. Suppose $x_n \to x$. Passing to the limit as $n \to \infty$ in (7), we obtain $x = Tx + f$ in view of the continuity of T and the relation $c_n \to 0$. This proves the validity of (6).

(b) Putting $A_0 = I$, $B_0 = 0$ in (8) and assuming that $y_n = A_n y_0 + B_n f$ with $A_n \in G_n$ and $B_n \in H_n$ for some $n \geqq 0$, we obtain

$$A_{n+1} = (1 - c_n)TA_n + c_n T, \qquad B_{n+1} = (1 - c_n)TB_n + I. \tag{11}$$

Then $A_{n+1} \in G_{n+1}$ and $B_{n+1} \in H_{n+1}$ whenever $A_n \in G_n$ and $B_n \in H_n$. It can be easily verified that $(I - T)B_{n+1} = I - A_{n+1}$ whenever $(I - T)B_n = I - A_n$. Hence the iteration (8) is based on an admissible matrix $A = [a_{nj}]$ with

$$a_{00} = 1; \qquad a_{10} = 0, \qquad a_{11} = 1;$$

$$a_{n+1,0} = 0, \qquad a_{n+1,1} = c_n, \qquad a_{n+1,j} = (1 - c_n)a_{n,j-1} \quad (2 \leqq j \leqq n + 1)$$

for all $n \geqq 1$. Suppose $c_n \to 0$. Then $a_{n+1,0} \to 0$, and $\lim_n a_{n+1,j} = \lim_n (1 - c_n) \cdot a_{n,j-1}$, so that $\lim_n a_{nj} = 0$ for each $j \geqq 0$ by induction, and A is Toeplitz. Clearly, (6) holds for the process (8).

(c) To prove the assertion about the process (9) we follow the same pattern as in parts (a) and (b). We find that $z_n = A_n z_0 + B_n f$, where

$$(12) \qquad A_{n+1} = ((1 - c_n)T + c_n I)A_n, \qquad B_{n+1} = ((1 - c_n)T + c_n I)B_n + (1 - c_n)I,$$

and that $A_n \in G_n$, $B_n \in H_n$ for all $n \geqq 0$. Then the process (9) is based on an admissible matrix $A = [a_{nj}]$ with $a_{00} = 1$ and

$$a_{n+1,0} = c_n a_{n0}, \qquad a_{n+1,j} = c_n a_{nj} + (1 - c_n)a_{n,j-1} \quad (1 \leqq j \leqq n),$$
$$a_{n+1,n+1} = (1 - c_n)a_{nn}$$

for all $n \geqq 0$. By induction on j we establish that $\lim_n a_{nj} = 0$ for each $j \geqq 0$, whenever $\lim_n c_n = 0$. Statement (6) is obviously true.

COROLLARY 2.2. *Let $\{c_n\}_0^\infty$ be a real sequence such that $0 \leqq c_n \leqq 1$ and $\lim_n c_n = c$. Then* (9) *is an averaging process satisfying the condition* (6). (This follows immediately from Proposition 2.1.)

All the results of § 1 apply to the averaging iterations (7), (8) and (9).

3. The subspace $D(Q)$ in a Banach space. The development of § 1 suggests the importance of the subspace $D(Q)$ for the averaging iteration (3). Our objective in this section is to establish certain relations for $D(Q)$, and to find $D(Q)$ explicitly in some special cases.

X denotes a Banach space, T a continuous linear operator on X, and A an admissible matrix. By $a_n(t)$ we denote the polynomials with coefficients based on the entries of A introduced in the first section, and by Q the operator defined by $Qx = \lim_n a_n(T)x$ whenever $\{a_n(T)x\}$ converges in norm. In addition, Q_1 denotes the projection operator associated with the infinite unit matrix, i.e., the operator defined by $Q_1 x = \lim_n T^n x$ whenever $\{T^n x\}$ converges in norm. The spectral properties of operators used in this section can be found in [5, Chap. VII] or in [11, Chap. 5].

PROPOSITION 3.1. *Let $A = [a_{nj}]$ be an admissible matrix such that $\lim_n a_{nj}$ exists for each $j \geqq 0$. Then*

$$D(Q_1) \subset D(Q).$$

Proof. Suppose x is an element of $D(Q_1)$ with $T^n x \to z$. Then $\|T^n x - z\| \leqq K$ for some $K > 0$ and all $n \geqq 0$. Let ε be a real positive number, and N a fixed positive integer such that $\|T^n x - z\| < \varepsilon$ for all $n > N$. There exists an integer p, $p > N$, such that the inequality $|a_{mj} - a_{nj}| < \varepsilon/(N + 1)$ is satisfied for all m,

$n > p$ and all j with $0 \leqq j \leqq N$. If $m, n > p$, then

$$\begin{aligned}
&\|a_n(T)x - a_m(T)x\| \\
&\quad = \left\| \sum_{j=0}^{n} a_{nj}T^jx - \sum_{j=0}^{m} a_{mj}T^jx \right\| \\
&\quad = \left\| \sum_{j=0}^{n} a_{nj}(T^jx - z) - \sum_{j=0}^{m} a_{mj}(T^jx - z) \right\| \\
&\quad \leqq \sum_{j=0}^{N} |a_{nj} - a_{mj}|\, \|T^jx - z\| + \sum_{j=N+1}^{n} a_{nj}\|T^jx - z\| + \sum_{j=N+1}^{m} a_{mj}\|T^jx - z\| \\
&\quad < K\varepsilon + \varepsilon + \varepsilon = (K+2)\varepsilon,
\end{aligned}$$

and $\{a_n(T)x\}$ converges in norm in the Banach space X. Observe that in the case when A is Toeplitz, i.e., when $\lim_n a_{nj} = 0$ for each $j \geqq 0$, the inequality

$$\|a_n(T)x - z\| \leqq \sum_{j=0}^{N} a_{nj}\|T^jx - z\| + \sum_{j=N+1}^{n} a_{nj}\|T^jx - z\|, \qquad n > N,$$

can be used to prove that $a_n(T)x \to z$ whenever $T^nx \to z$.

3.2. As an illustration we consider an operator T whose spectrum $\sigma(T)$ is the union $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$ of pairwise disjoint spectral sets σ_i of T, i.e. sets both closed and open in the relative topology of $\sigma(T)$; σ_1 and σ_4 are contained in the interior and exterior of the unit circle respectively, $\sigma_2 = \{1\}$, where 1 is a simple pole of $(\lambda I - T)^{-1}$, and σ_3 is a finite collection of poles of $(\lambda I - T)^{-1}$ lying on the unit circle. Let E_i be the projection associated with σ_i [5, p. 573], and let X_i be the range of E_i, $1 \leqq i \leqq 4$. Then $X = \sum_i \oplus X_i$, and T is completely reduced by X_i's. Furthermore, the spectrum $\sigma(T_i)$ of the restriction T_i of T to X_i is the set σ_i. Let Q_1 be the projection operator associated with the unit matrix, $Q_1x = \lim_n T^nx$. Then

$$N(Q_1) = X_1, \qquad D(Q_1) = X_1 \oplus X_2. \tag{13}$$

First, $\|T_1^n\| \to 0$ since $\sigma(T_1)$ is contained in the open unit disc [8]. Hence $X_1 \subset N(Q_1)$. Next we show that $N(Q_1) \cap X_i = \{0\}$ for $i = 2, 3, 4$. According to [5, p. 573], $X_2 = N(I - T)$, and

$$X_3 = \sum_{\lambda \in \sigma_3} \oplus N((\lambda I - T)^{\nu(\lambda)}), \tag{14}$$

where $\nu(\lambda)$ is the order of the pole λ. If $x \in X_2$, $T^nx = x$ for all $n \geqq 0$, and $\{T^nx\}$ converges to 0 if and only if $x = 0$. Let $x \in X_3$. In view of the decomposition (14) we may assume that $x \in N((\lambda I - T)^{\nu(\lambda)})$ for some $\lambda \in \sigma_3$. If $x \neq 0$, there is a positive integer $p \leqq \nu(\lambda)$ such that $(\lambda I - T)^{p-1}x \neq 0$ and $(\lambda I - T)^px = 0$. The vectors $u_1, \cdots, u_p$ defined by $u_{i+1} = (T - \lambda I)^ix$, where $0 \leqq i \leqq p - 1$, are linearly independent. We can easily verify the formula

$$T^nx = \sum_{i=1}^{p} C_{i-1}^{n}\lambda^{n-i+1}u_i, \qquad n \geqq p. \tag{15}$$

Then $T^nx \not\to 0$ since $|C_{i-1}^n\lambda^{n-i+1}| = C_{i-1}^n \geqq 1$ for all $n \geqq p$ and all i, $1 \leqq i \leqq p$.

Hence $T^n x \to 0$ if and only if $x = 0$. This proves the first formula in (13); the second formula follows from Lemma 1.3 and the equality $X_2 = N(I - T)$. It was proved in [8] that $\{T^n\}$ converges in the uniform operator topology if (and only if) $\sigma(T) - \{1\}$ is contained in the open unit disc and 1 is a pole of $(\lambda I - T)^{-1}$ of order $\leqq 1$. Therefore $\|T_1^n \oplus T_2^n - Q_1\| \to 0$ in the uniform operator topology of $D(Q_1) = X_1 \oplus X_2$, and $Q_1 = 0 \oplus I$ in $X_1 \oplus X_2$.

Let Q be the projection operator associated with the Cesàro matrix, i.e.,

$$Qx = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} T^j x,$$

whenever the limit exists in the strong topology of X. Then $N(Q_1)$ is a proper subset of $N(Q)$ since

$$\sum_{\lambda\in\sigma_3} \oplus N(\lambda I - T) \subset N(Q).$$

To prove this inclusion we select $x \in N(\lambda I - T)$ for some $\lambda \in \sigma_3$. Since $\lambda \neq 1$ and $|\lambda| = 1$,

$$\frac{1}{n+1} \sum_{j=0}^{n} T^j x = \frac{1}{n+1} \sum_{j=0}^{n} \lambda^j x = \frac{1}{n+1} \frac{1 - \lambda^{n+1}}{1 - \lambda} x \to 0 \quad \text{as } n \to \infty.$$

In the following proposition we give a complete description of $D(Q)$ in the case when T is an operator with rational resolvent.

PROPOSITION 3.3. *Let T be an operator with rational resolvent and A an admissible matrix satisfying the condition* (5). *Then*

$$(16) \qquad N(Q) = \sum_{\lambda\in\sigma(T)} \oplus N((\lambda I - T)^{\mu(\lambda)}), \qquad D(Q) = N(I - T) \oplus N(Q),$$

where $\mu(\lambda)$ is either the smallest nonnegative integer μ for which the sequence of the μ-th derivatives $\{a_n^{(\mu)}(\lambda)\}$ does not converge to zero or the order of the pole λ if such μ does not exist.

Proof. If $(\lambda I - T)^{-1}$ is rational, $\sigma(T)$ is a finite collection of poles of $(\lambda I - T)^{-1}$. Then [11, p. 317]

$$X = \sum_{\lambda\in\sigma(T)} \oplus N((\lambda I - T)^{\nu(\lambda)}),$$

where $\nu(\lambda)$ is the order of the pole λ, and T is completely reduced by the subspaces occurring in this direct sum. Suppose $x \in N((\lambda I - T)^{\nu(\lambda)})$ for some $\lambda \in \sigma(T)$, $x \neq 0$. As in the preceding paragraph, $T^n x$ is given by the formula (15) with $u_1, \cdots, u_p$ linearly independent, $p \leqq \nu(\lambda)$. From (15) we obtain the following explicit expression for $a_n(T)x$:

$$a_n(T)x = \sum_{i=1}^{p} \left(\sum_{j=0}^{n} a_{nj} C_{i-1}^{j} \lambda^{j-i+1} \right) u_i = \sum_{i=1}^{p} \frac{a_n^{(i-1)}(\lambda)}{(i-1)!} u_i, \qquad n \geqq p.$$

Hence $a_n(T)x \to 0$ if and only if $a_n^{(i-1)}(\lambda) \to 0$ for each i with $1 \leqq i \leqq p$, and the conditions

(a) $a_n(T)x \to 0$,

(b) $p \leqq \mu(\lambda)$,

(c) $x \in N((\lambda I - T)^{\mu(\lambda)})$

are equivalent. If x is an arbitrary element of X, the decomposition $x = \sum_\lambda x_\lambda$ with $x_\lambda \in N((\lambda I - T)^{\nu(\lambda)})$ shows that $a_n(T)x \to 0$ if and only if $a_n(T)x_\lambda \to 0$ for each $\lambda \in \sigma(T)$. This completes the proof.

The formula (16) is, in particular, valid in the case when X is finite-dimensional. The following result follows immediately from Proposition 3.3.

COROLLARY 3.4. *Let T be an operator with rational resolvent and let Q_1 be the projection operator associated with the infinite unit matrix and with the operator T. Then*

$$N(Q_1) = \sum_{\lambda \in \sigma(T), |\lambda| < 1} \oplus N((\lambda I - T)^{\nu(\lambda)}), \qquad D(Q_1) = N(I - T) \oplus N(Q_1),$$

where $\nu(\lambda)$ is the order of the pole λ.

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DOUBLY ASYMPTOTIC SERIES FOR nth ORDER DIFFERENTIAL EQUATIONS IN UNBOUNDED DOMAINS*

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Abstract. This paper is concerned with the asymptotic behavior of the solution, as $\varepsilon \to 0^+$ or z tends to infinity in the complex z-plane, of the differential equation

$$\varepsilon^n u^{(n)}(z) - \varepsilon^{n-1} p_{n-1}(z) u^{(n-1)}(z) - \varepsilon^{n-2} p_{n-2}(z) u^{(n-2)}(z) - \cdots - p_0(z) u(z) = 0,$$

where $p_i(z)$, $i = 0, 1, \cdots, n-1$, are polynomials in z with some restrictions on their degrees.

A series of transformations of the dependent variable transforms the equation into a convenient almost diagonal system. Then, asymptotic series solutions are found by iterative solution of an equivalent integral equation. A careful study of the regions of validity in the z-plane of the asymptotic series shows that for every sufficiently narrow sector a full fundamental system of n asymptotically known solutions can be found.

1. Introduction. This paper is concerned with a study of the asymptotic behavior of the solution, as $\varepsilon \to 0^+$, or z tends to infinity in the complex z-plane, of the differential equation

$$\varepsilon^n u^{(n)}(z) - \varepsilon^{n-1} p_{n-1}(z) u^{(n-1)}(z) - \varepsilon^{n-2} p_{n-2}(z) u^{(n-2)}(z) \cdots - p_0(z) u(z) = 0, \tag{1.1}$$

where $n \geqq 3$, $p_i(z)$, $i = 0, 1, \cdots, n-1$, are polynomials in z, $p_0(z)$ has degree $m > 0$, and $p_j(z)$ has degree less than $(m/n)(n-j)$ for $j = 1, 2, \cdots, n-1$.

Evgrafov and Fedoryuk have studied the equation

$$\varepsilon^2 u''(z) - p(z) u(z) = 0, \tag{1.2}$$

where $p(z)$ is a polynomial of degree m. They construct certain unbounded regions in the complex z-plane, called canonical domains, in which they calculate a fundamental system of solutions. They use an iterative method to construct a "doubly asymptotic" series for the solutions such that successive terms of the series become smaller in the sense of increasing orders of magnitude both as $\varepsilon \to 0^+$ or $z \to \infty$.

To solve equation (1.1), a series of transformations of the dependent variable is made to transform (1.1) into a convenient system of integral equations. Then series solutions are found by iterative methods similar to that of Evgrafov's and Fedoryuk's. Such series are valid, however, only in certain sectors tending to infinity in the z-plane. In this paper such series solutions will be constructed corresponding to any sector with central angle less than $(\pi/n)(m/n + 1)^{-1}$. This results from the fact that only in such sectors can we construct suitable paths satisfying the key properties necessary for our iteration procedure. More general results of the region of validity may be possible by a global study of Stokes curves [2]. Such studies were made by Evgrafov and Fedoryuk, and Wasow [6]. The extension of the global theory to nth order equations, is, however, beyond the scope of the present paper.

The following main theorem will be proved in § 4.

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THEOREM 1.1 (Main Theorem). *Let ω_1, ω_2 be angles such that* $0 < \omega_2 - \omega_1 < (\pi/n)((m/n) + 1)^{-1}$; *then for L sufficiently large, $\bar{\varepsilon}$ sufficiently small,* (1.1), *where* $n \geqq 3$, *has in the region*

$$S(L;\omega_1,\omega_2) = \{z \| z| \geqq L, \omega_1 \leqq \arg z \leqq \omega_2\}, \qquad 0 < \varepsilon < \bar{\varepsilon}, \tag{1.3}$$

n linearly independent solutions $u_j(z,\varepsilon)$, $j = 1, \cdots, n$, satisfying the following asymptotic relation for each j:

$$\begin{aligned} &\left| u_j(z,\varepsilon) - z^{-(m/n)(n-1)/2} \exp\left\{\frac{1}{\varepsilon}\int_b^z \lambda_j(\xi)\,d\xi\right\} \sum_{k=0}^{h} u_{jk}(z)\varepsilon^k \right| \\ &\qquad \leqq \varepsilon^{h+1} M_{jh}(z,\varepsilon)|z|^{-(m/n)\cdot(n-1)/2} \left| \exp\left\{\frac{1}{\varepsilon}\int_b^z \lambda_j(\xi)\,d\xi\right\}\right| \end{aligned} \tag{1.4}$$

in region (1.3) *for each nonnegative integer h. Here b is an arbitrary fixed point in $S(L;\omega_1,\omega_2)$, and the $\lambda_j(z)$, $j = 1, \cdots, n$, are n distinct roots of the equation*

$$\lambda^n - \sum_{k=1}^{n} p_{n-k}(z)\lambda^{n-k} = 0 \tag{1.5}$$

in the region $S(L;\omega_1,\omega_2)$. The functions $u_{jk}(z)$ can be explicitly calculated and are of the order $O(|z|^{-k(m/n+1)})$ as $z \to \infty$ in $S(L;\omega_1,\omega_2)$. Both $\lambda_j(z)$ and $u_{jk}(z)$, $j = 1, \cdots, n$, $k = 0, 1, 2, \cdots$, possess convergent expansions in decreasing powers of $z^{1/n}$ in $S(L;\omega_1,\omega_2)$. The functions $u_{j0}(z)$ tend to 1 as $z \to \infty$ in $S(L;\omega_1,\omega_2)$ for $j = 1, \cdots, n$. The functions $M_{jh}(z,\varepsilon)$ are of the order $O(|z|^{-(h+1)(m/n+1)})$, uniformly in ε, in region (1.3).

Most of the iterative techniques in this paper are related to those of Turrittin [4] and of Evgrafov and Feforyuk [2].

2. Preliminary transformations. Consider the differential equation (1.1) where $p_0(z)$ is a polynomial in z of degree $m > 0$ and $p_j(z)$ is a polynomial in z of degree less than $(m/n)(n - j)$ for $j = 1, 2, \cdots, n - 1$. The reason for this last condition will become apparent presently. An equivalent system, with $u = y_1$, $\varepsilon u' = y_2$, $\varepsilon^2 u'' = y_3, \cdots, \varepsilon^{n-1}u^{(n-1)} = y_n$ is

$$\varepsilon Y' = A(z)Y, \tag{2.1}$$

where

$$A(z) = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ & & & & 0 & 1 \\ p_0 & p_1 & \cdot & \cdot & p_{n-2} & p_{n-1} \end{bmatrix}, \qquad Y = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}.$$

The characteristic equation of $A(z)$ is

$$\phi(z,\lambda) = \lambda^n - p_{n-1}(z)\lambda^{n-1} - p_{n-2}(z)\lambda^{n-2} \cdots - p_0(z) = 0. \tag{2.2}$$

Let

$$S_p = \{z | \theta_1 < \arg z < \theta_2, \theta_2 - \theta_1 < (m/n + 1)^{-1}\pi, p < |z| < \infty\}.$$

(S_p depends on the choice of θ_1 and θ_2 and the reason for restricting z to such sectors will appear later.) Employing a technique due to Newton [1], we find that the n roots, $\lambda_j(z)$, of the characteristic equation (2.2) are meromorphic functions of $z^{-1/n}$ and are distinct in a neighborhood of $z = \infty$. For p sufficiently large, the functions $z^{-m/n}\lambda_j, j = 1, \cdots, n$, have uniformly convergent series in S_p of the form

$$z^{-m/n}\lambda_j = \sum_{k=0}^{\infty} a_{kj} z^{-k/n}. \tag{2.3}$$

We assign a fixed choice of $z^{1/n}$ in S_p for the n equations in (2.3), and $a_{0j}, j = 1, \cdots, n$, are the n distinct roots of the leading coefficient of $p_0(z)$. The distinctness of the n roots λ_j and the simplicity of the equations (2.3) are the results of our assumption on the degrees of the n polynomials $p_i(z)$.

Let $T(z)$ be the $n \times n$ matrix whose kth column, $1 \leqq k \leqq n$, is the column vector function $\operatorname{col}(1, \lambda_k, \lambda_k^2, \cdots, \lambda_k^{n-1})$ which are eigenvectors of $A(z)$. The transformation

$$Y = T(z)W \tag{2.4}$$

takes (2.1) into the system

$$\varepsilon W' = [\operatorname{diag}(\lambda_1(z), \cdots, \lambda_n(z)) - \varepsilon T^{-1}T'(z)]W \tag{2.5}$$

whose coefficient is diagonal to within terms of order $O(\varepsilon)$.

The matrix $T(z)$ can be assumed to have the form

$$T(z) = \operatorname{diag}(1, (a_{01}z^{m/n}), \cdots, (a_{01}z^{m/n})^{n-1})\Omega(\omega)[I + T_0(z^{-1/n})], \tag{2.6}$$

where $\omega = e^{(2\pi i)/n}$, $\Omega(\omega)$ is the $n \times n$ matrix whose (i, j) entry is $(\omega^{j-1})^{i-1}$, $1 \leqq i, j \leqq n$, and $T_0(x)$ is a matrix holomorphic in a neighborhood of $x = 0$, $T_0(0) = 0$. Thus

$$\begin{aligned} T'(z) = {} & \operatorname{diag}\left(0, \frac{m}{nz}(a_{01}z^{m/n}), \frac{2m}{nz}(a_{01}z^{m/n})^2, \cdots, \frac{(n-1)m}{nz}(a_{01}z^{m/n})^{n-1}\right)\Omega(\omega) \\ & \cdot [I + T_0(z^{-1/n})] - \frac{1}{n} z^{-1/n-1} \operatorname{diag}(1, (a_{01}z^{m/n}), \cdots, (a_{01}z^{m/n})^{n-1}) \\ & \cdot \Omega(\omega) T_0'(z^{-1/n}). \end{aligned} \tag{2.7}$$

Using equations (2.6) and (2.7), we see that

$$zT^{-1}(z)T'(z) = [\Omega(\omega)]^{-1} \operatorname{diag}\left(0, \frac{m}{n}, \frac{2m}{n}, \cdots, \frac{(n-1)m}{n}\right)\Omega(\omega) + \tilde{T}_0(z^{-1/n}), \tag{2.8}$$

where $\tilde{T}_0(x)$ is holomorphic in a neighborhood of $x = 0$, $\tilde{T}_0(0) = 0$.

We now write equation (2.5) in the form

$$\varepsilon W' = z^{m/n}[H - (\varepsilon z^{-m/n-1})M]W, \tag{2.9}$$

where $H(z) = z^{-m/n} \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, $M(z) = zT^{-1}T'$. The matrices $H(z)$ and $M(z)$ both tend to finite limits as $z \to \infty$ in S_p, from our computations above.

After having diagonalized the leading term of the coefficient matrix in (2.1) by transformation (2.4), we next diagonalize the remaining terms successively by a variant of a method of Turrittin.

THEOREM 2.1. *There exists a transformation*

$$W = R(z, \mu)V, \qquad \text{with } \mu = \varepsilon z^{-m/n-1}, \tag{2.10}$$

which changes the differential equation (2.9) *into the form*

$$\varepsilon V' = z^{m/n} D(z, \mu)V, \tag{2.11}$$

and has the following properties:

(a) *$D(z, \mu)$ is holomorphic in both variables z, μ for*

$$z \in S_p, \qquad 0 < \varepsilon < \varepsilon_0, \tag{2.12}$$

(b) $$D(z, \mu) \sim \sum_{r=0}^{\infty} D_r(z)\mu^r \text{ as } \mu \to 0,$$

uniformly in the region (2.12).

(c) *The matrices $D_r(z)$ are diagonal and holomorphic for $z \in S_p$, possess convergent expansions in descending powers of $z^{1/n}$ for $|z| > p$, and tend to finite limits as $z \to \infty$ in S_p; $D_0(z) = H(z)$.* (Observe that this does not imply that $D(z, \mu)$ is diagonal.)

(d) *The matrix $R(z, \mu)$ is holomorphic in the region* (2.12) *and possesses there a uniformly asymptotic expansion*:

$$R(z, \mu) \sim \sum_{r=0}^{\infty} R_r(z)\mu^r \qquad \text{as } \mu \to 0, \tag{2.13}$$

with $R_0(z) = I$. The $R_r(z)$ are holomorphic and tend to finite limits, as $z \to \infty$ in S_p. The $R_r(z)$ also have convergent expansions in descending powers of $z^{1/n}$ for $|z| > p$.

Proof. We begin by setting

$$W = [I + \mu Q_1(z)]V_1, \tag{2.14}$$

where $Q_1(z)$ is the matrix whose (i, j) entry is

$$[Q_1(z)]_{ij} = (1 - \delta_{ij})z^{m/n} m_{ij}(z)(\lambda_i(z) - \lambda_j(z))^{-1}.[1] \tag{2.15}$$

Here, δ_{ij} is the Kronecker delta and $m_{ij}(z)$ is the (i, j) entry of $M(z)$. The transformation (2.14) takes (2.9) into

$$\varepsilon V_1' = z^{m/n}\{H + \mu[-M - Q_1 H + HQ_1] + \cdots\}V_1, \tag{2.16}$$

where the matrix $D_1 = -M - Q_1 H + HQ_1$ is diagonal (exactly the diagonal part of $-M(z)$), and the dots indicate a convergent series in powers of μ if $|\mu|$ is small, beginning with terms of $O(\mu^2)$.

The differential equation (2.16) is next subjected to a sequence of transformations of the form

$$V_{j-1} = [I + \mu^j Q_j(z)]V_j, \qquad j = 2, 3, \cdots. \tag{2.17}$$

[1] This means $[Q_1(z)]_{ii} = 0$, $i = 1, \cdots, n$.

Each of them leaves the coefficients of $\mu^0, \mu^1, \cdots, \mu^{j-1}$ unchanged. $Q_j(z)$ is determined so as to diagonalize the coefficient of μ^j; it can be made as $Q_1(z)$ to be holomorphic in S_p and tend to finite values, as $z \to \infty$ in S_p. Finally, multiplying all these transformations, we arrive at the formal matrix

$$R(z, \mu) = \prod_{i=1}^{\infty} (I + \mu^i Q_i(z)) = \sum_{r=0}^{\infty} R_r(z)\mu^r.$$

Apply the "Borel–Ritt" theorem [3] to conclude that there exists a holomorphic function $R(z, \mu)$ which satisfies property (d). This proves the theorem.

We now perform a transformation whose effect is to annul one diagonal entry in $D_0(z)$ and also the whole matrix $D_1(z)$. Let $z_0 \in S_p$ such that $\arg z_0 = (\theta_1 + \theta_2)/2$. Write

$$F = F(z) = \operatorname{diag}\left(\exp\left\{-\int_{z_0}^{z} m_{11}(\xi)\xi^{-1}\,d\xi\right\}, \cdots, \exp\left\{-\int_{z_0}^{z} m_{nn}(\xi)\xi^{-1}\,d\xi\right\}\right).$$

Set

$$V = FX \exp\left\{\frac{1}{\varepsilon}\int_{z_0}^{z} \lambda_1(\xi)\,d\xi\right\}. \tag{2.18}$$

We obtain

$$X' = \left\{\frac{1}{\varepsilon}[\operatorname{diag}(\lambda_1, \cdots, \lambda_n) - \lambda_1(z)I] + \varepsilon G(z, \varepsilon)\right\}X, \tag{2.19}$$

where $G(z, \varepsilon)$ is holomorphic in z, μ in region (2.12) and

$$G(z, \varepsilon) \sim \sum_{r=0}^{\infty} (D_{r+2}(z)z^{-m/n-2})\mu^r, \tag{2.20}$$

as $\mu \to 0$ uniformly in the region (2.12). We have such a convenient expansion for $G(z, \varepsilon)$ because $F(z)$ and $D_r(z)$ are diagonal and commute with each other. The matrices $D_k(z)z^{-m/n-2}$, $k = 2, 3, \cdots$, are holomorphic in S_p, of the order $O(|z|^{-m/n-2})$ as $z \to \infty$, and possess convergent expansions in descending powers of $z^{1/n}$ for $|z| > p$.

3. An associated integral equation. We shall transform the differential equation (2.19) into an integral equation by the method of variation of parameters. We begin with a careful definition of an unbounded subdomain in S_p, and construct paths in it extending to infinity. These paths satisfy other essential properties which are necessary for our iteration procedure later on.

Let $S(\theta_1, \theta_2) = \{z | \theta_1 < \arg z < \theta_2\}$, $c_j = (m/n + 1)^{-1}(a_{0j} - a_{01})$, $\pi_{j1}(z) = c_j z^{m/n+1}$. Let $S^{j1}(\theta_1, \theta_2)$ be the image of $S(\theta_1, \theta_2)$ with respect to π_{j1}, $j = 2, \cdots, n$. (The root $z^{1/n}$ is taken to be the same as that in equation (2.3), and we shall follow this convention in the remaining part of this paper.) Assume $S(\theta_1, \theta_2)$ satisfies the following hypothesis.

[$\mathrm{H}_1$0]. $S^{j1}(\theta_1, \theta_2)$ contains a ray γ_j in the right half-plane, for each integer $j \neq 1$, $1 \leqq j \leqq n$.

(Conditions under which [$\mathrm{H}_1$0] is satisfied will be discussed in the proof of the main theorem.) There exists a small $\delta > 0$ such that the rays γ with $|\arg \gamma$

$-\arg\gamma_j| < \sin^{-1}(2\delta)$, $j = 2, \cdots, n$, are contained in the intersection of the right open half-plane and $S^{j1}(\theta_1, \theta_2)$, for the corresponding j. Let

$$h_k(z) = -1 + (a_{0k} - a_{01})z^{m/n}\{\lambda_k(z) - \lambda_1(z)\}^{-1}, \qquad k = 2, \cdots, n.$$

Then $|h_k(z)| < \delta$, if $z \in S_p$ for a sufficiently large constant p. Recall that we let $z_0 \in S_p$ with $\arg z_0 = \frac{1}{2}(\theta_1 + \theta_2)$. Let ζ_2 be the image of z_0 in the π_{21}-plane. From ζ_2 construct two rays to infinity parallel to the images of the rays $\arg z = \theta_1, \theta_2$ respectively. (Refer to Fig. 3.1.) Denote the unbounded open domain,

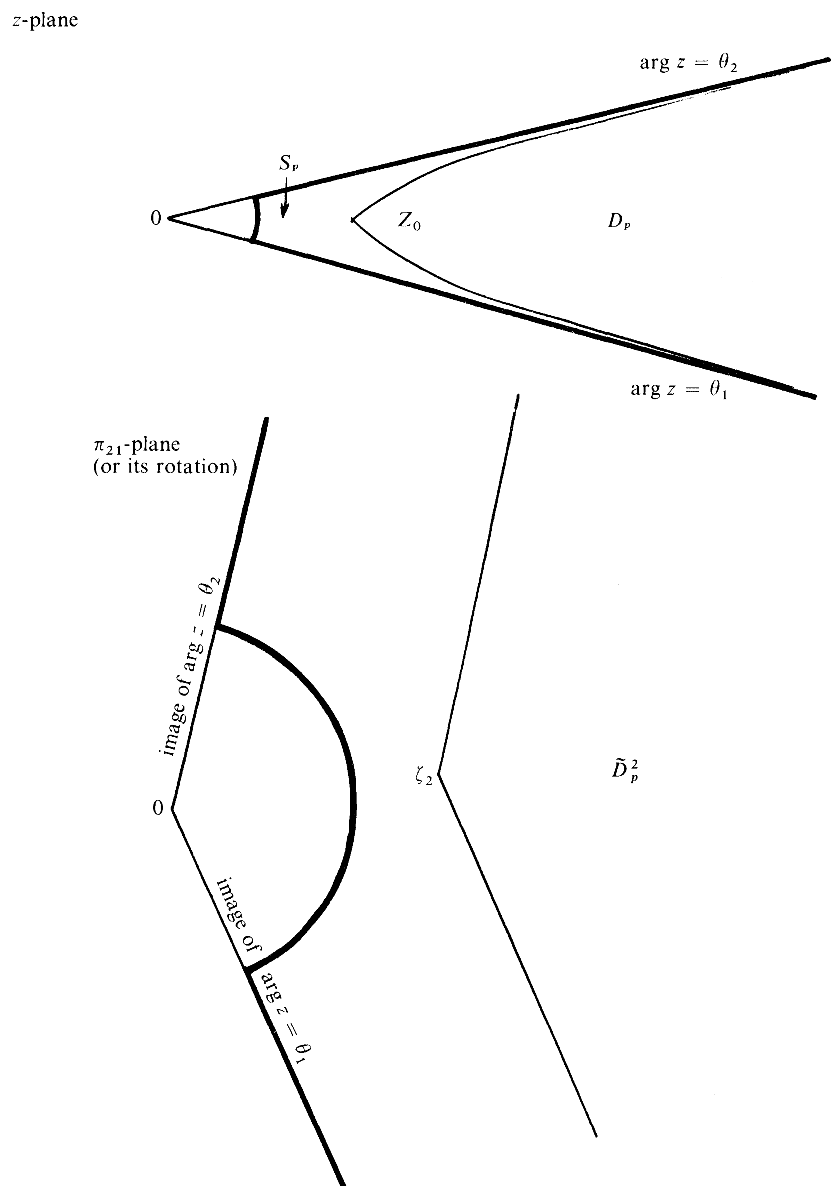

FIG. 3.1

bounded by these two rays from ζ_2, by $\tilde{D}_p^2$. Define D_p to be the preimage of $\tilde{D}_p^2$ in the z-plane. (Note that the image $\tilde{D}_p^j$ of D_p in the π_{j1}-plane, $j = 3, \cdots, n$, is just the rotation and stretching of $\tilde{D}_p^2$.)

From each point $z \in D_p$, let $\sigma_{j1}(t, z)$ be the solution to the initial value problem:

$$(3.1) \qquad [\lambda_j(\sigma_{j1}(t,z)) - \lambda_1(\sigma_{j1}(t,z))]\frac{d}{dt}(\sigma_{j1}(t,z)) = b_j, \qquad \sigma_{j1}(0,z) = z,$$

$j = 2, \cdots, n$, where t is a nonnegative real parameter, b_j is a constant, $b_j = \exp\{i(\arg \gamma_j)\}$. We consider the image of the solutions $\sigma_{j1}(t, z)$ on the π_{j1}-plane, $j \neq 1$. In view of equation (2.3), we write (3.1) in the form

$$(3.2) \qquad \pi_{j1}(\sigma_{j1}(t,z)) = \pi_{j1}(z) + b_j\left\{t + \int_0^t h_j(\sigma_{j1}(u,z))\,du\right\}, \qquad j = 2, \cdots, n.$$

The image of $\sigma_{j1}(t, z)$, $z \in D_p$, in the π_{j1}-plane is confined to a sector $\mathscr{T}_{j1}(z)$ (with the image of z as vertex, central angle $2\sin^{-1}\delta$, and with angular bisector at an angle of $\arg \gamma_j$ with the horizontal positive direction). This is true because $|b_j| = 1$ and $|h_j(s)| < \delta$ for all $s \in D_p \subset S_p$. (Refer to Fig. 3.2.) Since the sector $\mathscr{T}_{j1}(z)$ always stays in $\tilde{D}_p^j$, we can continue $\sigma_{j1}(t, z)$ as $t \to \infty$ and it will remain in D_p.

DEFINITION. For each $z \in D_p$, $C_{j1}(z)$, $j = 2, \cdots, n$, is the path in the z-plane parametrized by $\sigma_{j1}(t, z)$, $0 \leqq t < \infty$ (as defined in (3.1)). For each z in the closure of S_p, $C_{11}(z)$ is the path in the z-plane parametrized by $\sigma_{11}(t, z) = z + \exp\{(i/2) \cdot (\theta_1 + \theta_2)\}t$, $0 \leqq t < \infty$.

By our construction, it is evident that the following hypothesis holds.

[H1]. The paths $C_{j1}(z)$, $j = 1, \cdots, n$, $z \in D_p$, are contained in D_p.

The paths $C_{j1}(z)$ begin at z and tend to ∞. Furthermore we have the following property.

[H2]. If $\theta_1 < \alpha < \beta < \theta_2$, then there exists a positive number B such that $S(B; \alpha, \beta) \subset D_p$, where

$$S(B; \alpha, \beta) = \{z | \alpha \leqq \arg \leqq \beta, |z| \geqq B\}.$$

From (3.1), we see that for $j \neq 1$, $z \in D_p$,

$$\int_{\sigma_{j1}(t,z)}^{z} [\lambda_j(u) - \lambda_1(u)]\,du = \int_t^0 [\lambda_j(\sigma_{j1}(s,z)) - \lambda_1(\sigma_{j1}(s,z))]\frac{d}{ds}(\sigma_{j1}(s,z))\,ds$$
$$= -b_j t,$$

where $\operatorname{Re} b_j > 0$. Furthermore, if $c_j s^{m/n+1} \in \mathscr{T}_{j1}(z)$ with $|s|$ large compared with $|z|$, the integral $\int_s^z [\lambda_j(u) - \lambda_1(u)]\,du$ is closely approximated by $c_j z^{m/n+1} - c_j s^{m/n+1}$. Thus, the orientation of the sector $\mathscr{T}_{j1}(z)$ implies that $\operatorname{Re}[\int_s^z (\lambda_j - \lambda_1)\,du] \to -\infty$ as $|s| \to \infty$ with $c_j s^{m/n+1} \in \mathscr{T}_{j1}(z)$. Let $\xi_{jk}(s, z) = \int_s^z [\lambda_j(u) - \lambda_k(u)]\,du$ for s, $z \in D_p$, $1 \leqq j, k \leqq n$. We have the following two important properties for our paths.

[H3]. For $2 \leqq j \leqq n$, $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_0$, the limit $\exp\{(1/\varepsilon)\xi_{j1}(s, z)\} \to 0$ as $|s| \to \infty$ with $c_j s^{m/n+1} \in \mathscr{T}_{j1}(z)$, (in particular, if $|s| \to \infty$ along $C_{j1}(z)$).

[H4]. If $\chi(z)$ is defined and holomorphic in z for $z \in D_p$ and $|\chi(z)| \leqq |z|^{-k}$, and if $z \in D_p$ for some real constant k, then the functions

$$\Phi_{j1}(z, \varepsilon) = \int_{C_{j1}(z)} \exp\{\varepsilon^{-1}\xi_{j1}(s,z)\}\chi(s)\,ds, \qquad j = 2, \cdots, n,$$

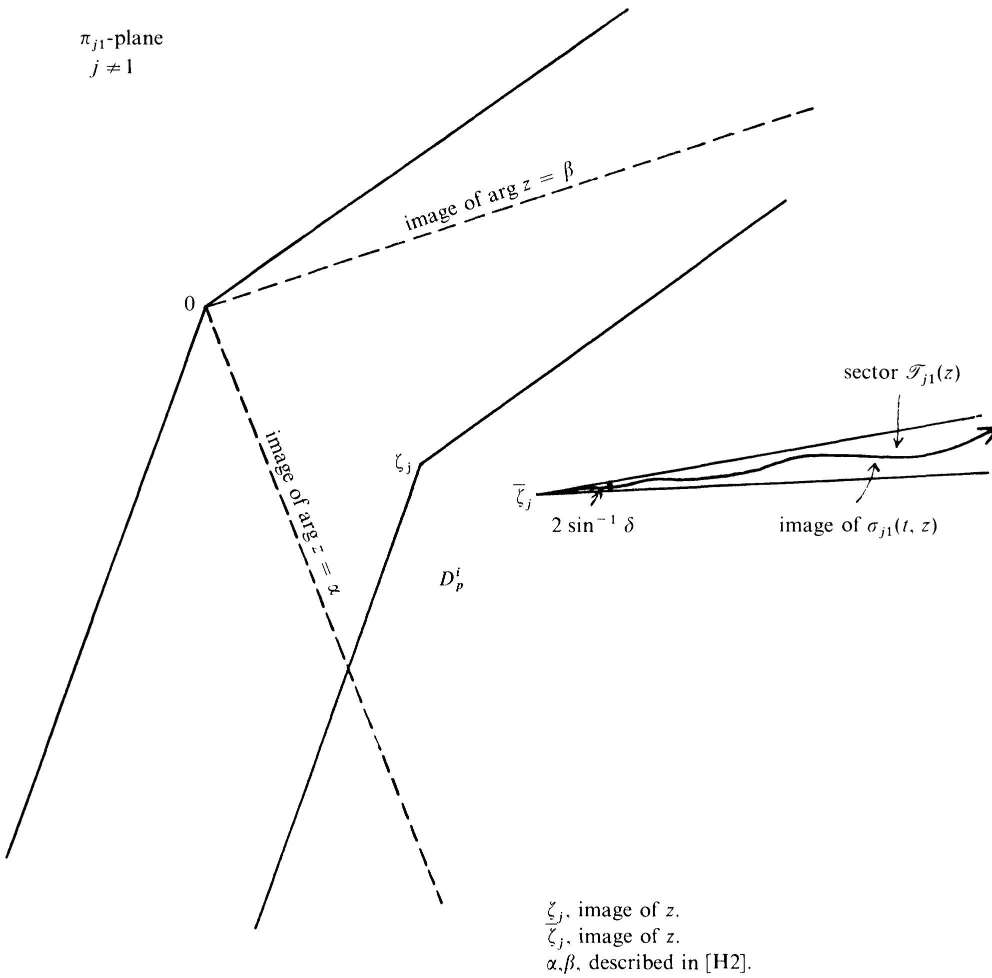

FIG. 3.2

are holomorphic in z on D_p for $0 < \varepsilon \leqq \varepsilon_0$ and $|\Phi_{j1}(z, \varepsilon)| \leqq K_1 |z|^{-k}$, $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_0$ for some constant K_1 independent of χ. If $-k < -1$, the assertion remains true for $j = 1$, if in the last estimate k is replaced by $k - 1$.

The details for proving the conditions [H1] to [H4] are tedious but straight forward, and will be omitted.

We can now consider the integral equation associated with the differential equation (2.19), for $z \in D_p$, $0 < \varepsilon < \varepsilon_0$:

$$X(z, \varepsilon) = X_0 - \varepsilon \int_{C_1(z)} E_1(s, z, \varepsilon) G(s, \varepsilon) X(s, \varepsilon)\, ds, \tag{3.3}$$

where $X_0 = \operatorname{col}(1, 0, 0, \cdots, 0)$, $E_1(s, z, \varepsilon) = \operatorname{diag}(\exp\{(1/\varepsilon)\xi_{11}(s, z)\}, \cdots, \exp \cdot \{(1/\varepsilon)\xi_{n1}(s, z)\})$; and $C_1(z)$ represents the system of n paths $C_{j1}(z)$, $j = 1, \cdots, n$, while we integrate the jth entry of the column vector along $C_{j1}(z)$.

For convenience, let $g_{jk}(z, \varepsilon)$, $1 \leqq j, k \leqq n$, be the (j, k) entry of the matrix function $G(z, \varepsilon)$. By (2.20) and Theorem 2.1, part (c), we have for $j \neq k$, $g_{jk}(z, \varepsilon) \sim 0$ as $\mu \to 0$ uniformly in the region (2.12).

4. Solutions of the integral and differential equations.

THEOREM 4.1. *Assume* $S(\theta_1, \theta_2)$ *satisfies hypothesis* $[\mathrm{H}_1 0]$. *There exists* ε_1, $0 < \varepsilon_1 < \varepsilon_0$, *such that there exists a bounded solution to* (2.19) *in the region* $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$.

Proof. Let $\delta_1 = \max_{1 \leqq j \leqq n} \{\hat{\delta}_{j1}\}$ where

$$\hat{\delta}_{j1} = \sup_{z \in D_p} \left\{ \sup_{0 < \varepsilon < \varepsilon_0} \int_{C_{j1}(z)} \left\{ |\exp[\varepsilon^{-1} \xi_{j1}(s, z)]| \cdot \sum_{k=1}^{n} |g_{jk}(s, \varepsilon)| \right\} ds \right\},$$

$j = 1, \cdots, n$. Properties [H4], Theorem 2.1 part (c), and equation (2.20) together imply that $\delta_1 < \infty$. Write (3.3) in the form $X(z, \varepsilon) = X_0 + KX(z, \varepsilon)$ for $z \in D_p$, $0 < \varepsilon < \varepsilon_0$, where KX is the integral operator

$$[KX](z, \varepsilon) = -\varepsilon \int_{C_1(z)} E_1(s, z, \varepsilon) G(s, \varepsilon) X(s, \varepsilon)\, ds.$$

Writing $K^j X_0 = \mathrm{col}(x_{j1}, \cdots, x_{jn})$, $j = 0, 1, 2, \cdots$, we can prove that we have the following inequality for all integers $t \geqq 0$:

$$(4.1) \qquad |x_{tk}(z, \varepsilon)| \leqq \left(\frac{1}{2}\right)^t \quad \text{for all } 0 < \varepsilon \leqq \varepsilon_1 < \varepsilon_0, \quad z \in D_p,$$

$k = 1, \cdots, n$. This can be clearly proved by induction after choosing $0 < \varepsilon_1 < \min\{1/(2\delta_1), \varepsilon_0\}$.

Although the paths of integration for the integrals $K[K^j X_0](z, \varepsilon)$ do not tend from z to a fixed point, the contribution for the integrals when $|s|$ is large compared with $|z|$ will be small if $[K^j X_0]$ is bounded for $s \in D_p$, $0 < \varepsilon < \varepsilon_0$. This is true by [H3] and the comments before it. Also, integrals of these same integrands along an arc of radius r inside $\mathscr{T}_{j1}(z)$, $z \in D_p$, can be made arbitrarily small for sufficiently large $r \gg |z|$. Thus the functions $K^j X_0$, $j = 0, 1, \cdots$, are holomorphic for $z \in D_p$, $0 < \varepsilon < \varepsilon_0$.

Inequality (4.1) implies that the series $\sum_{j=0}^{\infty} K^j X_0$ converges uniformly and absolutely for $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$, and $X = \sum_{j=0}^{\infty} K^j X_0$ is a holomorphic solution of (3.3) in this region. Finally,

$$\frac{d}{dz}[KX(z, \varepsilon)] = -\int_{C_1(z)} \{[\mathrm{diag}(\lambda_1(z), \cdots, \lambda_n(z)) - \lambda_1(z) I] E_1(s, z, \varepsilon) G(s, \varepsilon) X(s, \varepsilon)\}\, ds$$
$$+ \varepsilon G(z, \varepsilon) X(z, \varepsilon),$$

and X is therefore a solution of (2.19) for $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$, with its components x_k satisfying

$$(4.2) \qquad |x_k(z, \varepsilon)| \leqq 1 + \tfrac{1}{2} + \cdots = 2.$$

Before constructing an asymptotic series for $V(z, \varepsilon)$, related to $X(z, \varepsilon)$ by (2.18), we remark that the limit of $M(z) = zT^{-1}T'$ as $z \to \infty$ in S_p is given by (2.8) to be the matrix $[\Omega(\omega)]^{-1}\, \mathrm{diag}(0, m/n, \cdots, (n-1)m/n) \Omega(\omega)$, whose diagonal elements are all equal to $(m/n)\cdot(n-1)/2$ (see Wasow [7]). Therefore we have $m_{jj}(z) = (m/n)\cdot(n-1)/2 + O(|z|^{-1/n})$, as $z \to \infty$ in S_p, $j = 1, \cdots, n$. In reference to $F(z)$ in (2.18) we denote for simplicity

$$\tau_j(z) = \exp\left\{-\int_{z_0}^{z} m_{jj}(\xi)\xi^{-1}\,d\xi\right\}, \qquad j = 1, \cdots, n, \quad z \in S_p. \tag{4.3}$$

Then, we have

$$\tau_j(z) = z_0^{(m/n)(n-1)/2} z^{-(m/n)\cdot(n-1)/2} \exp\left\{\int_{z_0}^{z}\left[-m_{jj}(\xi)\xi^{-1} + \frac{m}{n}\cdot\frac{n-1}{2}\xi^{-1}\right]d\xi\right\},$$

where the last exponential factor on the right tends to 1 as $z \to \infty$ in S_p.

THEOREM 4.2. *Let $\alpha < \beta$, $\beta - \alpha < \pi(m/n + 1)^{-1}$. Assume $S(\alpha, \beta)$ satisfies hypothesis* [$H_1 0$]. *Then for $c > 0$ sufficiently large, and $\varepsilon_1 > 0$ sufficiently small,* (2.11) *has a solution $V(z, \varepsilon)$ satisfying the following inequality* (Eh) *for each integer $h \geqq 0$. Let $V = \operatorname{col}(v_1, \cdots, v_n)$.*

$$\text{(Eh)}\quad \begin{aligned} &\left| v_j(z, \varepsilon) - \tau_j(z)\exp\left\{\frac{1}{\varepsilon}\int_{z_0}^{z}\lambda_1(\xi)\,d\xi\right\}\sum_{k=0}^{h}\alpha_{jk}(z, \varepsilon)\varepsilon^k\right| \\ &\qquad \leqq \varepsilon^{h+1}|\tau_j(z)|\left|\exp\left\{\frac{1}{\varepsilon}\int_{z_0}^{z}\lambda_1(\xi)\,d\xi\right\}\right|K_h(z, \varepsilon) \end{aligned}$$

for $j = 1, \cdots, n$, $z \in S(c; \alpha, \beta)$, $0 < \varepsilon \leqq \varepsilon_1$, where the function $K_h(z, \varepsilon)$ is uniformly of the order $O(|z|^{-(h+1)(m/n+1)})$ for $z \in S(c; \alpha, \beta)$, $0 < \varepsilon \leqq \varepsilon_1$. The functions $\alpha_{jk}(z, \varepsilon)$ are defined for $z \in S(c; \alpha, \beta)$, $0 < \varepsilon \leqq \varepsilon_1$ recursively by the following formulas:

$$\alpha_{j0}(z, \varepsilon) = \alpha_{j1}(z, \varepsilon) = 0 \quad \textit{for } j = 2, 3, \cdots, n, \quad \alpha_{10}(z, \varepsilon) = 1, \tag{4.4}$$

$$\begin{aligned} \alpha_{jk+1}(z, \varepsilon) = -\sum_{i=0}^{k-1}\left[L_j^{k-i-1}\left(\sum_{q=1}^{n} g_{jq}\alpha_{qi}(z, \varepsilon)\right)\right][\lambda_j(z) - \lambda_1(z)]^{-1}& \\ \textit{for } k = 1, 2, \cdots, \quad j = 2, 3, \cdots, n&, \end{aligned} \tag{4.5}$$

$$\alpha_{1k+1}(z, \varepsilon) = -\int_{C_{11}(z)}\left[\sum_{q=1}^{n} g_{1q}\alpha_{qk}(s, \varepsilon)\,ds\right] \quad \textit{for } k = 0, 1, 2, \ldots, \tag{4.6}$$

where

$$L_j(f(x, \varepsilon)) = \frac{d}{dx}[f(x, \varepsilon)(\lambda_j(x) - \lambda_1(x))^{-1}],$$

$$L_j^{r+1}(f(x, \varepsilon)) = L_j(L_j^r(f(x, \varepsilon))).$$

$C_{11}(z)$ are paths as defined in § 3 with θ_1, θ_2 replaced respectively by α, β. Furthermore

$$\alpha_{jk}(z, \varepsilon) = O(|z|^{-k(m/n+1)}) \quad \textit{as} \quad z \to \infty, \quad j = 1, \ldots, n,$$

uniformly for $z \in S(c; \alpha, \beta)$, $0 < \varepsilon \leqq \varepsilon_1$.

Proof. Choose $\theta_1 < \alpha < \beta < \theta_2$ such that $\theta_2 - \theta_1 < \pi(m/n + 1)^{-1}$, $\theta_2 - \beta = \alpha - \theta_1$. Choose p as in § 3 and B as in [H2]. Choose ε_1 as in Theorem 4.1. We now prove the theorem with $S(c; \alpha, \beta)$ replaced by D_p. Then it holds on $S(B; \alpha, \beta)$ and consequently on $S(c; \alpha, \beta)$ for $c \geqq B$.

Let $\psi(z)$ be a holomorphic n-vector on D_p such that

$$|\psi(z)| \leqq |z|^{-d} \tag{4.7}$$

if $z \in D_p$. Here d is a constant, $d > 1$. Then define

$$[\tilde{A}\psi](z, \varepsilon) = \int_{C_1(z)} E_1(s, z, \varepsilon)\psi(s)\, ds. \tag{4.8}$$

By [H4], $[\tilde{A}\psi](z, \varepsilon)$ is holomorphic in z if $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$ and there exists a constant K_0 independent of ψ, z and ε such that

$$|[\tilde{A}\psi](z, \varepsilon)| \leqq K_0 |z|^{1-d} \tag{4.9}$$

if $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$. For notational simplicity, let

$$\begin{aligned} v_j(z) &= [\lambda_j(z) - \lambda_1(z)]^{-1} \quad \text{if } j \neq 1, \\ \alpha_k(z, \varepsilon) &= \operatorname{col}(\alpha_{1k}, \cdots, \alpha_{nk}), \qquad k = 0, 1, \cdots. \end{aligned} \tag{4.10}$$

If $f(z)$ is a scalar holomorphic function in D_p, denote the jth component of the vector $[\tilde{A}(fe_j)](z, \varepsilon)$ by $\tilde{A}_j f$ where e_j is the column n-vector whose ith component is the Kronecker delta δ_{ij}. Finally, denote the jth row of G by G_j.

Partial integration in (4.8) leads to

$$\tilde{A}_j\psi_j = \sum_{r=1}^{N-1} \varepsilon^r v_j L_j^{r-1}\psi_j + \varepsilon^N\{\tilde{A}_j L_j^N \psi_j + v_j L_j^{N-1}\psi_j\} \quad \text{if } j \neq 1, \tag{4.11}$$

where ψ_j is the jth component of ψ. The operator K defined in Theorem 4.1, when expressed in terms of the operator $\tilde{A}$, becomes $K\psi = -\varepsilon\tilde{A}G\psi$. Hence

$$X_N = \sum_0^N K^j X_0 = X_0 - \varepsilon\tilde{A}GX_{N-1}. \tag{4.12}$$

We shall prove

$$X_N = \sum_{h=0}^{N} \varepsilon^h \alpha_h + \varepsilon^{N+1} R_N, \qquad N = 0, 1, \cdots, \tag{4.13}$$

where α_h is defined by (4.4)–(4.6), (4.10) and the jth component R_{jN} of R_N satisfies

$$\begin{aligned} R_{1N}(z, \varepsilon) &= -\sum_{q=1}^{n} \int_{C_{11}(z)} g_{1q} R_{q,N-1}(s, \varepsilon)\, ds, \\ R_{jN} &= \sum_{h=0}^{N-1} \{\tilde{A}_j L_j^{N-h} G_j \alpha_h + v_j L_j^{N-h-1} G_j \alpha_h\} + \tilde{A}_j G_j R_{N-1} \end{aligned} \tag{4.14}$$

if $j = 2, \cdots, n$. ($R_{-1} \equiv 0$.) Moreover, there exist constants, $M_k, k = 0, 1, \cdots,$ independent of z and ε such that

$$\begin{aligned} |\varepsilon^{k+1} R_k(z, \varepsilon)| \leqq M_k |\mu^{k+1}|, \qquad |\varepsilon^k \alpha_k(z, \varepsilon)| \leqq M_k |\mu^k| \\ \text{if } z \in D_p, \quad 0 < \varepsilon \leqq \varepsilon_1. \end{aligned} \tag{4.15}$$

We prove these formulas by induction. If $N = 0$ and $k = 0$, then (4.13), (4.14), (4.15) are evident. Suppose the formulas are valid for N replaced by $N - 1$

and for $k \leqq N - 1$. Then (4.12) implies

$$X_N = X_0 - \sum_{h=1}^{N} \varepsilon^h \tilde{A} G \alpha_{h-1} - \varepsilon^{N+1} \tilde{A} G R_{N-1}.$$

From this, (4.4), (4.6) and (4.14) we deduce that the first components of both sides of (4.13) are equal. Using (4.11) we obtain

$$\begin{aligned} [X_N]_j = & - \sum_{h=1}^{N} \varepsilon^h \sum_{r=1}^{N-h} \varepsilon^r v_j L_j^{r-1} G_j \alpha_{h-1} \\ & - \varepsilon^{N+1} \sum_{h=1}^{N} \{\tilde{A}_j L_j^{N-h+1} G_j \alpha_{h-1} + v_j L_j^{N-h} G_j \alpha_{h-1}\} - \varepsilon^{N+1} \tilde{A}_j G_j R_{N-1} \\ & \qquad \text{if } j \neq 1. \end{aligned} \tag{4.16}$$

Using (4.4), (4.5) and (4.14) we deduce (4.13). Furthermore, (4.15) with $k = N$ follows from (4.15) with $k \leqq N - 1$, (2.20), (4.4)–(4.6), (4.14), (4.9) and the last two statements of Lemma 4.1.

Finally (4.12) implies

$$X - X_N = -\varepsilon \tilde{A} G (X - X_{N-1}).$$

From this, (4.2), (2.20) and (4.9) we deduce

$$X - X_N = O(\mu^{N+1}) \tag{4.17}$$

as $z \to \infty$ uniformly for $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$ for $N = 0, 1, 2, \cdots$ successively. The theorem now follows from (4.17), (4.13), (4.15), (2.18) and (4.3).

The following lemma has been used in the proof of Theorem 4.2 and will be used for proving Theorem 4.3. We omit the proof of the lemma because it involves merely straightforward induction using properties [H1] to [H4].

LEMMA 4.1. *The notation is that of Theorem* 4.2. *The functions* $\alpha_{jk}(z, \varepsilon)$, $j = 1, \cdots, n$, $k = 0, 1, \cdots$, *can be extended by means of formulas* (4.4)–(4.6) *to be defined in the set* $S(\hat{K}; \hat{\theta}_1, \hat{\theta}_2)$ *containing* D_p, *with* $\hat{K} < |z_0|$, $\hat{\theta}_1 < \theta_1 < \theta_2 < \hat{\theta}_2$, $\hat{\theta}_2 - \hat{\theta}_1 < \pi(m/n + 1)^{-1}$. *They satisfy*

$$z^{k(m/n+1)} \alpha_{jk}(z, \varepsilon) \sim \sum_{r=0}^{\infty} \alpha_{jkr}(z) \mu^r \quad \text{as } \mu \to 0 \tag{4.18}$$

uniformly for $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$, $j = 1, \cdots, n$, *where* $\alpha_{jkr}(z)$, $k \neq 0$, $r = 0, 1, \cdots$ *possess convergent expansions in descending powers of* $z^{1/n}$ *in* D_p *and are of the order* $O(1)$ *as* $z \to \infty$ *in* D_p. *Furthermore, asymptotic series in powers of* $\mu = \varepsilon z^{-m/n-1}$ *for any derivative of* $\alpha_{jk}(z, \varepsilon)$ *in* D_p *can be found by formally differentiating* (4.18) *termwise. For* $j \neq 1$, *the operator* L_j^s *reduces the order with respect to* z *of the coefficient of* μ^h *in the asymptotic expansion of the function* $\sum_{q=1}^{n} g_{jq} \alpha_{qk}(z, \varepsilon)$ *by* $-s(m/n + 1)$ *in* D_p. (For the asymptotic series of the derivatives of $\alpha_{jk}(z, \varepsilon)$, see W. Wasow [5, Theorem 9.4, p. 44].)

Remarks. Because g_{jq} are ~ 0 for $j \neq q$, the functions $\alpha_{jk}(z, \varepsilon) \sim 0$ as $\mu \to 0$ uniformly for $z \in D_p$, $0 < \varepsilon \leqq \varepsilon_1$, $k = 0, 1, \cdots$, for all $j \neq 1$ by (4.4), (4.5).

Remarks. Note that in (2.18), (2.19) the root $\lambda_1(z)$ plays a particular role among all other roots $\lambda_j(z)$. Actually, transformation analogous to (2.18) can be made with

$\lambda_1(\xi)$ replaced by any $\lambda_k(\xi)$, $k \neq 1$. Let $S^{jk}(\theta_1, \theta_2)$, $1 \leqq k, j \leqq n$, $k \neq j$ be the image of $S(\theta_1, \theta_2) = \{z | \theta_1 < \arg z < \theta_2\}$ on the $(m/n + 1)^{-1}(a_{0j} - a_{0k})z^{m/n+1}$-plane. Suppose $S(\theta_1, \theta_2)$ satisfies the following hypothesis.

[H_k0]. $S^{jk}(\theta_1, \theta_2)$ contains a ray in the right half-plane, for each integer $j \neq k$, $1 \leqq j \leqq n$.

We can construct paths $C_{jk}(z)$, $1 \leqq j \leqq n$, satisfying properties analogous to [H1] to [H4] with the role of λ_1 replaced by λ_k. Then we shall arrive at theorems analogous to Theorems 4.1, 4.2 with the role of $\lambda_1(z)$ replaced by $\lambda_k(z)$. Of course the formulas in Theorem 4.2 should be changed in the same manner.

THEOREM 4.3. *Let θ_1, θ_2 be angles such that $0 < \theta_2 - \theta_1 < \pi(m/n + 1)^{-1}$. Let r be an integer $1 \leqq r \leqq n$. Assume $S(\theta_1, \theta_2)$ satisfies hypothesis* [H_r0]. *Let α, β be angles $\theta_1 < \alpha < \beta < \theta_2$. Then there exist constants $c > 0$ sufficiently large and $\varepsilon_r > 0$ sufficiently small, such that for z, ε in the region*

$$z \in S(c; \alpha, \beta), \qquad 0 < \varepsilon \leqq \varepsilon_r, \tag{4.19}$$

equation (1.1) *has a solution $u_r(z, \varepsilon)$ with the following asymptotic property*:

$$\begin{aligned} &\left| u_r(z, \varepsilon) - z^{-(m/n)(n-1)/2} \exp\left\{ \frac{1}{\varepsilon} \int_a^z \lambda_r(\xi)\, d\xi \right\} \sum_{k=0}^{\lambda} u_{rk}(z) \varepsilon^k \right| \\ &\qquad \leqq \varepsilon^{\lambda+1} M_{r\lambda}(z, \varepsilon) |z|^{-(m/n)(n-1)/2} \left| \exp\left\{ \frac{1}{\varepsilon} \int_a^z \lambda_r(\xi)\, d\xi \right\} \right| \end{aligned} \tag{4.20}$$

in region (4.19) *for each nonnegative integer λ, where a is an arbitrary fixed point in $S(\theta_1, \theta_2)$. The functions $u_{rk}(z)$, $k = 0, 1, \cdots$, are of the order $O(|z|^{-k(m/n+1)})$ in $S(c; \alpha, \beta)$ and possess convergent expansions in descending powers of $z^{1/n}$ there*;

$$u_{r0}(z) = \exp\left\{ \int_a^z \left[- m_{rr}(\xi)\xi^{-1} + \frac{m}{n} \cdot \frac{n-1}{2} \xi^{-1} \right] d\xi \right\}$$

which tends to 1 *as $z \to \infty$ in $S(c; \alpha, \beta)$. The functions $M_{r\lambda}(z, \varepsilon)$, $\lambda = 0, 1, \cdots$, are of the order $O(|z|^{-(\lambda+1)(m/n+1)})$ in the region* (4.19).

Proof. By the remarks immediately above, it suffices to show the case when $r = 1$. Refer to (2.4), (2.10) and part (d) of Theorem 2.1, which together express u in terms of V. We only have to apply Theorem 4.2 and (4.18), rearrange terms, and express u in terms of V and R to obtain an asymptotic series for u in powers of μ^k, as $\mu \to 0$. Finally, let $u_1(z, \varepsilon)$ be an appropriate constant multiple of u to deduce formula (4.20).

COROLLARY 4.4. *The asymptotic formulas for $u_{rk}(z)$, $r = 1, \cdots, n$, $k = 0, 1, \cdots$, can be explicitly constructed by using the expansion for $R(z, \mu)$ in* (2.13), *the expansion for V in Theorem* 4.2 *and* (4.18).

COROLLARY 4.5. *The asymptotic series in powers of μ^k as $\mu \to 0$ for any derivative of $u_r(z, \varepsilon)$, $r = 1, \cdots, n$, with respect to z in region* (4.19) *can be found by formally differentiating the left of* (4.20).

Proof. Refer to W. Wasow [5, Theorem 9.4, p. 44].

Proof of the main theorem (*Theorem* 1.1). Let $\hat{\omega}_1, \hat{\omega}_2$ be angles such that $\hat{\omega}_1 < \omega_1 < \omega_2 < \hat{\omega}_2$, $\hat{\omega}_2 - \hat{\omega}_1 < (\pi/n)(m/n + 1)^{-1}$. There exist an angle ω_3 and an integer k such that

$$|\omega_3 - \hat{\omega}_j| < \frac{\pi}{2n}\left(\frac{m}{n} + 1\right)^{-1} \quad \text{for } j = 1, 2,$$

and

$$\frac{2k-1}{n}\pi < \arg\left\{ -a_{01} \exp\left[i\omega_3\left(\frac{m}{n} + 1\right)\right]\right\} < \frac{2k+1}{n}\pi.$$

Without loss of generality, we assume

$$a_{0g+1} = a_{01} \exp\frac{2g\pi i}{n}, \qquad g = 1, 2, \cdots, n-1,$$

and define

$$a_{0s} = a_{0r} \quad \text{if } s = r \,(\operatorname{mod} n).$$

If we consider two angles to be equal if they differ by an integral multiple of 2π, then

$$\arg(a_{0g+1+j} - a_{0g+1}) = \arg(ia_{01}) + \frac{2g+j}{n}\pi,$$

$j = 1, \cdots, n-1$. Let

$$\delta_{g+1} = \omega_3 - 2(k+g)\frac{\pi}{m+n}. \tag{4.21}$$

Then

$$\begin{aligned} &\arg\left\{(a_{0g+1+j} - a_{0g+1}) \exp\left[i\delta_{g+1}\left(\frac{m}{n} + 1\right)\right]\right\} \\ &\quad = -\frac{1}{2}\pi + \frac{j\pi}{n} + \arg\left\{ -a_{01} \exp\left[i\omega_3\left(\frac{m}{n} + 1\right)\right]\right\} - \frac{2k\pi}{n} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{aligned} \tag{4.22}$$

for $j = 1, 2, \cdots, n-1$. Let

$$S_1 \equiv S\left(\omega_3 - \pi\left(\frac{m}{n} + 1\right)^{-1}\left(1 - \frac{1}{2n}\right), \hat{\omega}_2\right).$$

Equations (4.21) and (4.22) imply that the π_{jg+1}-image of the ray $\arg z = \delta_{g+1}$ belongs to $\pi_{jg+1}(S_1) \cap \{\operatorname{Re} z > 0\}$, $j = 1, 2, \cdots, (n-1)$, whenever $0 \leqq k + g < n/2$, where

$$\pi_{ji}(z) = (m/n + 1)^{-1}(a_{0j} - a_{0i})z^{m/n+1}.$$

Thus, if $\tilde{g} + 1 = g + 1 \,(\operatorname{mod} n)$, $1 \leqq \tilde{g} + 1 \leqq n$, then S_1 satisfies $[\mathrm{H}_{\tilde{g}+1}0]$ when $0 \leqq k + g < \frac{1}{2}n$.

In the same way, let

$$S_2 \equiv S\left(\hat{\omega}_1, \omega_3 + \pi\left(\frac{m}{n} + 1\right)^{-1}\left(1 - \frac{1}{2n}\right)\right).$$

S_2 satisfies $[H_{\tilde{g}+1}0]$, if $\tilde{g} + 1 = g + 1 \pmod{n}$, $1 \leqq \tilde{g} + 1 \leqq n$, $-\frac{1}{2}n < k + g \leqq 0$. Since $S(\hat{\omega}_1, \hat{\omega}_2) \subset S_1 \cap S_2$, we may apply Theorem 4.3 and find n linear independent solutions $u_j(z)$, $j = 1, \cdots, n$, in $S(L; \omega_1, \omega_2)$ which satisfy (1.4).

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ON OBTAINING GENERATING FUNCTIONS OF BOAS AND BUCK TYPE FOR ORTHOGONAL POLYNOMIALS*

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Abstract. In this paper we introduce a method to obtain all generating functions of Boas and Buck type for any given orthogonal polynomial set. We also characterize the orthogonal polynomials that satisfy

$$xP_n'(x) = nP_n(x) + \sum_0^{n-1} (-1)^{n-k}\rho_k P_k(x) \quad \text{with } \rho_k \neq 0, \qquad k = 0, 1, \cdots.$$

1. Introduction. A polynomial set $\{P_n(x)\}_0^\infty$ is said to have a generating function of Boas and Buck type if there exists a sequence of nonzero numbers $\lambda_0, \lambda_1, \cdots, \lambda_n, \cdots$ such that

$$\sum_0^\infty \lambda_n t^n P_n(x) = A(t)\phi(xH(t)), \tag{1.1}$$

where

$$A(t) = \sum_0^\infty a_n t^n, \quad H(t) = \sum_1^\infty h_k t^k, \quad \phi(t) = \sum_0^\infty \phi_k t^k.$$

The condition $a_0 h_1 \phi_n \neq 0$ is necessary to make $P_n(x)$ of exact degree n.

Given any polynomial set $\{P_n(x)\}_0^\infty$ we can construct a sequence μ_n^n so that

$$xP_n'(x) = \sum_{k=0}^{n-1} \mu_k^n P_k(x) + nP_n(x). \tag{1.2}$$

From now on we restrict ourselves to orthogonal polynomials $P_n(x)$. Let

$$g_n \delta_{m,n} = \int_I P_n(x)P_m(x)\, d\alpha(x), \tag{1.3}$$

with $\alpha(x)$ of bounded variation on the interval I.

In § 2 we give a method for obtaining generating functions of Boas and Buck type for a given orthogonal polynomial set. Later in § 2 we apply this method to the classical polynomials. In § 3 we investigate polynomials that are orthogonal, have a generating function of the above type and for which

$$\mu_k^n = (-1)^{n-k}\rho_k, \qquad \rho_k \neq 0.$$

2. A method for obtaining generating functions. All the operations we perform are formal ones and we shall pay no attention to questions of convergence.

Define

$$k_n = \lambda_n^2 g_n, \quad f(x) = \sum_0^\infty k_n x^n, \quad \text{and} \quad G(x,t) = A(t)\phi(xH(t)).$$

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Clearly,

$$f(st) = \int_I G(x,t)G(x,s)\,d\alpha(x). \tag{2.1}$$

Differentiating (2.1) with respect to t and using

$$\frac{\partial G}{\partial t} = \frac{A'(t)}{A(t)}G(x,t) + x\frac{H'(t)}{H(t)}\frac{\partial G(x,t)}{\partial x},$$

we get

$$sf'(st) = \frac{A'(t)}{A(t)}f(st) + \frac{H'(t)}{H(t)}\int_I xG(x,s)\frac{\partial G(x,t)}{\partial x}\,d\alpha(x).$$

It is easy to see that

$$\left[\frac{sH(t)}{H'(t)} + \frac{tH(s)}{H'(s)}\right]f'(st) = \left[\frac{A'(t)H(t)}{A(t)H'(t)} + \frac{A'(s)H(s)}{A(s)H'(s)}\right]f(st) + g(s,t) \tag{2.2}$$

where

$$g(s,t) = \int_I x\frac{\partial}{\partial x}G(x,s)G(x,t)\,d\alpha(x). \tag{2.3}$$

Let

$$g(s,t) = \sum_0^\infty t^n\Gamma_n(s), \qquad \Gamma_n(s) = \sum_0^\infty \gamma_{n,k}s^k.$$

It is obvious that $\gamma_{n,k} = \gamma_{k,n}$. Equating coefficients of t and the constant terms in both sides of (2.2) we get

$$\frac{A'(s)H(s)}{A(s)H'(s)} = -\frac{\Gamma_0(s)}{k_0} \quad\text{and}\quad \frac{H(s)}{H'(s)} = -s - \frac{s\Gamma_0(s)}{k_0} + \frac{\Gamma_1(s) - \gamma_{1,0}}{k_1}, \tag{2.4}$$

as well as

$$\gamma_{0,0} = 0 \quad\text{and}\quad \gamma_{1,1} = 2k_1.$$

Eliminating$A'(s)/A(s)$ and $H(s)/H'(s)$ between (2.2) and (2.4) we obtain

$$\left\{\frac{\Gamma_1(t) - \gamma_{1,0}}{k_1 t} + \frac{\Gamma_1(s) - \gamma_{1,0}}{k_1 s} - 2\right\}\sum_2^\infty nk_n(st)^n + \Gamma_0(t) - \gamma_{0,1}t$$

$$+ s(\Gamma_1(t) - \gamma_{1,0} - \gamma_{1,1}t) = \frac{1}{k_0}\{\Gamma_0(s) + \Gamma_0(t)\}\sum_2^\infty (n-1)k_n(st)^n + \sum_2^\infty t^n\Gamma_n(s).$$

By straightforward induction we have

$$\gamma_{j,j} = 2jk_j, \tag{2.5}$$

$$\Gamma_j(s) = \sum_{l=0}^{j-1}\gamma_{l,j}s^l + k_js^j\left[\frac{j}{k_1}\left(\frac{\Gamma_1(s) - \gamma_{1,0}}{s}\right) - \frac{(j-1)}{k_0}\Gamma_0(s)\right] \tag{2.6}$$

and

$$\left\{\frac{\Gamma_1(s)-\gamma_{1,0}}{k_1 s}+\frac{\Gamma_1(t)-\gamma_{1,0}}{k_1 t}-2\right\}\sum_{n=j}^{\infty} nk_n(st)^n+\sum_{l=0}^{j-1} s^l \sum_{m=j}^{\infty}\gamma_{l,m}t^m$$

$$(2.7)\qquad =(\Gamma_0(s)+\Gamma_0(t))\sum_{n=j}^{\infty}(n-1)\frac{k_n}{k_0}(st)^n+\sum_{n=j}^{\infty}t^n\Gamma_n(s).$$

On the other hand,

$$(2.8)\qquad g(s,t)=\sum_{m,n=0}^{\infty}\lambda_m\lambda_n\Lambda_{m,n}s^m t^n$$

where, by (2.3) and (1.2),

$$\Lambda_{m,n}=(\mu_m^n g_m+\mu_n^m g_n),\qquad \mu_n^n=n,\quad \mu_m^n=0\quad \text{if } m>n.$$

Therefore,

$$(2.9)\qquad \Gamma_j(s)=\lambda_j\sum_{m=0}^{\infty}\lambda_m\Lambda_{j,m}s^m.$$

Comparing (2.6) and (2.9) we get

$$(2.10)\qquad \lambda_j\lambda_m\Lambda_{j,m}=\gamma_{m,j},\qquad m<j,$$

and

$$(2.11)\qquad \frac{\lambda_{m+j}}{\lambda_j}\Lambda_{j,m+j}=g_j\left[\frac{j}{g_1\lambda_1^2}\gamma_{1,m+1}-\frac{(j-1)}{\lambda_0^2 g_0}\gamma_{0,m}\right].$$

Equation (2.11) may be written as

$$(2.12)\qquad \frac{\lambda_{m+j}}{\lambda_j}\frac{\Lambda_{j,m+j}}{g_j}=\frac{j}{g_1\lambda_1}\Lambda_{1,m+1}\lambda_{m+1}-\frac{(j-1)}{\lambda_0 g_0}\lambda_m\Lambda_{0,m},$$

or equivalently,

$$(2.13)\qquad \frac{\lambda_{m+j}}{\lambda_j}\mu_j^{m+j}=j\mu_1^{m+1}\frac{\lambda_{m+1}}{\lambda_1}-(j-1)\mu_0^m\frac{\lambda_m}{\lambda_0}.$$

Therefore for any given set $\{P_n(x)\}_0^\infty$, the $\Lambda_{m,n}$'s and the g_n's can be evaluated. The equations (2.10) and (2.13), if solvable give the λ's and the γ's. Then the first order differential equations (2.4) allow us to calculate $A(t)$ and $H(t)$. To evaluate $\phi(x)$ we note that

$$\sum_0^\infty \lambda_j P_j\left(\frac{x}{h_1 t}\right)t^j=A(t)\phi\left(\frac{xH(t)}{h_1 t}\right),$$

and as $t\to 0$ we get

$$\phi(x)=\sum_{j=0}^{\infty}\lambda_j\left(\frac{x}{h_1}\right)^j\frac{q_j}{a_0},$$

where

$$P_j(x)=q_j x^j+\cdots.$$

We may assume, without loss of generality, that $h_1=a_0=1$.

I applied the above method to the classical polynomials, namely the Jacobi, Hermite, Laguerre and Bessel polynomials and the result is the following theorem.

THEOREM 1. *The only generating functions of Boas and Buck type for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x+1)$, Hermite polynomials $H_n(x)$, Laguerre polynomials $L_n^{(\alpha)}(x)$ and the Bessel polynomials $\phi_n(c, x)$ are*

$$\sum_0^\infty \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(1+x)t^n = (1-t)^{-\alpha-\beta-1} {}_2F_1\left[\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \alpha+1; \frac{2xt}{(1-t)^2}\right], \tag{2.14}$$

$$\sum_0^\infty \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\left(\frac{\alpha+\beta+1+2n}{\alpha+\beta+1}\right) t^n P_n^{(\alpha,\beta)}(x+1) = \frac{1+t}{(1-t)^{\alpha+\beta+2}} {}_2F_1\left[\frac{\alpha+\beta+3}{2}, \frac{\alpha+\beta+2}{2}; \alpha+1; \frac{2xt}{(1-t)^2}\right], \tag{2.15}$$

with the two corresponding limiting cases when $\alpha+\beta+1 \to 0$,

$$\lambda_0 \sum_0^\infty \frac{H_{2n}(x)}{(2n)!} t^{2n} + \lambda_1 \sum_0^\infty \frac{H_{2n+1}(x)}{(2n+1)!} t^{2n+1} = e^{-t^2}[\lambda_0 \cosh 2xt + \lambda_1 \sinh 2xt], \tag{2.16}$$

$$\lambda_0 \sum_0^\infty \frac{(c)_n}{(2n)!} H_{2n}(x)t^{2n} + \lambda_1 \sum_0^\infty \frac{(c+\frac{1}{2})_n}{(2n+1)!} H_{2n+1}(x)t^{2n+1} = (1+t^2)^{-c}\left\{\lambda_0\, {}_1F_1\left(c; \frac{1}{2}; \frac{x^2t^2}{t^2+1}\right) + \lambda_1 \frac{tx}{\sqrt{1+t^2}}\, {}_1F_1\left(c+\frac{1}{2}; \frac{3}{2}; \frac{x^2t^2}{t^2+1}\right)\right\}, \tag{2.17}$$

$$\sum_0^\infty \frac{(c)_n}{(\alpha+1)_n} L_n^{(\alpha)}(x)t^n = (1-t)^{-c}\, {}_1F_1\left(c; \alpha+1; \frac{-xt}{1-t}\right), \tag{2.18}$$

$$\sum_0^\infty L_n^{(\alpha)}(x) \frac{t^n}{(\alpha+1)_n} = e^t\, {}_0F_1(-; 1+\alpha; -xt), \tag{2.19}$$

$$\sum \phi_n(c, x)t^n = (1-t)^{-c}\, {}_2F_0\left(\frac{1}{2}c; \frac{1}{2}c+\frac{1}{2}; -; \frac{-4xt}{(1-t)^2}\right), \tag{2.20}$$

$$\sum_0^\infty (2n+c)\phi_n(c, x)t^n = \frac{c(1+t)}{(1-t)^{c+1}}\, {}_2F_0\left(\frac{1}{2}c+1, \frac{1}{2}c+\frac{1}{2}; -; \frac{-4xt}{(1-t)^2}\right). \tag{2.21}$$

Remarks. The class of generating functions (2.17) is contained in Brafman's class of peculiar generating functions [2]. All the other generating functions are essentially in [3]. The special Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, i.e., the Gegenbauer polynomials $C_n^{\alpha+1/2}(x)$ possess two more generating functions of the required type. They are

$$\lambda_0 \sum_0^\infty C_{2n}^\nu(x)t^{2n} + \lambda_1 \sum_0^\infty C_{2n+1}^\nu(x)t^{2n+1} = \tfrac{1}{2}(\lambda_0+\lambda_1)(1-2xt+t^2)^{-\nu} + \tfrac{1}{2}(\lambda_0-\lambda_1)(1+2xt+t^2)^{-\nu}, \tag{2.22}$$

$$\lambda_0 \sum_0^\infty (2n+\nu)C_{2n}^\nu(x)t^{2n} + \lambda_1 \sum_0^\infty (2n+\nu+1)C_{2n+1}^\nu(x)t^{2n+1}$$

$$= \frac{\nu}{2}(1-t^2)[(\lambda_0+\lambda_1)(1-2xt+t^2)^{-\nu-1} + (\lambda_0-\lambda_1)(1+2xt+t^2)^{-\nu-1}], \tag{2.23}$$

which are again essentially in [3].

3. Orthogonal polynomials with $\mu_k^n = (-1)^{n-k}\rho_k$, $\rho_k \neq 0$. Let $P_n(x)$ be such a set, so that

$$xP_n'(x) = nP_n(x) + \sum_0^{n-1} (-1)^{n-k}\rho_k P_k(x), \tag{3.1}$$

and assume that

$$P_{n+1}(x) = (xA_n + B_n)P_n(x) - \lambda_n P_{n-1}(x), \tag{3.2}$$
$$(P_{-1}(x) = 0, \quad P_0(x) = 1, \quad \lambda_n A_n \neq 0),$$

is the three term recurrence relation satisfied by $P_n(x)$. Differentiate (3.2), and use (3.1) to eliminate $P_n'(x)$. The result is

$$\begin{aligned} B_n\left\{P_n(x) - \sum_{k=0}^{n-1}(-1)^{n-k}\rho_k P_k(x)\right\} &= \sum_{k=0}^{n}(-1)^{n-k}\rho_k P_k(x) \\ &\quad + \lambda_n\left(2P_{n-1}(x) + \sum_0^{n-2}(-1)^{n-k}\rho_k P_k(x)\right) \\ &\quad + A_n \sum_0^{n-1} \frac{\rho_k}{A_k}(-1)^{n-k}[P_{k+1}(x) + \lambda_k P_{k-1}(x) \\ &\qquad - B_k P_k(x)], \end{aligned} \tag{3.3}$$

and hence,

$$\frac{B_n}{A_n} = \frac{\rho_n}{A_n} - \frac{\rho_{n-1}}{A_{n-1}}, \tag{3.4}$$

$$2\frac{\lambda_n}{A_n} = \frac{(1+\rho_n)}{A_n}\rho_{n-1} - \frac{2\rho_{n-1}^2}{A_{n-1}} + \frac{\rho_{n-2}}{A_{n-2}}(\rho_{n-1} - 1), \tag{3.5}$$

$$(1 + B_n + \lambda_n)\rho_k = A_n\left[\frac{\rho_{k-1}}{A_{k-1}} + \frac{\rho_{k+1}}{A_{k+1}}\lambda_{k+1} + \frac{\rho_k}{A_k}B_k\right]. \tag{3.6}$$

Equality (3.6) is equivalent to

$$\frac{1}{A_n} + \frac{B_n}{A_n} + \frac{\lambda_n}{A_n} = h, \tag{3.7}$$

and

$$\frac{\rho_{k-1}}{\rho_k A_{k-1}} + \frac{\rho_{k+1}}{\rho_k A_{k+1}}\lambda_{k+1} + \frac{B_k}{A_k} = h, \tag{3.8}$$

for some constant h, independent of k or n.

Eliminating B_n and λ_n from (3.4), (3.5) and (3.7), we get

$$\frac{(1+\rho_n)(2+\rho_{n-1})}{A_n} - 2\frac{\rho_{n-1}}{A_{n-1}}(1+\rho_{n-1}) + \frac{\rho_{n-2}}{A_{n-2}}(\rho_{n-1}-1) = 2h. \tag{3.9}$$

Similarly (3.8), with $k = n - 1$ implies

$$\frac{\rho_n(\rho_n+1)}{A_n} + 2\frac{\rho_{n-1}}{A_{n-1}}(1-\rho_n) + \frac{\rho_{n-2}(\rho_n-2)}{A_{n-2}\rho_{n-1}}(\rho_{n-1}-1) = 2h. \tag{3.10}$$

Subtracting (3.9) from (3.10) we get

$$\frac{1+\rho_n}{A_n} - 2\frac{\rho_{n-1}}{A_{n-1}} + \frac{\rho_{n-2}}{A_{n-2}}\frac{(\rho_{n-1}-1)}{\rho_{n-1}} = 0,$$

or

$$\rho_n - \rho_{n-1} = 2.$$

The first alternative is impossible by (3.5) since λ_n is never zero. Thus

$$\rho_n = c + 2n,$$

where c is an arbitrary constant. Now the solution of (3.9) is

$$\frac{(c+2n)(c+2n+1)}{A_n} = hn(n+1) + an + b, \tag{3.11}$$

where a and b are arbitrary constants. Note that $c \neq 0$ since $\rho_0 \neq 0$. Set

$$G(x,t) = \sum_0^\infty P_n(x)t^n. \tag{3.12}$$

It is easy to see that (3.1), with $\rho_k = c + 2k$, implies

$$x\frac{\partial G}{\partial x} = \left(\frac{1-t}{1+t}\right)\frac{\partial G}{\partial x} - \frac{c}{1+t}G(x,t). \tag{3.13}$$

If we set $G(x,t) = (1-t)^{-c}\phi(x,t)$, substitute in (3.13), change the variables x, t to $\xi = 4xt/(1-t)^2$ and $\eta = t$ and then solve the resulting simple equation, we obtain

$$\sum P_n(x)t^n = (1-t)^{-c}\phi\left(\frac{4xt}{(1-t)^2}\right), \tag{3.14}$$

where $\phi(\xi) = \sum \gamma_n \xi^n$, say, is an arbitrary function. Clearly,

$$A_n = 4(\gamma_{n+1}/\gamma_n). \tag{3.15}$$

Now (3.11) and (3.15) will imply

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{(c+2n)(c+2n+1)}{4\{h(n+1)n + an + b\}}. \tag{3.16}$$

If $h \neq 0$, then the solution of (3.16) is

$$\gamma_n = \frac{(c/2)_n}{(A_1)_n}\frac{(c/2+\frac{1}{2})_n}{(A_2)_n}\frac{\gamma_0}{h^n},$$

with

$$hn(n+1)+an+b=h(n+A_1)(n+A_2).$$

In this case, there is no loss of generality to assume $h=-1$ and $\gamma_0=1$. Consequently we have

$$\sum_0^\infty P_n(x)t^n=(1-t)^{-c}\,{}_3F_2\left[\frac{c}{2},\frac{c}{2}+\frac{1}{2},1;A_2,A_1;\frac{-4xt}{(1-t)^2}\right]. \tag{3.17}$$

The case $h=0$ can be treated similarly and both cases imply that (3.17) is valid if we adopt the convention that the ${}_3F_2$ means a ${}_3F_1$ (${}_3F_0$) if one (two) of the denominator parameters vanishes. The explicit form of $P_n(x)$ is

$$P_n(x)=\frac{(c)_n}{n!}\,{}_3F_2[-n,n+c,1;A_1,A_2;x].$$

If $A_1=1$ and $A_2=0$, then $P_n(x)$ reduces to the Bessel polynomial $\phi_n(c,x)$. If $A_1=1$, $A_2=\alpha+1$, $c=\alpha+\beta+1$, then $P_n(x)$ reduces to $(\alpha+\beta+1)_n/(1+\alpha)_n \cdot P_n^{(\alpha,\beta)}(1-2x)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order n.

Finally Theorem 1 of N. Al-Salam [1] shows that these are the only orthogonal polynomials in this class. Thus we proved the following characterization of the Bessel and Jacobi polynomials.

THEOREM 2. *The only orthogonal polynomials that satisfy* (3.1) *with* $\rho_k\neq 0$ *are the Bessel and Jacobi polynomials.*

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INTEGRAL REPRESENTATION, ANALYTIC CONTINUATION AND THE REFLECTION PRINCIPLE UNDER THE COMPLEMENTING BOUNDARY CONDITION FOR HIGHER ORDER ELLIPTIC EQUATIONS IN THE PLANE*

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Abstract. Let $a_k^{pq}(x, y)$ be analytic functions for x, y in the $z = x + iy$ plane. Consider the equation $\Delta^n u + \sum_{k=1}^n L_k(\Delta^{n-k}u) = 0$, where $L_k = \sum_{p,q=0}^{p+q\leqq k} a_k^{pq}(x, y)\partial^{p+q}/\partial x^p \partial y^q$. The following three areas will be investigated: (i) integral representation for solutions of the above equation to the boundary ∂G of a simply connected domain G, (ii) the reflection principle for solutions of the above equation under the complementing boundary conditions, (iii) analytic continuation and nonexistence for solutions of the Cauchy problem.

Introduction. The main purpose of this paper is to study the integral representation for solutions of elliptic equations of the type

$$(0.1)\qquad \begin{gathered} \Delta^n u + \sum_{k=1}^{n} L_k(\Delta^{n-k}u) = 0, \\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad L_k = \sum_{p,q=0}^{p+q\leqq k} a_k^{pq}(x, y)\frac{\partial^{p+q}}{\partial x^p \partial y^q}. \end{gathered}$$

in a closed domain $\bar{G}$ of the $z = x + iy$ plane, and its applications to the Cauchy problem and the reflection principle under the complementing boundary condition for solutions of (0.1), where the coefficients of (0.1) are real- or complex-valued analytic functions of two real variables x, y in a neighborhood of $\bar{G}$.

The theory described in the work of Vekua [14] has established the complex integral representation (1.1) for the solution of (0.1) in a simply connected domain G. However, his theorem cannot immediately be applied to $G \cup \partial G$, except under some very restrictive conditions, where ∂G is the boundary of G. In a variety of applications, due to the generality of the boundary conditions, the abovementioned representation is not general enough to be applied to some wider classes of solutions. These are also the main difficulties encountered in applying the method of Lewy [8] to study the reflection principle under complementing boundary conditions and the Cauchy problem for solutions of (0.1).

We shall overcome these difficulties by an elementary method in § 2.

Until now the main emphasis in the study of the Cauchy problem for elliptic equations has been on the question of uniqueness. Very little has been devoted to the question of existence of solutions, except for specific equations, data, and geometry (see, e.g., Payne [10]). The existence question is an extremely difficult one since, frequently, in order for a solution to exist, very strong compatibility relations must exist among the data.

By investigating the relations between $\Delta_2 u + \lambda u = 0$ and $\Delta_2 u = 0$, whose solutions share Cauchy data, Lewy [8] gave very interesting sufficient conditions

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for the analytic continuation and the nonexistence of the solution of the Cauchy problem for the equation $\Delta_2 u + \lambda u = 0$; but his results are local, and the domains of which continuation is possible are not made explicit. Yu later [17] extended these results to elliptic systems of two first order equations.

In the present paper the relations between two different differential equations of type (0.1) whose solutions share Cauchy data are discussed. We obtain explicit information of the domain of continuation and sufficient conditions for the non-existence of the solutions of the Cauchy problem for (0.1).

The classical reflection principle for the Cauchy–Riemann equations has been generalized by many authors to solutions of various types of elliptic equations with analytic coefficients under analytic boundary conditions in the plane. Second order equations have been investigated by Lewy [8], biharmonic equations by Poritsky [11], Duffin [4], Bramble [2], Sloss [12], polyharmonic equations by Huber [6], Kraft [7], and systems of two first order equations by Yu [16], [17].

Some higher order equations with constant coefficients have been treated by Brown [2] and Sloss [12]. Yu [15] demonstrated a reflection principle for (0.1) under analytic Dirichlet boundary conditions. Morrey [9] studied the analyticity at the boundary for general elliptic systems under the complementing boundary conditions. (See Agmon, Douglis and Nirenberg [1].)

In connection with these, it is interesting to consider the reflection principle for elliptic equations with the complementing boundary conditions. In this paper we shall study such a kind of reflection principle for (0.1).

In § 1 we introduce some notation and give a brief summary of Vekua's theory of the representation of the solutions of (0.1) in a domain G. In § 2 we extend Vekua's integral representation to the boundary ∂G of G. In § 3 we study the analytic continuation and the nonexistence of the solutions of the Cauchy problem for (0.1). In § 4 we establish the reflection principle for (0.1) under the complementing boundary condition.

1. Notation and integral representation on *G*. Throughout this paper let D denote a simply connected domain of the $z = x + iy$ plane whose boundary contains a segment σ, $\sigma = \{x : a < x < b\}$. It is assumed that σ contains the origin as an interior point. Also let G denote a simply connected domain such that $G \cup \partial G \subset D \cup \sigma \cup \bar{D}$ and $\sigma \subset G$, where $\bar{D} = \{z | z \in C, \bar{z} \in D\}$ and ∂G is the boundary of G.

According to the theory of functions of several complex variables, there corresponds a function called the analytic continuation of $a_k^{pq}(x, y)$ in (0.1), which is holomorphic in a domain of the space C^2 of the two complex variables x, y, and which coincides with $a_k^{pq}(x, y)$ when x, y are real. We shall denote the analytic continuation of $a_k^{pq}(x, y)$ again by $a_k^{pq}(x, y)$.

Let us introduce new variables z, ζ by the relations

$$z = x + iy, \qquad \zeta = x - iy;$$

the variables z and ζ are conjugate if and only if x and y are real. We put

$$A_k^{pq}(z, \zeta) = a_k^{pq}(x, y).$$

We further assume that $A_k^{pq}(z, \zeta)$ are holomorphic functions for $z, \zeta \in D \cup \sigma \cup \bar{D}$.

We now state Vekua's formula [14] for expressing the solution of (0.1) in G in terms of holomorphic functions of one complex variable.

THEOREM 1.1. *Every C^{2n}-solution $u(x, y)$ of (0.1) in the domain G can be analytically continued into a domain of complex values of x, y. Set $z = x + iy$, $\zeta = x - iy$. The resulting function $U(z, \zeta)$ is an analytic function in the domain $(G, \bar{G})$ and has the following integral representation formula*:

$$U(z,\zeta) = \sum_{k=0}^{n-1} g_k(z,\zeta)\phi_k(z) + \sum_{k=0}^{n-1} f_k(z,\zeta)\phi_k^*(\zeta) \tag{1.1}$$
$$- \sum_{k=0}^{n-1} \int_{z_0}^{z} B_k(t,z,\zeta)\phi_k(t)\,dt - \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{\zeta} A_k(\tau,z,\zeta)\phi_k^*(\tau)\,d\tau,$$

where

$$g_k(z,\zeta) = (-1)^k \left[\frac{\partial^k}{\partial t^k} G_k(t,\bar{z}_0,z,\zeta)\right]_{t=z},$$

$$f_k(z,\zeta) = (-1)^k \left[\frac{\partial^k}{\partial \tau^k} G_k(z_0,\tau,z,\zeta)\right]_{\tau=\zeta},$$

$$B_k(t,z,\zeta) = (-1)^k \frac{\partial^{k+1}}{\partial t^{k+1}} G_k(t,\bar{z}_0,z,\zeta),$$

$$A_k(\tau,z,\zeta) = (-1)^k \frac{\partial^{k+1}}{\partial \tau^{k+1}} G_k(z_0,\tau,z,\zeta), \qquad k = 0,1,\cdots,n-1,$$

the functions $\phi_k(z)$, $\phi_k^*(\zeta)$ are analytic for $z \in G$, $\zeta \in \bar{G}$, the functions G_k are analytic for $z, t, \zeta, \tau \in D \cup \sigma \cup \bar{D}$, depend only on the coefficients A_k^{pq}, and are given explicitly in Vekua [14, *p.* 188]. *Conversely, the representation* (1.1) *gives the analytic continuation of a solution of* (0.1) *in domain G. Moreover, if the coefficients of* (0.1) *are real functions, then the $G_k(z_0, \bar{z}_0, z, \bar{z})$, $k = 0, 1, \cdots, n-1$, are real, $G_k(t, \bar{z}_0, z, \bar{z})$, $G_k(z_0, \bar{t}, z, \bar{z})$ are conjugate functions, and all the real solutions of* (0.1) *are given by* (1.1), *where $\phi_k^*(\bar{z}) = \overline{\phi_k(z)}$, $k = 0, 1, \cdots, n-1$.*

THEOREM 1.2. *The following relations hold for the indicated Wronskians* (*Vekua* [14, *p.* 223]):

$$W_1(z,\zeta) = \left|\frac{\partial^k g_i(z,\zeta)}{\partial \zeta^k}\right| = \exp\left[-\int_{\bar{z}_0}^{\zeta} B_{n,n-1}(z,\tau)\,d\tau\right] \neq 0, \tag{1.2}$$

$$W_2(z,\zeta) = \left|\frac{\partial^k f_i(z,\zeta)}{\partial z^k}\right| = \exp\left[-\int_{z_0}^{z} B_{n-1,n}(t,\zeta)\,dt\right] \neq 0, \tag{1.3}$$

$$i = 0,1,\cdots,n-1; \quad k = 0,1,\cdots,n-1,$$

where $B_{n,n-1}(z, \zeta)$, $B_{n-1,n}(z, \zeta)$ are analytic functions for $z, \zeta \in D \cup \sigma \cup \bar{D}$, and are given explicitly in Vekua [14, *p.* 184].

2. Integral representation on $G \cup \partial G$. One of the main difficulties encountered in studying the reflection principle is that the complex representation (1.1) established by Vekua for the solution $u(x, y)$ of (0.1) in the domain D cannot immediately be applied to $D \cup \sigma$. This means that the properties of $\phi_0(z), \cdots, \phi_{n-1}(z), \phi_0^*(\zeta), \cdots, \phi_{n-1}^*(\zeta)$ near σ are unclear.

We are going to overcome this difficulty by studying some properties of Volterra integral equations.

LEMMA 2.1. *If $K(z, t)$, $f(t)$ are holomorphic for $z, t \in D \cup \sigma \cup \bar{D}$, then*

$$\begin{aligned}\int_0^z K(z,t)f(t)\,dt &= f(0)\int_0^z K(z,t)\,dt + f'(0)\int_0^z K_1(z,t)\,dt \\ &\quad + \cdots + f^{(n-1)}(0)\int_0^z K_{n-1}(z,t)\,dt + \int_0^z K_n(z,t)f^{(n)}(t)\,dt,\end{aligned} \tag{2.1}$$

where

$$K_i(z,t) = \int_t^z K_{i-1}(z,t')\,dt'.$$

COROLLARY 2.1. *If $f(z)$ is holomorphic for $z \in D \cup \sigma \cup \bar{D}$, then*

$$\begin{aligned}f(z) &= f(0) + f'(0)z + \cdots + f^{(n-1)}(0)\int_0^z \frac{(z-t)^{n-2}}{(n-2)!}\,dt \\ &\quad + \int_0^z \frac{(z-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt.\end{aligned} \tag{2.2}$$

LEMMA 2.2. *Let G be a simply connected domain in z-plane. If $w(z)$ is uniformly continuous on G, then there exists a unique continuous function $g(z)$ on $G \cup \partial G$ such that $g(z) = w(z)$ on G.*

Proof. See Graves [5, p. 117].

THEOREM 2.1. *Let $u(x, y)$ be a solution of the differential equation* (0.1) *in D, continuous along with its $(n - 1 + m)$-th derivatives in $D \cup \sigma$, $m \geqq n$. Then its analytic continuation $U(z, \zeta)$ has the complex integral representation* (1.1) *for $(z, \zeta) \in (D \cup \sigma, \bar{D} \cup \sigma)$. Furthermore, the functions $\phi_0(z), \cdots, \phi_{n-1}(z), \phi_0^*(\zeta), \cdots, \phi_{n-1}^*(\zeta)$, which are given in* (1.1), *and their m-th derivatives are continuous for $z \in D \cup \sigma$, $\zeta \in \bar{D} \cup \sigma$.*

Proof. It is easy to see that for $i + j \leqq n - 1 + m$, the functions

$$h_{ij}(z) \equiv h_{ij}(x,y) = \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} u(x,y) \tag{2.3}$$

are continuous for $z \in D \cup \sigma$, where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.4}$$

Let us differentiate (1.1); for $j \leqq n - 1$, we get

$$\begin{aligned}h_{mj}(z) &= \frac{\partial^{m+j}}{\partial z^m \partial \bar{z}^j} U(z,\bar{z}) \\ &= \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^m \partial \bar{z}^j}[g_k(z,\bar{z})\phi_k(z)] + \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^m \partial \bar{z}^j}[f_k(z,\bar{z})\phi_k^*(\bar{z})] \\ &\quad - \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^m \partial \bar{z}^j}\int_{z_0}^z B_k(t,z,\bar{z})\phi_k(t)\,dt - \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^m \partial \bar{z}^j}\int_{\bar{z}_0}^{\bar{z}} A_k(t,z,\bar{z})\phi_k^*(t)\,dt\end{aligned} \tag{2.5}$$

and

$$h_{jm}(z) = \frac{\partial^{m+j}}{\partial z^j \partial \bar{z}^m} U(z, \bar{z})$$

$$(2.6) \qquad = \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^j \partial \bar{z}^m}[g_k(z, \bar{z})\phi_k(z)] + \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^j \partial \bar{z}^m}[f_k(z, \bar{z})\phi_k^*(\bar{z})]$$
$$- \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^j \partial \bar{z}^m} \int_{z_0}^{z} B_k(t, z, \bar{z})\phi_k(t)\, dt - \sum_{k=0}^{n-1} \frac{\partial^{m+j}}{\partial z^j \partial \bar{z}^m} \int_{\bar{z}_0}^{\bar{z}} A_k(t, z, \bar{z})\phi_k^*(t)\, dt$$

for $z = x + iy \in D$.

By Lemma 2.1, Corollary 2.1 and Leibniz's rule, we may rewrite (2.5) and (2.6) as

$$(2.7) \qquad \begin{aligned} h_{mj}(z) = &\sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial \bar{z}^j} g_k(z, \bar{z})\right] \phi_k^{(m)}(z) + \sum_{k=0}^{n-1} \int_{z_0}^{z} F_{kmj}(t, z, \bar{z})\phi_k^{(m)}(t)\, dt \\ &+ \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{\bar{z}} G_{kmj}(t, z, \bar{z})\phi_k^{*(m)}(t)\, dt + H_{mj}(z, \bar{z}) \end{aligned}$$

and

$$(2.8) \qquad \begin{aligned} h_{jm}(z) = &\sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial z^j} f_k(z, \bar{z})\right] \phi_k^{*(m)}(\bar{z}) + \sum_{k=0}^{n-1} \int_{z_0}^{z} F_{kjm}(t, z, \bar{z})\phi_k^{(m)}(t)\, dt \\ &+ \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{\bar{z}} G_{kjm}(t, z, \bar{z})\phi_k^{*(m)}(t)\, dt + H_{jm}(z, \bar{z}), \end{aligned}$$

where the $g_k(z, \zeta)$, $f_k(z, \zeta)$, $F_{kmj}(t, z, \zeta)$, $G_{kmj}(t, z, \zeta)$, $F_{kjm}(t, z, \zeta)$, $G_{kjm}(t, z, \zeta)$, $H_{mj}(z, \zeta)$ and $H_{jm}(z, \zeta)$ are holomorphic for $t, z, \zeta \in D \cup \sigma \cup \bar{D}$.

Let ${}^jW_i(z, \bar{z})$ denote the matrix obtained from the Wronskians $W_i(z, \bar{z})$ by replacing the jth column of $W_i(z, \bar{z})$ by the vectors

$$\begin{pmatrix} h_{m0}(z) - H_{m0}(z, \bar{z}) \\ \vdots \\ h_{m(n-1)}(z) - H_{m(n-1)}(z, \bar{z}) \end{pmatrix}, \qquad \begin{pmatrix} h_{0m}(z) - H_{0m}(z, \bar{z}) \\ \vdots \\ h_{(n-1)m}(z) - H_{(n-1)m}(z, \bar{z}) \end{pmatrix},$$

respectively, for $i = 1, 2$. Set

$$(2.9) \qquad K_j(z) = \frac{{}^jW_1(z, \bar{z})}{W_1(z, \bar{z})}, \qquad K_j^*(z) = \frac{{}^jW_2(z, \bar{z})}{W_2(z, \bar{z})}.$$

According to Theorem 1.2, the Wronskian $W_1(z, \zeta) \neq 0$ and $W_2(z, \zeta) \neq 0$ for $z, \zeta \in D \cup \sigma \cup \bar{D}$; hence $K_j(z)$ and $K_j^*(z)$ are continuous for $z \in D \cup \sigma$. Therefore (2.7) and (2.8) can be written in the forms

$$(2.10) \qquad \begin{aligned} &\phi_j^{(m)}(z) + \sum_{k=0}^{n-1} \int_{z_0}^{z} F_{jk}(z, \bar{z}, t)\phi_k^{(m)}(t)\, dt \\ &\qquad + \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{\bar{z}} G_{jk}(z, \bar{z}, t)\phi_k^{*(m)}(t)\, dt = K_j(z), \end{aligned}$$

$$\begin{aligned}\phi_j^{*(m)}(\bar{z}) + \sum_{k=0}^{n-1} \int_{z_0}^{z} F_{jk}^*(z, \bar{z}, t)\phi_k^{(m)}(t)\, dt \\ + \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{\bar{z}} G_{jk}^*(z, \bar{z}, t)\phi_k^{*(m)}(t)\, dt = K_j^*(z),\end{aligned} \tag{2.11}$$

$j = 0, 1, \cdots, n-1$, where $F_{jk}(z, \zeta, t)$, $G_{jk}(z, \zeta, t)$, $F_{jk}^*(z, \zeta, t)$ and $G_{jk}^*(z, \zeta, t)$ are certain functions holomorphic for $z, \zeta, t \in D \cup \sigma \cup \bar{D}$, $\phi_j(z)$ and $\phi_j^*(\zeta)$ are holomorphic for $z \in D$, $\zeta \in \bar{D}$.

Define $\psi_j(z) = \phi_j^{(m)}(z)$, $\psi_{j+n}(z) = \phi_j^{*(m)}(\bar{z})$, $k_{j+n}(z) = k_j^*(z)$, $F_{j(k+n)}(z, \zeta, t) = G_{jk}(\bar{z}, \bar{\zeta}, \bar{t})$, $F_{(j+n)k}(z, \zeta, t) = F_{jk}^*(z, \zeta, t)$, $F_{(j+n)(k+n)}(z, \zeta, t) = G_{jk}^*(\bar{z}, \bar{\zeta}, \bar{t})$, for $j = 0, 1, \cdots, n-1$, $k = 0, 1, \cdots, n-1$.

Using matrix notation, $\psi = (\psi_j)$, $F(z, t) = (F_{ij}(z, \bar{z}, t))$, $K = (k_j)$, we have that systems (2.10) and (2.11) may be written as

$$\psi(z) + \int_{z_0}^{z} F(z, t)\psi(t)\, dt = k(z). \tag{2.12}$$

Recall that (2.10) and (2.11) are independent of path in D. Let D_0 be an arbitrary open half disc in D, whose boundary ∂D_0 contains a segment σ_0, $\sigma_0 \subset \sigma$, and $D_0 \cup \partial D_0 \subset D \cup \sigma$. Let G_1 be a simply connected compact subset of $D \cup \sigma$, which contains D_0 and z_0. Then the function $F(z, t)$ is bounded by a constant M for $(z, t) \in (G_1, G_1)$, that is, $|F(z, t)| < M$ for $(z, t) \in (G_1, G_1)$.

If we can prove $\psi(z)$ is continuous on $D_0 \cup \sigma_0$, then the proof will be complete.

Let z_1, z_2 be any two arbitrary points in D_0. Let $t = \lambda(\theta)$, $0 \leqq \theta \leqq 1$, $|\lambda'(\theta)| \leqq q$, denote a simple smooth path in G_1 for which $z_0 = \lambda(0)$, $z = \lambda(\beta)$, $0 \leqq \beta \leqq 1$, and this path connects z_1 and z_2. We also can choose a path such that q is independent of z_1 and z_2 for $z_1, z_2 \in D_0$. Hence (2.12) is equivalent to

$$\psi(\lambda(\beta)) + \int_0^{\beta} F(\lambda(\beta), \lambda(\theta))\psi(\lambda(\theta))|\lambda'(\theta)|\, d\theta = K(\lambda(\beta)). \tag{2.13}$$

In view of the method of successive approximation, $\psi(\lambda(\beta))$ has the representation

$$\psi(\lambda(\beta)) = k(\lambda(\beta)) + \int_0^{\beta} \Gamma(\lambda(\beta), \lambda(\theta))k(\lambda(\beta))|\lambda'(\theta)|\, d\theta. \tag{2.14}$$

This means along the path $t = \lambda(\theta)$,

$$\psi(z) = k(z) + \int_{z_0}^{z} \Gamma(z, t)k(t)\, dt, \tag{2.15}$$

and

$$|\Gamma(\lambda(\beta), \lambda(\theta))| \leqq M \exp(M|\lambda(\beta) - \lambda(\theta)|) \leqq M_1. \tag{2.16}$$

Therefore, we have

$$\psi(z_2) - \psi(z_1) = k(z_2) - k(z_1) + \int_{z_1}^{z_2} \Gamma(z, t)k(t)\, dt. \tag{2.17}$$

Finally, we use (2.16) to estimate

$$(2.18) \qquad |\psi(z_2) - \psi(z_1)| \leqq |k(z_2) - k(z_1)| + M_1 p|z_2 - z_1|,$$

where $p = \max_{t \in G_1} |k(t)|$.

Since $k(z)$ is uniformly continuous in D_0, by (2.18), $\psi(z)$ is uniformly continuous in D_0; therefore, $\psi(z)$ has a continuous extension to $D_0 \cup \partial D_0$ (Lemma 2.2). This completes the proof.

By a similar argument we have the following generalization.

THEOREM 2.2. *Let G be a simply connected domain whose boundary ∂G is a piecewise C^{n-1+m}, simple, closed curve and $G \cup \partial G \subset D \cup \sigma \cup \bar{D}$. Then every solution $u(x, y)$ of the differential equation* (0.1) *in G, continuous with its $(n-1+m)$-th derivatives in $G \cup \partial G$, $m \geqq n$, has the integral representation* (1.1), *where the functions $\phi_0(z), \cdots, \phi_{n-1}(z)$, $\phi_0^*(\zeta), \cdots, \phi_{n-1}^*(\zeta)$ and their m-th derivatives are continuous for $z \in G \cup \partial G$, $\zeta \in \bar{G} \cup \partial \bar{G}$.*

3. Analytic continuation and the nonexistence of the solutions of the Cauchy problem. Let us now consider the following two differential equations:

$$(\mathrm{M}_i) \qquad \begin{gathered} \Delta^n u + \sum_{k=1}^{n} L_{ki}(\Delta^{n-k} u) = 0, \\ L_{ki} = \sum_{p,q=0}^{p+q \leqq k} A_{ki}^{pq}(z, \bar{z}) \frac{\partial^{p+q}}{\partial x^p \partial y^q}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \end{gathered}$$

$i = 1, 2$, where $A_{ki}^{pq}(z, \zeta)$ is holomorphic for $z, \zeta \in D \cup \sigma \cup \bar{D}$.

The following theorem is concerned with the analytic continuation and the nonexistence of the solutions of (M_1) and (M_2).

THEOREM 3.1. *If $\delta_i(x) \in C^{2n-1-i}(\sigma)$, $0 \leqq i \leqq 2n-1$, are the Cauchy data of a solution $u_1(x, y)$ of (M_1) in D, and also are Cauchy data of a solution $u_2(x, y)$ of (M_2) in $\bar{D}$, then $\delta_i(x)$ is analytic on σ, and its analytic continuation $\delta_i(z)$ is holomorphic in $D \cup \sigma \cup \bar{D}$. Furthermore, $u_1(x, y)$ and $u_2(x, y)$ can be continued analytically into the whole of $D \cup \sigma \cup \bar{D}$ as solutions of (M_1) and (M_2) respectively.*

The following corollary of Theorem 3.1 is a generalization of well-known criteria for the analytic continuation of a holomorphic function.

COROLLARY 3.1. *If $\delta_i(x) \in C^{2n-1-i}(\sigma)$, $0 \leqq i \leqq 2n-1$, are the Cauchy data of a solution $u_1(x, y)$ of* (0.1) *in D, and also are the Cauchy data of a solution $u_2(x, y)$ of* (0.1) *in $\bar{D}$, then there is a solution $u(x, y)$ of* (0.1) *in $D \cup \sigma \cup \bar{D}$ such that*

$$u(x, y) = \begin{cases} u_1(x, y) & \text{in } D, \\ u_2(x, y) & \text{in } \bar{D}, \end{cases}$$

$$\left.\frac{\partial^i u(x, y)}{\partial y^i}\right|_{y=0} = \delta_i(x) \quad \text{on } \sigma, \qquad i = 0, 1, \cdots, 2n-1.$$

If we further assume that $\delta_i(x)$, $i = 0, 1, \cdots, 2n-1$, is nowhere analytic on σ, then Theorem 3.1 is equivalent to the following theorem.

THEOREM 3.2. *Let $\delta_i(x) \in C^{2n-1-i}(\sigma)$, $0 \leqq i \leqq 2n-1$, be nowhere analytic on σ. If $\delta_i(x)$, $i = 0, \cdots, 2n-1$, are Cauchy data of a solution $u_2(x, y)$ of (M_2) in $\bar{D}$, then the Cauchy problem for (M_1) has no solution in D.*

Proof of Theorem 3.1. It is easy to see that the functions

$$h_{ij}(x) = \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} u_1(x, y)|_{y=0} = \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} u_2(x, y)|_{y=0}, \tag{3.1}$$

$i + j \leqq 2n - 1$, can be expressed in terms of $\delta_k(x)$ and their derivatives, $0 \leqq k \leqq 2n - 1$.

According to Theorem 2.1, the analytic continuations $U_1(z, \zeta)$ of $u_1(x, y)$ and $U_2(z, \zeta)$ of $u_2(x, y)$ have the complex representation

$$\begin{aligned} U_i(z, \zeta) &= \sum_{k=0}^{n-1} g_{ki}(z, \zeta)\phi_{ki}(z) + \sum_{k=0}^{n-1} f_{ki}(z, \zeta)\phi_{ki}^*(\zeta) \\ &\quad - \sum_{k=0}^{n-1} \int_{z_0}^{z} B_{ki}(t, z, \zeta)\phi_{ki}(t)\, dt - \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{\zeta} A_{ki}(t, z, \zeta)\phi_{ki}^*(t)\, dt, \\ &\qquad z \in D \cup \sigma, \quad \zeta \in \bar{D} \cup \sigma, \quad i = 1, 2, \end{aligned} \tag{3.2}$$

where $g_{ki}(z, \zeta)$, $f_{ki}(z, \zeta)$, $B_{ki}(t, z, \zeta)$, $A_{ki}(t, z, \zeta)$ are holomorphic for $t, z, \zeta \in D \cup \sigma \cup \bar{D}$, $\phi_{k1}(z)$, $\phi_{k2}^*(z)$, $\phi_{k1}^*(\zeta)$, $\phi_{k2}(\zeta)$ are holomorphic for $z \in D$, $\zeta \in \bar{D}$, and continuous with their nth derivatives for $z \in D \cup \sigma$, $\zeta \in \bar{D} \cup \sigma$.

Let us differentiate (3.2), as we did in the proof of Theorem 2.1. We get

$$\begin{aligned} h_{nj}(x) &= \frac{\partial^{n+j}}{\partial z^n \partial \bar{z}^j} U_i(z, \bar{z})|_{z=x} \quad (i = 1, 2) \\ &= \sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial \bar{z}^j} g_{k1}(z, \bar{z}) \right]_{z=x} \phi_{k1}^{(n)}(x) + \sum_{k=0}^{n-1} \int_{z_0}^{x} F_{knj1}(t, x, x)\phi_{k1}^{(n)}(t)\, dt \\ &\quad + \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{x} G_{knj1}(t, x, x)\phi_{k1}^{*(n)}(t)\, dt + H_{nj1}(x, x) \\ &= \sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial \bar{z}^j} g_{k2}(z, \bar{z}) \right]_{z=x} \phi_{k2}^{(n)}(x) + \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{x} F_{knj2}(t, x, x)\phi_{k2}^{(n)}(t)\, dt \\ &\quad + \sum_{k=0}^{n-1} \int_{z_0}^{x} G_{knj2}(t, x, x)\phi_{k2}^{*(n)}(t)\, dt + H_{nj2}(x, x), \end{aligned} \tag{3.3}$$

where the $g_{ki}(z, \zeta)$, $F_{knji}(t, z, \zeta)$, $G_{knji}(t, z, \zeta)$, $H_{nji}(z, \zeta)$ are holomorphic for $t, z, \zeta \in D \cup \sigma \cup \bar{D}$, $i = 1, 2$.

Hence, (3.3) suggests the relation

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial \bar{z}^j} g_{k1}(z, \bar{z}) \right]_{z=x} \phi_{k1}^{(n)}(x) &+ \sum_{k=0}^{n-1} \int_{0}^{x} F_{knj1}(t, x, x)\phi_{k1}^{(n)}(t)\, dt \\ &- \sum_{k=0}^{n-1} \int_{0}^{x} G_{knj2}(t, x, x)\phi_{k2}^{*(n)}(t)\, dt = F_j(x), \end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
F_j(x) = {} & \sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial \bar{z}^j} g_{k2}(z, \bar{z}) \right]_{z=x} \phi_{k2}^{(n)}(x) + \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{x} F_{knj2}(t, x, x) \phi_{k2}^{(n)}(t)\, dt \\
& + \sum_{k=0}^{n-1} \int_{z_0}^{0} G_{knj2}(t, x, x) \phi_{k2}^{*(n)}(t)\, dt + H_{nj2}(x, x) \\
& - \sum_{k=0}^{n-1} \int_{z_0}^{0} F_{knj1}(t, x, x) \phi_{k1}^{(n)}(t)\, dt \\
& - \sum_{k=0}^{n-1} \int_{\bar{z}_0}^{x} G_{knj1}(t, x, x) \phi_{k1}^{*(n)}(t)\, dt - H_{nj1}(x, x).
\end{aligned}
$$

Similarly, by applying the differential operator $\partial^{n+j}/\partial z^j \partial \bar{z}^n$ to (3.2) we get the relation

$$
\begin{aligned}
\sum_{k=0}^{n-1} \left[\frac{\partial^j}{\partial z^j} f_{k2}(z, \bar{z}) \right]_{z=x} \phi_{k2}^{*(n)}(x) & - \sum_{k=0}^{n-1} \int_0^x F_{kjn1}(t, x, x) \phi_{k1}^{(n)}(t)\, dt \\
& + \sum_{k=0}^{n-1} \int_0^x G_{kjn2}(t, x, x) \phi_{k2}^{*(n)}(t)\, dt = F_{j+n}(x).
\end{aligned} \tag{3.5}
$$

According to Theorem 1.2, we know the Wronskians

$$
W_{11}(z) = \left| \left[\frac{\partial^j}{\partial \zeta^j} g_{k1}(z, \zeta) \right]_{\zeta = z} \right| \neq 0,
$$

$$
W_{22}(z) = \left| \left[\frac{\partial^j}{\partial z^j} f_{k2}(z, \zeta) \right]_{\zeta = z} \right| \neq 0.
$$

It is also clear that $F_j(z)$, $F_{j+n}(z)$ are holomorphic for $z \in \bar{D}$, and continuous in $\bar{D} \cup \sigma$, and these facts suggest a system of Volterra integral equations (3.4), (3.5) for the unknowns $\phi_{01}^{(n)}(z), \cdots, \phi_{(n-1)1}^{(n)}(z), \phi_{02}^{*(n)}(z), \cdots, \phi_{(n-1)2}^{*(n)}(z)$ in the domain $\bar{D} \cup \sigma$. Therefore, a unique solution $(\phi_{01}^{(n)}(z), \cdots, \phi_{(n-1)1}^{(n)}(z), \phi_{02}^{*(n)}(z), \cdots, \phi_{(n-1)2}^{*(n)}(z))$ exists that is holomorphic in $\bar{D}$ and continuous in $\bar{D} \cup \sigma$ (cf. Vekua [14]). Hence the $\phi_{01}^{(n)}(z), \cdots, \phi_{(n-1)1}^{(n)}(z), \phi_{02}^{*(n)}(z), \cdots, \phi_{(n-1)2}^{*(n)}(z)$ are defined and holomorphic for $z \in D \cup \sigma \cup \bar{D}$. By the same argument, we can continue $\phi_{01}^{*(n)}(z), \cdots, \phi_{(n-1)1}^{*(n)}(z)$, $\phi_{02}^{(n)}(z), \cdots, \phi_{(n-1)2}^{(n)}(z)$ analytically into $D \cup \sigma \cup \bar{D}$.

In consequence of these extensions, (3.2) holds for $z \in D \cup \sigma \cup \bar{D}$, $\zeta \in D \cup \sigma \cup \bar{D}$. This means that $u_1(x, y) = U_1(z, \bar{z})$ and $u_2(x, y) = U_2(z, \bar{z})$ are defined and satisfy (M_1) and (M_2) for $z \in D \cup \sigma \cup \bar{D}$, respectively. This completes the proof.

Proof of Corollary 3.1 *and Theorem* 3.2. These are immediate consequences of Theorem 3.1.

4. The reflection principle under the complementing boundary conditions. We now introduce n differential operators $\{B_k\}_{k=1}^n$ of respective orders n_k, given by

$$
B_k\left(z, \bar{z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \sum_{i+j \leqq n_k} B_{ij}^k(z, \bar{z}) \frac{\partial^{i+j}}{\partial x^i \partial y^j},
$$

and also let B_k' denote the highest order part of B_k.

We shall impose the following conditions.

Analytic condition. The functions $B_{ij}^k(z, \zeta)$, $i + j \leqq n_k$, are holomorphic for $z, \zeta \in D \cup \sigma \cup \bar{D}$.

Complementing condition. For every point $z \in D \cup \sigma \cup \bar{D}$ and every fixed real $\xi \neq 0$ the polynomials $B_k'(z, z, \xi, \tau)$ (in τ) are linearly independent mod $(\tau - i|\xi|)^n$.

The following theorem is concerned with the reflection principle for the solution of (0.1).

THEOREM 4.1. *Let $\delta_i(z)$ be holomorphic functions in $D \cup \sigma \cup \bar{D}$, $i = 1, \cdots, n$. Also let $u(x, y)$ be a solution of* (0.1) *in D, continuous with its derivatives up to order* $\max(2n + 1, n_k + 1)$ *in $D \cup \sigma$. If $u(x, y)$* satisfies

$$B_k\left(x, x, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u = \delta_k(x), \qquad x \in \sigma, \tag{4.1}$$

where the boundary operators B_k, $k = 1, \cdots, n$, satisfy the analytic condition and the complementing condition, then $u(x, y)$ can be continued analytically from D into the rest of the domain $D \cup \sigma \cup \bar{D}$; that is, there exists a unique $u(x, y)$ which is a solution of the differential equation (0.1) *in $D \cup \sigma \cup \bar{D}$ and which agrees with the given $u(x, y)$ in $D \cup \sigma$.*

LEMMA 4.1. *If the boundary operators $\{B_k(x, x, \partial/\partial x, \partial/\partial y)\}_{k=1}^n$ satisfy the complementing condition, then the boundary operators $\{(\partial^{m_k}/\partial x^{m_k})B_k(x, x, \partial/\partial x, \partial/\partial y)\}_{k=1}^n$ again satisfy the complementing condition.*

Proof. For every $\xi > 0$, let

$$\sum_{j=1}^n b_{jk}(z, z)\xi^{n_k - j + 1}\tau^{j-1} = B_k'(z, z, \xi, \tau) \pmod{(\tau - i\xi)^n},$$

and also let $A_k(x, x, \partial/\partial x, \partial/\partial y)$ be the leading part of $(\partial^{m_k}/\partial x^{m_k})B_k(x, x, \partial/\partial x, \partial/\partial y)$. Then

$$A_k(x, x, \xi, \tau) = \sum_{i+j=n_k} B_{ij}^k(x, x)\xi^{i+m_k}\tau^j. \tag{4.2}$$

Moreover,

$$\sum_{j=1}^n b_{jk}(z, z)\xi^{m_k + n_k - j + 1}\tau^{j-1} = A_k(z, z, \xi, \tau) \pmod{(\tau - i\xi)^n}. \tag{4.3}$$

Therefore the A_k are linearly independent mod $(\tau - i\xi)^n$ for every $\xi > 0$.

Similarly we can prove A_k are linearly independent mod $(\tau + i\xi)^n$ for every $\xi < 0$. This completes the proof.

Assume that the boundary operators B_k are of the same order K and satisfy the complementing condition. Let

$$\sum_{j=1}^n b_{kj}^1(z, z)\xi^{K-j+1}\tau^{j-1} = \sum_{i+j=K} B_{ij}^k(z, z)\xi^i\tau^j \pmod{(\tau + i\xi)^n}, \quad \xi < 0, \tag{4.4}$$

and let

$$\sum_{j=1}^n b_{kj}^2(z, z)\xi^{K-j+1}\tau^{j-1} = \sum_{i+j=K} B_{ij}^k(z, z)\xi^i\tau^j \pmod{(\tau - i\xi)^n}, \quad \xi > 0. \tag{4.5}$$

Then the complementing condition implies the following.

LEMMA 4.2.
$$d_1(z) = \det \|b_{kj}^1(z, z)\| \neq 0, \qquad z \in D \cup \sigma \cup \bar{D},$$
$$d_2(z) = \det \|b_{kj}^2(z, z)\| \neq 0, \qquad z \in D \cup \sigma \cup \bar{D}.$$

Introducing the differential operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\qquad \frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),$$

we can write

$$\begin{aligned} B_k\left(x,x,\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) &= \sum_{i+j\leqq K} B_{ij}^k(x,x)\frac{\partial^{i+j}}{\partial x^i\partial y^j} \\ &= \sum_{j=1}^n a_{kj}^1(x,x)\frac{\partial^K}{\partial z^{K-j+1}\partial \bar z^{j-1}} + A_{1k}\left(x,x,\frac{\partial}{\partial z},\frac{\partial}{\partial \bar z}\right)\frac{\partial^n}{\partial \bar z^n} \\ &\quad + C_{1k}\left(x,x,\frac{\partial}{\partial z},\frac{\partial}{\partial \bar z}\right) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} B_k\left(x,x,\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) &= \sum_{j=1}^n a_{kj}^2(x,x)\frac{\partial^K}{\partial \bar z^{K-j+1}\partial z^{j-1}} \\ &\quad + A_{2k}\left(x,x,\frac{\partial}{\partial z},\frac{\partial}{\partial \bar z}\right)\frac{\partial^n}{\partial z^n} + C_{2k}\left(x,x,\frac{\partial}{\partial z},\frac{\partial}{\partial \bar z}\right), \end{aligned} \tag{4.7}$$

where A_{1k}, A_{2k} are differential operators of order $K-n$ and having only terms of highest order, and C_{1k}, C_{2k} are differential operators of order $K-1$.

LEMMA 4.3.

$$\det\|a_{kj}^1(z,z)\|\neq 0,\qquad z\in D\cup\sigma\cup\bar D,$$
$$\det\|a_{kj}^2(z,z)\|\neq 0,\qquad z\in D\cup\sigma\cup\bar D.$$

Proof. The formulas (4.4) and (4.5) are equivalent to the formulas

$$\sum_{i+j=K} B_{ij}^k(z,z)\xi^i\tau^j = P_k^1(z,z,\xi,\tau)(\tau+i\xi)^n + \sum_{j=1}^n b_{kj}^1(z,z)\xi^{K-j+1}\tau^{j-1}$$

and

$$\sum_{i+j=K} B_{ij}^k(z,z)\xi^i\tau^j = P_k^2(z,z,\xi,\tau)(\tau-i\xi)^n + \sum_{j=1}^n b_{kj}^2(z,z)\xi^{K-j+1}\tau^{j-1},$$

where P_k^1, P_k^2 are polynomials for τ, ξ of order $K-n$. We set

$$X=\tfrac{1}{2}(\tau-i\xi),\qquad Y=\tfrac{1}{2}(\tau+i\xi);$$

then

$$\begin{aligned} &P_k^1(z,z,\xi,\tau)(\tau+i\xi)^n + \sum_{j=1}^n b_{kj}^1(z,z)\xi^{K-j+1}\tau^{j-1} \\ &\quad = P_k^1(z,z,i(X-Y),X+Y)2^nY^n \\ &\qquad +(i)^{K-n+1}(X-Y)^{K-n+1}\sum_{j=1}^n b_{kj}^1(z,z)(i)^{n-j}(X-Y)^{n-j}(X+Y)^{j-1} \\ &\quad = 2^nP_k^1(z,z,i(X-Y),X+Y)Y^n + \sum_{j=1}^n a_{kj}^1(z,z)X^{K-j+1}Y^{j-1} \\ &\quad = A_{1k}(z,z,X,Y)Y^n + \sum_{j=1}^n a_{kj}^1(z,z)X^{K-j+1}Y^{j-1}, \end{aligned}$$

where $a_{ki}^1(z, z) =$ a linear combination of the functions $b_{k1}^1(z, z), \cdots, b_{kn}^1(z, z)$. Similarly,

$$P_k^2(z, z, \xi, \tau)(\tau - i\xi)^n + \sum_{j=1}^{n} b_{kj}^2(z, z)\xi^{K-j+1}\tau^{j-1}$$

$$= A_{2k}(z, z, X, Y)X^n + \sum_{j=1}^{n} a_{kj}^2(z, z)Y^{K-j+1}X^{j-1},$$

where $a_{ki}^2(z, z) =$ a linear combination of the functions $b_{k1}^1(z, z), \cdots, b_{kn}^2(z, z)$.

Since the polynomials $\sum_{j=1}^{n} b_{kj}^1(z, z)\xi^{n-j}\tau^{j-1}$, $k = 1, \cdots, n$, are linearly independent, the polynomials

$$\sum_{j=1}^{n} d_{kj}^1(z, z)X^{n-j}Y^{j-1}\left(= \sum_{j=1}^{n} b_{kj}^1(z, z)(i)^{n-j}(X - Y)^{n-j}(X + Y)^{j-1}\right),$$
$$k = 1, \cdots, n,$$

also, are linearly independent. By some elementary calculations, it is easy to see $\det \|a_{kj}^i(z, z)\| \neq 0$ for $z \in D \cup \sigma \cup \bar{D}$, $i = 1, 2$; $k = 1, \cdots, n$; $j = 1, \cdots, n$. This completes the proof.

Proof of Theorem 4.1. By a well-known result of Agmon, Douglis and Nirenberg [1], we can assume that $u \in C^\infty(D \cup \sigma)$. And by Lemma 4.1 we shall assume that B_k are of the same order K, that is, $n_k = K$, $k = 1, 2, \cdots, n$.

From (4.6) and (4.7), it follows that u satisfies

$$\sum_{j=1}^{n} a_{kj}^1(x, x)\frac{\partial^K u}{\partial z^{K-j+1}\partial \bar{z}^{j-1}} = \delta_k(x) - A_{1k}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)\frac{\partial^n u}{\partial \bar{z}^n} - C_{1k}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)u \tag{4.8}$$

and

$$\sum_{j=1}^{n} a_{kj}^2(x, x)\frac{\partial^k u}{\partial \bar{z}^{K-j+1}\partial z^{j-1}} = \delta_k(x) - A_{2k}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)\frac{\partial^n u}{\partial z^n} - C_{2k}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)u \tag{4.9}$$

on σ.

Using Lemma 4.3, we see that (4.8) and (4.9) become

$$\frac{\partial^K u}{\partial z^{K-j}\partial \bar{z}^j} = h_{1j}(x) - H_{1j}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)\frac{\partial^n u}{\partial \bar{z}^n} - K_{1j}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)u \tag{4.10}$$

and

$$\frac{\partial^K u}{\partial \bar{z}^{K-j}\partial z^j} = h_{2j}(x) - H_{2j}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)\frac{\partial^n u}{\partial z^n} - K_{2j}\left(x, x, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)u, \tag{4.11}$$

where $h_{1j}(z)$, $h_{2j}(z)$ are holomorphic in $D \cup \sigma \cup \bar{D}$, H_{1j}, H_{2j} are differential operators of order $K - n$ and having only terms of highest order, and K_{1j}, K_{2j} are differential operators of order $K - 1$.

By Theorem 2.1, (4.10) becomes

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\left[\frac{\partial^j}{\partial \bar{z}^j} g_k(z, z)\right]\phi_k^{(K-j)}(z) \\
&\quad = \sum_{k=0}^{n-1}\sum_{i=0}^{K-j-1} (\text{given analytic function})\ \phi_k^{(i)}(z) \\
&\quad + \sum_{k=0}^{n-1} (\text{Volterra integral of } \phi_k(t)) \\
&\quad + \sum_{k=0}^{n-1} (\text{Volterra integral of } \phi_k^*(t)) \\
&\quad + \sum_{k=0}^{n-1}\sum_{i=0}^{j} (\text{given analytic function})\ \phi_k^{*(i)}(t) \\
&\quad + \text{given analytic function},
\end{aligned}
\tag{4.12}
$$

for $z \in \sigma$, that is, $z = x \in \sigma$.

Applying differential operator $\partial^j/\partial x^j$ on (4.12) for $z \in \sigma$, we have

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left[\frac{\partial^j}{\partial \bar{z}^j} g_k(z, \bar{z})\right]\phi_k^{K}(z) &= \sum_{k=0}^{n-1}\sum_{i=0}^{K-1} (\text{given analytic function})\ \phi_k^{(i)}(z) \\
&+ \sum_{k=0}^{n-1} (\text{Volterra integral of } \phi_k(t)) \\
&+ \sum_{k=0}^{n-1} (\text{Volterra integral of } \phi_k^*(t)) \\
&+ \sum_{k=0}^{n-1}\sum_{i=0}^{2j} (\text{given analytic function})\ \phi_k^{*(i)}(t) \\
&+ \text{given analytic function}
\end{aligned}
\tag{4.13}
$$

for $z = x \in \sigma$.

By Lemma 2.1, Corollary 2.1, and Leibniz's rule, (4.13) may be interpreted as

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left[\frac{\partial^j}{\partial \bar{z}^j} g_k(z, \bar{z})\right]\phi_k^{(K)}(z) &= \sum_{k=0}^{n-1}\int_{z_0}^{z} F_{kj1}(t, z, \bar{z})\phi_k^{(K)}(t)\, dt \\
&+ \sum_{k=0}^{n-1}\int_{\bar{z}_0}^{\bar{z}} G_{kj1}(t, z, \bar{z})\phi_k^{*(2n-1)}(t)\, dt + H_{j1}(z)
\end{aligned}
\tag{4.14}
$$

for $z \in \sigma$, where $F_{kj1}(t, z, \zeta)$, $G_{kj1}(t, z, \zeta)$ are holomorphic for $z, \zeta \in D \cup \sigma \cup \bar{D}$, and $H_{j1}(z)$ is holomorphic in $\bar{D}$ and continuous in $\bar{D} \cup \sigma$.

Similarly, (4.11) can be interpreted as

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left[\frac{\partial^j}{\partial z^j} f_k(z, \bar{z})\right]\phi_k^{*(K)}(\bar{z}) &= \sum_{k=0}^{n-1}\int_{z_0}^{z} F_{kj2}(t, z, \bar{z})\phi_k^{(2n-1)}(t)\, dt \\
&+ \sum_{k=0}^{n-1}\int_{\bar{z}_0}^{\bar{z}} G_{kj2}(t, z, \bar{z})\phi_k^{*(K)}(t)\, dt + H_{j2}(z)
\end{aligned}
\tag{4.15}
$$

for $z \in \sigma$, where $F_{kj2}(t, z, \zeta)$, $G_{kj2}(t, z, \zeta)$ are holomorphic for z, $\zeta \in D \cup \sigma \cup \bar{D}$, and $H_{j2}(z)$ is holomorphic in D and continuous in $D \cup \sigma$.

According to Theorem 1.2, the Wronskians $W_1(z, \zeta) \neq 0$ and $W_2(z, \zeta) \neq 0$ for $z, \zeta \in D \cup \sigma \cup \bar{D}$. Hence (4.14) and (4.15) can be written in the forms

$$\phi_j^{(K)}(z) + \sum_{k=0}^{n-1} \int_0^z F_{jk}(z, \tau)\phi_k^{(K)}(\tau)\, d\tau = \rho_j(z), \tag{4.16}$$

$$\phi_j^{*(K)}(z) + \sum_{k=0}^{n-1} \int_0^z G_{jk}(z, \tau)\phi_k^{*(K)}(\tau)\, d\tau = \eta_j(z) \tag{4.17}$$

for $z \in \sigma$, where $F_{jk}(z, \tau)$ and $G_{jk}(z, \tau)$ are holomorphic for $z, \tau \in D \cup \sigma \cup \bar{D}$, $\rho_j(z)$ are holomorphic in $\bar{D}$, continuous in $\bar{D} \cup \sigma$, and $\eta_j(z)$ are holomorphic in D and continuous in $D \cup \sigma$.

Therefore, a unique solution $(\phi_0^{(K)}(z), \cdots, \phi_{n-1}^{(K)}(z))$ of (4.16) exists that is holomorphic in $\bar{D}$ and continuous in $\bar{D} \cup \sigma$. Hence $\phi_0(z), \cdots, \phi_{n-1}(z)$ are defined and holomorphic for $z \in D \cup \sigma \cup \bar{D}$.

Similarly, by (4.17), the $\phi_j^*(z)$ can be defined as holomorphic functions in $D \cup \sigma \cup \bar{D}$.

After these extensions, (1.1) holds for $z, \zeta \in D \cup \sigma \cup \bar{D}$. This means that $u(x, y) = U(z, \bar{z})$ is defined and satisfies the differential equation (0.1) for $z \in D \cup \sigma \cup \bar{D}$. This completes the proof.

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LOCAL UNIQUENESS THEOREMS FOR A CLASS OF HIGHER ORDER ELLIPTIC EQUATIONS*

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Abstract. Local uniqueness theorems are obtained for higher order elliptic equations using integral inequalities and monotonicity properties for eigenvalues.

1. Introduction. Let u be a solution of the boundary value problem

$$\begin{aligned} \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0 \quad &\text{in } D, \\ u = 0 \quad &\text{on } \partial D, \end{aligned} \tag{1.1}$$

where D is a bounded domain in R^n, and all coefficients are uniformly bounded. If D is sufficiently small, then it follows from the generalized maximum principle (see [6, pp. 73–74]), that the only solution of (1.1) is $u \equiv 0$. Alternately, as is indicated in [6, p. 74], if the positive maximum of c is sufficiently small, then the only solution of (1.1) is $u \equiv 0$.

In this paper we prove analogous results for boundary value problems involving higher order elliptic equations. In particular, boundary value problems involving the equations

$$\Delta^k u + c(x)u = 0, \qquad k \geqq 1,$$

and

$$\alpha(x)\Delta^2 u + 2\beta(x)\Delta u + \gamma(x)u = 0$$

will be considered. The latter equation has been chosen to illustrate that our method can be extended to classes of non-self-adjoint equations. (Here Δ^k is the kth iterated n-dimensional Laplacian.)

Due to the lack of a well-defined maximum principle for higher order elliptic equations, our method will depend upon some elementary integral inequalities and the monotonicity property of eigenvalues with respect to domains.

2. Preliminaries. Our results will depend upon the following lemma.

LEMMA 2.1. *If*

$$u = \Delta u = \cdots = \Delta^{m-1}u = 0 \quad \text{on } \partial D,$$

then

$$\int_D u^2\,dx \leqq \frac{1}{\lambda_1^{2m}}\int_D (\Delta^m u)^2\,dx \tag{2.1}$$

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and

$$\int_D u^2\,dx \leqq \frac{1}{\lambda_1^{2m-1}}\int_D |\operatorname{grad}\Delta^{m-1}u|^2\,dx, \tag{2.2}$$

where λ_1 is the first eigenvalue of the fixed membrane problem.

Proof. We establish the inequality (2.1), using induction. If $u = 0$ on ∂D, it is well known [2] that

$$\int_D u^2\,dx \leqq \frac{1}{\lambda_1}\int_D |\operatorname{grad} u|^2\,dx. \tag{2.3}$$

Applying Green's identity

$$\int_D [u\Delta u + |\operatorname{grad} u|^2]\,dx = \int_{\partial D} u\frac{\partial u}{\partial n}\,ds \tag{2.4}$$

and Schwarz's inequality on the right-hand side of (2.3) and recalling that $u = 0$ on ∂D, we obtain

$$\left(\int_D u^2\,dx\right)^2 \leqq \frac{1}{\lambda_1^2}\left(\int_D u\Delta u\,dx\right)^2 \leqq \frac{1}{\lambda_1^2}\int_D u^2\,dx\int_D (\Delta u)^2\,dx.$$

Consequently,

$$\int_D u^2\,dx \leqq \frac{1}{\lambda_1^2}\int_D (\Delta u)^2\,dx, \tag{2.5}$$

which establishes (2.1) in the case $m = 1$. Assume (2.1) is true for all positive integers less than or equal to m. Suppose

$$u = \Delta u = \cdots = \Delta^m u = 0 \quad \text{on } \partial D. \tag{2.6}$$

Since $\Delta^m u = 0$ on ∂D, it follows from (2.5) that

$$\int_D (\Delta^m u)^2\,dx \leqq \frac{1}{\lambda_1^2}\int_D (\Delta^{m+1}u)^2\,dx. \tag{2.7}$$

Moreover, in view of the induction hypothesis, it follows from (2.1), (2.6) and (2.7) that

$$\int_D u^2\,dx \leqq \frac{1}{\lambda_1^{2(m+1)}}\int_D (\Delta^{m+1}u)^2\,dx,$$

which establishes (2.1) for all positive integers.

In order to establish (2.2), we note that since $\Delta^{m-1}u = 0$ on ∂D, (2.3) yields

$$\int_D (\Delta^{m-1}u)^2\,dx \leqq \frac{1}{\lambda_1}\int_D |\operatorname{grad}\Delta^{m-1}u|^2\,dx,$$

which combined with (2.1) implies the desired result:

$$\int_D u^2\,dx \leqq \frac{1}{\lambda_1^{2m-2}}\int_D (\Delta^{m-1}u)^2\,dx \leqq \frac{1}{\lambda_1^{2m-1}}\int_D |\operatorname{grad}\Delta^{m-1}u|^2\,dx.$$

We shall also have need for the following integral identities:

$$\int_D u\Delta^{2m}u\,dx = \int_D (\Delta^m u)^2\,dx + \sum_{i=0}^{m-1}\int_{\partial D}\left[\Delta^i u\frac{\partial(\Delta^{2m-1-i}u)}{\partial n} - \Delta^{m+i}u\frac{\partial(\Delta^{m-1-i}u)}{\partial n}\right]ds \tag{2.8}$$

and

$$\int_D [u\Delta^{2m-1}u + |\operatorname{grad}\Delta^{m-1}u|^2]\,dx = \sum_{i=0}^{m-1}\int_{\partial D}\Delta^i u\frac{\partial(\Delta^{2m-2-i}u)}{\partial n}\,ds + \sum_{i=1}^{m-1}\int_{\partial D}\Delta^{m-1+i}u\frac{\partial(\Delta^{m-1-i}u)}{\partial n}\,ds. \tag{2.9}$$

The identity (2.8) is obtained by setting $k = m$ and $v = \Delta^m u$ in the following Green's formula [4]:

$$\int_D (u\Delta^k v - v\Delta^k u)\,dx = \sum_{i=0}^{k-1}\int_{\partial D}\left[\Delta^i u\frac{\partial(\Delta^{k-1-i}u)}{\partial n} - \Delta^i v\frac{\partial(\Delta^{k-1-i}u)}{\partial n}\right]ds. \tag{2.10}$$

The identity (2.9) is obtained by setting $k = m$ and $v = \Delta^{m-1}u$ in (2.10) and then adding to it the obvious identity

$$\int_D [|\operatorname{grad}\Delta^{m-1}u|^2 + \Delta^{m-1}u\Delta^m u]\,dx = \int_{\partial D}\Delta^{m-1}u\frac{\partial(\Delta^{m-1}u)}{\partial n}\,ds.$$

3. Main results.

THEOREM 3.1. *Let $c_1(x)$ and $c_2(x)$ be continuous in $\overline{D^*}$, where D^* is a bounded domain in R^n. If the domain $D \subseteq D^*$ is sufficiently small, then $u \equiv 0$ is the unique solution of the problem*

$$\begin{aligned} &\Delta^{2m}u + c_1(x)u = 0 && \text{in } D,\\ &u = \Delta u = \cdots = \Delta^{m-1}u = 0 && \text{on } \partial D,\\ &\int_{\partial D}\Delta^{i+m}u\frac{\partial(\Delta^{m-i-1}u)}{\partial n}\,ds = 0, && i = 0, 1, \cdots, m-1. \end{aligned} \tag{3.1}$$

Similarly, if the domain $D \subseteq D^*$ is sufficiently small, then $u \equiv 0$ is the unique solution of the problem

$$\begin{aligned} &\Delta^{2m-1}u + c_2(x)u = 0 && \text{in } D,\\ &u = \Delta u = \cdots = \Delta^{m-1}u = 0 && \text{on } \partial D,\\ &\int_{\partial D}\Delta^{m-1+i}u\frac{\partial(\Delta^{m-1-i}u)}{\partial n}\,ds = 0, && i = 1, \cdots, m-1. \end{aligned} \tag{3.2}$$

Proof. Let

$$m_1 = \min_{x \in \overline{D^*}} c_1 > -\infty.$$

Since the eigenvalue λ_1 increases to infinity as D^* shrinks to the empty set [2], it follows that there exists a domain $D \subseteq D^*$, sufficiently small, such that

$$c_1 \geqq m_1 > -\lambda_1^{2m} \tag{3.3}$$

for all $x \in D$. From the identity (2.8) and the boundary conditions in (3.1) it follows that

$$\int_D [(\Delta^m u)^2 + c_1 u^2]\, dx = 0$$

which combined with (2.1) yields

$$0 \leqq \int_D (\lambda_1^{2m} + c_1) u^2\, dx. \tag{3.4}$$

In view of (3.3), equality holds in (3.4) and therefore $u \equiv 0$ in D.

In a similar fashion, the result is easily seen to be true for the system (3.2). In this case one uses the identity (2.9) and the fact that there exists a domain $D \subseteq D^*$, sufficiently small, such that

$$c_2 \leqq m_2 < \lambda_1^{2m-1}, \tag{3.5}$$

where

$$m_2 = \max_{x \in \overline{D^*}} c_2.$$

The details will be omitted.

Remark. If inequality (3.3) ((3.5)) holds in the original domain D^*, i.e., if the negative (positive) minimum (maximum) is sufficiently large (small), then we have uniqueness in D^* which is an improvement over the usual uniqueness criteria, namely $c_1 \geqq 0$ ($c_2 \leqq 0$) in D^* for (3.1) ((3.2)).

4. A non-self-adjoint problem. The preceding method can also be applied to non-self-adjoint problems. As an illustration we consider the following theorem.

THEOREM 4.1. *If the domain $D \subseteq D^*$ is sufficiently small, then $u \equiv 0$ is the unique solution of the problem*

$$\begin{aligned} \alpha(x)\Delta^2 u + 2\beta(x)\Delta u + \gamma(x)u &= 0 \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D, \\ \int_{\partial D} \Delta u \frac{\partial u}{\partial n}\, ds &= 0, \end{aligned} \tag{4.1}$$

where $\alpha(x) > 0$, $\beta(x)$ and $\gamma(x)$ are continuous in $\overline{D^*}$.

The proof of Theorem 4.1 depends upon the following lemmas.

LEMMA 4.1. *Consider the eigenvalue problem*

$$\Delta v + \frac{\beta(x)}{\alpha(x)} v + \mu v = 0 \quad in\ D^*, \qquad v = 0 \quad on\ \partial D^*. \tag{4.2}$$

Then, as D^* shrinks to the empty set, the first eigenvalue μ_1 increases to infinity.

Proof. Suppose to the contrary that μ_1 is bounded above by some constant K for all subdomains of D^*. Let

$$m = \max_{x \in \overline{D^*}} (\beta/\alpha).$$

Consider the eigenvalue problem

$$\Delta w + \omega w = 0 \quad \text{in } D, \qquad w = 0 \quad \text{on } \partial D, \tag{4.3}$$

where $D \subseteq D^*$ is sufficiently small such that the first eigenvalue ω_1 of (4.3) satisfies $\omega_1 > m + K$. Consequently, by Sturm's comparison theorem [3], applied to the problems (4.2) and (4.3) in D, the first eigenfunction w_1 must have a zero in D. This, however, contradicts the well-known property that $w_1 > 0$ in D.

LEMMA 4.2. *Let D be as determined in the preceding lemma. If $u = 0$ on ∂D, and if μ_1 is the first eigenvalue of* (4.2), *then*

$$\int_D u^2\,dx \leqq \frac{1}{\mu_1^2} \int_D \left(\Delta u + \frac{\beta}{\alpha} u\right)^2 dx. \tag{4.4}$$

Proof. By Rayleigh's characterization of the first eigenvalue of (4.2) we have

$$\int_D u^2\,dx \leqq \frac{1}{\mu_1} \int_D \left[|\text{grad}\, u|^2 - \frac{\beta}{\alpha} u^2\right] dx. \tag{4.5}$$

Applying Green's identity (2.4) and Schwarz's inequality on the right-hand side of (4.5), we obtain

$$\int_D u^2\,dx \leqq \frac{1}{\mu_1} \int_D \left[-u\left(\Delta u + \frac{\beta}{\alpha} u\right)\right] dx \leqq \frac{1}{\mu_1} \left(\int_D u^2\,dx\right)^{1/2} \left(\int_D \left(\Delta u + \frac{\beta}{\alpha} u\right)^2 dx\right)^{1/2}$$

which gives (4.4).

Proof of Theorem 4.1. Let

$$m_3 = \min_{x \in \overline{D^*}} \frac{\alpha\gamma - \beta^2}{\alpha^2} > -\infty.$$

Arguing as before and taking into account Lemma 4.1, there exists a domain $D \subseteq D^*$, sufficiently small, such that

$$\frac{\alpha\gamma - \beta^2}{\alpha^2} \geqq m_3 > -\mu_1^2 \tag{4.6}$$

for all $x \in D$.

If u is a solution of (4.1), we have

$$0 = \int_D u\left[\Delta^2 u + 2\frac{\beta}{\alpha}\Delta u + \frac{\gamma}{\alpha}u\right] dx$$

$$= \int_D \left[(\Delta u)^2 + \frac{2\beta}{\alpha}u\Delta u + \frac{\gamma}{\alpha}u^2\right] dx$$

$$= \int_D \left(\Delta u + \frac{\beta}{\alpha}u\right)^2 dx + \int_{D_3} \frac{\alpha\gamma - \beta^2}{\alpha^2}u^2\, dx.$$

Therefore, in view of (4.4) we obtain

$$0 \geqq \int_D \left(\mu_1^2 + \frac{\alpha\gamma - \beta^2}{\alpha^2}\right)u^2\, dx$$

which together with (4.6) implies $u \equiv 0$ in D.

Remark. In [1], Bremekamp considered similar results for (4.1), basing his proof upon a device due to Picard [5]. However, Bremekamp made the assumption that $\alpha\gamma - \beta^2$ has a fixed sign throughout the domain under consideration. In the case in which (4.6) holds in the original domain D^*, (which implies uniqueness in D^*) we have a definite improvement over Bremekamp's results.

5. Final remark. The preceding results are readily generalized to the case in which the operator Δ is replaced by a more general second order elliptic operator.

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REMARKS ON SINGULAR PERTURBATIONS WITH TURNING POINTS*

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Abstract. Boundary value problems are considered for the differential equation $\varepsilon y'' + f(x,\varepsilon)y' + g(x,\varepsilon)y' + g(x,\varepsilon)y = 0$, where $f(x,\varepsilon)x < 0$ *for* $x \neq 0$. Let $\{y(x,\varepsilon_n)\}$ converge to $\bar{y}(x)$ as $\varepsilon_n \to 0+$. In rare situations $\bar{y}(x) \not\equiv 0$. This report studies such a phenomenon.

1. Introduction. In this paper we consider solutions of the boundary value problem

$$\varepsilon y''(x) + f(x,\varepsilon)y'(x) + g(x,\varepsilon)y(x) = 0, \qquad -a \leqq x \leqq b, \tag{1.1}$$

$$y(-a) = A, \qquad y(b) = B, \tag{1.2}$$

where $a, b > 0$, $0 < \varepsilon$, and $f(x,\varepsilon)$ has a single simple zero in $[-a, b]$. Without loss of generality we assume $f(0,\varepsilon) = 0$ (hereafter referred to as the turning point). Many authors (Wasow [14], Cochran [2], Sibuya [13], O'Malley [10]) have studied asymptotic solutions of (1.1) as $\varepsilon \to 0+$. However, the recent work of Pearson [12] and Ackerberg and O'Malley [1] is of particular interest to us and motivated the present study. More recently A. M. Watts [15] and W. D. Lakin [7] have also discussed these problems.

We restrict our attention to the case where f and g are Lipschitz continuous and

$$\begin{cases} f(x,\varepsilon) > 0, & -a \leqq x < 0, \\ f(x,\varepsilon) < 0, & 0 < x \leqq b, \end{cases} \tag{1.3}$$

and (uniformly)

$$f'(0,\varepsilon) \leqq -\alpha < 0, \qquad 0 \leqq \varepsilon \leqq \varepsilon_0, \tag{1.4}$$

for some $\alpha > 0$ and some $\varepsilon_0 > 0$.

In the case where $f(x,\varepsilon)$, $g(x,\varepsilon)$ are analytic in (x,ε), Pearson [12] and Ackerberg and O'Malley [1] proved the following basic result: Let

$$-g(0,0)/f'(0,0) \equiv l. \tag{1.5}$$

Suppose $l \neq 0, 1, 2, \cdots$ and $\{y(x,\varepsilon_n)\}_{n=1}^{\infty}$ is a sequence of solutions of (1.1), (1.2) which converges (pointwise, as $\varepsilon_n \to 0+$) to a function $Y(t)$ for $t \in (-a, 0) \cup (0, b)$. Then

$$Y(t) \equiv 0, \qquad t \in (-a, 0) \cup (0, b). \tag{1.6}$$

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However, when l is a nonnegative integer the situation is "cloudy." Pearson [12] seems to have ignored these cases; Ackerberg and O'Malley [1] have an analysis for them, but do not make a precise statement of their hypothesis or results. And, as we shall see, their results are incomplete.

In § 2 of this paper we establish certain basic estimates on the regularity of solutions $y(t, \varepsilon)$ of (1.1) and (1.2).

In § 3 we discuss some results and examples in the exceptional case when l is a nonnegative integer.

Let us consider an example to show how delicate the situation can be. The solutions of

$$\varepsilon y'' - 2xy' = 0, \qquad -1 \leqq x \leqq 1/2, \tag{1.7}$$

$$y(-1) = \alpha, \qquad y(\tfrac{1}{2}) = \beta, \tag{1.8}$$

can easily be computed and are given by

$$y(x) = \alpha + (\beta - \alpha)A(x)/A(\tfrac{1}{2}), \qquad A(x) = \varepsilon^{-1}\int_{-1}^{x} e^{-\varepsilon^{-1}(1-\xi^2)}\,d\xi.$$

For $\varepsilon \to 0$ these solutions are uniformly bounded and only the left boundary condition is lost.

Consider now the modified problem

$$\varepsilon y'' - a(x)y' + b(x, \varepsilon)y = 0, \qquad -1 < x < 1/2, \tag{1.9}$$

$$y(-1) = \alpha, \qquad y(\tfrac{1}{2}) = \beta, \tag{1.10}$$

where

$$a(x) = \begin{cases} 2x & \text{for } \ -1 \leqq x \leqq \tfrac{1}{4}, \\ \tfrac{1}{2} & \text{for } \ x \geqq \tfrac{1}{4}, \end{cases}$$

and

$$b(x, \varepsilon) = \begin{cases} 0 & \text{for } \ -1 \leqq x \leqq \tfrac{1}{4}, \\ b_0 = b_0(\varepsilon) & \text{independent of } x \text{ for } \ x > \tfrac{1}{4}. \end{cases}$$

Let

$$\begin{aligned} \kappa_1 &= \frac{1}{4\varepsilon}(1 + \sqrt{1 - 16b_0\varepsilon}) = \frac{1}{2\varepsilon} + O(b_0), \\ \kappa_2 &= \frac{1}{4\varepsilon}(1 - \sqrt{1 - 16b_0\varepsilon}) = 2b_0 + O(\varepsilon b_0^2) \end{aligned} \tag{1.11}$$

be the solutions of the characteristic equation

$$\varepsilon\kappa^2 - \tfrac{1}{2}\kappa + b_0 = 0.$$

The solutions of (1.9), (1.10) are given by

$$y(x) = \begin{cases} \alpha + \lambda_2 A(x), & -1 \leqq x \leqq \tfrac{1}{4}, \\ \mu_1 e^{\kappa_1(x - 1/2)} + \mu_2 e^{\kappa_2(x - 1/4)}, & x \geqq \tfrac{1}{4}, \end{cases}$$

where the parameters λ_2, μ_1, μ_2 are determined from

$$\mu_1 e^{-\kappa_1/4} + \mu_2 - \lambda_2 A(\tfrac{1}{4}) = \alpha,$$
$$\mu_1\kappa_1 e^{-\kappa_1/4} + \mu_2\kappa_2 = \lambda_2 A'(\tfrac{1}{4}) = \lambda_2\varepsilon^{-1} e^{-15/16\varepsilon},$$
$$\mu_1 + \mu_2 e^{\kappa_2/4} = \beta.$$

After a direct but unpleasant calculation, we are led to the following result. If $\varepsilon_v \to 0+$ and

$$\lim_{\varepsilon\to 0} 4\varepsilon_v b_0(\varepsilon_v)\, e^{1/8\varepsilon_v} = \begin{cases} \infty, \\ \gamma \neq 1, \\ 1, \end{cases}$$

then we have in general for the corresponding solutions of the differential equation,

$$\lim y(x, \varepsilon_v) = \begin{cases} 0, \\ \text{finite}, \\ \infty, \end{cases}$$

in every subinterval $-1 < -a \leqq x \leqq b < 1/2$.

If we choose, for example,

$$b_0 = \tfrac{1}{4}\varepsilon^{-1} e^{-1/8\varepsilon}(\sin 1/\varepsilon)^2 + e^{-1/10\varepsilon}(\sin 2/\varepsilon)^2,$$

then there are subsequences $\varepsilon_v \to 0$ for which any of the above situations hold.

The coefficients of this example (1.9) are not smooth. However, it is an easy matter to use this example and smooth out the coefficients near $x = 1/4$ to obtain an example with coefficients $a(x, \varepsilon)$, $b(x, \varepsilon)$ which are in $C^\infty[-1, 1/2]$ for all $\varepsilon > 0$.

This example emphasizes two facts about this problem.

Fact 1. It is not reasonable to demand general results as $\varepsilon \to 0+$. It is necessary to consider sequential limits. That is, it is necessary to consider sequences of solutions $\{y(x, \varepsilon_n)\}$ which have certain properties.

Fact 2. Exponentially small perturbations of the coefficients *which occur outside of a neighborhood of the turning point* can cause large changes in the limiting behavior of the solutions.

2. Regularity. Let $y(x, \varepsilon)$ be the solution of (1.1), (1.2). Suppose $f(x, \varepsilon)$, $g(x, \varepsilon) \in C^k[-a, b]$ as functions of x, uniformly in ε. That is, there is a constant $L > 0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^j f(x, \varepsilon)\right| \leqq L, \qquad 0 \leqq j \leqq k, \quad 0 \leqq \varepsilon \leqq \varepsilon_0,$$
$$\left|\left(\frac{\partial}{\partial x}\right)^j g(x, \varepsilon)\right| \leqq L, \qquad 0 \leqq j \leqq k, \quad 0 \leqq \varepsilon \leqq \varepsilon_0. \tag{2.1}$$

Let

$$v_j(x,\varepsilon) \equiv \left(\frac{\partial}{\partial x}\right)^j \cdot y(x,\varepsilon), \qquad 0 \leqq j \leqq k. \tag{2.2}$$

Then a simple induction shows that

$$\varepsilon v_j''(x) + f(x,\varepsilon)v_j'(x) + \{g(x,\varepsilon) + jf'(x,\varepsilon)\}v_j = \sum_{s=0}^{j-1} A_{js}v_s(x,\varepsilon), \tag{2.3a}$$

where

$$A_{js} = -\left\{\binom{j}{s-1} f^{(j+1-s)}(x,\varepsilon) + \binom{j}{s} g^{(j-s)}(x,\varepsilon)\right\}. \tag{2.3b}$$

We now recall some basic estimates. For any $\psi(x) \in C[\alpha,\beta]$, let

$$\|\psi\|_{\alpha,\beta} \equiv \max\{|\psi(x)|; \alpha \leqq x \leqq \beta\}. \tag{2.4}$$

LEMMA 2.1. *Let $\varphi(x) \in C^N[\alpha,\beta]$. Let $t > 0$ be given. There exist constants $C_j(t) < \infty, j = 1, 2, \cdots, N-1$, such that*

$$\|\varphi^{(j)}\|_{\alpha,\beta} \leqq t\|\varphi^{(N)}\|_{\alpha,\beta} + C_j(t)\|\varphi^0\|_{\alpha,\beta}, \quad j = 1, 2, \cdots, N-1. \tag{2.5}$$

Proof. This result is well known. See [8] for a very general statement of this theorem.

LEMMA 2.2. *Let l be defined by* (1.5). *Then in the neighborhood of the origin* (*say* $-\Delta \leqq x \leqq \Delta$) *the solution of the reduced equation*

$$f(x,0)u'(x) + g(x,0)u = 0 \tag{2.6}$$

can be written in the form

$$\begin{array}{ll} \lambda_1 x^l \exp\left\{\int_0^x \psi(t)\,dt\right\}, & -\Delta \leqq x < 0, \\ \lambda_2 x^l \exp\left\{\int_0^x \psi(t)\,dt\right\}, & 0 < x \leqq \Delta. \end{array} \tag{2.7}$$

Moreover, suppose k is a natural number (*i.e., a nonnegative integer*) *with $k > l$ and $u(x) \in C^k(-\Delta,\Delta)$. Then $\lambda_1 = \lambda_2$ and*:

(i) *if l is not a natural number, then*

$$\lambda_1 = \lambda_2 = 0 \quad \textit{and} \quad u(x) \equiv 0; \tag{2.8}$$

(ii) *if l is a natural number and $u(x) \not\equiv 0$ in $[-\Delta,\Delta]$, then $u(x) \neq 0$ for $x \neq 0$ and*

$$\left(\frac{d}{dx}\right)^l u(x)\bigg|_{x=0} \neq 0. \tag{2.9}$$

Proof. We rewrite (2.6) in the form

$$[xf'(0,0) + x^2\tilde{f}(x)]u'(x) + [g(0,0) + x\tilde{g}(x)]u(x) = 0. \tag{2.6'}$$

Set

$$u(x) = x^l\omega(x).$$

Then (2.6′) takes the form

$$[f'(0,0) + x\tilde{f}(x)]\omega'(x) + [\tilde{g}(x) + l\tilde{f}(x)]\omega = 0.$$

Thus (2.7) follows with

$$\psi(x) = -[\tilde{g}(x) + l\tilde{f}(x)]/[f'(0,0) + x\tilde{f}(x)].$$

The remaining parts of the lemma follow from the representation (2.7).

For the remainder of this section we shall always assume:

H.1. $f(x, \varepsilon)$ and $g(x, \varepsilon) \in C^{k+1}[-a, b]$ as a function of x uniformly in ε with $k > \max(l, 0)$.

LEMMA 2.3. *Let $0 < \delta < a$. There exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, for all ε satisfying $0 < \varepsilon \leqq \varepsilon_0$ and all $F(x) \in C[-a, -\delta/2]$, the boundary value problem*

$$\varepsilon\omega'' + f(x,\varepsilon)\omega' + g(x,\varepsilon)\omega = F(x), \qquad -a \leqq x \leqq -\delta/2,$$
$$\omega(-a,\varepsilon) = \omega(-\delta/2,\varepsilon) = 0$$

has a unique solution. Moreover, there is a constant K such that

$$\|\omega(\cdot,\varepsilon)\|_{-a,-\delta/2} \leqq K\|F\|_{-a,-\delta/2}. \tag{2.10}$$

And if $\omega(x, \varepsilon)$ satisfies the homogeneous equations

$$\varepsilon\omega'' + f(x,\varepsilon)\omega' + g(x,\varepsilon)\omega = 0, \qquad -a \leqq x \leqq -\delta/2,$$
$$\omega(-a,\varepsilon) = \tilde{A}, \qquad \omega(-\delta/2,\varepsilon) = \tilde{B},$$

then there is a constant K_0 such that

$$\left\|\frac{d^j\omega}{dx^j}\right\|_{-a+\delta,-\delta/2} \leqq K_0[|\tilde{B}| + \varepsilon|\tilde{A}|], \qquad j = 0, 1, 2, \cdots, k+1. \tag{2.11}$$

Proof. These results follow easily from standard estimates on singular perturbation problems; see [10], [14]. For a complete proof see [6].

THEOREM 2.1. *Let $0 < \delta < \min(a, b)$. There exist an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and a constant $K_0(\delta)$ such that, for $0 < \varepsilon \leqq \varepsilon_0$, the solutions of* (1.1), (1.2) *belong to $C^{k+1}[-a, b]$ and satisfy*

$$\|y\|_{-a,b} \leqq K_0[\|y\|_{-\delta,\delta} + |A| + |B|], \tag{2.12}$$

$$\left\|\frac{d^j}{dx^j}y\right\|_{-a+\delta,b-\delta} \leqq K_0[\|y\|_{-\delta,\delta} + \varepsilon(|A| + |B|)], \qquad j = 0, 1, 2, \cdots, k+1. \tag{2.13}$$

Proof. Consider the equation (2.3a) for $j = k + 1$. Without loss of generality we may assume that ε and δ are so small that

$$g(x,\varepsilon) + kf'(x,\varepsilon) < 0, \qquad -\delta \leqq x \leqq \delta.$$

Thus, the maximum principle implies

$$\left\|\frac{d^{k+1}y}{dx^{k+1}}\right\|_{-\delta,\delta} \leqq C_1\left[\left|\frac{d^{k+1}}{dx^{k+1}}y(\delta,\varepsilon)\right| + \left|\frac{d^{k+1}}{dx^{k+1}}y(-\delta,\varepsilon)\right| + \sum_{j=0}^{k}\left\|\frac{d^jy}{dx^j}\right\|_{-\delta,\delta}\right].$$

The inequalities (2.12), (2.13) now follow from (2.10), (2.11) (and the corresponding results for $\delta/2 \leqq x \leqq b$) and Lemma 2.1.

This theorem emphasizes the fact that the global behavior of $y(x, \varepsilon)$ depends on the local behavior of $\|y(x, \varepsilon)\|_{-\delta,\delta}$. Thus if $\{y(x, \varepsilon_n)\}$ is a family of solutions of equation (1.1) which are uniformly bounded, there is a subsequence which together with their derivatives of order $1, 2, \cdots, k$ converges uniformly on every subinterval $[-a + \delta, b - \delta]$. Moreover, using the identity of "weak" and "strong" derivatives (see [4], [5]), the limit function, say $Y(x)$, satisfies the reduced equation (2.6).

Suppose $\{y(x, \varepsilon_n)\}$ is such a convergent sequence and $Y(x) \not\equiv 0$. Then there are constants m, M such that $0 < m \leqq |y(-a + \delta, \varepsilon_n)| \leqq M$. Thus on the interval $[-a + \delta, b - \delta]$ the functions

$$\omega_1(x, \varepsilon_n) = y(x, \varepsilon_n)/y(-a + \delta, \varepsilon_n)$$

satisfy:

(i) $\omega_1(-a + \delta, \varepsilon_n) = 1,$

(ii) $\|\omega_1'\|_{-a+\delta, b-\delta} \leqq K_1.$

Indeed, suppose that there is a $\beta(\varepsilon_n)$ such that

$$(\alpha) \qquad \beta(\varepsilon_n) \to \bar{\beta} \quad \text{as} \quad \varepsilon_n \to 0+$$

and the solutions $\{y(x, \varepsilon_n)\}$ of (1.1) together with boundary conditions

$$(\beta) \qquad y(-a, \varepsilon_n) = 1, \qquad y(b, \varepsilon_n) = \beta(\varepsilon_n)$$

converge to a nontrivial solution $Y(x) \not\equiv 0$ of the reduced equation. Suppose also that this convergence is uniform on all of $[-a, b]$, i.e., there is no boundary layer. Then these functions satisfy:

(i') $y(-a, \varepsilon_n) = 1,$

(ii') $\|y'\|_{-a,b} \leqq K_1.$

The bounds in (ii') over $[-a + \delta, b - \delta]$ follow from Theorem 2.1. In the two end intervals $[-a, -a + \delta], [b - \delta, b]$ the bounds follow from standard theory [10], [3] for the case without turning points.

These remarks lead to the following definition and lemma.

Definition 2.1. A sequence $\varepsilon_n \to 0+$ satisfies Condition B (for "bounded") if there exist a constant $K_1 > 0$ and functions $\{\omega_1(x, \varepsilon_n)\}$ which satisfy (1.1) and

$$\omega_1(-a, \varepsilon_n) = 1, \qquad \|\omega_1'\|_{-a,b} \leqq K_1. \tag{2.14}$$

Lemma 2.4. *Let $\{\varepsilon_n\}$ satisfy Condition* B. *Then $l(= -g(0, 0)/f'(0, 0))$ is a natural number and there is a unique solution $\hat{u}$ of the reduced equation* (2.6) *with*

$$\lim_{n \to \infty} \|\omega_1(\cdot, \varepsilon_n) - \hat{u}\|_{-a,b} = 0 \tag{2.15}$$

and

$$\hat{u}(-a) = 1, \qquad \hat{u}(b) \neq 0. \tag{2.16}$$

Proof. The functions $\omega_1(x, \varepsilon_n)$ form a compact set. Therefore we can find a convergent subsequence which converges to solution $\hat{u}$ of (2.6) for which $\hat{u}(-a) = 1$.

Furthermore, by Theorem 2.1 and (2.14) we see that

$$\left\| \left(\frac{d}{dx}\right)^{k+1} \omega_1(\cdot, \varepsilon_n) \right\|_{-a/2, b/2} \leqq C_2$$

for some constant C_2. Thus $\hat{u} \in C^k[-a/2, b/2]$. Therefore, Lemma 2.2 implies that l is a natural number (since $\hat{u}(x) \not\equiv 0$) and (2.16) holds. Thus $\hat{u}$ is uniquely determined and the entire sequence $\omega_1(x, \varepsilon_n)$ converges to $\hat{u}(x)$.

LEMMA 2.5. *Let* $\{\varepsilon_n\}$ *satisfy Condition* B. *Assume that*

$$I = \int_{-a}^{b} f(x, 0)\, dx > 0. \tag{2.17}$$

Then there is a corresponding sequence $\{\omega_2(x, \varepsilon_n)\}$ *of solutions of* (1.1) (*with* $\varepsilon = \varepsilon_n$) *for which*

$$\lim_{n \to \infty} \left\| \omega_2(x, \varepsilon_n) - \frac{1}{f(-a, 0)} \exp\left\{ -\frac{1}{\varepsilon} f(-a, 0)(x + a) \right\} \right\|_{-a,b} = 0. \tag{2.18}$$

Proof. Let $\omega_1(x, \varepsilon_n)$ be the solution of (1.1) as described in Lemma 2.4. All other solutions $\omega_2(x, \varepsilon_n)$ of (1.1) are solutions of the first order equation

$$\omega_2'\omega_1 - \omega_2\omega_1' = \frac{\lambda}{\varepsilon} \exp\left\{ -\frac{1}{\varepsilon} \int_{-a}^{x} f(x, \varepsilon)\, ds \right\}. \tag{2.19}$$

Let η be a constant which satisfies $0 < \eta < \min(a/2, b/2)$ and choose $\lambda = 1$. Then it follows from Lemma 2.4 and (2.19) that there is a solution $\omega_2(x, \varepsilon_n)$ which satisfies (2.18) on the restricted interval $-a \leqq x \leqq -\eta$. Furthermore,

$$\lim_{n \to \infty} \left\| \frac{d^j}{dx^j} \omega_2(\cdot, \varepsilon_n) \right\|_{-a/2, -\eta} = 0, \quad j = 0, 1, 2, \cdots, l + 2. \tag{2.20}$$

Therefore, Theorem 2.1 and Taylor expansions give

$$\begin{aligned} \limsup_{n \to \infty} \|\omega_2(\cdot, \varepsilon_n)\|_{-\eta, \eta} &\leqq (2\eta)^{l+1} \limsup_{n \to \infty} \left\| \frac{d^{l+1}}{dx^{l+1}} \omega_2(\cdot, \varepsilon_n) \right\|_{-\eta, \eta} \\ &\leqq (2\eta)^{l+1} \limsup_{n \to \infty} \left\| \frac{d^{l+1}}{dx^{l+1}} \omega_2(\cdot, \varepsilon_n) \right\|_{-a/2, b/2} \\ &\leqq K_0(\Delta) \cdot (2\eta)^{l+1} \limsup \|\omega_2(\cdot, \varepsilon_n)\|_{-a/2, b}, \end{aligned} \tag{2.21}$$

where $\Delta = \min(a/2, b/2)$ and $K_0(\Delta)$ is independent of η.

For $x \geqq \eta$ we may write this solution $\omega_2(x, \varepsilon_n)$ in the form

$$\omega_2(x, \varepsilon_n) = \omega_H(x, \varepsilon_n) + \omega_p(x, \varepsilon_n),$$

where

$$\omega_H(x, \varepsilon_n) = \frac{\omega_2(\eta, \varepsilon_n)}{\omega_1(\eta, \varepsilon_n)} \omega_1(x, \varepsilon_n)$$

and $\omega_p(x, \varepsilon_n)$ is the solution of (2.19) with $\lambda = 1$ and $\omega_p(\eta, \varepsilon) = 0$. By Lemma 2.4 the function $\omega_1(x, \varepsilon_n)$ converges to the solution $\hat{u} \not\equiv 0$ of the reduced equation and therefore Lemma 2.2 implies that there is a constant $C_3 > 0$ such that, for all sufficiently small ε_n,

$$|\omega_1(\eta, \varepsilon_n)| \geqq C_3\eta^l.$$

Furthermore, for every fixed $\eta > 0$,

$$\limsup_{n\to\infty} \|\omega_p(\cdot, \varepsilon_n)\|_{\eta,b} = 0.$$

Thus (2.20) and (2.21) imply

$$\begin{aligned}\limsup_{n\to\infty} \|\omega_2(\cdot, \varepsilon_n)\|_{-a/2,b} &\leqq \limsup_{n\to\infty} \|\omega_2(\cdot, \varepsilon_n)\|_{-\eta,\eta} + \limsup_{n\to\infty} \|\omega_2(\cdot, \varepsilon_n)\|_{\eta,b} \\ &\leqq K_0(2\eta)^{l+1} \limsup_{n\to\infty} \|\omega_2(\cdot, \varepsilon_n)\|_{-a,b} + \|\hat{u}\|_{-a,b} \cdot \limsup_{n\to\infty} \frac{|\omega_2(\eta, \varepsilon_n)|}{C_3\eta^l} \\ &\leqq K_1(\eta) \limsup \|\omega_2(\cdot, \varepsilon_n)\|_{-a/2,b},\end{aligned}$$

where

$$K_1(\eta) = 2K_0[(2\eta)^l + 2^l C_3^{-1}\|\hat{u}\|_{-a,b}] \cdot \eta.$$

Choosing η so small that $K_1(\eta) < 1/2$ we see that

$$\limsup \|\omega_2(\cdot, \varepsilon_n)\|_{-a/2,b} = 0$$

and the lemma is proved.

In exactly the same way we obtain our next result.

LEMMA 2.6. *Let $\{\varepsilon_n\}$ satisfy Condition* B. *Assume that*

$$I = \int_{-a}^{b} f(x, 0)\, dx < 0. \tag{2.22}$$

Then there is a corresponding sequence of solutions $\{\omega_2(x, \varepsilon_n)\}$ of (1.1) *(with $\varepsilon = \varepsilon_n$) for which*

$$\lim \left\| \omega_2(\cdot, \varepsilon_n) - \frac{1}{f(b, 0)\hat{u}(b)} \exp\left\{+\frac{1}{\varepsilon} f(b, 0)(b - x)\right\}\right\|_{-a,b} = 0. \tag{2.23}$$

Finally if

$$I = \int_{-a}^{b} f(x, b)\, dx = 0, \tag{2.24}$$

then there is a corresponding sequence of solutions $\{\omega_2(x, \varepsilon_n)\}$ of (1.1) *(with $\varepsilon = \varepsilon_n$) for which*

$$\begin{aligned}\lim_{n\to\infty} &\left\| \omega_2(\cdot, \varepsilon_n) - \frac{1}{f(-a, 0)} \exp\left\{-\frac{1}{\varepsilon} f(-a, 0)(x + a)\right\}\right. \\ &\left. - \frac{1}{f(b, 0)\hat{u}(b)} \exp\left\{\frac{1}{\varepsilon} f(b, 0)(b - x)\right\}\right\|_{-a,b} = 0.\end{aligned} \tag{2.25}$$

A consequence of the last two lemmas is the following theorem.

THEOREM 2.2. *Let $\{\varepsilon_n\}$ satisfy Condition* B. *Let*

$$I = \int_{a}^{b} f(x, 0)\, dx.$$

Then there exists an $\bar{\varepsilon} > 0$ such that:

(i) *for all* A *and* B *and all* $\varepsilon_n \leqq \bar{\varepsilon}$ (ε_n *in the given sequence*), (1.1) *and* (1.2) *have a unique solution* $y(x, \varepsilon_n)$,

(ii) *there is a solution* $u(x)$ *of the reduced equation* (2.10) *such that*

$$\lim_{n\to\infty} \|y(x,\varepsilon_n) - u\|_{-a+\delta,b-\delta} = 0,$$

(iii) *if* $I > 0$, *then* $u(b) = B$ *and there is no boundary layer near* $x = b$,

(iv) *if* $I < 0$, *then* $u(a) = A$ *and there is no boundary layer near* $x = a$,

(v) *if* $I = 0$, *then* $u(x) = \lambda\hat{u}(x)$, *where* $\hat{u}$ *is described by* (2.15), (2.16) *and*

$$\lambda = \frac{Af(-a,0) - Bf(b,0)\hat{u}(b)}{f(-a,0) - f(b,0)|\hat{u}(b)|^2}.$$

Proof. The general solution of (1.1) can be written in the form

$$y(x,\varepsilon_n) = \lambda\omega_1(x,\varepsilon_n) + \alpha\omega_2(x,\varepsilon_n),$$

where $\omega_2(x, \varepsilon_n)$ satisfies one of the inequalities (2.18), (2.23) or (2.25) and $\omega_1(x, \varepsilon_n) \to \hat{u}(x)$. The theorem follows without difficulty.

The results of this theorem should be compared to the claims of Ackerberg and O'Malley [1]. These results are consistent with their results in cases (iii) and (iv) and yield a different value of λ in case (v).

Motivated by the results of Ackerberg and O'Malley [1] and the results of Watts [15], we now consider sequences ε_n for which bounded sequences $\{y(x, \varepsilon_n)\}$ must converge to zero.

DEFINITION 2.2. A sequence $\varepsilon_n \to 0+$ will be said to satisfy Condition ZB relative to the interval $[-a, b]$ if, for any sequence of uniformly bounded solutions $y(x, \varepsilon_n)$ of (1.1), we have

$$\lim_{n\to\infty} \|y(\cdot,\varepsilon_n)\|_{-a+\delta,b-\delta} = 0. \tag{2.26}$$

DEFINITION 2.3. A sequence $\varepsilon_n \to 0+$ will be said to satisfy Condition Z relative to the interval $[-a, b]$ if there exists an $\bar{\varepsilon} > 0$ such that, for all choices of A and B and all $\varepsilon_n < \bar{\varepsilon}$, there exists a unique solution $y(x, \varepsilon_n; A, B)$ of (1.1) and (1.2) and there is a constant $C_4 = C_4(A, B) > 0$ such that

$$|y(x,\varepsilon_n; A, B)| \leqq C_4, \tag{2.27}$$

$$\lim_{n\to\infty} \|y(\cdot,\varepsilon_n; A, B)\|_{-a+\delta,b-\delta} = 0. \tag{2.28}$$

THEOREM 2.3. *Suppose* $\varepsilon_n \to 0$ *is a sequence which satisfies Condition* ZB *relative to the interval* $[-a, b]$. *Then the sequence* ε_n *also satisfies Condition* Z *relative to the interval* $[-a, b]$.

Proof. Suppose that for some value of ε there are two solutions $y_1(x, \varepsilon)$ and $y_2(x, \varepsilon)$ of the boundary value problem (1.1), (1.2). Then there are infinitely many solutions, for example,

$$y_\alpha(x,\varepsilon) = y_1(x,\varepsilon) + \alpha[y_2(x,\varepsilon) - y_1(x,\varepsilon)].$$

Moreover, the solutions of (1.1), (1.2) are not bounded. Thus, if we show that all solutions of (1.1), (1.2) (with $\varepsilon = \varepsilon_n \leqq \bar{\varepsilon}$) are bounded we shall have established the existence of $y(x, \varepsilon_{n'}, A, B)$ for all A and B. Finally if the sequence $\{\varepsilon_n\}$ satisfies Condition ZB we shall also have proved (2.28). Thus it suffices to prove (2.27).

Suppose there is a subsequence $\varepsilon_{n'} \to 0+$ such that the associated solutions $y(x, \varepsilon_{n'})$ of (1.1), (1.2) are unbounded. Let

$$Z(x, \varepsilon_{n'}) = y(x, \varepsilon_{n'})/\|y(\cdot, \varepsilon_{n'})\|_{-a,b}. \tag{2.29}$$

Using Theorem 2.1 we may extract a subsequence $\{Z(x, \varepsilon_{n''})\}$ which converges uniformly to a function $u(x) \in C^k[-a/2, b/2]$ which is a solution of the reduced equation (2.6). However, the sequence $\{Z(x, \varepsilon_{n''})\}$ also satisfies Condition ZB. Thus $u(x) \equiv 0$. Now consider $Z(x, \varepsilon_{n''})$ on the intervals $[-a, -a/2], [b/2, b]$. When n'' is large enough, $|Z(x, \varepsilon_{n''})|$ assumes its maximum on one of these intervals. Thus,

$$\|Z(\cdot, \varepsilon_{n''})\|_{-a,-a/2} + \|Z(\cdot, \varepsilon_{n''})\|_{b/2,b} \geqq 1 \tag{2.30}$$

while

$$|Z(-a, \varepsilon_{n''})| + |Z(-a/2, \varepsilon_n'')| + |Z(b/2, \varepsilon_{n''})| + |Z(b, \varepsilon_n'')| \to 0. \tag{2.31}$$

However, (2.30) and (2.31) are impossible in view of Lemma 2.3.

Suppose one has a problem (1.1), (1.2) for which one can show that every sequence $\{\varepsilon_n\}$ satisfies Condition ZB. Then every sequence satisfies Condition Z. Using a simple contradiction argument one sees that, without recourse to sequences, we have: there is an $\varepsilon_0 > 0$ such that, for all ε satisfying $0 < \varepsilon \leqq \varepsilon_0$, the boundary value problem (1.1), (1.2) possesses a unique solution $y(x, \varepsilon, A, B)$. Moreover, there is a constant $C(A, B)$ such that

$$|y(x, \varepsilon, A, B)| \leqq C(A, B). \tag{2.32}$$

In particular, we obtain the following theorem.

THEOREM 2.4. *If $l \neq 0, 1, 2, \cdots$, then there exists an $\varepsilon_0 > 0$ such that, for all ε satisfying $0 < \varepsilon \leqq \varepsilon_0$, there is a unique solution of the boundary value problem* (1.1), (1.2), *say $y(x, \varepsilon, A, B)$. Moreover, there exists a constant $C(A, B)$ such that* (2.32) *holds. Finally,*

$$\lim_{\varepsilon \to 0} \|y(\cdot, \varepsilon)\|_{-a+\delta, b-\delta'} = 0.$$

Proof. Following the remarks above it is merely necessary to prove that every sequence $\{\varepsilon_n\}$ satisfies Condition ZB. Let $\{y(x, \varepsilon_n)\}$ be a uniformly bounded sequence. Then the estimates (2.12), (2.13) of Theorem 2.1 hold. Suppose there is a point $x_0 \in (a, b)$ and there is a subsequence $y(x_0, \varepsilon_{n'}) \to \bar{y} \neq 0$. Then there is a subsubsequence which is convergent to a function $Y(x) \in C^k[-a, b]$. Moreover, $Y(x)$ is given by equation (2.7) of Lemma 2.2. However, because of the restrictions on l, we see that $Y(x) \equiv 0$. But $\bar{y} = Y(x_0)$.

We close this section with a discussion of the effect of the "size" of $[-a, b]$ on this behavior. Clearly, the estimates of Theorem 2.1 show that if $\{\varepsilon_n\}$ satisfies Condition ZB relative to the interval $[-a, b]$, then $\{\varepsilon_n\}$ satisfies Condition ZB relative to every larger interval $[-a - \delta, b + \delta']$ with $\delta \geqq 0, \delta' \geqq 0$.

THEOREM 2.5. *Suppose $\{y(x, \varepsilon_n)\}$ is a sequence of solutions of* (1.1), (1.2) *which is unbounded on $[-a, b]$. Let $0 < \delta, 0 < \delta'$. Then:*

(i) *on every strictly smaller interval* $[-a+\delta, b-\delta']$ *there is a subsequence* $\varepsilon_{n'} \to 0+$ *and a sequence of solutions* $\omega_1(x, \varepsilon_{n'})$ *of* (1.1) *which satisfy Condition* B, *and*

(ii) *on every strictly larger interval* $[-a-\delta, b+\delta']$ *there is a subsequence* $\varepsilon_{n'} \to 0+$ *which satisfies Condition Z relative to the interval* $[-a-\delta, b+\delta']$.

Proof. Let $Z(x, \varepsilon_n) = y(x, \varepsilon_n)/\|y(\cdot, \varepsilon_n)\|_{-a,b}$. Using Theorem 2.1 we may extract a subsequence ε_n, and a solution $u(x)$ of the reduced equation (2.6) such that

$$\lim_{n' \to \infty} \|Z(\cdot, \varepsilon_{n'}) - u\|_{-a+\delta, b-\delta'} = 0.$$

The argument of Theorem 2.3 shows that

$$u(-a+\delta) \neq 0.$$

The functions

$$\omega_1(x, \varepsilon_{n'}) = \frac{Z(x, \varepsilon_{n'})}{Z(-a+\delta, \varepsilon_{n'})}$$

satisfy Condition B on $[-a+\delta, b-\delta']$.

On the other hand, consider any larger interval $[-a-\delta, b+\delta']$. Suppose there exists a sequence of solutions $\{\omega(x, \varepsilon_n)\}$ of (1.1) which also satisfy $y(-a-\delta, \varepsilon_n) = A_0, y(b+\delta, \varepsilon_n) = B_0$. If this family is unbounded on $[-a-\delta, b+\delta']$ part (i) shows that there exists a family $\{\omega_1(x, \varepsilon_n)\}$ satisfying (2.14). That is, on the interval $[-a, b]$, the sequence $\varepsilon_n \to 0+$ satisfies Condition B. Applying Theorem 2.2 we see that the solutions of (1.1), (1.2) on the interval $[-a, b]$ are bounded. Thus, we may assume these functions are bounded. If any subsequence $\{\omega(x, \varepsilon_{n'})\}$ were to converge to a nonzero limit solution that sequence (using Theorem 2.1) would also lead to functions which satisfy Condition B relative to $[-a, b]$ and, using Theorem 2.2, the original sequence $\{y(x, \varepsilon_n)\}$ is bounded on $[-a, b]$.

3. Some special cases and examples. In [1] Ackerberg and O'Malley and in [11] O'Malley observed that there is a whole class of equations of the type (1.1) for which one always obtains nontrivial limit functions. Interestingly enough these are the "simplest" equations of the type (1.1). These are our first examples.

Example 1. Consider the boundary value problem:

$$\varepsilon y'' - xy' + ny = 0, \qquad -a \leqq x \leqq b, \tag{3.1}$$

$$y(-a) = A, \quad y(b) = B, \tag{3.2}$$

where n is a natural number.

In this case the exact solution is given in terms of parabolic cylinder functions (see [1], [11], [15]). A complete discussion is given in [11], [7]. Given n, there is an $\varepsilon_0 > 0$ such that, for all ε satisfying $0 < \varepsilon \leqq \varepsilon_0$, there exists a unique solution of the boundary value problem (3.1), (3.2). Moreover, there is a constant C, determined by Theorem 2.2, such that

$$\|y(x, \varepsilon) - Cx^n\|_{-a+\delta, b-\delta} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Example 2. Consider the equations

$$\varepsilon y'' - xy' + \frac{x}{1+x}\, y = 0, \qquad -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2},$$

$$\varepsilon y'' - x(1+x)y' + xy = 0, \qquad -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}.$$

In both of these cases

$$y(x, \varepsilon) = \sigma(1 + x)$$

is a solution of both the second order equation and the reduced equation. Thus, employing Theorem 2.2, we have: for all $\varepsilon > 0$, $0 < \varepsilon \leqq \varepsilon_0$, the boundary value problem (1.1), (1.2) has a unique solution $y(x, \varepsilon)$. For an appropriate constant σ determined by Theorem 2.2,

$$\|y(x, \varepsilon) - \sigma(1 + x)\|_{-a+\delta, b-\delta} \to 0 \quad \text{and} \quad \varepsilon \to 0+.$$

Example 3. Let $g(x) \equiv 0$. Then the constant function $W_1(x, \varepsilon) \equiv 1$ satisfies both the second order equation and the reduced equation. Applying Theorem 2.2 we have: for all $\varepsilon > 0$ the boundary value problem (1.1), (1.2) has a unique solution $y(x, \varepsilon)$. Moreover, if

$$I = \int_{-a}^{b} f(x, 0)\, dx$$

and

$$C \equiv \begin{cases} \dfrac{f(-a)A - f(b)B}{f(-a) - f(b)}, & I = 0, \\ A, & I < 0, \\ B, & I > 0, \end{cases}$$

then

$$\|y(\cdot, \varepsilon) - C\|_{-a+\delta, b-\delta} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Example 4. Suppose $g(x, \varepsilon) \in C[-a, b]$ uniformly in ε and $g(x, \varepsilon) \leqq 0$. Suppose there exist two points $x-$, $x+$ with

$$-a \leqq x- < 0 < x+ \leqq b$$

such that

$$g(x-, 0)g(x+, 0) \neq 0.$$

Then applying the maximum principle, we see that:

(i) for each $\varepsilon > 0$ there is a unique solution $y(x, \varepsilon)$ of (1.1), (1.2). Moreover,

$$|y(x, \varepsilon)| \leqq \max(|A|, |B|).$$

Finally, using the argument of [3, Theorem 3.6] we have

(ii) $\lim\limits_{\varepsilon \to 0+} \|y(\cdot, \varepsilon)\|_{-a+\Delta, b-\Delta} = 0$ for all $\Delta > 0$.

Remark. This example once more illustrates that an analysis which is based *only* on the behavior of $f(x, \varepsilon)$ and $g(x, \varepsilon)$ "near" the turning point may not be adequate.

THEOREM 3.1. *Suppose*

$$g(x, \varepsilon) = x^2 b(x), \tag{3.3a}$$

where

$$b(x) \geqq b_0 > 0. \tag{3.3b}$$

Let

$$f(x, \varepsilon) = -xa(x), \tag{3.4a}$$

where

$$a(x) \geqq a_0 > 0. \tag{3.4b}$$

Then there exists an $\varepsilon_0 > 0$ *such that, for all* ε *satisfying* $0 < \varepsilon \leqq \varepsilon_0$, *the boundary value problem* (1.1), (1.2) *has a unique solution. Moreover,*

$$\|y(\cdot, \varepsilon)\|_{-a+\delta, b-\delta} \to 0 \quad as \quad \varepsilon \to 0. \tag{3.5}$$

Proof. Using the remark following Theorem 2.3 it is only necessary to show that all bounded sequences converge to zero. Indeed, it is sufficient to show that all convergent sequences converge to zero.

The solution of the reduced equation (2.6) is given by

$$Y(x) = Y(0) \exp \int_0^x \frac{tb(t)}{a(t)} dt. \tag{3.6}$$

If $Y(0) > 0$, then $Y(x)$ has a relative minimum at $x = 0$. And, if $Y(0) < 0$, then $Y(x)$ has a relative maximum at $x = 0$. Suppose $\{y(x, \varepsilon_n)\}_{n=0}^{\infty}$ is a sequence of solutions of (1.1), (1.2) such that

$$\lim_{\varepsilon_n \to 0} \|y(\cdot, \varepsilon_n) - Y\|_{-\Delta, \Delta} = 0.$$

Suppose $Y(0) > 0$. Then for ε_n small enough we have that $y(x, \varepsilon_n) > 0$ and, in the interval $[-\Delta, \Delta]$, $y(x, \varepsilon_n)$ has an interior relative minimum. But

$$\varepsilon_n y''(x, \varepsilon_n) + f(x, \varepsilon_n) y'(x, \varepsilon_n) = -x^2 b(x) y(x, \varepsilon_n) \leqq 0.$$

Applying the maximum principle, we see that $y(x, \varepsilon_n)$ *cannot* have an interior relative minimum. This contradiction shows that

$$Y(0) \leqq 0.$$

A similar argument shows that

$$Y(0) \geqq 0$$

and hence, using (3.6),

$$Y(x) \equiv 0.$$

THEOREM 3.2. *Let us assume that*

(a) $f'(0,\varepsilon) = -1$,

(b) $g(x,\varepsilon) = g(x)$ *is independent of* ε,

(c) $g(0) = l = 0$,

(d) $g(x) = g(-x)$,

(e) *there is a* $\Delta > 0$ *such that* $g(x) \in C^\infty[-\Delta,\Delta]$ *and* $f(x,\varepsilon) \in C^\infty[-\Delta,\Delta]$ *as a function of* x.

Suppose also that for some k, *a positive integer*,

$$g^{(k)}(0) \neq 0.$$

Then there is an $\varepsilon_0 > 0$ *such that, for all* ε *satisfying* $0 < \varepsilon \leqq \varepsilon_0$, *there is a unique solution of the boundary value problem* (1.1), (1.2) *and*

$$\lim_{\varepsilon \to 0} \|y(\cdot,\varepsilon)\|_{-a+\delta,b-\delta} = 0.$$

Proof. Once more, it is only necessary to show that the sequence $y(x,\varepsilon_n)$ converges to zero.

Let $\{y(x,\varepsilon_n)\}_{n=1}^\infty$ be a sequence of solutions of (1.1) which are uniformly bounded and converge to a function $Y(x) \not\equiv 0$. As before, let

$$v_j(x,\varepsilon_n) = \left(\frac{\partial}{\partial x}\right)^j y(x,\varepsilon_n).$$

Applying Theorem 2.1 we see that there are constants C_j, independent of ε_n, such that

$$\|v_j(\cdot,\varepsilon_n)\|_{-\Delta,\Delta} \leqq C_j.$$

Let

$$V_j = V_j(\varepsilon) = v_j(0,\varepsilon).$$

Then $Y(x) \not\equiv 0$ implies

$$V_0(\varepsilon_n) \nrightarrow 0. \tag{3.7}$$

Consider (2.3a), (2.3b) with $j = 0$. Then for every ε,

$$\varepsilon V_2(\varepsilon) = 0.$$

Thus, $V_2(\varepsilon) = 0$. Consider (2.3a), (2.3b) with $j = 2$. Then

$$\varepsilon V_4 - 2V_2 = -g''(0)V_0.$$

Thus (3.7) implies

$$g''(0) = 0, \quad V_4(\varepsilon) = 0. \tag{3.8}$$

We proceed by induction. Suppose that

$$g^{(2s)}(0) = 0, \quad V_{2s+2}(\varepsilon) = 0, \qquad s = 1,2,\cdots,j. \tag{3.9}$$

Then

$$\varepsilon V_{2j+4} - (2j+2)V_{2j+2} = -\binom{2j+2}{0} g^{(2j+2)}(0)V_0$$

or

$$\varepsilon V_{2j+4} = -g^{(2j+2)}(0)V_0.$$

Once more, (3.7) implies

$$g^{(2j+2)}(0) = 0, \quad V_{2j+4}(\varepsilon) = 0 \quad \text{for all } j.$$

The recursion relationships provided by (2.3a) and (2.3b) *evaluated at* $x = 0$ lead to additional necessary conditions for "resonance". However, the analysis of these recursion relationships is technically complicated. Hence we collect the necessary results in the Appendix and proceed with statements of results and remarks.

Consider the examples

$$\varepsilon y'' - xy' + (x + \alpha\varepsilon)y = 0, \tag{3.10}$$

where α is a fixed constant. In all cases the solution of the reduced equation is

$$\bar{y}(x) = c\, e^x.$$

However, evaluating (3.10) at $x = 0$ we find

$$\varepsilon y''(0, \varepsilon) + \alpha\varepsilon y(0, \varepsilon) = 0,$$

that is,

$$y''(0, \varepsilon) = -\alpha y(0, \varepsilon).$$

Thus, employing our usual argument based on the remarks following Theorem 2.3, and Theorem 2.2, we obtain the following theorem.

THEOREM 3.3. *There is an* $\varepsilon_0 > 0$ *such that, for all* ε *satisfying* $0 < \varepsilon \leqq \varepsilon_0$, *the boundary value problem* (3.10), (1.2) *has a unique solution. Moreover, there exists a unique constant* C *such that*

$$\|y(x, \varepsilon) - C\, e^x\|_{-a+\delta, b-\delta} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

In addition, if

$$\alpha = -1,$$

then the constant C *is determined by Theorem* 2.2. *If* $\alpha \neq -1$, *then*

$$C = 0.$$

This result is closely related to the discussion by Watts [15]. Our next result is also related to the examples of Watts.

LEMMA 3.1. *Consider the equation*

$$\varepsilon y'' - xy' + g(x)y = 0, \qquad -a \leqq x \leqq b. \tag{3.11}$$

Suppose that

$$g(0) = n \tag{3.12}$$

and that

$$(n + 1/2)g''(0) \neq -[g'(0)]^2. \tag{3.13}$$

Then there exists an $\varepsilon_0 > 0$ *such that, for all* ε *satisfying* $0 < \varepsilon \leqq \varepsilon_0$, *the boundary value problem* (1.1), (1.2) *has a unique solution* $y(x, \varepsilon)$. *Moreover,*

(3.14) $$\|y(\cdot, \varepsilon)\|_{-a+\delta, b-\delta} \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. See Appendix.

COROLLARY. *Assume that*

(3.15a) $$g(x) = n + \alpha x, \quad \alpha \neq 0,$$

or

(3.15b) $$g(x) = n + \alpha x^2, \quad \alpha \neq 0,$$

where α *is a constant. Then the conclusions of the lemma apply.*

Remark. The special case (3.15a) is discussed by Watts [13].

THEOREM 3.4. *Consider the equation* (3.11) *where*:

(a) *there is a* $\Delta > 0$ *such that* $g(x) \in C^{\infty}[-\Delta, \Delta]$,

(b) $g(0) = n$,

(c) $g(x) = g(-x)$.

Suppose that for some k, a positive integer,

$$g^{(k)}(0) \neq 0.$$

Then there is an $\varepsilon_0 > 0$ *such that, for all* ε *satisfying* $0 < \varepsilon \leqq \varepsilon_0$, *there is a unique solution of the boundary value problem* (3.11), (1.2) *and*

$$\|y(\cdot, \varepsilon)\|_{-a+\delta, b-\delta} \to 0 \quad as \quad \varepsilon \to 0+.$$

Proof. Let $\{y(x, \varepsilon_n)\}$ be a uniformly bounded sequence of solutions of (1.1) which converge on $[-a + \delta, b - \delta]$ to a nontrivial solution $Y(x)$ of reduced equation (2.6). As we know

(3.16) $$V_n(\varepsilon_s) \nrightarrow 0.$$

From Lemma 3.1 we know that

$$g^{(2)}(0) = 0.$$

Assume that

$$g^{(2j)}(0) = 0, \qquad j = 1, 2, \cdots, k - 1.$$

Combining Lemma A.3 with Lemma A.4 we obtain

$$\frac{g^{(2k)}(0)}{2k} V_n(\varepsilon_s) \left[\sum_{r=0}^{k-1} (-1)^r \binom{n + 2r}{n + 2r - 2k} \binom{k}{k - r} + (-1)^k \binom{n + 2k}{n} \binom{k}{0} \right] = 0.$$

Applying (3.16) and Lemma A.5 we see that

(3.17) $$g^{(2k)}(0) = 0.$$

Thus we have established (3.17) for all k.

THEOREM 3.5. *Consider the equation*

(3.18) $$\varepsilon y'' + f(x)y' + ny = 0, \qquad -a \leqq x \leqq b,$$

where

(a) $f'(0) = -1, f(0) = 0$,

(b) $f(x) = -f(-x)$,

(c) *there is a* $\Delta > 0$ *such that* $f(x) \in C^\infty[-\Delta, \Delta]$.

Suppose there is an integer $k > 1$ *such that*

$$f^{(k)}(0) \neq 0.$$

Then there is an $\varepsilon_0 > 0$ *such that, for all* ε *satisfying* $0 < \varepsilon \leqq \varepsilon_0$, *there is a unique solution for the boundary value problem* (3.18), (1.2) *and*

$$\|y(\cdot, \varepsilon)\|_{-a+\delta, b-\delta} \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. The proof follows the same lines as the proof of Theorem 3.4. A special case of this equation is the equation

$$\varepsilon y'' - x(1 + x^2)y' + 2y = 0, \tag{3.19}$$

discussed by Ackerberg and O'Malley [1]. They assert a contrary result in this case.

In the particular case of (3.19) one can use (2.3a), (2.3b) directly to establish

$$v_4(0, \varepsilon) = 0.$$

Hence, if there is a limit function $Y(x)$, then

$$Y^{(\mathrm{iv})}(0) = 0.$$

But

$$Y(x) = C\frac{x^2}{1 + x^2}.$$

Thus,

$$C = 0, \quad Y \equiv 0.$$

Appendix.

LEMMA A.1. *Let* $f'(0, \varepsilon) = -1$, $V_j = V_j(\varepsilon) = v_j(0, \varepsilon)$, *and let*

$$l = n = g(0, \varepsilon). \tag{A.1}$$

Let $\{y(x, \varepsilon_n)\}_{n=1}^\infty$ *be a sequence of solutions of* (1.1) *which are uniformly bounded and converge to a function* $Y(x)$. *If*

$$0 \leqq j \leqq n + 1 - 2k, \quad k \geqq 1, \tag{A.2}$$

then

$$V_j(\varepsilon_s) = O(\varepsilon_s^k). \tag{A.3}$$

Proof. We proceed by induction on k and j. Take $k = 1$. Suppose that $0 \leqq n - 1$. Then

$$V_0 = -\frac{\varepsilon_s}{n}, \quad V_2 = O(\varepsilon_s).$$

Next, suppose that

$$V_r = O(\varepsilon_s), \quad 0 \leqq r \leqq j \leqq n - 2.$$

Consider (2.3a), (2.3b) with j replaced by $j + 1$. We obtain

$$\varepsilon V_{j+3}(\varepsilon_s) + [n - (j + 1)]V_{j+1}(\varepsilon_s) = O(\varepsilon_s).$$

Hence

$$V_{j+1} = O(\varepsilon_s).$$

Thus the lemma is established for $k = 1$.

Suppose the lemma has been established for k and $2 \leqq n - 2k + 1$. Hence, by the inductive hypothesis,

$$V_2(\varepsilon_s) = O(\varepsilon_s^k)$$

and

$$V_0(\varepsilon_s) = -\frac{\varepsilon_s}{n} V_2(\varepsilon_s) = O(\varepsilon_s^{k+1}).$$

Now assume that the lemma has been established for $k + 1$ and all $j = j_0$ which satisfy $0 \leqq j_0 \leqq n - 2(k + 1)$ or $(j_0 + 1) + 2 \leqq n - 2k + 1$. Then

$$\begin{aligned}[n - (j_0 + 1)]V_{j_0+1} &= -\varepsilon_s V_{j_0+3} + O(\varepsilon_s^{k+1}) \\ &= -\varepsilon_s O(\varepsilon_s^k) + O(\varepsilon_s^{k+1}),\end{aligned}$$

and the lemma is proved.

Proof of Lemma 3.1. As usual, it suffices to consider the sequence $\varepsilon_n \to 0+$ for which $\{y(x, \varepsilon_n)\}$ is convergent.

Using Lemma A.1 and (2.3a), (2.3b) with $j = n - 2$ we obtain

$$V_{n-2}(\varepsilon_s) = -\frac{\varepsilon_s}{2} V_n(\varepsilon_s) + O(\varepsilon_s^2). \tag{A.4}$$

Letting $j = n - 1$ and using Lemma A.1 we obtain

$$V_{n-1}(\varepsilon_s) = -\varepsilon_s\left[V_{n+1}(\varepsilon_s) - \frac{1}{2}\binom{n-1}{n-2}g'(0)V_n(\varepsilon_s)\right] + O(\varepsilon_s^2). \tag{A.5}$$

Letting $j = n + 1$ we obtain

$$\varepsilon_s V_{n+3} - V_{n+1} = -g'(0)\binom{n+1}{n}V_n + O(\varepsilon_s) \tag{A.6}$$

or

$$V_{n+1}(\varepsilon_s) = g'(0)(n + 1)V_n(\varepsilon_s) + O(\varepsilon_s).$$

Combining (A.5) and (A.6) we obtain

$$V_{n-1}(\varepsilon_s) = \varepsilon_s g'(0)[\tfrac{1}{2}(n - 1) - (n + 1)]V_n(\varepsilon_s) + O(\varepsilon_s^2). \tag{A.7}$$

Letting $j = n$ we have

$$\varepsilon V_{n+2}(\varepsilon_s) = -ng'(0)V_{n-1}(\varepsilon_s) - \frac{n(n-1)}{2}g''(0)V_{n-2} + O(\varepsilon_s^2).$$

Using (A.4) and (A.7) we obtain

$$V_{n+2}(\varepsilon_s) = \{[(n+1)n - \tfrac{1}{2}n(n-1)][g'(0)]^2 + \tfrac{1}{4}n(n-1)g''(0)\}V_n(\varepsilon_s) + O(\varepsilon_s). \tag{A.8}$$

On the other hand, letting $j = n + 2$ we obtain

$$\varepsilon V_{n+4}(\varepsilon_s) - 2V_{n+2}(\varepsilon_s) = -(n+2)g'(0)V_{n+1}(\varepsilon_s) - \tfrac{1}{2}(n+2)(n+1)g''(0)V_n(\varepsilon_s) + O(\varepsilon_s).$$

On using (A.6) this gives

$$V_{n+2}(\varepsilon_s) = \tfrac{1}{2}\{(n+2)(n+1)[g'(0)]^2 + \tfrac{1}{2}(n+2)(n+1)g''(0)\}V_n(\varepsilon_s) + O(\varepsilon_s). \tag{A.9}$$

However, $Y \not\equiv 0$ together with the basic representation of $Y(x)$ (see Lemma 2.2) implies

$$V_n(\varepsilon_s) \not\to 0.$$

Comparing (A.8) and (A.9) we obtain that if $Y(x) \not\equiv 0$, then

$$-[g'(0)]^2 = (n + 1/2)g''(0).$$

LEMMA A.2. *Consider the equation* (3.11) *under the additional hypotheses*:
(a) *there is a* $\Delta > 0$ *such that* $g(x) \in C^\infty[-\Delta, \Delta]$,
(b) $g(0) = n$,
(c) $g(x) = g(-x)$.
Let $\{y(x, \varepsilon_s)\}_{s=1}^\infty$ *be a sequence of uniformly bounded solutions of* (3.11) *which converges to a function* $Y(x)$. *Then*

$$V_{n-2j}(\varepsilon_s) = \frac{(-\varepsilon_s)^j}{2^j(j!)}V_n(\varepsilon_s) + O(\varepsilon_s^{j+1}), \qquad n - 2j \geqq 0. \tag{A.10}$$

Proof. Using (2.3a), (2.3b) and Lemma A.1 we proceed by induction. Suppose $n - 2 \geqq 0$. Then

$$\varepsilon_s V_n(\varepsilon_s) + 2V_{n-2}(\varepsilon_s) = -\sum_{r=0}^{n-3}\binom{n-2}{r}g^{(n-2-r)}(0)V_r(\varepsilon_s) = O(\varepsilon_s^2).$$

Hence,

$$V_{n-2}(\varepsilon_s) = -\frac{\varepsilon_s}{2}V_n(\varepsilon_s) + O(\varepsilon_s^2).$$

Suppose the lemma has been established for $j = j_0$ and $n - 2j_0 - 2 \geqq 0$. Then

$$\varepsilon_s V_{n-2j_0}(\varepsilon_s) + [2j_0 + 2]V_{n-2j_0-2}(\varepsilon_s) = O(\varepsilon_s^{j_0+2}).$$

That is,

$$V_{n-2(j_0+1)} = \frac{(-\varepsilon_s)V_{n-2j_0}}{2[j_0+1]} + O(\varepsilon_s^{(j_0+1)+1}).$$

Using the inductive hypothesis we obtain

$$V_{n-2(j_0+1)}(\varepsilon_s) = \frac{(-\varepsilon_s)^{j_0+1}V_n(\varepsilon_s)}{2^{j_0+1}[(j_0+1)!]} + O(\varepsilon_s^{(j_0+1)+1})$$

and the lemma is proved.

LEMMA A.3. *Under the same hypothesis as in Lemma* A.2, *suppose*

$$g^{(2j)}(0) = 0, \qquad j = 0, 1, \cdots, k-1. \tag{A.11}$$

Then, for $1 \leqq j \leqq k$, *we have*

$$\begin{aligned} V_{n+2j}(\varepsilon_s) = {} & \frac{g^{(2k)}(0)(-1)^{k+1}\varepsilon_s^{k-j}[(j-1)!]V_n(\varepsilon_s)}{2^{k+1-j}(k!)} \\ & \cdot \sum_{s=0}^{j-1}(-1)^s\binom{n+2s}{n-2k+2s}\binom{k}{k-s} + O(\varepsilon_s^{k+1-j}). \end{aligned} \tag{A.12}$$

Proof. Once more, we proceed by induction. Let $j = n$ in (2.3a), (2.3b). Using Lemma A.1 we obtain

$$\varepsilon_s V_{n+2}(\varepsilon_s) = -\binom{n}{n-2k}g^{(2k)}(0)V_{n-2k}(\varepsilon_s) + O(\varepsilon_s^{k+1}).$$

Using Lemma A.2 we next obtain

$$\varepsilon_s V_{n+2}(\varepsilon_s) = \frac{g^{(2k)}(0)(-1)^{k+1}\varepsilon_s^{k-1}[(1-1)!]}{2^k k!}V_n(\varepsilon_s)\binom{n}{n-2k}\binom{k}{k} + O(\varepsilon_s^k).$$

That is, the lemma is true for $j = 1$. Suppose the lemma is true for $j = 1, 2, \cdots, j_0 - 1$ and $j_0 \leqq k$. Then using (2.3a) and (2.3b) with $j = n + 2(j_0 - 1)$ we obtain

$$\begin{aligned} \varepsilon_s V_{n+2j_0} = {} & 2(j_0-1)V_{n+2(j_0-1)} - \binom{n+2(j_0-1)}{n-2k+2(j_0-1)} \\ & \cdot g^{(2k)}(0)V_{n-2k+2(j_0-1)}(\varepsilon_s) + O(\varepsilon_s^{k-j_0+2}). \end{aligned} \tag{A.13}$$

Using the inductive assumption in (A.13) we obtain

$$\begin{aligned} \varepsilon_s V_{n+2j_0} = {} & \frac{2(j_0-1)(-1)^{k+1}\varepsilon_s^{k-(j_0-1)}[(j_0-2)!]}{2^{k+1-(j_0-1)}(k!)}g^{(2k)}(0)V_n \\ & \cdot \sum_{r=0}^{j_0-2}(-1)^r\binom{n+2r}{n-2k+2r}\binom{k}{k-r} - \binom{n+2(j_0-1)}{n-2k+2(j_0-1)} \\ & \cdot g^{(2k)}(0)V_{n-2k+2(j_0-1)} + O(\varepsilon_s^{k-j_0+2}). \end{aligned}$$

Applying Lemma A.2 we have

$$\begin{aligned} \varepsilon_s V_{n+2j_0} = {} & \frac{g^{(2k)}(0)(-1)^{k+1}\varepsilon_s^{k-j_0+1}[(j_0-1)!]}{2^{k+1-j_0}(k!)}V_n\sum_{r=0}^{j_0-2}(-1)^r\binom{n+2r}{n-2k+2r}\binom{k}{k-r} \\ & + \left[(-1)^{j_0-1}\binom{n+2(j_0-1)}{n-2k+2(j_0-1)}\binom{k}{k-(j_0-1)}\right] \\ & \cdot \frac{g^{(2k)}(0)(-1)^{k+1}\varepsilon_s^{k-j_0+1}[(j_0-1)!]V_n}{2^{k+1-j_0}(k!)} + O(\varepsilon_s^{k-j_0+2}). \end{aligned} \tag{A.14}$$

Combining the terms on the right-hand side of (A.14) and dividing by ε_s (since $k \geqq j_0$) we obtain (A.12).

COROLLARY.

$$V_{n+2k}(\varepsilon_s) = \frac{g^{2k}(0)(-1)^{k+1}[(k-1)!]V_n(\varepsilon_s)}{2(k!)} \sum_{r=0}^{k-1} (-1)^r \binom{n+2r}{n-2k+2r}\binom{k}{k-r} + O(\varepsilon_s). \tag{A.15}$$

LEMMA A.4. *Under the hypothesis of Lemma* A.2, *assume that*

$$g^{(2j)}(0) = 0, \qquad j = 1, 2, \cdots, k-1.$$

Then

$$V_{n+2k}(\varepsilon_s) = \frac{1}{2k}\binom{n+2k}{n} g^{(2k)}(0)V_n(\varepsilon_s) + O(\varepsilon_s).$$

Proof. Apply (3.2a), (3.2b) with $j = n + 2k$.

LEMMA A.5. *Let*

$$J \equiv \sum_{r=0}^{k} (-1)^r \binom{n+2r}{n-2k+2r}\binom{k}{k-r}. \tag{A.16}$$

Then $J > 0$.

Proof. Let ∇ denote the backward difference operator with step size 1 and let $\nabla(2)$ denote the backward difference operator with step size 2. Let k be fixed and $r - 2k \geqq 0$. Let

$$\varphi(r) \equiv \binom{r}{r-2k} = \frac{r!}{(2k)!(r-2k)!}.$$

Then

$$J = \nabla^k(2)\varphi(r)|_{r=n+2k}.$$

As is well known (see [9, p. 6]),

$$\nabla^k\varphi(r)|_{r \geqq n+k} > 0.$$

Thus, the lemma follows from the identity

$$\nabla^k(2)A(r)\Big|_{r=r_0} = \sum_{m=0}^{k} \binom{k}{m} \nabla^k A(r)\Big|_{r_0-s},$$

which is easily established by induction.

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ANALYTIC CONTINUATION OF THE EULER TRANSFORM*

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Abstract. The Euler transform is defined by an integral over the unit interval [0, 1] if the transform variables have positive real parts. If the function to be transformed is holomorphic on a neighborhood of [0, 1], its transform can be represented for all complex values of the transform variables by an integral around a contour which encircles [0, 1]. The integrand contains a ${}_2F_1$-function. This type of contour integral represents the principal branch of Appell's double hypergeometric function F_1 for all values of the parameters and variables.

1. Introduction. If the function g is piecewise continuous on [0, 1], its Euler transform is the integral

$$h(x, y) = [\Gamma(x)\Gamma(y)]^{-1} \int_0^1 u^{x-1}(1-u)^{y-1} g(u)\, du, \tag{1.1}$$

where $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$. If the right-half-complex-plane is denoted by $\mathbb{C}_> = \{x \in \mathbb{C} : \operatorname{Re} x > 0\}$, it can be shown that h is holomorphic on $\mathbb{C}_>^2$. Provided that g is holomorphic on an open set in the complex plane which contains the interval [0, 1], a holomorphic continuation of h is obtained by replacing the integral in (1.1) by a Pochhammer double loop integral. The function h can then be shown to be an entire function on $\mathbb{C}^2$ which is represented for almost all values of x and y by

$$h(x, y) = (2\pi i)^{-2} e^{-i\pi(x+y)} \Gamma(1-x)\Gamma(1-y) \int_\delta s^{x-1}(1-s)^{y-1} g(s)\, ds, \tag{1.2}$$

where δ is a double loop slung around the points 0 and 1. One sometimes writes $\delta = (1+, 0+, 1-, 0-)$ to indicate the nature of the contour [6, p. 257], [3, p. 14]. The representation (1.2) fails when x or y is a positive integer, for the integrand then has no singularity at 0 or 1, respectively, and the contour δ can be shrunk to a point.

In § 2 we show that h can be represented on $\mathbb{C}^2$ by a second contour integral,

$$h(x, y) = [2\pi i \Gamma(x + y)]^{-1} \int_\gamma s^{-1}\, {}_2F_1(1, x; x + y; s^{-1}) g(s)\, ds, \tag{1.3}$$

where ${}_2F_1$ is Gauss' hypergeometric function. The contour $\gamma = (1+, 0+)$ is a single loop encircling the line segment with endpoints 0 and 1, and the representation is valid for all x and y without exception.

In § 3 we apply (1.3) to obtain an integral representation of Appell's double hypergeometric function F_1. This new representation is the only one which does not fail for any values of the parameters or variables in the domain on which F_1 is holomorphic.

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2. Representation by a contour integral. We give first an informal argument leading to (1.3) and subsequently a formal proof that (1.3) is the holomorphic continuation of (1.1). If g is holomorphic on a neighborhood of the line segment joining 0 and 1, we may use Cauchy's integral formula,

$$g(u) = (2\pi i)^{-1} \int_\gamma (s-u)^{-1} g(s)\, ds, \tag{2.1}$$

where u is any point of the line segment and the contour γ encircles the line segment in the positive direction. Substituting in (1.1) and changing the order of integration (which is easily justified), we meet the integral [4, p. 240]

$$\begin{aligned}[]
&[\Gamma(x)\Gamma(y)]^{-1} \int_0^1 u^{x-1}(1-u)^{y-1}(s-u)^{-1}\, du \\
&\qquad = [\Gamma(x+y)]^{-1} s^{-1} {}_2F_1(1, x; x+y; s^{-1}),
\end{aligned} \tag{2.2}$$

where $s \notin [0, 1]$. Provided $(x, y) \in \mathbb{C}^2_>$, (1.3) follows immediately. We must show next that the right side of (1.3) is an entire function of x and y and so provides the holomorphic continuation of h to $\mathbb{C}^2$. This is done in the proof of Theorem 1 below.

For the formal proof we use a different procedure which is less straightforward but gives a better understanding of the relation between (1.1) and (1.3). It requires several properties of ${}_2F_1$ which we state at the outset. Let $\mathbb{C}'$ be the complex plane cut along the real axis from 0 to 1. If $(x, y) \in \mathbb{C}^2$ and $s \in \mathbb{C}'$, let

$$f(x, y, s) = [\Gamma(x+y)]^{-1} s^{-1} {}_2F_1(1, x; x+y; s^{-1}). \tag{2.3}$$

From [4, (9.5.1)] it follows that

$$f(x, y, s) = -f(y, x, 1-s). \tag{2.4}$$

We shall want the following properties:

(a) f is holomorphic on $\mathbb{C}^2 \times \mathbb{C}'$.

(b) If $\operatorname{Re} x > 0$, then $sf(x, y, s) \to 0$ as $s \to 0$ in $\mathbb{C}'$.

(c) If $\operatorname{Re} y > 0$, then $(s-1)f(x, y, s) \to 0$ as $s \to 1$ in $\mathbb{C}'$.

(d) If $u \in \mathbb{R}$ and $0 < u < 1$, then

$$\lim_{\varepsilon \to 0+} [f(x, y, u - i\varepsilon) - f(x, y, u + i\varepsilon)] = 2\pi i [\Gamma(x)\Gamma(y)]^{-1} u^{x-1}(1-u)^{y-1}. \tag{2.5}$$

For property (a) see [4, pp. 239, 245] and note that $s \in \mathbb{C}'$ implies that s^{-1} lies in the complex plane cut along $[1, +\infty]$. Property (b) is a consequence of [4, (9.5.9) and (9.7.7)], and property (c) follows from (b) by (2.4). For (d) see [4, p. 276].

THEOREM 1. *Let $J = \{u \in \mathbb{R} : 0 \leqq u \leqq 1\}$, let $\Omega \subset \mathbb{C}$ be a simply connected open set such that $J \subset \Omega$, and let $g : \Omega \to \mathbb{C}$ be holomorphic. Let γ be a positively oriented rectifiable Jordan curve in Ω with inner region $I(\gamma) \supset J$. If $(x, y) \in \mathbb{C}^2$ define $h(x, y)$ by* (1.3). *Then h is an entire function of (x, y), and h is represented on $\mathbb{C}^2_>$ by* (1.1).

Proof. By (1.3) and (2.3),

$$h(x, y) = (2\pi i)^{-1} \int_\gamma f(x, y, s) g(s)\, ds. \tag{2.6}$$

Let γ^* denote the set of points lying on the curve γ. Property (a) implies that the integrand is entire in x and y for each $s \in \gamma^*$, integrable on γ^* for each $(x, y) \in \mathbb{C}^2$, and continuous on $\mathbb{C}^2 \times \gamma^*$. It follows that h is entire (see for example [5, p. 283]).

If the s-plane is cut along the unit interval J, we may deform γ by Cauchy's theorem into a contour which consists of two small circles of radius ρ and centers 0 and 1, joined by the upper and lower edges of the cut. By properties (b) and (c) the integrals around the two circles tend to zero with ρ if $(x, y) \in \mathbb{C}^2_>$. The integrals along the two edges of the cut combine by property (d) to give (1.1).

3. Integral representation of Appell's F_1. The double hypergeometric function $F_1(a, \beta, \beta', c; \xi, \eta)/\Gamma(c)$ can be represented [3, p. 231] as an Euler transform (1.1) with $(x, y) = (a, c - a)$ and $g(u) = (1 - u\xi)^{-\beta}(1 - u\eta)^{-\beta'}$. It is required that $\operatorname{Re} c > \operatorname{Re} a > 0$ and that ξ and η be points in the complex plane cut along the segment $[1, +\infty]$ of the real axis. The parameters β and β' may be any complex numbers. By Theorem 1, with no restriction on a and c,

$$\begin{aligned}(3.1)\qquad &[\Gamma(c)]^{-1}F_1(a, \beta, \beta', c; \xi, \eta)\\ &= [2\pi i\Gamma(c)]^{-1}\int_\gamma s^{-1}\,{}_2F_1(1, a; c; s^{-1})(1 - s\xi)^{-\beta}(1 - s\eta)^{-\beta'}\,ds,\end{aligned}$$

where γ is a closed contour which contains the interval $[0, 1]$ of the real axis in its inner region and the points ξ^{-1} and η^{-1} in its outer region. If $c \neq 0, -1, -2, \cdots$, the factors $\Gamma(c)$ may be cancelled.

Except for representations which contain another F_1 in the integrand, (3.1) is the only representation of F_1 which is valid for all complex values of the parameters and all values of ξ and η in the cut plane.

The generalization of (3.1) to Lauricella's function F_D in several variables presents no difficulty. We give instead the corresponding representation of the R-function, a variant of F_D which has properties of homogeneity and permutation symmetry [1], [2]. By [2, (4.15)], (2.3) becomes

$$(3.2)\qquad f(x, y, s) = [\Gamma(x + y)]^{-1}R_{-1}(x, y; s - 1, s).$$

Let $\mathbb{C}_0$ be the complex plane cut along the nonpositive real axis, and let z be a k-tuple with components in $\mathbb{C}_0$. Let $b \in \mathbb{C}^k$ and define $c = \sum_{i=1}^k b_i$. If $\operatorname{Re} c > \operatorname{Re} a > 0$, $R_{-a}(b, z)/\Gamma(c)$ can be represented [1, (4.22)] as an Euler transform (1.1) with $(x, y) = (a, c - a)$ and $g(u) = \prod_{i=1}^k (1 - u + uz_i)^{-b_i}$. By Theorem 1, with no restrictions on a and c,

$$\begin{aligned}(3.3)\qquad &[\Gamma(c)]^{-1}R_{-a}(b, z)\\ &= [2\pi i\Gamma(c)]^{-1}\int_\gamma R_{-1}(a, c - a; s - 1, s - 0)\prod_{i=1}^k (1 - s + sz_i)^{-b_i}\,ds,\end{aligned}$$

where γ is a closed contour which contains the interval $[0, 1]$ of the real axis in its inner region and the points $(1 - z_i)^{-1}$, $i = 1, \cdots, k$, in its outer region.

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ON GENERALIZED BERNSTEIN POLYNOMIALS*

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Abstract. The generalized Bernstein polynomials of Jakimovski and Leviatan and the generalized Euler summability method of Wood are considered in the general context of Gronwall-like transformations. It is shown under general circumstances that, for bounded sequences, generalized Euler summability is equivalent to Euler summability. A class of generalized Bernstein polynomials which are generated by certain Gronwall methods is defined and the members of this class which possess the uniform approximation property are characterized.

1. Introduction. The Euler summation matrix $E = (E_{mn})$ is defined by

$$E_{mn} = \binom{m}{n} x^n(1 - x)^{m-n}, \qquad n \leqq m,$$

where x is a parameter, $0 < x \leqq 1$. Given a sequence $\{s_n\}$, its Euler transform is the sequence $\{E_m\}$ defined by

$$E_m = \sum_{n=0}^{m} E_{mn}s_n, \qquad m = 0, 1, \cdots,$$

and $\{s_n\}$ is said to be E-summable to s if $\lim E_m = s$.

It seems to be not very well known that Euler summability belongs to a general class of summation methods which we shall call the $[f, \gamma]$ methods. The notion of $[f, \gamma]$-summability was introduced by T. H. Gronwall [2] (see also [1]). Given functions $f(w)$ and $\gamma(w)$ of certain types, Gronwall defined the transform of a sequence $\{s_n\}$ to be the sequence $\{U_n\}$ defined by the formal power series identity

$$(1.1) \qquad (1 - f(w))\gamma(w) \sum_{n=0}^{\infty} s_n[f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n,$$

where

$$\gamma(w) = \sum_{n=0}^{\infty} b_n w^n.$$

It is not difficult to see that

$$U_n = \sum_{k=0}^{n} \lambda_{nk}s_k,$$

where the matrix elements λ_{nk} are generated by the identity

$$(1.2) \qquad (1 - f(w))\gamma(w)[f(w)]^n = \sum_{m=n}^{\infty} \lambda_{mn}b_m w^m.$$

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Euler summability is obtained by taking

$$f(w) = \frac{xw}{1-(1-x)w} \quad \text{and} \quad \gamma(w) = (1-w)^{-1}.$$

Specifically, for Euler summability (1.1) and (1.2) become after routine computations,

$$\sum_{n=0}^{\infty} s_n \frac{[xw]^n}{[1-(1-x)w]^{n+1}} = \sum_{n=0}^{\infty} E_n w^n \tag{1.1'}$$

and

$$\frac{[xw]^n}{[1-(1-x)w]^{n+1}} = \sum_{m=n}^{\infty} E_{mn} w^m, \tag{1.2'}$$

respectively.

The Euler matrix plays a central role in the classical theory of approximation. In fact, Bernstein's proof of the Weierstrass theorem asserts that the polynomials

$$B_m(h, x) = \sum_{n=0}^{m} E_{mn} h\left(\frac{n}{m}\right)$$

converge uniformly to $h(x)$ as $m \to \infty$ for each $h \in C[0, 1]$. The Bernstein polynomials were generalized in [4] and the generalization was extended in [3]. We shall now give a brief account of this last generalization.

Suppose that the functions

$$g_m(z) = \sum_{n=0}^{\infty} (-1)^n a_{mn} z^n$$

are analytic in $|z| < R$ $(R > 1)$ and that as $m \to \infty$ they converge uniformly on compact subsets of $|z| < r \leqq R$ $(r > 1)$ to the function

$$g(z) = \sum_{n=0}^{\infty} (-1)^n a_n z^n.$$

We assume that $g(z) \neq 0$ for $-1 \leqq z \leqq 0$. Define a sequence of polynomials $\{\zeta_n^{(m)}(x)\}$ by the equation

$$g_m(u)(1+u)^x = \sum_{n=0}^{\infty} \zeta_n^{(m)}(x) u^n.$$

The generalized Bernstein operator of Jakimovski and Leviatan is then defined by

$$L_m(h, x) = \frac{1}{g(x-1)} \sum_{n=0}^{m} (-1)^{m-n} \zeta_{m-n}^{(m)}(-n-1) x^n (1-x)^{m-n} h\left(\frac{n}{m}\right). \tag{1.3}$$

In [3] various approximation theoretic properties of the linear operators (1.3) are investigated under the assumption that

$$(-1)^{m-n} \zeta_{m-n}^{(m)}(-n-1) \geqq 0 \quad \text{for } m \geqq m_0 \quad \text{and} \quad n_0 \leqq n \leqq m. \tag{1.4}$$

The generalized Euler summability transform of Wood [6], whose matrix representation we shall denote by $E^* = (b_{mn})$, is then defined by

$$L_m(h, x) = \sum_{n=0}^{m} b_{mn} h\left(\frac{n}{m}\right). \tag{1.5}$$

Note that if $g_m(u) \equiv g(u) \equiv 1$, then $L_m(h, x) = B_m(h, x)$ and $E^* = E$.

In this paper we shall clarify the relationship between E and E^* by considering both methods in the general context of Gronwall-like transforms. In particular we shall show that E^* is equivalent to E for bounded sequences if g has no zeros in the unit disc. As a consequence of these summability results we shall show that the approximation property of the operators (1.3) can be simplified and that the condition (1.4) can be replaced by a more natural condition on the function g.

2. The generalized Euler matrix. It is natural to ask if Wood's generalization of Euler summability falls within the class of $[f, \gamma]$-methods. That is, is there a generating relation like (1.1) for E^*? We shall show that although E^* is not quite an $[f, \gamma]$-method it does satisfy a certain power series identity. This identity will make clear the relationship between E and E^*.

First we observe that the matrix elements (b_{mn}) are generated by a power series identity similar to (1.2). Suppose we define the functions $h_n[(1 - x)w]$ by

$$h_n[(1 - x)w] = \frac{1}{g(x - 1)} \sum_{p=0}^{\infty} \sum_{k=0}^{p} \binom{n + p - k}{n} (a_{n+p,k} - a_k)[(1 - x)w]^p. \tag{2.1}$$

A computation with the series involved shows that

$$\sum_{m=n}^{\infty} b_{mn} w^m = \frac{x^n w^n}{[1 - (1 - x)w]^{n+1}} \frac{g[(x - 1)w]}{g(x - 1)} + x^n w^n h_n[(1 - x)w]. \tag{2.2}$$

When $g_m(z) \equiv g(z) \equiv 1$ equation (2.2) reduces to (1.2′). Let the first and second terms on the right of (2.2) have the respective power series expansions

$$\sum_{m=n}^{\infty} c_{mn} w^m \quad \text{and} \quad \sum_{m=n}^{\infty} \varepsilon_{mn} w^m.$$

Then (2.2) implies that

$$b_{mn} = c_{mn} + \varepsilon_{mn}. \tag{2.3}$$

Given a sequence $\{s_n\}$, let

$$b_m = \sum_{n=0}^{m} b_{mn} s_n, \quad c_m = \sum_{n=0}^{m} c_{mn} s_n \quad \text{and} \quad \varepsilon_m = \sum_{n=0}^{m} \varepsilon_{mn} s_n.$$

Then $b_m = c_m + \varepsilon_m$ and by (2.2) we have

$$\frac{g[(x - 1)w]}{g(x - 1)} \sum_{n=0}^{\infty} s_n \frac{x^n w^n}{[1 - (1 - x)w]^{n+1}} = \sum_{m=0}^{\infty} c_m w^m \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} s_n x^n w^n h_n[(1 - x)w] = \sum_{m=0}^{\infty} \varepsilon_m w^m.$$

First we shall prove that for bounded sequences the summability properties of (b_{mn}) depend entirely on the matrix (c_{mn}) because if $\{s_n\}$ is bounded then $\{\varepsilon_n\}$ converges to zero.

THEOREM 2.1. *The matrix (ε_{mn}) sums every bounded sequence to zero.*

Proof. We need only show that $\sum_{n=0}^{m} |\varepsilon_{mn}| \to 0$ as $m \to \infty$. We have that g_m converges uniformly on compacta to g in some disc of radius $R > 1$. Consequently, given $\varepsilon > 0$ there is an M with $|a_{mk} - a_k| < \varepsilon R^{-k}$ for $m > M$ (see [3]). Then for $m > M$ we have

$$\sum_{n=0}^{m} |\varepsilon_{mn}| \leqq \sum_{n=0}^{m} x^n(1-x)^{m-n} \sum_{k=0}^{m-n} \binom{m-k}{n} |a_{mk} - a_k|$$

$$< \varepsilon \sum_{n=0}^{m} x^n(1-x)^{m-n} \binom{m}{n} \sum_{k=0}^{m} R^{-k} < \frac{\varepsilon R}{R-1},$$

which completes the proof.

Theorem 2.1 shows that when restricted to bounded sequences E^* is equivalent to (c_{mn}).

THEOREM 2.2. *E^* is equivalent to E for bounded sequences if $g(z) \neq 0$ for $|z| \leqq 1$.*

Proof. By Theorem 2.1, $\{s_n\}$ is E^*-summable if and only if $\{s_n\}$ is (c_{mn})-summable. The generating identities give

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{m} c_{mn} s_n w^m &= \frac{g[(x-1)w]}{g(x-1)} \frac{1}{1-(1-x)w} \sum_{n=0}^{\infty} s_n [f(w)]^n \\ &= \frac{g[(x-1)w]}{g(x-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{m} E_{mn} s_n w^m. \end{aligned} \tag{2.5}$$

Let $g[(x-1)w]/g(x-1) = \sum_{n=0}^{\infty} q_n w^n$. Since $g(w)$ is analytic in a disc $|w| < R$ $(R > 1)$ we have for some constant $M > 0$,

$$|q_n| < M(1-x)^n R^{-n}. \tag{2.6}$$

From (2.5) we get that

$$c_m = \sum_{k=0}^{m} q_{m-k} E_k.$$

We suppose that $E_k \to 0$ and choose m_0 so that $|E_k| < \varepsilon$ for $k > m_0$. Then for $m > m_0$ we have

$$|c_m| \leqq \sum_{k=0}^{m_0} |q_{m-k}| \, |E_k| + \varepsilon \sum_{k=m_0+1}^{m} |q_{m-k}|.$$

Letting $Q = \max\{|E_k| : k \leqq m_0\}$, we have

$$|c_m| \leqq Q \sum_{k=0}^{m_0} |q_{m-k}| + \varepsilon \sum_{k=m_0+1}^{m} |q_{m-k}|.$$

The second sum above is bounded and by (2.6) the first sum converges to zero as $m \to \infty$. Hence $c_m \to 0$ if $E_m \to 0$. This proves that bounded E-summable sequences are E^* - summable. For the converse we note that from (2.5),

$$\frac{g(x-1)}{g[(x-1)w]}\sum_{m=0}^{\infty} c_m w^m = \sum_{m=0}^{\infty} E_m w^m.$$

Now, if $g(w)$ has no zero in $|w| \leqq 1$, we can use the argument above to conclude that $c_m \to 0$ implies $E_m \to 0$.

3. Uniform approximation. In our discussion of results on uniform approximation we shall use the basic simplification due to Korovkin [5, p. 14] which states that for monotone linear operators L_n on $C[0, 1]$ the condition $L_n h \to h$ (uniformly) for all $h \in C[0, 1]$ is equivalent to $L_n h \to h$ (uniformly) for each of the functions $h(t) = 1, t, t^2$ respectively.

In this section we consider the linear operators defined by

$$B_m^{(\gamma)}(h, x) = \sum_{n=0}^{m} a_{mn} h\left(\frac{n}{m}\right), \tag{3.1}$$

where the matrix (a_{mn}) is generated by the $[f, \gamma]$-transform with

$$f(w) = \frac{xw}{1-(1-x)w} \quad \text{and} \quad \gamma(w) = \sum_{n=0}^{\infty} b_n w^n.$$

That is, the transform of a sequence $\{s_n\}$ by the matrix (a_{mn}) is the sequence $\{U_m\}$ defined by (1.1). Note that if $\gamma(w) = (1-w)^{-1}$, then $B_m^{(\gamma)}(h, x) = B_m(h, x)$ and $(a_{mn}) = (E_{mn})$. It is not difficult to see that a_{mn} is a polynomial in x and $a_{mn} \geqq 0$ for each m and n if the power series coefficients of $\gamma(w)$ are nonnegative. Thus if $b_n \geqq 0$, then the operators $B_m^{(\gamma)}(h, x)$ are monotone and Korovkin's theorem can be applied to characterize those operators of type (3.1) which have the uniform approximation property.

THEOREM 3.1. *Suppose that $b_n > 0$ for each n. Then $B_m^{(\gamma)}(h, x) \to h(x)$ as $m \to \infty$ uniformly in x for each $h \in C[0, 1]$ if and only if $b_m^{-1}\sigma_{m-k}^{(k)} \to 1$ as $m \to \infty$ for $k = 1, 2$, where $\{\sigma_m^{(k)}\}$ is the (C, k) transform of the sequence $\{b_m\}$.*

Proof. Setting $s_n = 1$ in (1.1) gives $B_m^{(\gamma)}(1, x) = U_m = 1$ for all m. If $s_n = n$, then (1.1) gives

$$\gamma(w)\frac{xw}{1-w} = \sum_{m=0}^{\infty} b_m U_m w^m.$$

Therefore,

$$B_m^{(\gamma)}(t, x) = \frac{U_m}{m} = \frac{x}{mb_m}\sum_{j=0}^{m-1} b_j = xb_m^{-1}\sigma_{m-1}^{(1)}.$$

Finally, to complete the proof we note that if $s_n = n^2$, then by (1.1),

$$\gamma(w)\left\{\frac{2x^2w^2}{(1-w)^2} + \frac{xw}{1-w}\right\} = \sum_{m=0}^{\infty} b_m U_m w^m.$$

Hence,

$$B_m^{(\gamma)}(t^2, x) = \frac{U_m}{m^2} = 2x^2 b_m^{-1} m^{-2} \sum_{j=0}^{m-2} b_j(m-1-j) + xm^{-1} b_m^{-1} \sigma_{m-1}^{(1)}$$

$$= x^2 \frac{m-1}{m} b_m^{-1} \sigma_{m-2}^{(2)} + \frac{x}{m} b_m^{-1} \sigma_{m-1}^{(1)}.$$

Our next result shows that a uniform approximation theorem of Jakimovski and Leviatan [3] can be obtained as a simple corollary of the above theorem. By (1.5) and (2.3) we have that for any $h \in C[0, 1]$,

$$L_m(h, x) = \sum_{n=0}^{m} c_{mn} h\left(\frac{n}{m}\right) + \sum_{n=0}^{m} \varepsilon_{mn} h\left(\frac{n}{m}\right).$$

Since $h(n/m)$ is uniformly bounded it follows from Theorem 2.1 that

$$\lim_m L_m(h, x) = \lim_m \sum_{n=0}^{m} c_{mn} h\left(\frac{n}{m}\right). \tag{3.2}$$

Now (2.4) can be written

$$\frac{g[(x-1)w]}{g(x-1)} \frac{1-f(w)}{1-w} \sum_{n=0}^{\infty} s_n [f(w)]^n = \sum_{m=0}^{\infty} c_m w^m.$$

Thus by (3.2) and (1.1) we have that

$$\lim_m L_m(h, x) = \lim_m B_m^{(\gamma)}(h, x), \tag{3.3}$$

where

$$\gamma(w) = \frac{g[(x-1)w]}{g(x-1)} (1-w)^{-1}. \tag{3.4}$$

COROLLARY 3.2 (see [3]). *Suppose that the power series coefficients of* $g[(x-1)w]/g(x-1)$ *are nonnegative and* $g(z) \neq 0$ *for* $-1 \leqq z \leqq 0$. *Then* $\lim_m L_m(h, x) = h(x)$ *uniformly in* x *for each* $h \in C[0, 1]$.

Proof. The power series coefficients of $\gamma(w)$ in (3.4) are clearly positive. Also if

$$\frac{g[(x-1)w]}{g(x-1)} = \sum_{k=0}^{\infty} q_k w^k,$$

then

$$\gamma(w) = \sum_{m=0}^{\infty} b_m w^m = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} q_k \right) w^m.$$

Hence

$$\lim_m b_m = \sum_{k=0}^{\infty} q_k = \frac{g(x-1)}{g(x-1)} = 1.$$

Therefore the corollary follows by (3.3) and Theorem 3.1.

An interesting class of Gronwall methods is generated by the functions

$$f(w) = \frac{x - x(1-w)^{\beta}}{x + (1-x)(1-w)^{\beta}}, \qquad 0 < \beta \leqq 1, \tag{3.5}$$

and

$$\gamma(w) = (1-w)^{-\alpha} = \sum_{n=0}^{\infty} b_n^{(\alpha)} w^n, \qquad 0 < \alpha. \tag{3.6}$$

For appropriate choices of x, α and β this class includes the Cesàro methods, the Euler method and the method of de la Vallée Poussin. One may define linear operators on $C[0, 1]$ generated by these $[f, \gamma]$ methods in analogy with (3.1) and the question arises whether these operators have the uniform approximation property. It turns out that the Euler method is unique in this class with respect to having this property.

THEOREM 3.3. *The linear operators* $B_m^{(\gamma)}$ *generated by the functions* (3.5) *and* (3.6) *have the uniform approximation property if and only if* $\alpha = \beta = 1$.

Proof. If $\alpha = \beta = 1$, then $B_m^{(\gamma)}(h, x)$ are the classical Bernstein polynomials and the approximation property is well known.

Conversely if $s_n = n$, then by (1.1),

$$\gamma(w)\frac{f(w)}{1 - f(w)} = \sum_{n=0}^{\infty} b_n^{(\alpha)} U_n w^n.$$

Equations (3.5) and (3.6) give

$$x[(1-w)^{-\alpha-\beta} - (1-w)^{-\alpha}] = \sum_{n=0}^{\infty} b_n^{(\alpha)} U_n w^n,$$

and therefore,

$$B_m^{(\gamma)}(t, x) = \frac{U_m}{m} = x\left[\frac{b_m^{(\alpha+\beta)}}{m b_m^{(\alpha)}} - \frac{1}{m}\right].$$

The asymptotic estimate $b_m^{(\delta)} \sim \Gamma(\delta)^{-1} m^{\delta-1}$ now gives

$$\lim_m \frac{U_m}{m} = x \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \lim_m m^{\beta-1}.$$

Therefore, $B_m^{(\gamma)}(t, x) \to x$ only if $\alpha = \beta = 1$.

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THE NONCHARACTERISTIC CAUCHY PROBLEM FOR PARABOLIC EQUATIONS IN ONE SPACE VARIABLE*

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Abstract. An integral operator is constructed which maps ordered pairs of analytic functions onto analytic solutions of linear second order parabolic equations in one space variable with analytic coefficients. This operator is then used to construct a solution to the noncharacteristic Cauchy problem for parabolic equations in one space variable. Applications are made to the inverse Stefan problem and the analytic continuation of solutions to parabolic equations.

1. Introduction. Consider a thin block of ice at 0°C occupying the interval $0 \leqq x < \infty$ and suppose at $x = 0$ the temperature is given by a prescribed function $\varphi(t) > 0$ where $t \geqq 0$ denotes time. Then the ice will begin to melt and for $t > 0$ the water will occupy an interval $0 \leqq x < s(t)$. If $u(x, t)$ is the temperature of the water we have

$$\begin{aligned} \frac{k}{\rho c} u_{xx} - u_t &= 0 \qquad \text{for } 0 < x < s(t), \\ u(0, t) &= \varphi(t) \quad \text{for } t > 0, \\ u(s(t), t) &= 0 \qquad \text{for } t > 0, \end{aligned} \tag{1.1}$$

where c denotes heat capacity, ρ the density, and k the conductivity of the water. In (1.1) it is assumed that c, ρ and k are constants. The curve $x = s(t)$ is a free boundary and is not given a priori. However, from the law of conservation of energy we have

$$u_x(s(t), t) = \frac{-\lambda\rho}{k} \frac{ds(t)}{dt}, \tag{1.2}$$

where λ is the latent heat of fusion. Equations (1.1) and (1.2) constitute a free boundary problem (the Stefan problem) for the heat equation. In the more general case when c, ρ and k are not constants, but are functions of x and t, we arrive at a free boundary problem for a parabolic equation in one space variable with variable coefficients.

Free boundary problems for parabolic equations are in general quite difficult to solve, and in recent years attention has been given to a study of the inverse problem, i.e., given $s(t)$ to find $\varphi(t)$ (c.f., [2], [3], [4], [6, pp. 71–80]). In physical terms this means we are asking how to heat the water in order to melt the ice along a prescribed curve, and in certain situations (e.g., the growing of crystals) it is in this inverse problem that we are primarily interested. Such an inverse approach leads mathematically to the problem of solving a noncharacteristic Cauchy problem for a parabolic equation and difficulties arise due to the fact that this problem is improperly posed in the sense of Hadamard (c.f., [4], [5]). However, as a

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consequence of the Cauchy–Kowalewski theorem, the noncharacteristic Cauchy problem is well-posed in the complex domain, and hence we are led to impose the requirement that $s(t)$ be an analytic function of t.

However, even after making the assumption that $s(t)$ is analytic, we are still left with serious problems in providing a constructive approach for solving the inverse Stefan problem for parabolic equations with (possibly) variable coefficients. For example, even though a local solution can always be constructed via the Cauchy–Kowalewski theorem, such an approach is far too tedious for practical application, and (more seriously) may not converge in the full region in which the solution is needed, i.e., in a region containing (a portion of) the positive t-axis. On the other hand, in the special case when the coefficients of the parabolic equation are independent of time (e.g., the heat equation), a constructive method for solving the inverse Stefan problem has been given by C. D. Hill [4]. In theory Hill's approach also applies when the coefficients are time-dependent. However, in practice, this is not the case, since Hill's work is based on the construction of a fundamental solution $S(x, t; \xi, \tau)$ given by the series expansion

$$S(x, t; \xi, \tau) = \sum_{j=0}^{\infty} S_j(x, t; \xi) \frac{j!}{(t-\tau)^{j+1}}, \tag{1.3}$$

where in the case of time-dependent coefficients each $S_j(x, t; \xi)$, $j = 0, 1, 2, \cdots$, is in turn a solution of a nonhomogeneous, noncharacteristic Cauchy problem for a parabolic equation with time-dependent coefficients. To construct $S(x, t; \xi, \tau)$ via this method (and to determine its domain of regularity) is as tedious and impractical as using the Cauchy–Kowalewski theorem, and hence in this general case it is desirable to derive new methods for solving the noncharacteristic Cauchy problem.

Our approach to this problem is based on the construction of an integral operator which maps noncharacteristic Cauchy data onto solutions of (a canonical form of) the parabolic equation being investigated. The kernel of this operator can be expanded in an infinite series, each term of which is determined by a simple three term recursion relation. To guarantee the global existence of our operator we will make the assumption that the coefficients of the differential equation are entire functions of x and analytic in t for $|t| < t_0$ where t_0 is some positive constant. We will show that as a consequence of this assumption every solution of a linear parabolic equation in one space variable (with analytic coefficients) which is analytic in some (complex) neighborhood of the origin has an automatic analytic continuation into an infinite strip parallel to the x axis containing this neighborhood. This theorem generalizes analogous results obtained by Widder for the heat equation [7] and Hill for parabolic equations with time-independent coefficients [4].

2. Integral operators for parabolic equations. Consider the general linear homogeneous parabolic equation of the second order in one space variable written in normal form

$$u_{xx} + a(x, t)u_x + b(x, t)u - c(x, t)u_t = 0. \tag{2.1}$$

We shall make the assumption that the coefficients $a(x,t)$, $b(x, t)$ and $c(x, t)$ are analytic functions of the (complex) variables x and t for $|x| < \infty$ and $|t| < t_0$. By making the change of dependent variable

$$u(x,t) = v(x,t)\exp\left\{-\frac{1}{2}\int_0^x a(\xi,t)\,d\xi\right\} \tag{2.2}$$

we arrive at an equation for $v(x,t)$ of the same form as (2.1) but with $a(x,t) = 0$. Hence without loss of generality we can restrict our attention to equations of the form

$$L[u] \equiv u_{xx} + b(x,t)u - c(x,t)u_t = 0, \tag{2.3}$$

where $b(x,t)$ and $c(x,t)$ are analytic functions of x and t for $|x| < \infty$, $|t| < t_0$.

We now look for a solution of (2.3) in the form

$$u(x,t) = -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau)f(\tau)\,d\tau - \frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau)g(\tau)\,d\tau, \tag{2.4}$$

where $t_0 - |t| > \delta > 0$ and $f(\tau)$ and $g(\tau)$ are arbitrary analytic functions of τ for $|\tau| < t_0$. We shall furthermore ask that $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$ satisfy the initial conditions

$$E^{(1)}(0,t,\tau) = \frac{1}{t-\tau}, \tag{2.5a}$$

$$E^{(1)}_x(0,t,\tau) = 0, \tag{2.5b}$$

$$E^{(2)}(0,t,\tau) = 0, \tag{2.6a}$$

$$E^{(2)}_x(0,t,\tau) = \frac{1}{t-\tau} \tag{2.6b}$$

and be analytic functions of their independent variables for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$, $t \neq \tau$. We shall first construct the function $E^{(1)}(x,t,\tau)$. Setting $g(\tau) = 0$ and substituting (2.4) into the differential equation shows that, as a function of x and t, $E^{(1)}(x,t,\tau)$ must be a solution of $L[u] = 0$ for $t \neq \tau$. We now assume that $E^{(1)}(x,t,\tau)$ has the expansion

$$E^{(1)}(x,t,\tau) = \frac{1}{t-\tau} + \sum_{n=2}^{\infty} x^n P^{(n)}(x,t,\tau), \tag{2.7}$$

where the $P^{(n)}(x,t,\tau)$ are (analytic) functions to be determined. Note that if termwise differentiation is permitted the series (2.7) satisfies the initial conditions (2.5a) and (2.5b). Observe that, in contrast to the kernel (1.3) of Hill's integral operator, we are expanding the kernel $E^{(1)}(x,t,\tau)$ (and later the kernel $E^{(2)}(x,t,\tau)$) in powers of x (instead of powers of $1/(t-\tau)$). This will allow us to determine the coefficients $P^{(n)}(x,t,\tau)$ via a simple three term recursion relation instead of being forced to determine each coefficient as a solution of a noncharacteristic Cauchy problem for a parabolic equation as in Hill's work [4]. Indeed, if we substitute (2.7) into

$L[u] = 0$, we are immediately led to the following recursion formula for the $P^{(n)}(x, t, \tau)$:

$$
\begin{aligned}
&P^{(1)} = 0,\\
&P^{(2)} = -\frac{c}{2(t-\tau)^2} - \frac{b}{2(t-\tau)},\\
&P^{(k+2)} = -\frac{2}{k+2}P_x^{(k+1)} - \frac{1}{(k+2)(k+1)}[P_{xx}^{(k)} + bP^{(k)} - cP_t^{(k)}], \quad k \geqq 1.
\end{aligned} \tag{2.8}
$$

We now let

$$\tilde{P}^{(k)}(x, t, \tau) = P^{(k)}(x, t, t - \tau). \tag{2.9}$$

Then (2.8) becomes

$$
\begin{aligned}
&\tilde{P}^{(1)} = 0,\\
&\tilde{P}^{(2)} = -\frac{c}{2\tau^2} - \frac{b}{2\tau},\\
&\tilde{P}^{(k+2)} = -\frac{2}{k+2}\tilde{P}_x^{(k+1)} - \frac{1}{(k+2)(k+1)}[\tilde{P}_{xx}^{(k)} + b\tilde{P}^{(k)} - c\tilde{P}_t^{(k)} - c\tilde{P}_\tau^{(k)}], \quad k \geqq 1.
\end{aligned} \tag{2.10}
$$

If we now define $Q^{(k)}(x, t, \tau)$ by the equation

$$\tilde{P}^{(k)}(x, t, \tau) = \tau^{-k}Q^{(k)}(x, t, \tau), \tag{2.11}$$

then (2.10) yields the following recursion formula for the $Q^{(k)}(x, t, \tau)$:

$$
\begin{aligned}
&Q^{(1)} = 0,\\
&Q^{(2)} = -\tfrac{1}{2}[c + \tau b],\\
&Q^{(k+2)} = -\frac{2\tau}{k+2}Q_x^{(k+1)} - \frac{2\tau}{(k+2)(k+1)}[\tau Q_{xx}^{(k)} + \tau bQ^{(k)} - \tau cQ_t^{(k)}\\
&\qquad\qquad + ckQ^{(k)} - \tau cQ_\tau^{(k)}], \qquad k \geqq 1.
\end{aligned} \tag{2.12}
$$

Now let M_0 be a positive constant such that

$$
\begin{aligned}
&c(x, t) \ll M_0(1 - x/r)^{-1}(1 - t/t_0)^{-1},\\
&b(x, t) \ll M_0(1 - x/r)^{-1}(1 - t/t_0)^{-1}
\end{aligned} \tag{2.13}
$$

for $|x| < r$ and $|t| < t_0$. In (2.13) the symbol "$\ll$" means "is dominated by" (c.f., [1]). The main properties of dominants we will use are the following: If $f(x) \ll g(x)$ for $|x| < r$, then

$$\frac{df(x)}{dx} \ll \frac{dg(x)}{dx} \quad \text{for } |x| < r, \tag{2.14a}$$

$$f(x) \ll g(x)(1 - x/r)^{-1} \quad \text{for } |x| < r. \tag{2.14b}$$

Similar properties apply to functions of several complex variables. Using the property

$$\tau \ll 2t_0\left(1 - \frac{\tau}{2t_0}\right)^{-1} \tag{2.15}$$

for $|\tau| < 2t_0$ we shall now show by induction that there exist positive constants M_n, $n = 1, 2, \cdots$, and ε (where ε can be chosen arbitrarily small and is independent of n, and M_n is a bounded function of n) such that for $|x| < r$, $|t| < t_0$, $|\tau| < 2t_0$, we have

$$\begin{aligned} Q^{(n+1)} &\ll M_{n+1}4^{n+1}t_0^{n+1}(3/2 + \varepsilon)^{n+1} \\ &\quad \cdot \left(1 - \frac{x}{r}\right)^{-(n+1)}\left(1 - \frac{t}{t_0}\right)^{-(n+1)}\left(1 - \frac{\tau}{2t_0}\right)^{-(2n+2)} r^{-(n+1)}, \\ &\qquad n = 0, 1, 2, \cdots. \end{aligned} \tag{2.16}$$

Equation (2.16) is clearly true for $n = 0$ and $n = 1$. Assume now that it is true for $n = k - 1$ and $n = k$. Then from (2.12)–(2.15) we have

$$\begin{aligned} Q^{(k+2)} &\ll \left\{\frac{(k+1)}{(k+2)}M_{k+1}4^{k+2}t_0^{k+2}(3/2+\varepsilon)^{k+1}\right. \\ &\quad \left. + \frac{M_k 4^{k+1}t_0^{k+1}(3/2+\varepsilon)^k}{(k+2)(k+1)}(2t_0k(k+1) + 2M_0t_0r^2 + 4M_0kr^2)\right\} \\ &\quad \cdot \left(1 - \frac{x}{r}\right)^{-(k+2)}\left(1 - \frac{t}{t_0}\right)^{-(k+2)}\left(1 - \frac{\tau}{2t_0}\right)^{-(2k+4)} r^{-(k+2)} \\ &\ll 4^{k+2}t_0^{k+2}(3/2+\varepsilon)^{k+1}\left[M_{k+1} + \frac{kM_k}{2(k+2)(3/2+\varepsilon)}\right. \\ &\quad \left. + \frac{M_0r^2M_k}{2(k+2)(k+1)(3/2+\varepsilon)} + \frac{kM_0r^2M_k}{(k+2)(k+1)t_0(3/2+\varepsilon)}\right] \\ &\quad \cdot \left(1 - \frac{x}{r}\right)^{-(k+2)}\left(1 - \frac{t}{2t_0}\right)^{-(k+2)}\left(1 - \frac{\tau}{2t_0}\right)^{-(2k+4)} r^{-(k+2)}. \end{aligned} \tag{2.17}$$

If we now set

$$\begin{aligned} M_{k+2} = (3/2+\varepsilon)^{-1}\left[M_{k+1} + \frac{M_k}{(3/2+\varepsilon)}\left(\frac{k}{2(k+2)} + \frac{M_0r^2}{2(k+2)(k+1)}\right.\right. \\ \left.\left. + \frac{kM_0r^2}{(k+2)(k+1)t_0}\right)\right], \end{aligned} \tag{2.18}$$

we have shown that (2.16) is true for $n = k + 1$, thus completing the induction step. It remains to be shown from (2.18) that M_k is a bounded function of k. For $k \geqq k_0 = k_0(\varepsilon)$ we have from (2.18) that

$$M_{k+2} \leqq (3/2+\varepsilon)^{-1}\left[M_{k+1} + \frac{M_k}{(3/2+\varepsilon)}(1/2 + \varepsilon/2)\right], \qquad k \geqq k_0. \tag{2.19}$$

If $M_{k+1} \leqq M_k$ for $k \geqq k_0$ we are done, for then we have $M_k \leqq \max\{M, M_2, \cdots, M_{k_0}\}$. Suppose then that there exists $k_1 \geqq k_0$ such that $M_{k_1+1} > M_{k_1}$. Then from (2.19) we have

$$\begin{aligned} M_{k_1+2} &< (3/2+\varepsilon)^{-1}\left[M_{k_1+1} + M_{k_1+1}\frac{(1/2+\varepsilon/2)}{(3/2+\varepsilon)}\right] \\ &= \frac{(2+3/2\varepsilon)}{(3/2+\varepsilon)(3/2+\varepsilon)}M_{k_1+1} < M_{k_1+1}, \end{aligned} \tag{2.20}$$

and by induction

$$M_{k_1+m} \leqq M_{k_1+1} \tag{2.21}$$

for $m = 1, 2, 3, \cdots$. Hence $M_k \leqq \max\{M_1, M_2, \cdots, M_{k_1+1}\}$ and we can conclude that M_k is a bounded function of k.

We now return to the convergence of the series (2.7). Let δ_0, δ_1 and $\alpha > 1$ be positive numbers and let

$$\begin{aligned} &|x| \leqq r/\alpha, && |\tau| \leqq t_0, \\ &|t| \leqq t_0/(1+\delta_1), && |t-\tau| \geqq \delta_0. \end{aligned} \tag{2.22}$$

Then

$$\begin{aligned} &\left(1-\frac{x}{r}\right) \geqq \frac{\alpha-1}{\alpha}, && \left(1-\frac{\tau}{2t_0}\right) \geqq \frac{1}{2}, \\ &\left(1-\frac{t}{t_0}\right) \geqq \frac{\delta_1}{1+\delta_1}, && |t-\tau| \leqq t_0\left(\frac{2+\delta_1}{1+\delta_1}\right) < 2t_0. \end{aligned} \tag{2.23}$$

From (2.9) and (2.11) we have

$$P^{(k)}(x,t,\tau) = (t-\tau)^{-k}Q^{(k)}(x,t,t-\tau). \tag{2.24}$$

Hence for x, t and τ restricted as in (2.22) we have from (2.16) and (2.24) that the series (2.7) is majorized by

$$\frac{1}{\delta_0} + \sum_{n=2}^{\infty} \frac{M_n 16^n t_0^n (3/2+\varepsilon)^n(\alpha-1)^n(1+\delta_1)^n}{\alpha^{2n}\delta_0^n\delta_1^n}. \tag{2.25}$$

Owing to the fact that M_n is a bounded function of n it is seen that if α is chosen sufficiently large then the series (2.25) converges. Since δ_0, δ_1 and ε are arbitrarily small (and independent of r) and r can be chosen arbitrarily large, we can now conclude that the series (2.7) converges uniformly and absolutely for $|x| \leqq r$, $|t| \leqq t_0/(1+\delta_1)$, $|\tau| \leqq t_0$ and $|t-\tau| \geqq \delta_0$ for δ_0 and δ_1 arbitrarily small and r arbitrarily large. Since each term of the series (2.7) is an analytic function of the variables x, t and τ for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$, $\tau \neq t$, we can conclude that $E^{(1)}(x, t, \tau)$ exists and is an analytic function of its independent variables for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$ and $t \neq \tau$. At the point $t = \tau$, $E^{(1)}(x,t,\tau)$ has an essential singularity. It is clear from our majorization argument that termwise differentiation of the series (2.7) is permissible and hence $E^{(1)}(x,t,\tau)$ satisfies the differential equation (2.3) and the initial conditions (2.5a), (2.5b).

We now turn our attention to the construction of the function $E^{(2)}(x, t, \tau)$. Setting $f(\tau) = 0$ in (2.4) and substituting this equation into (2.3) shows that, as a function of x and t, $E^{(2)}(x, t, \tau)$ must be a solution of $L[u] = 0$ for $t \neq \tau$. We now assume that $E^{(2)}(x, t, \tau)$ has the expansion

$$E^{(2)}(x,t,\tau) = \frac{x}{t-\tau} + \sum_{n=3}^{\infty} x^n p^{(n)}(x,t,\tau), \tag{2.26}$$

where the $p^{(n)}(x, t, \tau)$ are (analytic) functions to be determined. We again note that if termwise differentiation is permitted the series (2.26) satisfies the initial conditions (2.6a), (2.6b). Substituting (2.26) into (2.3) leads to the following recursion formulas for the $p^{(n)}(x, t, \tau)$:

$$\begin{aligned} p^{(2)} &= 0, \\ p^{(3)} &= -\frac{c}{6(t-\tau)^2} - \frac{b}{6(t-\tau)}, \\ p^{(k+2)} &= -\frac{2}{k+2}p_x^{(k+1)} - \frac{1}{(k+2)(k+1)}[p_{xx}^{(k)} + bp^{(k)} - cp_t^{(k)}], \quad k \geqq 2. \end{aligned} \tag{2.27}$$

The recursion scheme (2.27) is essentially identical to the scheme given in (2.8) and following our previous analysis showing the convergence of the series (2.7) we can again verify that the series (2.26) defines an analytic function of x, t and τ for $|x| < \infty, |t| < t_0, |\tau| < t_0, t \neq \tau$, which satisfies $L[u] = 0$ and the initial data (2.6a), (2.6b). At the point $t = \tau$, $E^{(2)}(x, t, \tau)$ has an essential singularity.

We have now shown that the integral operator defined by (2.4) exists and maps ordered pairs of analytic functions onto analytic solutions of $L[u] = 0$. It is a simple matter to show that in fact every solution of $L[u] = 0$ which is analytic for $|t| < t_0$, $|x| < x_0$, can be represented in the form of (2.4). For let $u(x, t)$ be an analytic solution of $L[u] = 0$ and set $u(0, \tau) = f(\tau)$, $u_x(0, \tau) = g(\tau)$. Then $f(\tau)$ and $g(\tau)$ are analytic for $|\tau| < t_0$. Define

$$\begin{aligned} w(x,t) = &-\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau)f(\tau)\,d\tau \\ &-\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau)g(\tau)\,d\tau. \end{aligned} \tag{2.28}$$

Then $w(x, t)$ is an analytic solution of $L[u] = 0$ and from (2.5a), (2.5b), (2.6a), (2.6b) we have

$$\begin{aligned} w(0,t) &= -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} \frac{f(t)}{t-\tau}\,d\tau = f(t), \\ w_x(0,t) &= -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} \frac{g(\tau)}{t-\tau}\,d\tau = g(t); \end{aligned} \tag{2.29}$$

i.e., the Cauchy data for $w(x, t)$ and $u(x, t)$ agree on the noncharacteristic curve $x = 0$. From the Cauchy–Kowalewski theorem we can now conclude that $u(x, t) = w(x, t)$, i.e., $u(x, t)$ can be represented in the form of (2.4).

3. The noncharacteristic Cauchy problem. Consider the parabolic equation (2.1) where the coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$ are analytic functions of the (complex) variables x and t for $|x| < \infty$ and $|t - t_0| < t_0$. Suppose we wish to construct a solution of this equation which satisfies the Cauchy data

$$u(s(t),t) = f(t),$$
$$u_x(s(t),t) = g(t), \tag{3.1}$$

where $x = s(t)$ is a noncharacteristic curve and $f(t)$, $g(t)$ and $s(t)$ are analytic for $|t - t_0| < t_0$. We note that the inverse Stefan problem is of this form where $f(t) = 0$ and $g(t) = -(\lambda\rho/k)\, ds(t)/dt$. By making the nonsingular change of variables

$$\xi_1 = x - s(t), \qquad \xi_2 = t - t_0, \tag{3.2}$$

we arrive at an equation of the same form as (2.1) with the coefficients analytic for $|\xi_1| < \infty$ and $|\xi_2| < t_0$. Under the transformation (3.2) the curve $x = s(t)$ is transformed into the straight line $\xi_1 = 0$. If we now apply the change of variables (2.2) we arrive at an equation of the form (2.3) in the variables ξ_1 and ξ_2 with Cauchy data prescribed along $\xi_1 = 0$. As shown at the end of the last section, this problem can be solved by using the operator defined by (2.4). Hence, if the coefficients and interphase boundary are analytic in appropriate regions, we have a constructive method for solving the noncharacteristic Cauchy problem (2.1), (3.1). Note that due to the factor of $(k + 2)^{-1}$ which appears in each term of the recursion formulas (2.8) and (2.27), the convergence of the series expansions of $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$ is in general quite rapid. Hence close approximations can usually be made by truncating these series after a few terms and using the resulting approximate E-functions in (2.4).

As a simple example of the above method we shall now construct a solution to the (normalized) inverse Stefan problem for the heat equation

$$u_{xx} = u_t, \tag{3.3}$$

$$u(s(t),t) = 0, \tag{3.4a}$$

$$u_x(s(t),t) = -ds(t)/dt \tag{3.4b}$$

in the special case when $s(t) = t$. The transformation (3.2) (with $t_0 = 0$) reduces this problem to

$$w_{\xi_1\xi_1} + w_{\xi_1} = w_{\xi_2}, \tag{3.5}$$

$$w(0,\xi_2) = 0, \tag{3.6a}$$

$$w_{\xi_1}(0,\xi_2) = -1, \tag{3.6b}$$

where $w(\xi_1,\xi_2) = u(\xi_1 + \xi_2, \xi_2) = u(x,t)$. If we now set

$$w(\xi_1,\xi_2) = v(\xi_1,\xi_2)\exp\left(-\tfrac{1}{2}\xi_1\right), \tag{3.7}$$

equations (3.5), (3.6a), (3.6b) become

$$v_{\xi_1\xi_1} - \tfrac{1}{4}v_{\xi_1} = v_{\xi_2}, \tag{3.8}$$

$$v(0, \xi_2) = 0, \tag{3.9a}$$

$$v_{\xi_1}(0, \xi_2) = -1. \tag{3.9b}$$

From the form of the differential equation (3.8) and the initial conditions (3.9a), (3.9b) it is seen that we only need to compute the coefficient of $(\xi_2 - \tau)^{-1}$ in the series expansion for $E^{(2)}(\xi_1, \xi_2, \tau)$. From (2.26) we have

$$\begin{aligned} E^{(2)}(\xi_1, \xi_2, \tau) &= \frac{2}{\xi_2 - \tau}\left[\frac{\xi_1}{2} + \sum_{k=1}^{\infty} \frac{(\frac{1}{2}\xi_1)^{2k+1}}{(2k+1)!}\right] \\ &\quad + \text{terms involving higher powers of } (\xi_2 - \tau)^{-1} \\ &= \frac{2\sinh(\frac{1}{2}\xi_1)}{\xi_2 - \tau} \\ &\quad + \text{terms involving higher powers of } (\xi_2 - \tau)^{-1}, \end{aligned} \tag{3.10}$$

and hence, from (2.4) we have

$$\begin{aligned} v(\xi_1, \xi_2) &= \frac{1}{2\pi i}\oint_{|\xi_2 - \tau| = \delta} \frac{2\sinh(\frac{1}{2}\xi_1)}{\xi_2 - \tau}\, d\tau \\ &= -2\sinh(\tfrac{1}{2}\xi_1). \end{aligned} \tag{3.11}$$

Then $w(\xi_1, \xi_2) = e^{-\xi_1} - 1$ and the solution of (3.3), (3.4a), (3.4b) is given by

$$u(x, t) = e^{t-x} - 1. \tag{3.12}$$

In particular we see from (3.12) that the temperature distribution $\varphi(t)$ needed at $x = 0$ in order to make the ice melt along the curve $x = t$ is given by

$$\varphi(t) = u(0, t) = e^{t} - 1. \tag{3.13}$$

We now conclude by stating a result on the analytic continuation of solutions to parabolic equations with analytic coefficients which generalizes those obtained by Widder for the heat equation [7] and Hill for parabolic equations with time-independent coefficients [4]. The theorem follows immediately from the transformation (2.2), the representation (2.4), and the fact that $E^{(1)}(x, t, \tau)$ and $E^{(2)}(x, t, \tau)$ are analytic for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$, $t \neq \tau$.

THEOREM. *Let $u(x, t)$ be a solution of* (2.1) *which is an analytic function of the complex variables x and t for $|t| < t_0$, $|x| < x_0$. Suppose the coefficients $a(x, t)$, $b(x, t)$ and $c(x, t)$ are analytic functions of the complex variables x and t for $|x| < \infty$, $|t| < t_0$. Then $u(x, t)$ can be analytically continued into the strip $|x| < \infty$, $|t| < t_0$.*

An important application of this theorem is the conclusion that the solution of the inverse Stefan problem can always be analytically continued into a domain containing the line $x = 0$, provided the coefficients of the parabolic equation are analytic for $|x| < \infty$, $|t - t_0| < t_0$, and the interphase boundary is an analytic function of t for $|t - t_0| < t_0$. In particular the above theorem implies that $u(0, t) = \varphi(t)$ is an analytic function of t for $|t - t_0| < t_0$. Thus we can conclude that if $u(0, t) = \varphi(t)$ is not analytic then neither is the interphase boundary $s(t)$. This partially answers the problem posed by Rubinstein in [6, p. 353].

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ON THE ASYMPTOTIC BEHAVIOR OF PERTURBED VOLTERRA INTEGRAL EQUATIONS*

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Abstract. The asymptotic behavior of the solution of the perturbed system of Volterra integral equations $x(t) = f(t) - \int_0^t a(t, s)\{x(s) + g(s, x(s))\}\,ds$ is compared to that of the solution of the unperturbed system $y(t) = f(t) - \int_0^t a(t, s)y(s)\,ds$. We show that under suitable restrictions on the resolvent kernel $r(t, s)$, $|x(t) - y(t)| \to 0$ as $t \to \infty$ whenever $|g(t, x)| \leqq \lambda(t)(1 + |x|)$ with λ bounded and diminishing. This establishes and generalizes a recent conjecture of J. L. Kaplan.

1. Introduction and statement of results. We wish to compare solutions of the following systems of Volterra integral equations:

$$y(t) = f(t) - \int_0^t a(t, s)y(s)\,ds, \tag{1.1}$$

$$x(t) = f(t) - \int_0^t a(t, s)\{x(s) + g(s, x(s))\}\,ds, \tag{1.2}$$

where x, y, f and g are vectors in n-dimensional Euclidean space E_n, and $a(t, s)$ is an $n \times n$ matrix. Let $|z|$ denote any vector norm in E_n, and for a vector function $z(t)$ bounded on $0 \leqq t < \infty$, define $\|z\| = \sup_{t \geqq 0} |z(t)|$.

We assume throughout that the solution of the resolvent system

$$r(t, s) = a(t, s) - \int_s^t a(t, u)r(u, s)\,du, \qquad 0 \leqq s \leqq t < \infty,$$

corresponding to system (1.1) exists and that systems (1.1) and (1.2) may be rewritten in the equivalent forms

$$y(t) = f(t) - \int_0^t r(t, s)f(s)\,ds, \tag{1.3}$$

$$x(t) = y(t) - \int_0^t r(t, s)g(s, x(s))\,ds. \tag{1.4}$$

It is well known [2, Chap. 4] that this is the case when $a(t, s)$ and $r(t, s)$ are locally L^1 in (t, s). In addition, we assume sufficient hypotheses on $f(t)$, $a(t, s)$ and $g(t, x)$ to guarantee the local existence and uniqueness of solutions of (1.1) and (1.2), and to insure that the solution of (1.2) exists on $0 \leqq t < \infty$ whenever $|g(t, x)| = O(1 + |x|)$. (See [2], [4].)

In a recent paper [1, Thm. 3.5], Kaplan proves the following theorem comparing the solution of (1.1) with that of (1.2).

THEOREM A. *Let $y(t)$ and $x(t)$ denote the solutions of* (1.1) *and* (1.2) *respectively. Suppose $r(t, s) = r(t - s)$ and $r \in L^1(0, \infty)$. Let*

$$|g(t, x)| \leqq \lambda(t), \qquad 0 \leqq t < \infty, \quad |x| < \infty, \tag{1.5}$$

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where λ is bounded and diminishing, that is,

$$\int_T^{T+1} \lambda(s)\,ds \to 0 \quad \text{as } T \to \infty. \tag{1.6}$$

If $\|y\| < \infty$, then

$$\frac{1}{T}\int_0^T |x(t) - y(t)|\,dt \to 0 \quad \text{as } T \to \infty. \tag{1.7}$$

Kaplan also conjectures that Theorem A remains valid when (1.5) is replaced by

$$|g(t, x)| \leqq \lambda(t)(1 + |x|), \qquad 0 \leqq t < \infty, \quad |x| < \infty, \tag{1.8}$$

and that, in fact, $|x(t) - y(t)|$ is diminishing. Our Theorem 1 shows that actually the even stronger conclusion

$$|x(t) - y(t)| \to 0 \quad \text{as } t \to \infty \tag{1.9}$$

holds for more general resolvent kernels.

THEOREM 1. *Let $x(t)$ and $y(t)$ be as in Theorem* A. *Assume that $r(t, s)$ satisfies*

$$\int_0^t |r(t, s)|\,ds \leqq B < \infty \quad \text{for } t \geqq 0, \tag{1.10}$$

$$\sup_{t \geqq T} \int_0^{t-T} |r(t, s)|\,ds \to 0 \quad \text{as } T \to \infty, \tag{1.11}$$

and has the additional property:

(1.12) *given $\varepsilon > 0$ there exists $\delta > 0$ such that if $t > 0$ and A is any set contained in $[0, t]$ with $m(A) < \delta$, then*

$$\int_A |r(t, s)|\,ds < \varepsilon.$$

If (1.6) *and* (1.8) *hold with $\|\lambda\| < \infty$ and if $\|y\| < \infty$, then* (1.9) *holds.*

Observe that all the hypotheses (1.10)–(1.12) are satisfied when $r(t, s) = r(t - s)$ with $r \in L^1(0, \infty)$. For an example of a kernel $a(t, s)$ of nonconvolution type whose resolvent $r(t, s)$ satisfies the hypotheses of Theorem 1 see [3, Thm. 6].

Kaplan's proof of Theorem A uses Laplace transform techniques as well as a tauberian theorem for Laplace transform due to Hardy and Littlewood [1, Lemma 3.7]. Our proof of Theorem 1 is of a more elementary nature and is similar to the proof of the following theorem of Kaplan [1].

THEOREM B. *Let $x(t)$ and $y(t)$ be as in Theorem* A *and suppose that $r(t, s)$ satisfies* (1.10) *and* (1.12) *as well as*

$$\lim_{t \to \infty} \int_0^T |r(t, s)|\,ds = 0 \quad \text{for each fixed } T > 0. \tag{1.13}$$

Assume that (1.8) *holds where* $\|\lambda\| < \infty$ *and* λ *satisfies*:

$$\text{(1.14)} \qquad \begin{gathered} \text{given } \alpha, \beta > 0, \text{ there exists } T = T(\alpha, \beta) \text{ such that} \\ m\{t : t \geqq T, |\lambda(t)| \geqq \alpha\} < \beta. \end{gathered}$$

Then (1.9) *holds provided* $\|y\| < \infty$.

Thus, by strengthening the hypothesis on $r(t, s)$ from (1.13) to (1.11), we can allow a larger class of perturbations and still deduce (1.9). Moreover, Theorem 1 is best possible in the sense that Theorem B is not valid for bounded diminishing perturbations as the following example shows.

Example 1. Define $\phi(t)$ on $[0, \infty)$ by

$$\phi(t) = \begin{cases} 1 & \text{for } 0 \leqq t \leqq 1 \quad \text{or} \quad n \leqq t \leqq n + 1/2n, \\ 0 & \text{for } n + (1/2n) + 1/2^{n+1} \leqq t \leqq n + 1 - 1/2^{n+1}, \\ \text{linear elsewhere,} \end{cases}$$

for $n = 1, 2, \cdots$. Let

$$\alpha(t) = \begin{cases} 1 & \text{for } 0 \leqq t \leqq 1, \\ \left[\displaystyle\int_0^t \phi^2(s)\, ds \right]^{-1} & \text{for } t \geqq 1. \end{cases}$$

Choose $f(t) \equiv c$, $g(t, x) = \lambda(t) = \phi(t)$, and

$$a(t, s) = \alpha(t)\phi(s) \exp \int_s^t \alpha(u)\phi(u)\, du.$$

Then $r(t, s) = \alpha(t)\phi(s)$. It is easy to see that $r(t, s)$ satisfies the hypotheses of Theorem B, but that (1.11) does not hold. Furthermore, $y(t) \to 0$ as $t \to \infty$ and $x(t) = y(t) - 1$ for $t \geqq 1$. Thus, the conclusion of Theorem B fails, and an additional hypothesis such as (1.11) is necessary in the statement of Theorem 1.

2. Proof of Theorem 1. The proof consists of two parts.

PROPOSITION 1. *If* $\|y\| < \infty$, *then* $\|x\| < \infty$.

Proof. Choose a positive integer T by (1.11) such that

$$\sup_{t \geqq T} \int_0^{t-T} |r(t, s)|\, ds < \frac{1}{4\|\lambda\|}.$$

Then using (1.12), choose $\delta > 0$ such that

$$\int_A |r(t, s)|\, ds < \frac{1}{4\|\lambda\|}$$

whenever $A \subseteq [0, t]$ and $m(A) < \delta$. Define

$$A(t) = \{s : t - T \leqq s \leqq t, \lambda(s) \geqq 1/(4B)\} \quad \text{for } t \geqq T.$$

It follows from (1.6) that there exists $T_1 > T$ so that $m\{A(t)\} < \delta$ whenever $t \geqq T_1$.

Since $x(t)$ exists on $[0, T_1]$, there exists $M > 1$ such that $|x(t)| \leqq M$ on $[0, T_1]$. Choose $P > 4\|y\| + 3M$. It follows that $|x(t)| < P$ on $[0, \infty)$. If not, there exists

$t > T_1$ such that $|x(s)| < P$ for $0 \leqq s < t$, but $|x(t)| = P$. Then

$$|x(t)| \leqq \|y\| + \int_0^{t-T} |r(t,s)||g(s,x(s))|\,ds$$

$$+ \int_{[t-T,t]-A(t)} |r(t,s)||g(s,x(s))|\,ds$$

$$+ \int_{A(t)} |r(t,s)||g(s,x(s))|\,ds$$

$$\leqq \|y\| + 3(1+P)/4 < P.$$

This contradiction shows that $|x(t)| < P$ on $[0, \infty)$.

PROPOSITION 2. *If* $\|x\| < \infty$, *then* $|x(t) - y(t)| \to 0$ *as* $t \to \infty$.

Proof. Let $\|x\| = M$ and fix $\varepsilon > 0$. Again using (1.11), choose a positive integer T such that

$$\sup_{t \geqq T} \int_0^{t-T} |r(t,s)|\,ds < \frac{\varepsilon}{3\|\lambda\|(1+M)}.$$

Then by (1.12) select $\delta > 0$ so that if $A \subseteq [0, t]$ and $m(A) < \delta$, then

$$\int_A |r(t,s)|\,ds < \frac{\varepsilon}{3\|\lambda\|(1+M)}.$$

Define for $t \geqq T$,

$$A(t) = \left\{ s : t - T \leqq s \leqq t, \lambda(s) \geqq \frac{\varepsilon}{3B(1+M)} \right\}.$$

As in the proof of Proposition 1, there exists $T_1 > T$ such that $m\{A(t)\} < \delta$ whenever $t \geqq T_1$.

Therefore, using (1.4) and (1.8), we have

$$|x(t) - y(t)| \leqq (1+M) \left\{ \int_0^{t-T} |r(t,s)|\lambda(s)\,ds + \int_{[t-T,t]-A(t)} |r(t,s)|\lambda(s)\,ds \right.$$

$$\left. + \int_{A(t)} |r(t,s)|\lambda(s)\,ds \right\}$$

$$< \varepsilon \quad \text{whenever } t \geqq T_1.$$

Since ε is arbitrary, this completes the proof of Proposition 2 and establishes Theorem 1.

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A HARMONIC MEAN INEQUALITY FOR THE GAMMA FUNCTION*

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Abstract. We prove that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to $\Gamma(1) = 1$ for arbitrary $x > 0$.

1. Introduction. V. R. Rao Uppuluri [2] brought the following conjectured inequality to the author's attention:

$$\frac{2}{1/(\Gamma(x)) + 1/(\Gamma(1/x))} \geqq 1 \quad \text{on } 0 < x < \infty. \tag{1.1}$$

It states that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is always larger than or equal to $\Gamma(1) = 1$, equality being assumed for $x = 1$. Because of the well-known inequalities between the harmonic, geometric and arithmetic means, the conjecture implies these other inequalities, $\Gamma(x)\Gamma(1/x) \geqq 1$ and $\Gamma(x) + \Gamma(1/x) \geqq 2$.

The proof of (1.1), given below in §§ 2–5, is "computational" in the sense that it relies on certain isolated numerical values of the psi function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

and its derivative. This deficiency, however, is removed in § 6, where numerical values of only standard constants, such as π, ln 2, and Euler's constant γ, are required.

It suffices to prove (1.1) for $1 < x \leqq x_0$, where $x_0 = 1.4616 \cdots$ is the positive minimum point of $\Gamma(x)$. In fact, the left-hand expression in (1.1) is clearly increasing on the interval (x_0, ∞). If we prove the inequality for $1 < x \leqq x_0$, it will hold for all $x > 1$, hence also for all positive $x < 1$, on account of its invariance under the substitution $x \to 1/x$.

2. Reformulation of the inequality. Letting

$$\phi(t) = \frac{1}{\Gamma(e^t)}, \qquad -\infty < t < \infty, \tag{2.1}$$

we may rewrite (1.1) in the form

$$\tfrac{1}{2}[\phi(t) + \phi(-t)] \leqq \phi(0), \tag{2.2}$$

which expresses a "symmetric concavity" property for ϕ. We must prove (2.2) for $0 < t \leqq \ln x_0$.

Using Taylor's theorem, we have for $t > 0$,

$$\tfrac{1}{2}[\phi(t) + \phi(-t)] - \phi(0) = \frac{t^2}{4}[\ddot{\phi}(\tau_1) + \ddot{\phi}(\tau_2)],$$

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where dots denote derivatives with respect to t, and

$$0 < \tau_1 < t, \quad -t < \tau_2 < 0. \tag{2.3}$$

We will show that

$$\ddot{\phi}(\tau_1) + \ddot{\phi}(\tau_2) < 0 \quad \text{on } 0 < t \leqq \ln x_0. \tag{2.4}$$

3. The second derivative of ϕ. Differentiating (2.1) we obtain

$$\dot{\phi}(t) = -e^t \frac{\Gamma'(e^t)}{[\Gamma(e^t)]^2} = -xy(x),$$

where

$$x = e^t, \quad y(x) = \frac{\psi(x)}{\Gamma(x)}.$$

Another differentiation gives

$$\ddot{\phi}(t) = \left(\frac{d}{dx}\dot{\phi}\right)\frac{dx}{dt} = x(-xy)' = -x(y + xy'),$$

where primes indicate differentiation with respect to x. Noting that

$$y\Gamma = \psi,$$
$$y'\Gamma = \psi' - y\Gamma' = \psi' - y\Gamma\psi = \psi' - \psi^2,$$

we may express the second derivative of ϕ in terms of ψ and ψ',

$$\ddot{\phi}(t) = -\frac{1}{\Gamma(x)}(x\psi + x^2\psi' - x^2\psi^2), \qquad x = e^t. \tag{3.1}$$

4. Some monotonicity properties. We now observe that both functions $x\psi(x)$ and $x^2\psi'(x)$ are monotonically increasing on the interval $1/x_0 < x < x_0$. To see this for the first, we use the known expansion [1]

$$x\psi(x) = -1 + (1 - \gamma)x + \sum_{m=1}^{\infty} \frac{x(x-1)}{(m+1)(m+x)}, \tag{4.1}$$

where $\gamma = .5772 \cdots$ is Euler's constant. One checks that for $m > 0$,

$$\frac{x(x-1)}{m+x}$$

is monotonically increasing for $x > (1 + \sqrt{1 + 1/m})^{-1}$, hence in particular for $x > 1/x_0$. Monotonicity of $x\psi$ thus follows from (4.1). Since $\psi(x_0) = 0$, we also see that

$$x\psi(x) < 0 \quad \text{on } 1/x_0 < x < x_0.$$

For $x^2\psi'$, our assertion follows directly from

$$x^2\psi'(x) = \sum_{m=0}^{\infty} \left(\frac{x}{m+x}\right)^2,$$

$x/(m + x)$, for each $m \geqq 1$, being monotonically increasing for $x > 0$. We also note that

$$x^2\psi'(x) > 0 \quad \text{on } 1/x_0 < x < x_0.$$

5. Conclusion of the proof. We are now in a position to estimate the second derivative of ϕ in (3.1), first on the interval $1/x_0 < x < x_0$, then on $1 < x < x_0$.

On the first interval we have by the monotonicity properties of § 4,

$$x\psi + x^2\psi' - x^2\psi^2 \geqq x_0^{-1}\psi(1/x_0) + x_0^{-2}\psi'(1/x_0) - x_0^{-2}\psi^2(1/x_0). \tag{5.1}$$

Using linear interpolation in [1, Table 6.1] we find $\psi(1/x_0) = \psi(1 + 1/x_0) - x_0 = -1.2657\cdots$, $\psi'(1/x_0) = \psi'(1 + 1/x_0) + x_0^2 = 2.9392\cdots$, so that the lower bound in (5.1) is $-.2400\cdots$. From (3.1), since $\Gamma(x_0) = .8856\cdots$, we thus obtain

$$\ddot{\phi}(t) \leqq \frac{.2400\cdots}{\Gamma(x_0)} < .272 \quad \text{on } -\ln x_0 < t < \ln x_0.$$

On the second interval, similarly,

$$x\psi + x^2\psi' - x^2\psi^2 \geqq \psi(1) + \psi'(1) - \psi^2(1) = .7345\cdots, \tag{5.2}$$

where we have used [1]

$$\psi(1) = -\gamma, \quad \psi'(1) = \zeta(2) = \pi^2/6.$$

Thus,

$$\ddot{\phi}(t) \leqq \frac{-.7345\cdots}{\Gamma(1)} < -.734 \quad \text{on } 0 < t < \ln x_0.$$

The proof is now completed by recalling from (2.3) that

$$0 < \tau_1 < \ln x_0, \quad -\ln x_0 < \tau_2 < 0,$$

and hence

$$\ddot{\phi}(\tau_1) + \ddot{\phi}(\tau_2) < -.734 + .272 = -.462 < 0,$$

as we set out to show in (2.4).

6. A less computational variant of the proof. Reference to numerical values of $\psi(1/x_0)$ and $\psi'(1/x_0)$ in (5.1) can be avoided by observing that $x_0^{-1} > \frac{1}{2}$ and that $x\psi$ and $x^2\psi'$ are monotonically increasing on $\frac{1}{2} < x < x_0$. Using [1]

$$\psi(\tfrac{1}{2}) = -\gamma - 2\ln 2, \quad \psi'(\tfrac{1}{2}) = 3\zeta(2) = \pi^2/2,$$

we can thus write in place of (5.1),

$$x\psi + x^2\psi' - x^2\psi^2 \geqq \tfrac{1}{2}\psi(\tfrac{1}{2}) + \tfrac{1}{4}\psi'(\tfrac{1}{2}) - \tfrac{1}{4}\psi^2(\tfrac{1}{2}) = -.7118\cdots.$$

Together with the companion inequality (5.2), and (3.1), this gives

$$\ddot{\phi}(t)\Gamma(e^t) \leqq .712 \quad \text{on } -\ln x_0 < t < \ln x_0,$$
$$\ddot{\phi}(t)\Gamma(e^t) \leqq -.734 \quad \text{on } 0 < t < \ln x_0.$$

It follows, in particular, that

$$(6.1)\quad \ddot{\phi}(\tau_1) < 0 \quad \text{and} \quad \ddot{\phi}(\tau_1)\Gamma(e^{\tau_1}) + \ddot{\phi}(\tau_2)\Gamma(e^{\tau_2}) \leqq -.734 + .712 = -.022 < 0.$$

Were $\ddot{\phi}(\tau_2)$ negative or zero, our assertion (2.4) would follow immediately from the first inequality in (6.1). If $\ddot{\phi}(\tau_2)$ were positive, then $\ddot{\phi}(\tau_2)\Gamma(e^{\tau_2}) > \ddot{\phi}(\tau_2)\Gamma(e^{\tau_1})$, and the second inequality in (6.1) would give

$$0 > \ddot{\phi}(\tau_1)\Gamma(e^{\tau_1}) + \ddot{\phi}(\tau_2)\Gamma(e^{\tau_2}) > \Gamma(e^{\tau_1})[\ddot{\phi}(\tau_1) + \ddot{\phi}(\tau_2)],$$

that is again (2.4). Thus (2.4) is true in either case, and the proof, once more, is completed.

REFERENCES

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[2] V. R. RAO UPPULURI, Personal communication, April, 1972.

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SOME MEAN VALUE INEQUALITIES FOR THE GAMMA FUNCTION*

In Memory of George E. Forsythe

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Abstract. We determine the infimum of the harmonic mean of $\Gamma(x_1), \Gamma(x_2), \cdots, \Gamma(x_n)$ under the constraints $\prod_{k=1}^{n} x_k = 1$, all $x_k > 0$. We present numerical evidence for this infimum to be equal to $\Gamma(1) = 1$ if $n \leqq 8$, and show it to be less than 1 when $n > 8$. We also prove that the geometric mean of $\Gamma(x_1), \Gamma(x_2), \cdots, \Gamma(x_n)$ is always $\geqq 1$ under the same constraints, and that the geometric mean is the power mean with the smallest exponent for which this is true.

1. Introduction. In a recent note [1] we proved that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ for $x > 0$ is never smaller than $\Gamma(1) = 1$, that is,

$$\frac{2}{1/\Gamma(x) + 1/\Gamma(1/x)} \geqq 1 \quad \text{for } 0 < x < \infty. \tag{1.1}$$

Equality, of course, is assumed when $x = 1$. We report here on attempts at generalizing (1.1) to more variables. A natural generalization would be $n/\sum_{k=1}^{n} 1/\Gamma(x_k) \geqq \Gamma((x_1 x_2 \cdots x_n)^{1/n})$, which, however, is readily dismissed as false by considering the case $n = 2$, $x_1 = 1$, x_2 large. More promising is the conjecture

$$\frac{n}{\sum_{k=1}^{n} 1/\Gamma(x_k)} \geqq 1 \quad \text{for all } x_k > 0 \text{ with } x_1 x_2 \cdots x_n = 1. \tag{1.2}$$

We present evidence that this inequality is in fact valid for $n = 1, 2, \cdots, 8$, but prove it to be false for $n \geqq 9$. We also determine the infimum of the expression on the left of (1.2) under the constraints listed in (1.2). We next show that for all $n \geqq 1$ we have

$$\left[\prod_{k=1}^{n} \Gamma(x_k)\right]^{1/n} \geqq 1 \quad \text{for all } x_k > 0 \text{ with } x_1 x_2 \cdots x_n = 1. \tag{1.3}$$

In terms of the power means

$$M_n^{[r]}(a_i) = \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{1/r}, \tag{1.4}$$

the last inequality may be restated as $M_n^{[0]}(\Gamma(x_i)) \geqq 1$, for all $n \geqq 1$, and all $x_i > 0$ with $x_1 x_2 \cdots x_n = 1$. Since $M_n^{[r]}$ increases monotonically with r, the same statement holds for any power mean with $r \geqq 0$. We show, on the other hand, that the statement is false for any power mean with $r < 0$.

2. Main results. We denote by R^n the space of real vectors $\mathbf{x}^T = [x_1, x_2, \cdots, x_n]$ and by R_+^n the positive orthant $R_+^n = \{\mathbf{x} \in R^n : x_k > 0, k = 1, 2, \cdots, n\}$.

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The constraints in (1.2) can then be written as

$$\mathbf{x} \in S_n, \quad \text{where } S_n = \left\{ \mathbf{x} \in R_+^n : \prod_{k=1}^n x_k = 1 \right\}.$$

Our main results are as follows.

THEOREM 1. *For $n = 1, 2, 3, \cdots$ we have*

$$\inf_{\mathbf{x} \in S_n} \frac{n}{\sum_{k=1}^n 1/\Gamma(x_k)} = \frac{1}{\gamma_n}, \tag{2.1}$$

where

$$\gamma_n = \max_{1 \leqq \nu \leqq n-1} \{ \max_{0 \leqq x \leqq 1} g_{n,\nu}(x) \}, \qquad g_{n,\nu}(x) \overset{\text{def}}{=} \frac{1}{n}\left[\frac{\nu}{\Gamma(x)} + \frac{n-\nu}{\Gamma(x^{-\nu/(n-\nu)})} \right]. \tag{2.2}$$

Moreover,

$$\gamma_n \to \frac{1}{\Gamma(x_0)} = 1.1291 \cdots \quad \text{as } n \to \infty, \tag{2.3}$$

where $x_0 = 1.4616 \cdots$ is the unique point at which $\Gamma(x)$ attains its minimum on the positive x-axis.

Equation (2.1), for $n = 1$ with $\gamma_1 = 1$, is trivial, since the only point x_1 satisfying the constraints is $x_1 = 1$. For $n > 1$, the maxima in (2.2) can easily be computed with the aid of a digital computer. It turns out (cf. § 5) that $\gamma_n = 1$ for $1 \leqq n \leqq 8$, and it will be shown that $\gamma_n > 1$ for $n \geqq 9$. The conjecture (1.2) thus seems true for $n \leqq 8$, but is certainly false for all $n \geqq 9$.

We also note from (2.3) that in the (obvious) inequality

$$\frac{n}{\sum_{k=1}^n 1/\Gamma(x_k)} \geqq \Gamma(x_0) = .88560 \cdots, \qquad n = 1, 2, 3, \cdots, \tag{2.4}$$

the constant on the right is best possible under the constraints $\mathbf{x} \in S_n$.

THEOREM 2.[1] *For $n = 1, 2, 3, \cdots$, we have*

$$\left[\prod_{k=1}^n \Gamma(x_k) \right]^{1/n} \geqq 1 \quad \text{for all } \mathbf{x} \in S_n. \tag{2.5}$$

THEOREM 3. *For the power means $M_n^{[r]}$ defined in* (1.4) *we have*

$$M_n^{[r]}(\Gamma(x_i)) \geqq 1 \quad \text{on } S_n \quad \text{for all } n \geqq 1 \tag{2.6}$$

if and only if $r \geqq 0$.

3. Auxiliary propositions. We need a few elementary properties of the psi function $\psi(x) = \Gamma'(x)/\Gamma(x)$ and some related functions.

PROPOSITION 1. *The function $x\psi(x)$ is convex for $x > 0$.*

Proof. We have

$$(x\psi)'' = (1/x^2)(x^3\psi'' + 2x^2\psi').$$

[1] An examination of inequality (2.5) was suggested to the author by Professor R. A. Askey.

From the known expansion

$$\psi(x) = -\frac{1}{x} + 1 - \gamma + \sum_{m=1}^{\infty} \frac{x-1}{(m+1)(m+x)}, \tag{3.1}$$

where $\gamma = .57721 \cdots$ is Euler's constant, we obtain by two differentiations,

$$\begin{aligned} x^3\psi'' + 2x^2\psi' &= -2 - 2\sum_{m=1}^{\infty}\left(\frac{x}{m+x}\right)^3 + 2 + 2\sum_{m=1}^{\infty}\left(\frac{x}{m+x}\right)^2 \\ &= 2x^2 \sum_{m=1}^{\infty} \frac{m}{(m+x)^3} > 0, \end{aligned}$$

i.e., $(x\psi)'' > 0$ for $x > 0$.

The next result concerns the function

$$f(x) = x[\Gamma(x)]^r\psi(x) = \frac{x}{r}\frac{d}{dx}\{[\Gamma(x)]^r\}, \qquad r < 0, \tag{3.2}$$

where r is a fixed (negative) parameter. By x_0 we denote, as before, the abscissa of the minimum of $\Gamma(x)$.

PROPOSITION 2. *The function f in* (3.2) *vanishes at $x = 0$ and $x = x_0$, and is negative and unimodal on $0 < x < x_0$, i.e., there exists a ξ with $0 < \xi < x_0$ such that f decreases on $0 < x < \xi$ and increases on $\xi < x < x_0$.*

Proof. From the known power series expansion of $1/\Gamma(x)$, letting $\rho = |r|$, we find

$$f(x) = -x^\rho - (\rho+1)\gamma x^{\rho+1} + \cdots,$$

showing that $f(0) = 0$. (It is also seen, incidentally, that f need not be convex on $0 < x < x_0$; for example, if $\rho = 1$, then $f''(0) = -4\gamma < 0$.) By definition of x_0, we also have $f(x_0) = 0$.

To prove unimodality, we look at the derivative f'. A simple computation gives

$$x[\Gamma(x)]^{-r}f'(x) = x\psi(x) + rx^2\psi^2(x) + x^2\psi'(x). \tag{3.3}$$

Let

$$u(x) = x\psi(x) + rx^2\psi^2(x) + x^2\psi'(x). \tag{3.4}$$

From the power series expansion of $\psi(x+1)$, we obtain

$$x\psi(x) = x\psi(x+1) - 1 = -1 - \gamma x + \zeta(2)x^2 + \cdots.$$

This shows that the function $x\psi(x)$ decreases for small positive x; since it is convex by Proposition 1, and vanishes at $x = x_0$, it must have a unique minimum at some point ξ^* with $0 < \xi^* < x_0$. (In fact, $\xi^* \doteq .2161$.) As the derivative of $x\psi$ vanishes at this point, we have

$$\xi^*\psi'(\xi^*) + \psi(\xi^*) = 0. \tag{3.5}$$

We consider first the interval $0 < x < \xi^*$. On this interval, we have from (3.4), since $r < 0$ and $x^2\psi^2(x) > 1$,

$$u(x) < U(x), \quad U(x) = x\psi(x) + r + x^2\psi'(x), \qquad 0 < x < \xi^*. \tag{3.6}$$

Note that

(3.7) $$U(0) = -1 + r + 1 = r < 0, \qquad U(\xi^*) = r < 0,$$

the second relation being a consequence of (3.5). We now show that $U(x)$ is convex on $0 < x < \xi^*$. By Proposition 1, it suffices to show that the last term in $U(x)$ is convex, i.e.,

(3.8) $$(x^2\psi')'' = 2\psi' + 4x\psi'' + x^2\psi''' > 0, \qquad 0 < x < \xi^*.$$

Repeated differentiation of (3.1), however, gives

$$(x^2\psi')'' = \sum_{m=1}^{\infty} \frac{2m(m-2x)}{(m+x)^4},$$

which is certainly positive if $0 < x < \frac{1}{2}$, hence, in particular, if $0 < x < \xi^*$. From (3.6), (3.7), and the convexity of $U(x)$ just established, it now follows that $u(x) < r < 0$ on $0 < x < \xi^*$, i.e., by virtue of (3.3), (3.4),

(3.9) $$f'(x) < 0 \quad \text{on } 0 < x < \xi^*.$$

On the remaining interval $\xi^* < x < x_0$, the function $x\psi(x)$, while still negative, increases monotonically. Since also $x^2\psi'(x)$ increases monotonically for $x > 0$ (cf. [1]), it follows from (3.4) that $u(x)$ is monotonically increasing on $\xi^* < x < x_0$. Moreover, $u(x_0) = x_0^2\psi'(x_0) > 0$. Hence there is a unique point ξ, with $\xi^* < \xi < x_0$, such that $u(\xi) = 0$, and thus $u(x) < 0$ for $0 < x < \xi$ and $u(x) > 0$ for $\xi < x < x_0$. In view of (3.3), (3.4), this implies unimodality of f.

4. Proof of Theorem 1. We assume $n \geqq 2$, since the case $n = 1$, as we pointed out, is trivial. For short, let

$$\gamma(x_1, x_2, \cdots, x_n) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\Gamma(x_k)}.$$

Since obviously

$$\gamma(x_1, x_2, \cdots, x_n) \leqq \frac{1}{\Gamma(x_0)} \quad \text{for all } \mathbf{x} \in R_+^n,$$

the function γ is bounded from above in all of R_+^n, and hence, in particular, on S_n. We denote

(4.1) $$\sigma_n = \sup_{\mathbf{x} \in S_n} \gamma(x_1, x_2, \cdots, x_n) < \infty.$$

We want to prove that

$$\sigma_n = \gamma_n.$$

We distinguish two major cases (not a priori mutually exclusive):

Case I. The supremum σ_n is "assumed at infinity", i.e., there exists a sequence of vectors $\mathbf{x}^{(r)} \in S_n$ such that

(4.2) $$\|\mathbf{x}^{(r)}\| \to \infty, \quad \gamma(\mathbf{x}^{(r)}) \to \sigma_n \quad \text{as } r \to \infty.$$

By virtue of the first relation, and the fact that $\mathbf{x}^{(r)} \in S_n$, there must exist a subsequence of $\mathbf{x}^{(r)}$ for which at least one component tends to ∞ and another tends to 0. Let us write again $\mathbf{x}^{(r)}$ for this subsequence, and for definiteness, assume that

$$x_1^{(r)} \to \infty, \quad x_2^{(r)} \to 0 \quad \text{as } r \to \infty. \tag{4.3}$$

Since

$$\begin{aligned} n\gamma(\mathbf{x}^{(r)}) &= \frac{1}{\Gamma(x_1^{(r)})} + \frac{1}{\Gamma(x_2^{(r)})} + \sum_{k=3}^{n} \frac{1}{\Gamma(x_k^{(r)})} \\ &\leqq \frac{1}{\Gamma(x_1^{(r)})} + \frac{1}{\Gamma(x_2^{(r)})} + \frac{n-2}{\Gamma(x_0)}, \end{aligned}$$

we obtain from (4.2), (4.3), by letting $r \to \infty$ in this inequality,

$$\sigma_n \leqq \frac{1 - 2/n}{\Gamma(x_0)}. \tag{4.4}$$

We show that equality holds in (4.4). Define $\mathbf{x}(t)$ by

$$x_1(t) = tc, \quad x_2(t) = c/t, \quad x_3 = \cdots = x_n = x_0, \qquad c = x_0^{-(n-2)/2}.$$

Clearly, $\mathbf{x}(t) \in S_n$ for all $t > 0$, and

$$n\gamma(\mathbf{x}(t)) = \frac{1}{\Gamma(tc)} + \frac{1}{\Gamma(c/t)} + \frac{n-2}{\Gamma(x_0)}.$$

Letting $t \to \infty$ gives $\gamma(\mathbf{x}(t)) \to (1 - 2/n)/\Gamma(x_0)$, and therefore strict inequality cannot hold in (4.4).

Thus, in Case I, we conclude that

$$\sigma_n = \frac{1 - 2/n}{\Gamma(x_0)}. \tag{4.5}$$

Case II. The supremum σ_n is assumed at a finite point $\mathbf{x} = \mathbf{s}$ of S_n,

$$\gamma(\mathbf{x}) \leqq \gamma(\mathbf{s}) \quad \text{for all } \mathbf{x} \in S_n. \tag{4.6}$$

The function γ thus has on S_n a global maximum at $\mathbf{s}$.

Using Lagrange multipliers, it follows that $\mathbf{s}^T = [s_1, s_2, \cdots, s_n]$ must be a solution of the system of equations

$$\frac{\partial}{\partial x_i}\left[\gamma(x_1, x_2, \cdots, x_n) + \lambda\left(\prod_{k=1}^{n} x_k - 1\right)\right] = 0, \quad i = 1, 2, \cdots, n,$$

$$\prod_{k=1}^{n} x_k - 1 = 0,$$

that is,

$$-\frac{\Gamma'(s_i)}{[\Gamma(s_i)]^2} + \lambda n \prod_{\substack{k=1 \\ k \neq i}}^{n} s_k = 0, \qquad i = 1, 2, \cdots, n, \tag{4.7}$$

$$\prod_{k=1}^{n} s_k = 1. \tag{4.8}$$

Multiplying the ith equation in (4.7) by s_i, and taking note of (4.8), we find

$$\lambda n = f(s_1) = f(s_2) = \cdots = f(s_n), \tag{4.9}$$

where the function f is as in (3.2), with $r = -1$. Since $f(x) < 0$ for $0 < x < x_0$ and $f(x) \geqq 0$ for $x \geqq x_0$, it follows from (4.9) that either all s_k are between 0 and x_0, or all s_k are $\geqq x_0$. The latter, however, is excluded by (4.8), since $x_0 > 1$. Consequently,

$$0 < s_k < x_0, \qquad k = 1, 2, \cdots, n. \tag{4.10}$$

Now, using Proposition 2 of § 3, according to which f is unimodal on $0 < x < x_0$, we conclude from (4.9) and (4.10) that only one of two situations can arise:

IIa. All s_k are the same. By (4.8), this implies $s_1 = s_2 = \cdots = s_n = 1$, and so $\sigma_n = \gamma(\mathbf{s}) = 1$ in this case.

IIb. There are exactly *two* distinct s_k, say,

$$s_1 = s_2 = \cdots = s_\nu < s_{\nu+1} = s_{\nu+2} = \cdots = s_n, \qquad 1 \leqq \nu < n,$$
$$s_1^\nu s_n^{n-\nu} = 1.$$

We then have

$$\sigma_n = \gamma(\mathbf{s}) = \frac{1}{n}\left[\frac{\nu}{\Gamma(s_1)} + \frac{n-\nu}{\Gamma(s_n)}\right], \qquad 0 < s_1 < s_n < x_0.$$

Since $s_n = s_1^{-\nu/(n-\nu)}$ and $s_1 < s_n$, it follows that $0 < s_1 < 1$. (In fact, $x_0^{-(n-\nu)/\nu} < s_1 < 1$, by virtue of $s_n < x_0$.) According to the definition of $g_{n,\nu}$ in (2.2), we thus have

$$\sigma_n = g_{n,\nu}(s_1), \qquad 0 < s_1 < 1. \tag{4.11}$$

Furthermore, by (4.9), s_1 is a solution of the equation

$$f(x) = f(x^{-\nu/(n-\nu)}). \tag{4.12}$$

One checks readily that the roots of (4.12) are precisely the stationary points of $g_{n,\nu}(x)$. Since

$$\gamma(\underbrace{x, x, \cdots, x}_{\nu\text{-times}}, \underbrace{y, y, \cdots, y}_{(n-\nu)\text{-times}}) = g_{n,\nu}(x), \qquad y = x^{-\nu/(n-\nu)},$$

where the argument of γ is a point on S_n for each $x > 0$, and since σ_n is the global maximum of γ on S_n, the stationary point (4.11) cannot be other than a local maximum. There are now two possibilities:

IIba. For no integer ν with $1 \leqq \nu \leqq n-1$ does $g_{n,\nu}(x)$ have a local maximum on $(0, 1)$.

IIbb. There is at least one integer ν, $1 \leqq \nu \leqq n-1$, for which $g_{n,\nu}(x)$ has a local maximum on $(0, 1)$.

Case IIba is incompatible with Case IIb, so that IIa necessarily applies, and $\sigma_n = 1$. In Case IIbb we must look for the largest local maximum (if there are several, corresponding to different values of ν), which is then equal to σ_n if larger than 1. Otherwise, $\sigma_n = 1$ from Case IIa.

Summarizing Case II, we can write

$$\sigma_n = \max_{1 \leqq \nu \leqq n-1} \{\max_{0 \leqq x \leqq 1} g_{n,\nu}(x)\} = \gamma_n,$$

where the inner maximum picks up a local maximum of $g_{n,\nu}$, if it is larger than 1, or the value $g_{n,\nu}(1) = 1$, if it is less than 1 or nonexistent. With Case I, equation (4.5), taken into account, we thus have

$$\sigma_n = \max\left[\frac{1 - 2/n}{\Gamma(x_0)}, \gamma_n\right].$$

Observing, however, that

$$\gamma_n \geqq \max_{0 \leqq x \leqq 1} g_{n,1}(x) \geqq g_{n,1}(x_0^{-(n-1)}) = \frac{1}{n}\left[\frac{1}{\Gamma(x_0^{-(n-1)})} + \frac{n-1}{\Gamma(x_0)}\right]$$

$$> \frac{1 - 1/n}{\Gamma(x_0)} > \frac{1 - 2/n}{\Gamma(x_0)},$$

we see that in fact $\sigma_n = \gamma_n$, proving (2.1).

Noting further that

$$g_{n,\nu}(x) \leqq \frac{1}{n}\left[\frac{\nu}{\Gamma(x_0)} + \frac{n-\nu}{\Gamma(x_0)}\right] = \frac{1}{\Gamma(x_0)} \quad \text{on } 0 \leqq x \leqq 1,$$

we have $\gamma_n \leqq 1/\Gamma(x_0)$, and thus

$$\frac{1 - 1/n}{\Gamma(x_0)} < \gamma_n \leqq \frac{1}{\Gamma(x_0)}, \qquad n = 1, 2, 3, \cdots,$$

showing that $\lim_{n\to\infty} \gamma_n = 1/\Gamma(x_0)$, as claimed in (2.3). Theorem 1 is now proved.

5. Numerical results and graphs. In this section we present some information concerning the functions $g_{n,\nu}(x)$ in (2.2) which was obtained by extensive numerical computation, using the CDC 6500 computer.

First of all, we observe that for large n many of the functions $g_{n,\nu}(x)$ do in fact have local maxima in $0 < x < 1$. This can be seen by noting that

$$g_{n,\nu}(x_0^{-(n-\nu)/\nu}) = \frac{1}{n}\left[\frac{\nu}{\Gamma(x_0^{-(n-\nu)/\nu})} + \frac{n-\nu}{\Gamma(x_0)}\right] > \frac{1 - \nu/n}{\Gamma(x_0)},$$

so that $g_{n,\nu}(x_0^{-(n-\nu)/\nu}) > 1$ whenever $(1 - \nu/n)/\Gamma(x_0) > 1$, i.e., whenever

$$\frac{\nu}{n} < 1 - \Gamma(x_0) = .1143 \cdots. \tag{5.1}$$

Since $g_{n,\nu}(0) = 0$, $g_{n,\nu}(1) = 1$, the presence of a local maximum in the case of (5.1) is thus evident.

More detailed computations, covering the range $2 \leqq n \leqq 30$, $1 \leqq \nu \leqq n - 1$, revealed that:

(i) $g_{n,\nu}(x)$ is monotonically increasing on $0 \leqq x \leqq 1$ for $n \leqq 6$, $1 \leqq \nu \leqq n - 1$.

(ii) $g_{n,1}(x)$ for $n \geqq 7$ has a unique local maximum on (0, 1) which is less than 1 for $n = 7$ and $n = 8$, but larger than 1 for $n \geqq 9$.

(iii) $g_{n,\nu}(x)$ for $\nu = 2, 3, 4$ has a local maximum only for $n \geqq 14$, $n \geqq 21$, $n \geqq 28$, respectively, each being smaller than the respective maximum of $g_{n,1}$. As ν increases from 2 to 4, the maxima in question decrease.

(iv) $g_{n,\nu}(x)$ for $7 \leqq n \leqq 30$, $5 \leqq \nu \leqq n - 1$ is monotonically increasing on $0 \leqq x \leqq 1$.

The numerical results suggest the conjecture that the relative maxima of $g_{n,v}$ decrease as v increases (with n held fixed), but we do not have a proof for this. Some critical portions of the "dominant" curves $y = g_{n,1}(x)$, $7 \leqq n \leqq 10$, are shown in Fig. 1.

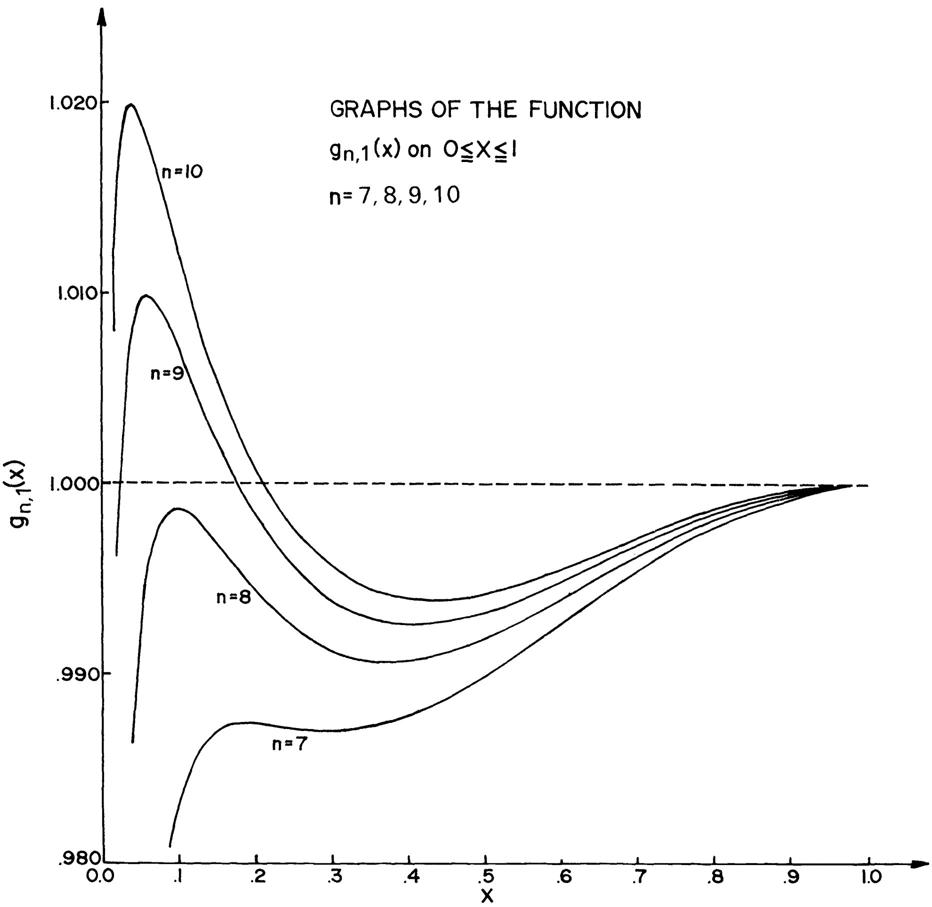

FIG. 1. *Graphs of* $y = g_{n,1}(x)$ *for* $7 \leqq n \leqq 10$

Based on the numerical evidence described above in (i) and (ii), we may infer with confidence that[2]

$$\gamma_n = 1 \quad \text{for } 1 \leqq n \leqq 8. \tag{5.2}$$

From (5.1) with $v = 1$, on the other hand, we see that $\gamma_n > 1$ whenever $n > 1/(1 - \Gamma(x_0)) = 8.741 \cdots$, i.e.,

$$\gamma_n > 1 \quad \text{for all } n \geqq 9. \tag{5.3}$$

[2] Equation (5.2) is trivial for $n = 1$, and established in [1] for $n = 2$.

The local maxima γ_n^* of $g_{n,1}$, $7 \leqq n \leqq 30$, were computed more accurately by applying Newton's method to the equation (4.12) with $\nu = 1$. A binary search method was used to obtain fairly accurate initial approximations. The results, believed to be accurate to all digits shown, are displayed in Table 1. (Observe that $\gamma_n^* = \gamma_n$ for $n \geqq 9$.)

TABLE 1[3]

Local maxima $\gamma_n^* = g_{n,1}(\xi_n^*)$ of $g_{n,1}(x)$ for $7 \leqq n \leqq 30$

n	ξ_n^*	γ_n^*	n	ξ_n^*	γ_n^*
7	1.900855126(−1)	.9874040859	19	1.087835945(−3)	1.069800731
8	9.819583769(−2)	.9986294355	20	7.425128883(−4)	1.072752220
9	6.005471800(−2)	1.009798259	21	5.071386946(−4)	1.075427804
10	3.840859500(−2)	1.019864207	22	3.465411640(−4)	1.077863530
11	2.512555627(−2)	1.028706548	23	2.368819775(−4)	1.080089658
12	1.666215274(−2)	1.036420644	24	1.619635354(−4)	1.082131717
13	1.114939565(−2)	1.043152112	25	1.107593956(−4)	1.084011358
14	7.506998631(−3)	1.049045683	26	7.575306970(−5)	1.085747033
15	5.076813299(−3)	1.054229841	27	5.181558830(−5)	1.087354549
16	3.444215694(−3)	1.058813803	28	3.544457248(−5)	1.088847512
17	2.341999266(−3)	1.062888730	29	2.424710624(−5)	1.090237691
18	1.595180690(−3)	1.066530189	30	1.658765573(−5)	1.091535309

[3] The integers in parentheses denote powers of 10 by which the preceding numbers are to be multiplied.

6. Proof of Theorem 2. The proof follows similar lines of reasoning as the proof of Theorem 1. We can therefore be brief. Letting

$$\gamma(x_1, x_2, \cdots, x_n) = \frac{1}{n} \sum_{k=1}^{n} \ln \Gamma(x_k),$$

we denote

$$\sigma_n = \inf_{\mathbf{x} \in S_n} \gamma(x_1, x_2, \cdots, x_n) > -\infty, \tag{6.1}$$

and propose to prove that

$$\sigma_n = 0.$$

The infimum in (6.1) cannot be assumed at infinity, since otherwise there would be a sequence of vectors $\mathbf{x}^{(r)} \in S_n$ satisfying (4.2), (4.3), hence

$$\begin{aligned} n\gamma(\mathbf{x}^{(r)}) &= \ln \Gamma(x_1^{(r)}) + \ln \Gamma(x_2^{(r)}) + \sum_{k=3}^{n} \ln \Gamma(x_k^{(r)}) \\ &\geqq \ln \Gamma(x_1^{(r)}) + \ln \Gamma(x_2^{(r)}) + (n-2)\Gamma(x_0) \to \infty \quad \text{as } r \to \infty. \end{aligned}$$

The function $\gamma(\mathbf{x})$ thus assumes a minimum on S_n at some finite point $\mathbf{x} = \mathbf{s} \in S_n$,

$$\gamma(\mathbf{x}) \geqq \gamma(\mathbf{s}) \quad \text{for all } \mathbf{x} \in S_n.$$

On using Lagrange multipliers, it follows that $\mathbf{s}^T = [s_1, s_2, \cdots, s_n]$ must satisfy

$$\phi(s_1) = \phi(s_2) = \cdots = \phi(s_n), \tag{6.2}$$

$$\prod_{k=1}^{n} s_k = 1, \tag{6.3}$$

where

$$\phi(x) \overset{\text{def}}{=} x\psi(x).$$

Since $\phi(x) < 0$ for $0 < x < x_0$, and $\phi(x) \geqq 0$ for $x \geqq x_0$, we conclude from (6.2), (6.3) that

$$0 < s_k < x_0, \qquad k = 1, 2, \cdots, n.$$

From Proposition 1 we know that $\phi(x)$ is convex for $x > 0$, and from the proof of Proposition 2, that

$$\phi(0) = -1, \quad \phi'(0) < 0, \quad \phi(x_0) = 0.$$

There are thus points ξ^*, ξ_0, with $0 < \xi^* < \xi_0$, such that $\phi(0) = \phi(\xi_0) = -1$ and $\phi(x)$ is monotonically decreasing on $0 \leqq x < \xi^*$ and monotonically increasing on $\xi^* < x \leqq x_0$. Since $\phi(1) = -\gamma > -1$, we have in fact $0 < \xi_0 < 1$.

From (6.2) we now conclude that only one of two situations can hold:

(a) All s_k are the same. Then $s_1 = s_2 = \cdots = s_n = 1$, giving $\sigma_n = 0$.

(b) There are exactly two distinct s_k, say,

$$0 < s_1 = s_2 = \cdots = s_\nu < s_{\nu+1} = s_{\nu+2} = \cdots = s_n, \qquad 1 \leqq \nu < n,$$

such that

$$0 < s_1 < \xi^* < s_n < \xi_0 < 1.$$

Since the last inequalities imply $s_1^\nu s_n^{n-\nu} < 1$, in contradiction to (6.3), case (b) is impossible, leaving us with case (a), i.e., $\sigma_n = 0$. Theorem 2 is proved.

7. Proof of Theorem 3. We have already observed in § 1 that (2.6) is true for all $r \geqq 0$. It suffices therefore to show that (2.6) is false for $r < 0$.

By an obvious adaptation of the proof of Theorem 1, one finds that

$$\inf_{\mathbf{x}\in S_n} M_n^{[r]}(\Gamma(x_i)) = \gamma_n^{1/r}, \qquad r < 0, \tag{7.1}$$

where

$$\gamma_n = \max_{1 \leqq \nu \leqq n-1} \{ \max_{0 \leqq x \leqq 1} g_{n,\nu}(x) \},$$

$$g_{n,\nu}(x) \overset{\text{def}}{=} \frac{1}{n}\{\nu[\Gamma(x)]^r + (n-\nu)[\Gamma(x^{-\nu/(n-\nu)})]^r\}.$$

Now arguing as in (5.3), we have

$$\begin{aligned}\gamma_n &\geqq \max_{0 \leqq x \leqq 1} g_{n,1}(x) \geqq g_{n,1}(x_0^{-(n-1)}) \\ &= \frac{1}{n}\{[\Gamma(x_0^{-(n-1)})]^r + (n-1)[\Gamma(x_0)]^r\} > \left(1 - \frac{1}{n}\right)[\Gamma(x_0)]^r,\end{aligned}$$

from which it follows that $\gamma_n > 1$ as soon as

$$n > \frac{1}{1 - [\Gamma(x_0)]^{-r}}.$$

For all these values of n, the infimum in (7.1) is <1, and thus the inequality (2.6) false. This proves Theorem 3.

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UNIQUENESS AND EXISTENCE FOR THE INTEGRAL EQUATION OF INTERREFLECTIONS*

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Abstract. The integral equation of interreflections, determining radiant energy exchange in cavities, is shown to have a unique solution in the space on nonnegative functions defined over the cavity surface. The result is established by employing the contraction mapping principle.

1. Background, definitions and objectives. In order to familiarize the reader with the integral equation of interreflections, this paper begins with a short informal description of its physical origin and basis.

We begin this description by supposing one has a source of radiation inside some cavity, e. g., a light bulb inside a closed room. The source emits particles of energy (call them photons) which collide with the cavity wall. At each point, $\mathbf{x}$, of the cavity wall, S, a certain fraction, $R(\mathbf{x})$, of all the photons incident upon an infinitesimal element of area surrounding $\mathbf{x}$, $dA_{\mathbf{x}}$, are reflected back into the cavity; the remaining fraction, $1 - R(\mathbf{x})$, permanently escapes the cavity by passing through the wall. A certain fraction of those photons which are reflected from $dA_{\mathbf{x}}$ travel to every other element of area $dA_{\mathbf{y}}$ of the cavity wall. The empirical law of diffuse reflection states that the fraction of photons which are reflected from $dA_{\mathbf{x}}$ to $dA_{\mathbf{y}}$ is proportional to the product of the solid angle subtended by $dA_{\mathbf{y}}$ at point $\mathbf{x}$ and the ratio of the area of $dA_{\mathbf{x}}$ projected on the plane perpendicular to the line joining $\mathbf{x}$ and $\mathbf{y}$ to $dA_{\mathbf{x}}$. This ratio is just the cosine of the angle between the normal to S at point $\mathbf{x}$ and the line in the direction $\mathbf{r} \equiv \mathbf{y} - \mathbf{x}$. Thus the fraction of reflected photons which travel from $dA_{\mathbf{x}}$ to $dA_{\mathbf{y}}$ is given by

$$K_0(\mathbf{y}, \mathbf{x}) = \frac{dA_{\mathbf{y}}(\mathbf{n}(\mathbf{y}) \cdot \mathbf{r})(-\mathbf{n}(\mathbf{x}) \cdot \mathbf{r})}{\pi \|\mathbf{r}\|^4}, \tag{1.1}$$

where $\mathbf{n}(\mathbf{y}), \mathbf{n}(\mathbf{x})$ [abbreviated $\mathbf{n}$] are the unit outward normals to S at points $\mathbf{y}$ and $\mathbf{x}$ respectively and $\|\mathbf{r}\|$ is the Euclidean length of $\mathbf{r}$. Throughout this paper $\mathbf{r}, \mathbf{y}, \mathbf{x}$ are vectors with the standard rectangular Cartesian coordinates (r_1, r_2, r_3) etc.; the Euclidean inner product $\mathbf{x} \cdot \mathbf{y} \equiv \sum_i x_i y_i$ is represented by a dot between vectors, as in (1.1). Whenever it is appropriate $\mathbf{n}(\mathbf{y}), dA_{\mathbf{y}}, \cdots$ are abbreviated by $\mathbf{n}$, dA etc.

It is easily seen that $K_0(\mathbf{y}, \mathbf{x})$ is also given by

$$K_0(\mathbf{y}, \mathbf{x}) = \frac{dA_{\mathbf{y}}(\cos \theta_{\mathbf{y}})(\cos \theta_{\mathbf{x}})}{\pi \|\mathbf{r}\|^2}, \tag{1.2}$$

where $\theta_{\mathbf{y}}, \theta_{\mathbf{x}}$ are the acute angles between the unit outward normals $\mathbf{n}(\mathbf{y}), \mathbf{n}(\mathbf{x})$ and the lines in the directions $\mathbf{r}$, $-\mathbf{r}$ respectively; see Fig. 1.

The quantity of physical interest is the steady state "total" density of incoming radiation power at each surface element $dA_{\mathbf{x}}$. This quantity is denoted by $H(\mathbf{x})$ and has the dimensions of energy per unit time and area. We proceed to derive the integral equation which determines $H(\mathbf{x}), \mathbf{x} \in S$. The total incoming

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power at any point $\mathbf{y}$ is composed of the "direct" power coming to $dA_{\mathbf{y}}$ directly from the source, say $H_0(\mathbf{y})\,dA_{\mathbf{y}}$, plus the "indirect" power coming from every other surface element $dA_{\mathbf{x}}$ that "can be seen" from $dA_{\mathbf{y}}$, i.e., the solid angle subtended at $\mathbf{x}$ by $dA_{\mathbf{y}}$ is not blocked by some obstruction (in this paper it is assumed that all

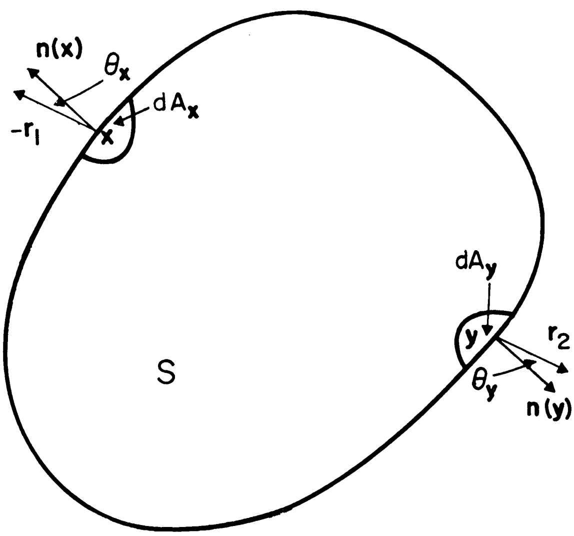

FIG. 1. *Kernel geometry*

surface elements see each other). From the definitions of $R(\cdot)$ and $K_0(\cdot,\cdot)$ it is seen that the indirect power at $dA_{\mathbf{y}}$ is given by

$$\sum_{dA_{\mathbf{x}}\in S} R(\mathbf{x})K_0(\mathbf{y},\mathbf{x})H(\mathbf{x})\,dA_{\mathbf{x}}. \tag{1.3}$$

Thus

$$H(\mathbf{y})\,dA_{\mathbf{y}} = H_0(\mathbf{y})\,dA_{\mathbf{y}} + \sum_{dA_{\mathbf{x}}\in S} R(\mathbf{x})K_0(\mathbf{y},\mathbf{x})H(\mathbf{x})\,dA_{\mathbf{x}}. \tag{1.4}$$

After dividing (1.4) by $dA_{\mathbf{y}}$ and passing to the limit $dA_{\mathbf{x}} \to 0$, one obtains the integral equation of interreflections

$$H(\mathbf{y}) = H_0(\mathbf{y}) + \iint_S R(\mathbf{x})K(\mathbf{y},\mathbf{x})H(\mathbf{x})\,dA_{\mathbf{x}}, \tag{1.5}$$

where $K(\mathbf{y},\mathbf{x}) \equiv K_0(\mathbf{y},\mathbf{x})/(dA_{\mathbf{y}})$. The derivation of (1.5) sketched above is a synthesis of ideas in [3] and [5].

From consideration of the physical meaning of (1.5) it seems plausible to seek its solution in the complete metric space, $\mathscr{M}$, consisting of the nonnegative continuous functions defined on S together with the metric

$$d(H,G) = \sup_{s\in S} |H(s) - G(s)|. \tag{1.6}$$

The first objective of this paper is to show, under suitable restrictions which will be specified later, that the mapping determined by

$$K: \iint_S R(\mathbf{x})K(\mathbf{y}, \mathbf{x})H(\mathbf{x})\, dA_{\mathbf{x}} = G(\mathbf{y}) \tag{1.7}$$

is a contraction mapping of the space $\mathscr{M}$ into itself. Our second objective is to prove that the integral equation of interreflections possesses a unique solution in $\mathscr{M}$. To attain both of these objectives it is necessary to make the physically reasonable restrictions:

$$R_{\max} = \max_{s \in S} R(s) < 1, \tag{1.8}$$

$$H_0(\mathbf{x}) \in \mathscr{M}. \tag{1.9}$$

The integral equation of interreflections is of current technological interest in connection with light bulb standardization at the National Bureau of Standards; for other current applications, see [7], [8, Chap. 3] and further references cited therein. Equation (1.5) has been discussed from a computational and physical point of view (but not with a formal view of establishing existence and uniqueness) in the papers [3], [5], [10]. In [3] the Fredholm approach was investigated while in [5] the Hilbert–Schmidt approach was followed. Some analytic solutions to equations involving ideal surfaces have been given in [9]. For a description of the paper [10], see § 4.

The results in this paper have importance beyond the fact that they establish existence and uniqueness of (1.5) rigorously and with a method much simpler than the Fredholm or Hilbert–Schmidt methods. Our results suggest and provide the basis for a powerful iterative numerical method for solving (1.5) and computing a posteriori error bounds [1, p. 38]. Finally the methods we use seem sufficiently precise and powerful enough to generalize our results to cover all physically realistic situations; see the remarks concerning the generality of our techniques in § 4 and Remark 3.1.

The proof of the second part of Lemma 1 (which is shorter and more analytical than the author's original proof) is due to A. J. Goldman.

2. Basic ideas and auxiliary constructions. The existence of a unique solution to the integral equation (1.5) is established by applying the contraction mapping principle [4, p. 43]. There are two prerequisites for employing this principle. First of all, the mapping K, see (1.7), must be shown to map the space $\mathscr{M}$ into itself; secondly it must be shown that there exists a positive constant, c, such that $c < 1$ and

$$d(K(H), K(G)) \leqq c\, d(H, G) \tag{2.1}$$

for all $H(\cdot), G(\cdot)$ in $\mathscr{M}$.

Before proceeding to establish these prerequisites it is convenient to introduce certain definitions and geometrical constructions that serve as a framework for defining precisely the improper integrals that we deal with. We note that all integrals with $K(\mathbf{y}, \mathbf{x})$ occurring in their integrand are formally improper because the quantity $\|\mathbf{r}\|$ in the denominator of $K(\mathbf{y}, \mathbf{x})$ vanishes when $\mathbf{y} = \mathbf{x}$. With the goal of defining these improper integrals in mind we denote the tangent plane to S at an arbitrary point $\mathbf{y}$ by $T(\mathbf{y})$. For each point $\mathbf{y}$ of S construct (see Fig. 2) a local rectangular Cartesian coordinate system with coordinates x_i, $i = 1, 2, 3$, and with origin at $\mathbf{y}$ in such a way that the x_1, x_2-plane coincides with $T(y)$ and the positive x_3-axis points into the interior of S. Furthermore, suppose $\{\varepsilon_j\}$, $j = 1, \cdots, \infty$, to be some strictly decreasing monotonic sequence of positive numbers that converges to zero, and define for each value of j: (i) $T^j(\mathbf{y})$ to be the plane parallel to $T(\mathbf{y})$ that intersects the local x_3-axis at $x_3 = \varepsilon_j$, (ii) C^j to be the intersection curve of S and $T^j(\mathbf{y})$, (iii) F^j and N^j to be the two closed surfaces that S is separated into by C^j,

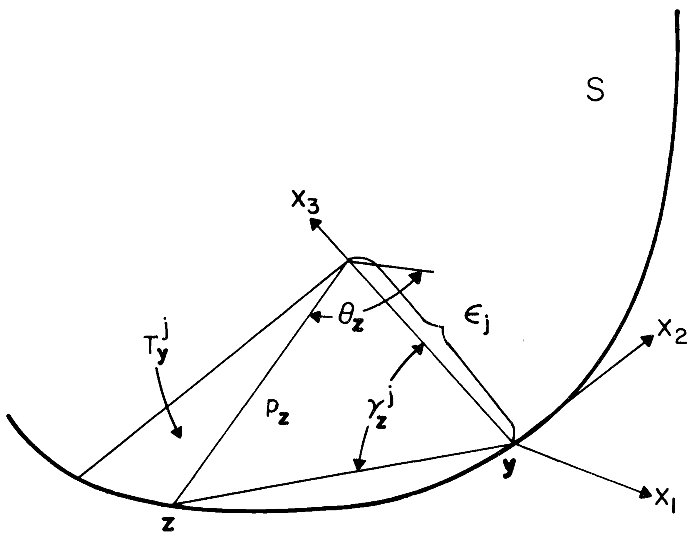

FIG. 2. *Local constructions at point* $\mathbf{y}$

N^j being the part that contains $\mathbf{y}$. With the aid of these constructions our improper integral can be defined, formally, by

$$\iint_S R(\mathbf{x})K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} \equiv \lim_{j\to\infty} \iint_{F^j} RK\, dA_{\mathbf{x}} \tag{2.2}$$

and

$$\iint_S R(\mathbf{x})K(\mathbf{y}, \mathbf{x})H(\mathbf{x})\, dA_{\mathbf{x}} \equiv \lim_{j\to\infty} \iint_{F^j} RKH\, dA_{\mathbf{x}}. \tag{2.3}$$

It will be shown later that the limits on the right-hand sides (abbreviated RHS) of (2.1), (2.2) exist independently of the particular sequence $\{\varepsilon_j\}$ whenever $H(\mathbf{x}) \in \mathscr{M}$; thus the integrals are meaningfully described. It is also convenient now to define for later use the sequence of functions on S:

$$H^j(\mathbf{y}) \equiv \iint_{F^j} RKH\, dA_{\mathbf{x}}, \qquad j = 1, \cdots, \infty. \tag{2.4}$$

Of course, certain hypotheses must be made regarding the surface S in order to guarantee that the constructions just introduced and the quantities appearing in $K(\mathbf{y}, \mathbf{x})$ exist and are uniquely definable at each point of S. To assure that S possesses these general properties we hypothesize as follows.

Hypothesis 1. The quantities $T^j(\mathbf{y})$, $T(\mathbf{y})$, $\mathbf{n}(\mathbf{y})$, C^j etc. defined above exist and are uniquely defined at each point of S and S is compact.

3. Outline and proof of lemmas and theorems. This section starts with a brief description of the relation between the contents of the lemmas and the objectives of our papers. The formal proofs of the lemmas and theorems follow the description.

The two prerequisites that are needed for establishing the contraction mapping principle are essentially consequences of the fundamental result

$$\lim_{j\to\infty} \iint_{F^j} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} = 1 \tag{3.1}$$

which is proved in Lemma 1. By using this result and Lemma 2 the limits in (2.2), (2.3) are shown to exist in Lemma 3. Lemma 4 is used in the proof of Lemma 5 to show that the mapping K transforms $\mathcal{M}$ into itself. Theorem 1 establishes the contractivity of the mapping K. Theorem 2 combines the result of Lemma 5 with the contraction mapping principle to obtain the result that the integral equation (1.5) possesses a unique solution in $\mathcal{M}$.

Before beginning the formal parts of our proofs we introduce some preliminary definitions and hypotheses that will be repeatedly referred to in our lemmas and theorems. Thus (see Fig. 2), let $(\rho_{\mathbf{z}}, \theta_{\mathbf{z}})$ be the polar coordinates of an arbitrary point $\mathbf{z}$ on C^j. Define $\gamma^j(\mathbf{z})$ to be the angle between the line connecting $\mathbf{z}$ to $\mathbf{y}$ and the x_3-axis of the local Cartesian frame at y. In our proofs we will refer to the following hypotheses.

Hypothesis 2. With reference to the previously defined quantities,

$$\lim_{j\to\infty} \gamma^j(\mathbf{z}) = \pi/2, \quad \text{uniformly in } \mathbf{z}. \tag{3.2}$$

Hypothesis 3. The surface S is convex.

Hypothesis 4. Each surface $S^j(\mathbf{y})$ can be represented [2, p. 159] by a vector-valued function $\mathbf{x}(u, v, \mathbf{y})$; and $\mathbf{x}(u, v, \mathbf{y})$ is assumed to be continuously differentiable in u and v and continuous in $\mathbf{y}$.

The statement and proofs of our lemma and theorems follow.

LEMMA 1. *If Hypotheses* 1, 2, 3, 4 *hold, then*

$$\lim_{j\to\infty} \iint_{F^j} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} \equiv \iint_{S} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} = 1 \tag{3.3}$$

for each $\mathbf{y}$ *in* S.

Proof. The proof is divided into two logically distinct phases. In the first phase the surface integrals on the LHS of (3.3) are transformed, by Stokes' theorem, into line integrals around C^j; this transformation of a surface integral of $K(\mathbf{y}, \mathbf{x})$ into a line integral is well known [5, p. 313]. In the second phase of the proof it is shown that in the limit $j \to \infty$ the value of these line integrals approaches one.

For ease of calculation [e.g., to verify (3.4)] it is convenient to choose the origin of the original Cartesian coordinate system at the point $\mathbf{y}$ [i.e., at the $\mathbf{y}$ appearing in (3.3)]; the quantities $\|\mathbf{r}\|$ and $\mathbf{r}$ then became $(x_1^2 + x_2^2 + x_3^2)^{1/2}$ and $(-x_1, -x_2, -x_3)$ respectively.

To begin the first phase of the proof we note that a trivial rearrangement of the factors in $K(\mathbf{y}, \mathbf{x})$ followed by application of the (directly verifiable) vector identity

$$\frac{\mathbf{r}_1(\mathbf{n}(\mathbf{y}) \cdot \mathbf{r}_1)}{\|\mathbf{r}\|^2} = \frac{1}{2}\operatorname{curl}\left[\frac{\mathbf{r}_1}{\|r\|} \times \mathbf{n}(\mathbf{y})\right], \tag{3.4}$$

where $\mathbf{r}_1 \equiv \mathbf{r}/\|\mathbf{r}\|$ and " $\times$ " is the standard vector cross product, yields

$$\iint_{F^j} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} = -\frac{1}{2} \iint_{F^j} \mathbf{n}(\mathbf{x}) \cdot \operatorname{curl}\left[\frac{\mathbf{r}_1}{\|\mathbf{r}\|} \times \mathbf{n}(\mathbf{y})\right] dA_{\mathbf{x}}. \tag{3.5}$$

Now, because Hypothesis 3 holds, it is legitimate to apply Stokes' theorem [1, p. 395] to the RHS of (3.5); there results

$$\iint_{F^j} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} = -\frac{1}{2\pi} \int_{C^j} \left[\frac{\mathbf{r}_1}{\|\mathbf{r}\|} \times \mathbf{n}(\mathbf{y})\right] \cdot \mathbf{ds}, \tag{3.6}$$

and the first phase of the proof is complete.

The RHS of (3.6) can be evaluated in the framework of the local Cartesian frame. For $\mathbf{r}_1/\|\mathbf{r}\|$, $\mathbf{n}(\mathbf{y})$ and $\mathbf{ds}$ are given in the local coordinate system by $(-x_1, -x_2, \varepsilon_j)/(x_1^2 + x_2^2 + \varepsilon_j^2)^{1/2}$, $(0, 0, -1)$ and $(dx_1/ds, dx_2/ds)\, ds$ (it being understood that the curve C^j is parametrized by the Euclidean arc length "s"). With this notation and with the proper calculation of the cross and dot products one finds

$$-\frac{1}{2\pi} \int_{C^j} \left[\frac{\mathbf{r}_1}{\|\mathbf{r}\|} \times \mathbf{n}(\mathbf{y})\right] \cdot \mathbf{ds} = \frac{1}{2\pi} \int_{C^j} (x_1^2 + x_2^2 + \varepsilon_j^2)^{-1}[-x_2\, dx_1 + x_1\, dx_2]. \tag{3.7}$$

It is worthwhile to introduce polar coordinates (see Fig. 2)

$$x_1 = \rho \cos\theta; \qquad dx_1 = d\rho \cos\theta - \rho \sin\theta,$$
$$x_2 = \rho \sin\theta; \qquad dx_2 = d\rho \sin\theta + \rho \cos\theta$$

into the RHS of (3.7); there results

$$-\frac{1}{2\pi} \int_{C^j} \left[\frac{\mathbf{r}_1}{\|\mathbf{r}\|} \times \mathbf{n}(\mathbf{y})\right] \cdot \mathbf{ds} = \frac{1}{2\pi} \int_{C^j} (\rho^2 + \varepsilon_j^2)^{-1} \rho^2\, d\theta. \tag{3.8}$$

Next observe from Fig. 2 and the definition of $\gamma^j(z)$,

$$\rho_z^2/(\rho_z^2 + \varepsilon_j^2) = \sin^2[\gamma^j(z)], \tag{3.9}$$

where $z = (\rho_z, \theta_z)$ is an arbitrary point on C^j. Now by combining equations (3.5)–(3.9) there results

$$\iint_{F^j} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} = \frac{1}{2\pi} \int_{C^j} \sin^2[\gamma^j]\, d\theta. \tag{3.10}$$

Finally, by taking the limit $j \to \infty$ in (3.10) and utilizing Hypothesis 2 we get (3.3), and the proof of Lemma 1 is complete. {Remark 3.1. Observe that since $\rho^2/(\rho^2 + \varepsilon_j^2) < 1$ it follows by combining (3.4)–(3.9) that

$$\iint_S K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}} \leqq 1 \tag{3.11}$$

provided only $\lim_{j\to\infty} \rho^2/(\rho^2 + \varepsilon_j^2)$ exists. Also note that if there is a conical shaped cusp with vertex angle $\propto$ at some point $\mathbf{y}$ on S then the RHS of (3.3) equals $\sin^2 \propto$ at $\mathbf{y}$.}

The positivity of $K(\mathbf{y}, \mathbf{x})$ and $\int_{F^j} K\, dA_{\mathbf{x}}$ can hold without requiring the smoothness properties in Hypothesis 4. Thus we prove the following lemma.

LEMMA 2. *Let Hypotheses* 1 *and* 3 *hold. Then* $K(\mathbf{y}, \mathbf{x})$ *is defined and nonnegative.*

Proof. From the definition $K \equiv K_0/dA_{\mathbf{y}}$ and (1.2) it is seen that it is only necessary to prove that the angles $\theta_{\mathbf{y}}, \theta_{\mathbf{x}}$ are not greater than $\pi/2$. This is proved true of $\theta_{\mathbf{y}}$ and an analogous proof works for $\theta_{\mathbf{x}}$. It is shown that if $\theta_{\mathbf{y}}$ is assumed greater than $\pi/2$ a contradiction to Hypothesis 3 results.

Suppose $\theta_{\mathbf{y}}$, the angle between $\mathbf{r}$ and $\mathbf{n}(\mathbf{y})$, is greater than $\pi/2$. This implies the angle between $-\mathbf{r}$ and $\mathbf{n}(\mathbf{y})$ is less than $\pi/2$. Consequently, $\mathbf{n}(\mathbf{y})$ and $-\mathbf{r}$ lie on the same side of the tangent plane at point $\mathbf{y}$. This implies that some points near $\mathbf{y}$ along the line in the direction of $-\mathbf{r}$ lie outside S, because $\mathbf{n}(\mathbf{y})$ by definition is the outward pointing normal at $\mathbf{y}$. But, by Hypothesis 3, all the points on the segment from $\mathbf{y}$ to $\mathbf{x}$ should lie in S, hence a contradiction.

LEMMA 3. *If Hypotheses* 1–4 *hold, then the sequences*

$$\left\{\iint_{F^j} K(\mathbf{y}, \mathbf{x})\, dA_{\mathbf{x}}\right\}, \quad \left\{\iint_{F^j} RK\, dA_{\mathbf{x}}\right\} \quad \textit{and} \quad \left\{\iint_{F^j} RKH\, dA_{\mathbf{x}}\right\}, \tag{3.12}$$

where $H(\mathbf{x}) \in \mathscr{M}$, *are positive, monotonic and convergent.*

Proof. Lemma 2 and the restrictions on the functions R and H imply that the integrands of all the integrals in (3.12) are positive. Furthermore $F^m > F^n$ if $m > n$. Therefore all the sequences in (3.12) are monotonic increasing. These sequences are also bounded from above and hence converge. We have already shown in Lemma 1 that the first sequence in (3.12) converges. To establish that the second and third sequences converge note first of all that there exist finite numbers $R_{\max}, H_{\max}$ that bound the functions $R(\mathbf{x})$, $H(\mathbf{x})$, respectively. This is because R and H are continuous functions defined on a compact surface. Therefore we have

$$\iint_{F^j} RK\, dA_x \leqq R_{\max} \iint_{F^j} K\, dA_{\mathbf{x}} \leqq R_{\max}, \tag{3.13}$$

$$\iint_{F^j} RKH\, dA_{\mathbf{x}} \leqq R_{\max} H_{\max} \iint_{F^j} K\, dA_{\mathbf{x}} \leqq R_{\max} H_{\max}, \tag{3.14}$$

where to obtain the second inequality in (3.13), (3.14) we have used the monotonicity of $\{\iint_{F^j} K\, dA_{\mathbf{x}}\}$ and the conclusion of Lemma 1. Thus the integrals on the left-hand side of (2.2) and (2.3) are well-defined.

The next lemma plays an important role in establishing the fact that K maps $\mathscr{M}$ into $\mathscr{M}$.

LEMMA 4. *If Hypotheses* 1–4 *hold, then the functions* $H^j(\mathbf{y})$, $\mathbf{y} \in S$ (see (2.4)), *are well-defined and continuous.*

Proof. It is convenient in the proof of Lemma 4 to lump the product of $R(\mathbf{x})$ and $H(\mathbf{x})$ that appears in the integrand on the RHS of (2.4) together, thus below $H(\mathbf{x})$ is considered to be absorbed into $R(\mathbf{x})$; also "j" is considered to be arbitrary but fixed.

It must be shown that for an arbitrary but fixed $\mathbf{y}_1, \mathbf{y}_1 \in S$, and any positive ε it is possible to choose a $\delta(\varepsilon)$ so that

$$(3.15) \qquad B(\mathbf{y}_1, \mathbf{y}_2) \equiv \left| \iint\limits_{F^j(\mathbf{y}_1)} RK(\mathbf{y}_1, \mathbf{x})\, dA_{\mathbf{x}} - \iint\limits_{F^j(\mathbf{y}_2)} RK(\mathbf{y}_2, \mathbf{x})\, dA_{\mathbf{x}} \right| < \varepsilon$$

whenever $\|\mathbf{y}_1 - \mathbf{y}_2\| < \delta(\varepsilon)$.

By defining the subdomains of S, $I(\mathbf{y}_1, \mathbf{y}_2) \equiv F^j(\mathbf{y}_1) \cap F^j(\mathbf{y}_2)$, $Z_i \equiv F^j(\mathbf{y}_i) \sim I(\mathbf{y}_1, \mathbf{y}_2)$, $i = 1, 2$, and using the triangle inequality it is possible to rewrite $B(\mathbf{y}_1, \mathbf{y}_2)$ in the more convenient form

$$(3.16) \qquad \begin{aligned} B(\mathbf{y}_1, \mathbf{y}_2) \leqq \sum_{i=1}^{2} \iint\limits_{Z_i} RK(\mathbf{y}_i, \mathbf{x})\, dA_{\mathbf{x}} \\ + \iint\limits_{I(\mathbf{y}_1, \mathbf{y}_2)} R|K(\mathbf{y}_1, \mathbf{x}) - K(\mathbf{y}_2, \mathbf{x})|\, dA_{\mathbf{x}}. \end{aligned}$$

From the postulated continuous dependence of $F^j(\mathbf{y})$ on $\mathbf{y}$ it is possible to make the areas of Z_i, $i = 1, 2$, arbitrarily small by choosing y_2 sufficiently close to y_1. Also, it is possible to choose $\mathbf{y}_2$ sufficiently close to $\mathbf{y}_1$ so that both $F^j(\mathbf{y}_i)$, $i = 1, 2$, are contained in a closed compact set $\mathscr{W}$ which contains neither $\mathbf{y}_1$ nor $\mathbf{y}_2$. Hence $K(\mathbf{y}_i, \mathbf{x})$, $i = 1, 2$, are continuous functions of $\mathbf{x}$ for $\mathbf{x} \in F^j(\mathbf{y}_i) \cap \mathscr{W}$, $i = 1, 2$, respectively; and hence since $Z_i \subset F^j(\mathbf{y}_i) \cap \mathscr{W}$, $i = 1, 2$, the functions $K(\mathbf{y}_i, \mathbf{x})$, $i = 1, 2$, possess upper bounds in Z_i, $i = 1, 2$, respectively.

From the previous discussion it follows that there exists a $\delta_1(\varepsilon)$ such that

$$(3.17) \qquad \sum_{i=1}^{2} \iint\limits_{Z_i} RK(\mathbf{y}_i, \mathbf{x})\, dA_x < \frac{\pi}{2}$$

whenever $\|\mathbf{y}_2 - \mathbf{y}_1\| = \delta_1(\varepsilon)$.

The last term on the RHS of (3.16) can also be bounded, for we know that $\mathbf{y}_i$, $i = 1, 2$, lie outside $\mathscr{W}$ and hence for each fixed $\mathbf{x} \in \mathscr{W}$ it is possible to find a $\delta_2(\mathbf{x}, \varepsilon)$ such that

$$(3.18) \qquad |K(\mathbf{y}_1, \mathbf{x}) - K(\mathbf{y}_2, \mathbf{x})| < \varepsilon/2R_{\max} \cdot (\text{area of } S)$$

whenever $\|\mathbf{y}_2 - \mathbf{y}_1\| < \delta_2(\mathbf{x}, \varepsilon)$. {Remark. If $H(\mathbf{x})$ is identically zero, then $R_{\max}$, which in this proof contains H as a factor, is zero and the RHS of (3.18) is meaningless. However, if $H(\mathbf{x})$ is identically zero, then Lemma 4 is clearly true.} On account of the compactness of $\mathscr{W}$ and $I \subset \mathscr{W}$ there is a nonzero $\delta_2(\varepsilon)$ (independent of $\mathbf{x}$) such that (3.18) holds whenever $\|\mathbf{y}_2 - \mathbf{y}_1\| < \delta_2(\varepsilon)$ and $\mathbf{x} \in I$. Finally, by combining (3.16)–(3.18), inequality (3.15) is established.

LEMMA 5. *If Hypotheses* 1–4 *hold, then the mapping* $H \to KH + H_0$, *where* $H_0 \in \mathscr{M}$, *maps the space* $\mathscr{M}$ *into itself.*

Proof. By combining Lemma 3 and the definition (2.3) we deduce that KH is well-defined and nonnegative. KH is also continuous, for by Lemma 3 the sequence $\{H^j(\mathbf{x})\}$ is monotonically convergent and therefore Dini's theorem [2, p. 101] implies the sequence is uniformly convergent. This and Lemma 4 imply that KH

is a limit of a uniformly convergent sequence of continuous functions and consequently is itself continuous. The proof of Lemma 5 is complete.

THEOREM 1. *If Hypotheses* 1–4 *hold, then the mappings* $H \to KH$ *and* $H \to KH + H_0$, $H_0 \in \mathscr{M}$, *are contractions on* $\mathscr{M}$.

Proof. The elementary inequality $|\int f\,dA_{\mathbf{x}}| \leqq \int |f|\,dA_{\mathbf{x}}$ implies

$$d(KH, KG) \leqq \left\{\iint_S RK\,dA_{\mathbf{x}}\right\} d(H, G). \tag{3.19}$$

The assumption (see (1.8)) $R_{\max} < 1$ and Lemma 1 imply

$$\iint_S RK\,dA_{\mathbf{x}} < 1. \tag{3.20}$$

Therefore K is a contraction and the same immediately follows for $H \to KH + H_0$.

THEOREM 2. *If Hypotheses* 1–4 *hold, then the integral equation specified by equation* (1.5) *has a unique solution in* $\mathscr{M}$.

Proof. It follows from the conclusions of Lemma 5 and Theorem 1 that the prerequisites for the contraction mapping principle [4] are satisfied and this principle implies the conclusion of Theorem 2.

4. Comments. For the purpose of presenting our principal ideas directly and clearly we have deliberately chosen to work with uncomplicated and perhaps overly restrictive hypotheses. It is clear that some of these restrictions can be removed. For example it is clear that our results hold for "piecewise" smooth surfaces. Even the convexity requirement seems to be removable but not without the stipulation that the domain of integration S in (1.5) be only over the points $\mathbf{x}$ from which it is possible "to see" (see § 1) $dA_{\mathbf{y}}$. For if S is not convex then for each $\mathbf{y}$ there are some points $\mathbf{x}$ on S at which $K(\mathbf{y}, \mathbf{x})$ is negative. By choosing $H(\mathbf{x})$ properly the image KH then becomes negative and therefore K is not an into mapping of $\mathscr{M}$.

Sydnor [10] establishes an error estimate for the remainder in the Neumann series solution of (1.5). His derivation of this result is based on an assumption similar to (3.3) and an assumption that the Neumann series converges. Both these assumptions are justified by plausibility arguments of a physical nature. Specifically Sydnor refers to [8, eq. 3.2] to justify our relation (3.3), and equation (3.2) of [8, Chap. 3] is justified by "energy" considerations. In this paper both of these assumptions are established mathematically; also the error bound cited in § 1 and which follows from the contraction mapping principle is better and more useful than Sydnor's remainder estimate. In addition, the other ingredients (Lemmas 4 and 5) necessary to establish the contraction mapping principle are established. These results and the fact that the integral on the RHS of (2.4) is formally improper are not mentioned by Sydnor.

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ASYMPTOTIC STABILITY FOR ORDINARY DIFFERENTIAL EQUATIONS WITH DELAYED PERTURBATIONS*

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Abstract. Asymptotic stability of the zero solution of

$$\dot{x}(t) = -a(t)x(t) + P(t, x_t)$$

is studied with a direct method of Razumikhin. If $|P(t, \varphi)| \leqq a(t)\theta\|\varphi\|$ for some $\theta < 1$, and $a(t) \geqq \alpha > 0$, then zero is exponentially stable; if the condition on $a(t)$ is weakened to $a(t) \geqq 0$ and $\int^\infty a(t)\,dt = \infty$, then zero is asymptotically stable.

1. Introduction. The problem of determining various stability properties of solutions of functional differential equations has been widely studied. Krasovskii [6] developed a theory of Lyapunov functionals to answer such questions for a large class of equations. As an application, he proved that the zero solution of

$$\dot{x}(t) = -ax(t) + b(t)x(t - h(t)), \tag{1.1}$$

where $a > 0$ and $0 \leqq h(t) \leqq r$, is asymptotically stable if $|b(t)| \leqq a\theta$ for some $\theta < 1$. With the same hypotheses, Hale [5], in fact, showed that zero is uniformly asymptotically stable, and hence exponentially stable by linearity [3]. By studying the norm of solutions directly, Razumikhin [8] was able to prove that zero is uniformly stable if $|b(t)| \leqq a$. More specifically, he showed that $\|x_t\|$ is a nonincreasing function of t. It is the purpose of this paper to use Razumikhin's method to investigate the asymptotic stability of zero for a similar class of equations.

We observe that under the above restrictions on the coefficient functions, the delayed term of (1.1) is, in some sense, dominated by the ordinary term. Thus, (1.1) may be viewed as a perturbation problem in which the stability properties of the unperturbed ordinary differential equation are preserved. The theorems in § 3 give sufficient conditions for exponential and asymptotic stability of the zero solution, and can be applied to nonlinear equations. For example, if the form of (1.1) is generalized to

$$\dot{x}(t) = -a(t)x(t) + b(t, x_t)x(t - h(t, x_t)), \tag{1.2}$$

we show that zero is exponentially stable if $|b(t, \varphi)| \leqq a(t)\theta$ for some $\theta < 1$, and $a(t) \geqq \alpha > 0$; if the condition on $a(t)$ is weakened to $\int^\infty a(t)\,dt = \infty$, then zero is asymptotically stable.

Grossman and Yorke [2] have applied Razumikhin's method to a class of equations closely related to the one dealt with in this paper. Other discussions of this method can be found in Halanay and Yorke [4], and Mikolajska [7]. The author is indebted to J. Yorke for several discussions which led to improvements of the results.

2. Preliminaries. Let R be the real line and $R^+ = [0, \infty)$. For $r > 0$, let C be the Banach space of continuous functions φ mapping $[-r, 0]$ into R with the

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usual supremum norm, $\|\cdot\|$; let $C(\eta) = \{\varphi \in C \mid \|\varphi\| < \eta\}$. If $x(u)$ is a continuous function on $[-r, T]$, where $T > 0$, then x_t denotes an element of C defined by $x_t(s) = x(t+s)$, $-r \leqq s \leqq 0$, for each t such that $0 \leqq t \leqq T$. A delay differential equation has the form

$$\dot{x}(t) = G(t, x_t), \tag{2.1}$$

where $G: R^+ \times C \to R$ and $\dot{x}(t)$ denotes the right-hand derivative. We write $x(t, \tau, \varphi)$ for a solution of (2.1) with initial value (τ, φ), and $x_t(\tau, \varphi)$ when considered as a curve in C. When the choice of initial value is obvious, we simply write $x(t)$ and x_t, respectively. We shall be concerned with equations where the right-hand side has the form $G(t, \varphi) = F(t, \varphi(0)) + P(t, \varphi)$, so that (2.1) becomes

$$\dot{x}(t) = F(t, x(t)) + P(t, x_t). \tag{2.2}$$

Henceforth, we shall assume there exists $\eta > 0$ such that a solution through any (τ, ξ) in $R^+ \times C(\eta)$ exists on $[\tau, \infty)$.

Suppose $x(t) = 0$ is a solution of (2.2).

DEFINITION 2.1. We say that zero is *stable* if for given $\varepsilon > 0$ and $\tau \geqq 0$, there exists $\delta = \delta(\varepsilon, \tau) > 0$ such that

$$\|\varphi\| < \delta \Rightarrow \|x_t(\tau, \varphi)\| < \varepsilon$$

for $t \geqq \tau$. If δ can be chosen independently of τ, then zero is *uniformly stable*.

DEFINITION 2.2. We say that zero is *asymptotically stable* if it is stable and for each $\tau \geqq 0$, there exists $\mu = \mu(\tau) > 0$ such that

$$\|\varphi\| < \mu \Rightarrow \lim_{t \to \infty} \|x_t(\tau, \varphi)\| = 0.$$

DEFINITION 2.3. We say that zero is *exponentially (asymptotically) stable* if it is stable and there exist positive constants K, λ, and μ such that

$$\|\varphi\| < \mu \Rightarrow \|x_t(\tau, \varphi)\| \leqq K\|\varphi\| e^{-\lambda(t-\tau)}$$

for $t \geqq \tau$.

We conclude this section with a theorem which is a special case of more general theorems of Razumikhin [8] and Yorke [4]. Because the proof employs arguments typical of those used throughout the paper, it is included.

THEOREM 2.4. *Assume $F: R^+ \times R \to R$ satisfies $F(t, 0) = 0$ for all $t \in R^+$. Suppose there exists $\eta > 0$ such that $\|\varphi\| < \eta$ implies*

$$\varphi(0)F(t, \varphi(0)) \leqq -a(t)\varphi^2(0), \tag{A}$$

where $a: R^+ \to R^+$, and

$$|P(t, \varphi)| \leqq a(t)\theta\|\varphi\|, \tag{B}$$

where $P: R^+ \times C \to R$, for some $\theta \leqq 1$. If $\|\varphi\| < \eta$, then $\|x_t(\tau, \varphi)\|$ is a nonincreasing function of t and, hence, the zero solution of (2.2) *is uniformly stable.*

Proof. Given $t_0 \geqq \tau \geqq 0$ and $\|\varphi\| < \eta$, we show that $\|x_t(\tau, \varphi)\|$ is nonincreasing on $[t_0, t_0 + r]$. If not, there exists t_1 such that $t_0 < t_1 < t_0 + r$ and $\|x_{t_0}\| < \|x_{t_1}\| < \eta$. Since $x(t)$ is continuous, there exists t_2 such that $\|x_{t_1}\| = |x(t_2)|$ and $t_1 - r \leqq t_2 \leqq t_1$. But $t_1 - r \leqq t \leqq t_0$ implies $|x(t)| \leqq \|x_{t_0}\| < \|x_{t_1}\|$ so that, in fact,

$t_0 < t_2 \leqq t_1$. We choose t_2 so that it is the first such point in $(t_0, t_1]$. Now

$$t_2 - r \leqq t \leqq t_0 \Rightarrow |x(t)| \leqq \|x_{t_0}\| < \|x_{t_1}\| = |x(t_2)|$$

and

$$t_0 < t \leqq t_2 \Rightarrow |x(t)| \leqq \|x_{t_1}\| = |x(t_2)|$$

together imply that $\|x_{t_2}\| = |x(t_2)|$. Since $\|x_{t_2}\| = \|x_{t_1}\| > \|x_{t_0}\|$, the choice of t_2 implies there exists $\delta > 0$ such that $\|x_{t_2^-}\| = |x(t_2^-)|$, where $t_2^- \in (t_2 - \delta, t_2)$. If $x(t_2) > 0$, then $\dot{x}(t_2^-) > 0$. But

$$\begin{aligned} \dot{x}(t_2^-) &= F(t_2^-, x(t_2^-)) + P(t_2^-, x_{t_2^-}) \\ &\leqq -a(t_2^-)x(t_2^-) + a(t_2^-)\theta\|x_{t_2^-}\| \\ &\leqq -a(t_2^-)x(t_2^-) + a(t_2^-)x(t_2^-) = 0, \end{aligned}$$

a contradiction. The assumption $x(t_2) < 0$ leads to an analogous contradiction. Thus $\|x_t\|$ is nonincreasing on $[t_0, t_0 + r]$. Q.E.D.

3. Results. Without loss of generality, we assume that $\tau = 0$ in this section. The following theorem deals with the problem of exponential stability.

THEOREM 3.1. *Assume the hypotheses of Theorem* 2.4. *If the conditions on* θ *and* $a(t)$ *are strengthened so that* $\theta < 1$ *and* $a(t) \geqq \alpha > 0$, *then the zero solution of* (2.2) *is exponentially stable.*

Proof. Choose $\lambda > 0$ so small that $\lambda < \alpha(1 - \theta e^{\lambda r})$ and $\lambda < \min(\alpha, -\log\theta/r)$. Let $y(t) = e^{\lambda t}x(t)$. Then

$$\begin{aligned} \dot{y}(t) &= \lambda e^{\lambda t}x(t) + e^{\lambda t}\dot{x}(t) \\ &= \lambda e^{\lambda t}x(t) + e^{\lambda t}(F(t, x(t)) + P(t, x_t)) \\ &= \lambda y(t) + e^{\lambda t}(F(t, e^{-\lambda t}y(t)) + P(t, e_t^{-\lambda}y_t)) \\ &= G(t, y(t)) + Q(t, y_t), \end{aligned}$$

where $G(t, y) = \lambda y + e^{\lambda t}F(t, e^{-\lambda t}y)$, $Q(t, \varphi) = e^{\lambda t}P(t, e_t^{-\lambda}\varphi)$, and $e_t^{-\lambda}\varphi$ is a member of C defined by $(e_t^{-\lambda}\varphi)(s) = e^{-\lambda(t+s)}\varphi(s)$, $-r \leqq s \leqq 0$. If $\|\varphi\| < \eta$, we obtain the estimates

$$\begin{aligned} \varphi(0)G(t, \varphi(0)) &= \lambda\varphi^2(0) + e^{\lambda t}\varphi(0)F(t, e^{-\lambda t}\varphi(0)) \\ &\leqq \lambda\varphi^2(0) + e^{2\lambda t}(-a(t)[e^{-\lambda t}\varphi(0)]^2) \\ &= -(a(t) - \lambda)\varphi^2(0), \end{aligned}$$

and

$$\begin{aligned} |Q(t, \varphi)| &= e^{\lambda t}|P(t, e_t^{-\lambda}\varphi)| \\ &\leqq e^{\lambda t}a(t)\theta\|e_t^{-\lambda}\varphi\|. \end{aligned}$$

But

$$\begin{aligned} \|e_t^{-\lambda}\varphi\| &= \sup_{-r \leqq s \leqq 0} |e^{-\lambda(t+s)}\varphi(s)| \\ &\leqq e^{-\lambda(t-r)} \sup_{-r \leqq s \leqq 0} |\varphi(s)| \\ &= e^{-\lambda t} e^{\lambda r}\|\varphi\|. \end{aligned}$$

Hence $|Q(t, \varphi)| \leqq a(t)\theta\, e^{\lambda r}\|\varphi\|$. Recalling the choice of λ, we have $0 < \alpha - \lambda \leqq a(t) - \lambda$ and $\theta\, e^{\lambda r} < 1$, which imply

$$\lambda < \alpha(1 - \theta\, e^{\lambda r}) \leqq a(t)(1 - \theta\, e^{\lambda r}),$$

so that

$$a(t)\theta\, e^{\lambda r} \leqq a(t) - \lambda.$$

On applying Theorem 2.4, it follows that for $t \geqq 0$,

$$\begin{aligned} &\|y_t\| \leqq \|\varphi\| \\ \Rightarrow\ &|y(t)| \leqq \|\varphi\| \\ \Rightarrow\ &|x(t)| \leqq \|\varphi\|\, e^{-\lambda t} \\ \Rightarrow\ &\|x_t\| \leqq K\|\varphi\|\, e^{-\lambda t}, \end{aligned}$$

where $K = e^{\lambda r}$. Q.E.D.

The question of asymptotic stability is studied in the remainder of the paper.

LEMMA 3.2. *Assume $F(t, 0) = 0$ for all $t \in R^+$. Suppose there exists $\eta > 0$ such that $\|\varphi\| < \eta$ implies either*

$$\text{(A)} \qquad \varphi(0)F(t, \varphi(0)) \leqq -a(t)\varphi^2(0),$$

where $a(t) > 0$, or

$$\text{(A}'\text{)} \qquad F(t, \varphi(0)) = -a(t)\varphi(0),$$

where $a(t) \geqq 0$, and $P(t, \varphi)$ satisfies condition (B) *for some $\theta < 1$. Moreover, suppose that for all $0 < T < \infty$ and $0 < v < \eta$, $F(t, x)$ is bounded on $[0, T] \times [0, v]$, and $a(t)$ is bounded on $[0, T]$. If $\|\varphi\| < \eta$ and $x(t)$ oscillates, that is, $\dot{x}(t)$ changes sign as $t \to \infty$, then $\lim_{t\to\infty} x(t) = 0$.*

Proof. Theorem 2.4 and the hypotheses imply that the slope of solutions $x(t)$ with initial functions φ such that $\|\varphi\| < \eta$, is bounded on $[0, T]$ for any $T < \infty$. Hence $x(t)$ cannot oscillate wildly on compact sets, and therefore the following points are well-defined:

$$\begin{aligned} t_0 &= \text{first point in } [0, \infty) \text{ such that } |x(t_0)| = \sup_{s \geqq t_0} |x(s)|, \\ t_n &= \text{first point in } [t_{n-1} + r, \infty) \text{ such that} \\ &\qquad |x(t_n)| = \sup_{s \geqq t_n} |x(s)|, \quad \dot{x}(t_n) = 0, \quad \text{and} \\ &\qquad x(t_n) > 0\,(<0) \Rightarrow \dot{x}(t_n^-) > 0\,(<0), \\ &\qquad \text{where } t_n^- \in (t_n - \delta_n, t_n) \text{ for some } \delta_n > 0. \end{aligned}$$

Geometrically speaking, these points are peaks and crests of $x(t)$. We note that $t_0 \leqq r$, and $\|x_{t_n+r}\| = |x(t_n)|$ for $n = 0, 1, 2, \cdots$. If $x(t_n) > 0$, then $\dot{x}(t_n^-) > 0$, which, in turn, implies

$$0 < -a(t_n^-)x(t_n^-) + a(t_n^-)\theta\|x_{t_n^-}\|.$$

But $t_{n-1}^- + r \leqq t_n^-$, so that $\|x_{t_n^-}\| \leqq \|x_{t_{n-1}^-+r}\| = |x(t_{n-1})|$ by Theorem 2.4 and the definition of t_n. Moreover, if condition (A′) is satisfied, then $\dot{x}(t) \neq 0$ implies that

$a(t) \neq 0$ and, hence, $a(t_n^-) > 0$ for $n = 1, 2, \cdots$. Thus

$$|x(t_n^-)| < \theta |x(t_{n-1})|.$$

The same inequality is obtained if $x(t_n) < 0$ and, hence, continuity yields

$$|x(t_n)| \leqq \theta |x(t_{n-1})|$$
$$\Rightarrow |x(t_n)| \leqq \cdots \leqq \theta^n |x(t_0)| \leqq \theta^n \|\varphi\|.$$

Therefore $\lim_{n\to\infty} x(t_n) = 0$. From the definition of $\{t_n\}$, it follows that $\lim_{t\to\infty} x(t) = 0$. Q.E.D.

THEOREM 3.3. *Assume the hypotheses of Lemma* 3.2. *If* $\int^\infty a(s)\,ds = \infty$, *then the zero solution of* (2.2) *is asymptotically stable.*

Proof. By Theorem 2.4 and Lemma 3.2, it only remains to show that non-oscillatory solutions tend to zero. If $x(t)$ is such a solution, then $\dot{x}(t)$ has constant sign (including zero) for large t and, thus, $x(t)$ must be monotonic as $t \to \infty$. Hence, $x(t)$ must be either identically zero or have constant sign for large t. Suppose $x(t) > 0$ for large t. By monotonicity, $\lim_{t\to\infty} x(t)$ exists so that $\lim_{t\to\infty} \|x_t\|$ also exists, and

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \|x_t\| = \gamma.$$

If $\gamma > 0$, then

$$\lim_{t\to\infty} \frac{x(t)}{\|x_t\|} = 1.$$

Choose $\varepsilon > 0$ so small that $\theta < 1 - \varepsilon < 1$. Then there exists $T > 0$ such that $t \geqq T$ implies

$$\frac{x(t)}{\|x_t\|} > \frac{\theta}{1-\varepsilon}$$
$$\Rightarrow x(t) - \theta\|x_t\| > \varepsilon x(t)$$
$$\Rightarrow \dot{x}(t) \leqq -\varepsilon a(t) x(t)$$
$$\Rightarrow x(t) \leqq x(T) \exp\left[-\varepsilon \int_T^t a(s)\,ds\right].$$

Thus, $\lim_{t\to\infty} x(t) = 0$, a contradiction. Similarly, $x(t) \not< 0$ for large t, and the result is proved. Q.E.D.

Remark. It is necessary that $\theta < 1$ in Theorem 3.3 because the equation

$$\dot{x}(t) = -x(t) + x(t-r)$$

has constant solutions corresponding to constant initial functions.

Remark. The restriction on the sign of $a(t)$ in Theorem 3.3 cannot be removed although $\int^\infty a(s)\,ds = \infty$ is necessary and sufficient for the zero solution of the unperturbed equation to be asymptotically stable. In fact, without this assumption, the theorem is not true even in the ordinary case, that is, when $r = 0$. For example, define

$$a(t) = \begin{cases} 2\sin t, & 0 \leqq t \leqq \pi, \\ \sin t, & \pi \leqq t \leqq 2\pi, \end{cases}$$

and extend $a(t)$ periodically. We consider

$$\dot{x}(t) = \left(-a(t) + \frac{|a(t)|}{2} \right) x(t),$$

so that $\theta = 1/2$. Then

$$\int_0^{2\pi} -a(t)\,dt = -2\int_0^{\pi} \sin t\,dt - \int_{\pi}^{2\pi} \sin t\,dt = -2,$$

but

$$\int_0^{2\pi} \left(-a(t) + \frac{|a(t)|}{2} \right) dt = -2 + \int_0^{\pi} \sin t\,dt + \frac{1}{2}\int_{\pi}^{2\pi} |\sin t|\,dt = 1.$$

Hence nontrivial solutions tend to infinity.

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LIE THEORY AND MEIJER'S G-FUNCTION*

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Abstract. It is shown how Lie algebra techniques can be used to derive and interpret expansions of G-functions in series of G-functions.

Introduction. This work represents a continuation of the author's program to uncover the Lie algebraic structure of the theory of special functions of hypergeometric type, [1], [2], [3], [4]. Here we use the known differential recurrence relations obeyed by the G-functions $G^{m,n}_{p,q}$ to construct an associated Lie algebra $\mathscr{G}_{p,q}$. We show how to characterize various generating functions for series of G-functions as simultaneous eigenfunctions of a set of commuting (or almost commuting) operators of $\mathscr{G}_{p,q}$. The basic concepts used here were already discussed in some detail in [2].

To illustrate our theory, we derive several identities most of which can be found in the works of Luke [5], [6]. Most of the expansions of G-functions in series of G-functions computed by Luke, Fields, Wimp and Meijer among others, can be obtained by these Lie algebraic methods. When compared with the usual inductive proofs of such identities, Lie algebraic methods are striking in their simplicity and elegance. Furthermore, once understood these techniques permit one to compute new identities of specified types at will.

The computations in this paper are mostly formal so as to save space and to keep from obscuring the basic ideas. Domains of validity for most of our examples can be found in [5] and [6]. Rigorous proofs can be devised by careful justification of each step in our method. Alternatively, once an identity is obtained by the formal method it may be easier to verify the result directly.

Clearly, the methods of this paper extend to G-functions of many variables.

1. Properties of G-functions. Meijer's G-function is defined by the integral

$$
\begin{aligned}
G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right) &= G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\\
&= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)z^s\,ds}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j - s)},\\
& 0 \leqq m \leqq q, \qquad 0 \leqq n \leqq p, \qquad a_h - b_l \neq 1, 2, \cdots,\\
& z \neq 0, \qquad 1 \leqq h \leqq n, \qquad 1 \leqq l \leqq m,
\end{aligned}
\tag{1.1}
$$

where L is one of three paths in the complex s-plane [5], [7]. For example, one path goes from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_l - s)$ lie on the right and all poles of $\Gamma(1 - a_h + s)$ lie on the left. The integral converges for some a_j, b_k if

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$2(m + n) - (p + q) > 0$. The remaining paths and convergence criteria are discussed in [5], [7] as are the other results in this section.

Evaluating (1.1) by residues one can show that $G^{m,n}_{p,q}$ can be expressed as a linear combination of m, (n) generalized hypergeometric functions ${}_pF_{q-1}$, $({}_qF_{p-1})$. In particular we have the special cases

$$G^{1,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = \frac{\prod_{j=1}^{n}\Gamma(1+b_1-a_j)z^{b_1}}{\prod_{j=2}^{q}\Gamma(1+b_1-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_1)}$$

$$\tag{1.2} \cdot {}_pF_{q-1}\left(\begin{matrix}1+b_1-a_p\\ 1+b_1-b_q^*\end{matrix}\,\middle|\,(-1)^{p-m-n}z\right), \qquad p < q \text{ or } p = q \text{ and } |z| < 1,$$

$$G^{m,1}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = \frac{\prod_{j=1}^{m}\Gamma(b_j-a_1+1)z^{a_1-1}}{\prod_{j=2}^{q}\Gamma(1+a_j-a_1)\prod_{j=m+1}^{p}\Gamma(a_j-b_1)}$$

$$\tag{1.3} \cdot {}_qF_{p-1}\left(\begin{matrix}1+b_q-a_1\\ 1+a_p-a_1^*\end{matrix}\,\middle|\,\frac{(-1)^{q-m-1}}{z}\right), \qquad q < p \text{ or } q = p \text{ and } |z| > 1.$$

Here, we are using the notation of Luke [6] for the ${}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\,\middle|\,z\right)$ and the term $b_j - b_k^*$ is omitted whenever $j = k$.

The function $G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$ is symmetric in the parameters $a_1, \cdots, a_n$, the parameters $a_{n+1}, \cdots, a_p$, the parameters $b_1, \cdots, b_m$ and the parameters $b_{m+1}, \cdots, b_q$ separately. It also obeys the recurrence formulas

$$z^\sigma G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p+\sigma\\ b_q+\sigma\end{matrix}\right), \qquad \sigma \in \mathbb{C},$$

$$\left(z\frac{d}{dz} - a_1 + 1\right)G^{m,n}_{p,q} = G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1-1, a_2, \cdots, a_p\\ b_q\end{matrix}\right), \qquad n \geqq 1,$$

$$\tag{1.4} \left(z\frac{d}{dz} - a_p + 1\right)G^{m,n}_{p,q} = -G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1, \cdots, a_{p-1}, a_p - 1\\ b_q\end{matrix}\right), \qquad n < p,$$

$$\left(z\frac{d}{dz} - b_1\right)G^{m,n}_{p,q} = -G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_1+1, b_2, \cdots, b_q\end{matrix}\right), \qquad m \geqq 1,$$

$$\left(z\frac{d}{dz} - b_q\right)G^{m,n}_{p,q} = G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_1, \cdots, b_{q-1}, b_q+1\end{matrix}\right), \qquad m < q.$$

We can use relations (1.4) and the methods of [2] to associate a Lie algebra with the G-functions. Namely, we introduce $p + q$ new variables $t_1, \cdots, t_p, u_1, \cdots, u_q$ and define basis functions

$$\tag{1.5} F^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}t_p\\ u_q\end{matrix}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)t_1^{a_1}\cdots t_p^{a_p}u_1^{b_1}\cdots u_q^{b_q}$$

and partial differential operators

$$
\begin{aligned}
&T_j = t_j\partial_{t_j}, \qquad L_j = t_j^{-1}(z\partial_z - t_j\partial_{t_j} + 1), \qquad 1 \leqq j \leqq p,\\
(1.6)\qquad &U_k = u_k\partial_{u_k}, \qquad R_k = u_k(z\partial_z - u_k\partial_{u_k}), \qquad 1 \leqq k \leqq q,\\
&V = zt_1 \cdots t_p u_1 \cdots u_q.
\end{aligned}
$$

These $2(p + q) + 1$ operators generate a Lie algebra $\mathcal{G}_{p,q}$ with nonzero commutation relations

$$
(1.7)\qquad \begin{aligned}
&[T_j, L_j] = -L_j, \qquad [U_k, R_k] = R_k,\\
&[T_j, V] = [U_k, V] = V, \qquad 1 \leqq j \leqq p, \qquad 1 \leqq k \leqq q.
\end{aligned}
$$

All other commutators of two generators of $\mathcal{G}_{p,q}$ are zero. Expressions (1.4) and (1.5) imply

$$
\begin{aligned}
&T_jF^{m,n}_{p,q} = a_jF^{m,n}_{p,q}, \qquad U_kF^{m,n}_{p,q} = b_kF^{m,n}_{p,q},\\
&VF^{m,n}_{p,q} = F^{m,n}_{p,q}\left(z\left|\begin{matrix} t_p \\ u_q\end{matrix}\right|\begin{matrix} a_p + 1 \\ b_q + 1\end{matrix}\right),\\
(1.8)\qquad &L_jF^{m,n}_{p,q} = \pm F^{m,n}_{p,q}\left(z\left|\begin{matrix} t_p \\ u_q\end{matrix}\right|\begin{matrix} a_1, \cdots, a_j - 1, \cdots, a_p \\ b_q\end{matrix}\right),\\
&\text{(plus sign if } 1 \leqq j \leqq n, \text{ minus if } n + 1 \leqq j \leqq p),\\
&U_kF^{m,n}_{p,q} = \mp F^{m,n}_{p,q}\left(z\left|\begin{matrix} t_p \\ u_q\end{matrix}\right|\begin{matrix} a_p \\ b_1, \cdots, b_j + 1, \cdots, b_q\end{matrix}\right),\\
&\text{(minus sign if } 1 \leqq k \leqq m, \text{ plus if } m + 1 \leqq k \leqq q).
\end{aligned}
$$

It follows that the generators (1.6) and basis functions (1.5) can be used to construct (reducible) representations of $\mathcal{G}_{p,q}$. Furthermore, the operator

$$
(1.9)\qquad C^{m,n}_{p,q} \equiv R_1 \cdots R_q + (-1)^l VL_1 \cdots L_p, \qquad l = p - m - n + 1,
$$

satisfies the identities

$$
(1.10)\qquad C^{m,n}_{p,q}F^{m,n}_{p,q} = 0
$$

for all $a_1, \cdots, a_p, b_1, \cdots, b_q$. Indeed, factoring out the quantity $t_1^{a_1} \cdots t_p^{a_p}u_1^{b_1} \cdots u_q^{b_q}$ from the partial differential equation (1.10) we are left with the ordinary differential equation in z satisfied by the function $G^{m,n}_{p,q}\left(z\left|\begin{matrix} a_p \\ b_q\end{matrix}\right.\right)$, see [5, p. 181]:

$$
(1.11)\qquad \begin{aligned}
&\{(z\partial_z - b_1) \cdots (z\partial_z - b_q) + (-1)^l z(z\partial_z - a_1 + 1) \cdots (z\partial_z - a_p + 1)\}\\
&\cdot G^{m,n}_{p,q}\left(z\left|\begin{matrix} a_p \\ b_q\end{matrix}\right.\right) = 0.
\end{aligned}
$$

We note for future reference that if no two of the b_j differ by an integer then a fundamental system of solutions of (1.11) in a neighborhood of $z = 0$ is given by

$$y_h(z) = \frac{\Gamma(1 + b_h - a_p)}{\Gamma(1 + b_h - b_q)} z^{b_h} {}_pF_{q-1}\left(\begin{matrix}1 + b_h - a_p \\ 1 + b_h - b_q^*\end{matrix}\middle| (-1)^{l+1}z\right), \tag{1.12}$$

$$h = 1, \cdots, q, \quad p \leqq q - 1 \text{ or } p = q \text{ and } |z| < 1,$$

see [5, p. 181]. Also, one can easily show that (1.3) is the only solution (1.11) of the form $\sum_{h=0}^{\infty} z^{a_1 - 1 - h} c_h$ near $z = \infty$, provided no two of the a_k differ by an integer.

At this point we could apply the method of [2] to derive generating functions for ${}_pF_q$ and $G_{p,q}^{m,n}$. However, deeper results can be obtained through use of the Mellin transform. We formally define the Mellin transform $\mathscr{M}G(s)$ of the function $G(z)$ by

$$B(s) = \mathscr{M}G(s) = \int_0^{\infty} G(z) z^{-s-1}\, dz \tag{1.13}$$

and the inverse Mellin transform by

$$\mathscr{M}^{-1}F(z) = G(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} B(s) z^s\, ds, \tag{1.14}$$

[8]. It follows from (1.1) and (1.14) that $G_{p,q}^{m,n}\left(z \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)$ can be viewed as the inverse Mellin transform of the function

$$B_{p,q}^{m,n}\left(s \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}. \tag{1.15}$$

An operator K acting on a space of functions $G(z)$ induces an operator $\mathscr{K} = \mathscr{M}K\mathscr{M}^{-1}$ on the space of Mellin transformed functions $\mathscr{M}G(s)$. It is easy to show that the transformed operators corresponding to the generators (1.6) are

$$\begin{aligned} &\mathscr{T}_j = T_j = t_j\partial_{t_j}, \quad \mathscr{L}_j = t_j^{-1}(s - t_j\partial_{t_j} + 1), \qquad 1 \leqq j \leqq p, \\ &\mathscr{U}_k = U_k = u_k\partial_{u_k}, \quad \mathscr{R}_k = u_k(s - u_k\partial_{u_k}), \qquad 1 \leqq k \leqq q, \\ &\mathscr{V} = t_1 \cdots t_p u_1 \cdots u_q \underline{L}, \end{aligned} \tag{1.16}$$

where $\underline{L}B(s) = B(s - 1)$. To derive these results we have integrated by parts and assumed that the functions $G(z)$ are sufficiently bounded near $z = 0$ and $z = +\infty$ so that all boundary terms are zero.

It is evident that relations (1.7) and (1.8) still hold where now T_j, L_j, U_k, R_k and V are replaced by script letters and $F_{p,q}^{m,n}$ is replaced by the basis function

$$H_{p,q}^{m,n}\left(s \middle| \begin{matrix} t_p \\ u_q \end{matrix} \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = B_{p,q}^{m,n}\left(s \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) t_1^{a_1} \cdots t_p^{a_p} u_1^{b_1} \cdots u_q^{b_q}. \tag{1.17}$$

In particular the relations

$$\begin{aligned} &\mathscr{C}_{p,q}^{m,n} H_{p,q}^{m,n} = 0, \\ &\mathscr{C}_{p,q}^{m,n} = \mathscr{R}_1 \cdots \mathscr{R}_q + (-1)^l \mathscr{V} \mathscr{L}_1 \cdots \mathscr{L}_p, \qquad l = p - m - n + 1, \end{aligned} \tag{1.18}$$

now hold. We note also that the null space of the operator $\mathscr{C}^{m,n}_{p,q}$ is invariant under $\mathscr{T}_j$, $\mathscr{L}_j$, $\mathscr{U}_k$, $\mathscr{R}_k$, and $\mathscr{V}$.

2. The Lie algebraic method. We now apply what is basically Weisner's method [9] to characterize generating functions for G-functions as simultaneous eigenvectors of $p + q$ independent operators constructed from the generators of $\mathscr{G}_{p,q}$ which also lie in the null space of $\mathscr{C}^{m,n}_{p,q}$. The basis functions $H^{m,n}_{p,q}$ themselves have such a description:

$$\begin{aligned} &\mathscr{T}_j H^{m,n}_{p,q} = a_j H^{m,n}_{p,q}, \qquad \mathscr{U}_k H^{m,n}_{p,q} = b_k H^{m,n}_{p,q}, \\ &\mathscr{C}^{m,n}_{p,q} H^{m,n}_{p,q} = 0. \end{aligned} \tag{2.1}$$

For our first simple example, we search for a solution $H(s, t_1, \cdots t_p, u_1, \cdots, u_q)$ of the equations

$$\begin{aligned} &(\mathscr{T}_1 + \beta\mathscr{L}_1)H = a_1 H, \qquad \mathscr{T}_j H = a_j H, \qquad j = 2, \cdots, p, \\ &\mathscr{U}_k H = b_k H, \qquad k = 1, \cdots, q, \qquad \mathscr{C}^{m,n}_{p,q} H = 0. \end{aligned} \tag{2.2}$$

The first $p + q$ equations have the solution

$$H = B(s)(1 - \beta/t_1)^{a_1 - s - 1} t_1^{a_1} \cdots t_p^{a_p} u_1^{b_1} \cdots u_q^{b_q}, \tag{2.3}$$

where $B(s)$ is an arbitrary function of s. Requiring that $\mathscr{C}^{m,n}_{p,q} H = 0$ we find a solution

$$B(s) = B^{m,n}_{p,q}\left(s \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right), \tag{2.4}$$

equation (1.15). Next we expand our solution (2.3) as a Laurent series in $t_1, \cdots, t_p, u_1, \cdots, u_q$. (Necessarily, each term in this series will be a simultaneous eigenfunction of the T_j and U_k and will lie in the null space of $\mathscr{C}^{m,n}_{p,q}$.) We easily obtain the expansion

$$B^{m,n}_{p,q}\left(s \,\middle|\, \begin{matrix} a_1, a_2, \cdots, a_p \\ b_q \end{matrix}\right)\left(1 - \frac{\beta}{t_1}\right)^{a_1 - s - 1} = \sum_{h=0}^{\infty} \frac{(\beta/t_1)^h}{h!} B^{m,n}_{p,q}\left(s \,\middle|\, \begin{matrix} a_1 - h, a_2, \cdots, a_p \\ b_q \end{matrix}\right). \tag{2.5}$$

Taking the inverse Mellin transform of both sides of this equation (or integrating along the path L) we obtain the identity

$$\begin{aligned} &(1 - \beta)^{a_1 - 1} G^{m,n}_{p,q}\left(\frac{z}{1 - \beta} \,\middle|\, \begin{matrix} a_1, a_2, \cdots, a_p \\ b_q \end{matrix}\right) \\ &\quad = \sum_{h=0}^{\infty} \frac{\beta^h}{h!} G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1 - h, a_2, \cdots, a_p \\ b_q \end{matrix}\right), \qquad |\beta| < 1, \end{aligned} \tag{2.6}$$

which is the multiplication theorem for the G-function [5, p. 157]. (The transition from (2.5) to (2.6) can be justified by Stirling's approximation for the gamma function. Alternatively, the computations could be carried out using the operators (1.6). However, the easiest proof follows from a consideration of the expression $(\exp \beta L_1) F^{m,n}_{p,q}$.)

For our second example we take the equations

$$\begin{aligned}(2.7)\quad &\mathcal{T}_j H = a_j H, \qquad 1 \leqq j \leqq p, \qquad \mathcal{R}_1 H = H, \qquad \mathcal{U}_k H = b_k H, \qquad 2 \leqq k \leqq q,\\ &\mathcal{C}^{m,n}_{p,q} H = 0.\end{aligned}$$

In this case we obtain a solution of the form

$$(2.8)\qquad H = B^{m-1,n}_{p,q-1}\left(s\,\middle|\,\begin{matrix} a_p \\ b_2, \cdots, b_q\end{matrix}\right) e^{1/u_1} u_1^s t_1^{a_1} \cdots t_p^{a_p} u_2^{b_2} \cdots u_q^{b_q}.$$

Due to the presence of the term u_1^s the method of the preceding example no longer applies. However, setting $n = 1$ and taking the inverse Mellin transform we see that the function

$$\begin{aligned}(2.9)\quad f = {}& e^{1/u_1}(u_1 z)^{a_1 - 1}\,{}_{q-1}F_{p-1}\left(\begin{matrix} 1 + b_2 - a_1, \cdots, 1 + b_q - a_1 \\ 1 + a_2 - a_1, \cdots, 1 + a_p - a_1\end{matrix}\,\middle|\,\frac{(-1)^{q-m-1}}{u_1 z}\right)\\ &\cdot t_1^{a_1} \cdots t_p^{a_p} u_2^{b_2} \cdots u_q^{b_q}\end{aligned}$$

is a solution of the equations

$$\begin{aligned}(2.10)\quad &T_j f = a_j f, \qquad 1 \leqq j \leqq p, \qquad R_1 f = f, \qquad U_k f = b_k f, \qquad 2 \leqq k \leqq q,\\ &C^{m,n}_{p,q} f = 0.\end{aligned}$$

Now we can apply the methods of [2] directly. Expanding (2.9) in a Laurent series in u_1 we necessarily obtain an identity of the form

$$\begin{aligned}(2.11)\quad & e^{\tau}\,{}_{q-1}F_{p-1}\left(\begin{matrix} 1 + b_2 - a_1, \cdots, 1 + b_q - a_1 \\ 1 + a_2 - a_1, \cdots, 1 + a_p - a_1\end{matrix}\,\middle|\, -z\tau\right)\\ &= \sum_{h=0}^{\infty} c_h \tau^h\,{}_qF_{p-1}\left(\begin{matrix} -h, 1 + b_2 - a_1, \cdots, 1 + b_q - a_1 \\ 1 + a_2 - a_1, \cdots, 1 + a_p - a_1\end{matrix}\,\middle|\, z\right).\end{aligned}$$

Setting $z = 0$ we find $c_h = (h!)^{-1}$. Thus,

$$(2.12)\qquad e^{\tau}\,{}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_q\end{matrix}\,\middle|\, -z\tau\right) = \sum_{h=0}^{\infty} \frac{\tau^h}{h!}\,{}_{p+1}F_q\left(\begin{matrix} -h, a_1, \cdots, a_p \\ b_q\end{matrix}\,\middle|\, z\right).$$

(Setting $m = 2$ in (2.8) and using (1.2) we could obtain another such identity.)

The equations

$$\begin{aligned}(2.13)\quad &\mathcal{T}_j H = a_j H, \qquad 1 \leqq j \leqq p, \qquad \{\mathcal{U}_1(\mathcal{U}_1 + \gamma - 1) + \mathcal{R}_1(\mathcal{U}_1 + \beta)\}H = 0,\\ &\mathcal{U}_k H = b_k H, \qquad 2 \leqq k \leqq q, \qquad \mathcal{C}^{m,n}_{p,q} H = 0\end{aligned}$$

lead to a less trivial identity. The first $p + q$ equations are easily shown to have a solution of the form

$$(2.14)\qquad H = c(s)\,{}_2F_1\left(\begin{matrix} -s, \beta \\ \gamma\end{matrix}\,\middle|\, u_1\right) t_1^{a_1} \cdots t_p^{a_p} u_2^{b_2} \cdots u_q^{b_q},$$

where $c(s)$ is arbitrary. The requirement that H lie in the null space of $\mathscr{C}^{m,n}_{p,q}$ leads to a solution

$$c(s) = B^{m,n}_{p,q}\left(s\,\middle|\,\begin{matrix} a_p \\ 0, b_2, \cdots, b_q \end{matrix}\right). \tag{2.15}$$

Expanding this solution in a power series in u_1 we find

$$\begin{aligned} &B^{m,n}_{p,q}\left(s\,\middle|\,\begin{matrix} a_p \\ 0, b_2, \cdots, b_q \end{matrix}\right) {}_2F_1\left(\begin{matrix} -s, \beta \\ \gamma \end{matrix}\,\middle|\,\tau\right) \\ &= \sum_{h=0}^{\infty} \frac{\tau^h}{h!}\frac{(\beta)_h}{(\gamma)_h} B^{m,n}_{p,q}\left(s\,\middle|\,\begin{matrix} a_p \\ h, b_2, \cdots, b_q \end{matrix}\right), \qquad \tau = u_1. \end{aligned} \tag{2.16}$$

The inverse Mellin transform of the left-hand side of (2.16) does not have a simple expression in terms of G-functions, unless $\gamma = \beta$. However, in the special case $\tau = 1$ we can use the well-known result [7],

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} -s, \beta \\ \gamma \end{matrix}\,\middle|\,1\right) = \frac{\Gamma(\gamma)\Gamma(\gamma + s - \beta)}{\Gamma(\gamma + s)\Gamma(\gamma - \beta)}, \\ &\gamma \neq 0, -1, -2, \cdots, \quad \operatorname{Re}(\gamma + s - \beta) > 0, \end{aligned} \tag{2.17}$$

to obtain the identity

$$\begin{aligned} &\frac{\Gamma(\gamma)}{\Gamma(\gamma - \beta)} G^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix} \beta - \gamma + 1, a_p \\ 0, b_2, \cdots, b_q, 1 - \gamma \end{matrix}\right) \\ &= \sum_{h=0}^{\infty} \frac{(\beta)_h}{(\gamma)_h h!} G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix} a_p \\ h, b_2, \cdots, b_q \end{matrix}\right). \end{aligned} \tag{2.18}$$

(Of course, except for the case where β is a negative integer the formal passage from (2.16) and (2.17) to (2.18) needs rigorous justification.) We can extend this result by applying the operator $\exp(\rho\mathscr{R}_1)$ to the solution (2.14), (2.15):

$$\begin{aligned} \exp(\rho\mathscr{R}_1)H = (1 + \tau\rho)^s &B^{m,n}_{p,q}\left(s\,\middle|\,\begin{matrix} a_p \\ 0, b_2, \cdots, b_q \end{matrix}\right) \\ &\cdot {}_2F_1\left(\begin{matrix} -s, \beta \\ \gamma \end{matrix}\,\middle|\,\frac{\tau}{1 + \rho\tau}\right) t_1^{a_1} \cdots t_p^{a_p} u_2^{b_2} \cdots u_q^{b_q}. \end{aligned} \tag{2.19}$$

Expanding in powers of τ and using a well-known generating function for the ${}_2F_1$, [1, p. 211], we find

$$\begin{aligned} &(1 + \tau\rho)^s B^{m,n}_{p,q}\left(s\,\middle|\,\begin{matrix} a_p \\ 0, b_2, \cdots, b_q \end{matrix}\right) {}_2F_1\left(\begin{matrix} -s, \beta \\ \gamma \end{matrix}\,\middle|\,\frac{\tau}{1 + \rho\tau}\right) \\ &= \sum_{h=0}^{\infty} \frac{\tau^h\rho^h}{h!} B^{m,n+1}_{p+1,q+1}\left(s\,\middle|\,\begin{matrix} 0, a_p \\ 0, b_2, \cdots, b_q, h \end{matrix}\right) {}_2F_1\left(\begin{matrix} -h, \beta \\ \gamma \end{matrix}\,\middle|\,\rho^{-1}\right) \quad |\tau\rho| < 1. \end{aligned} \tag{2.20}$$

In the special case $\tau = 1 + \rho\tau = \omega$ we can take the inverse Mellin transform to obtain

$$
\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)} G^{m,n+1}_{p+1,q+1}\left(\omega z\,\middle|\,\begin{matrix}1+\beta-\gamma, a_p\\ 0, b_2, \cdots, b_q, 1-\gamma\end{matrix}\right)\\
&\quad = \sum_{h=0}^{\infty} \frac{(-1)^h}{h!}\, {}_2F_1\left(\begin{matrix}-h, \gamma-\beta\\ \gamma\end{matrix}\,\middle|\,\omega\right) G^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}0, a_p\\ 0, b_2, \cdots, b_q, h\end{matrix}\right).
\end{aligned}
\tag{2.21}
$$

(We could also have obtained this result by replacing $\mathscr{U}_1$ with $\mathscr{U}_1 - \rho\mathscr{R}_1$ wherever it occurs in (2.13).) Note that the basic idea here was the use of $\mathscr{U}_1$ and $\mathscr{R}_1$ to construct the differential equation for the ${}_2F_1$. Similarly we could further generalize these results by using $\mathscr{U}_1$ and $\mathscr{R}_1$ to construct the differential equation for a general ${}_{k+1}F_k$. Equivalently, we could repeatedly use the Euler transform to "augment" the indices [6, p. 2].

In the special case $m = n = 1$, $p = q = 2$, we can explicitly compute the inverse Mellin transform of (2.20) with the result

$$
\begin{aligned}
&[(1+\tau\rho)z]^{a_1-1} \frac{\Gamma(1-a_1)}{\Gamma(a_1-b_2)\Gamma(a_2-a_1+1)}\\
&\quad \cdot F_2\left[1-a_1; \beta, 1+b_2-a_1; a_2-a_1+1, \gamma; \frac{\tau}{1+\rho\tau}, \frac{1}{z(1+\rho\tau)}\right]\\
&= \sum_{h=0}^{\infty} \frac{\tau^h\rho^h}{h!} \frac{\Gamma(1-a_1)}{\Gamma(1-b_2)\Gamma(a_2)} \frac{(1-a_1)_h(1-a_2)_h}{(1-b_2)_h}\\
&\quad \cdot z^h\, {}_2F_1\left(\begin{matrix}1-a_1+h, 1-a_2+h\\ 1-b_2+h\end{matrix}\,\middle|\,z\right) {}_2F_1\left(\begin{matrix}-h, \beta\\ \gamma\end{matrix}\,\middle|\,\rho^{-1}\right)
\end{aligned}
\tag{2.22}
$$

valid in a neighborhood of $\tau = 0$, see [10, p. 216]. Here F_2 is an Appell function.

For our final and much more general example, we consider complex variables $s, t_1, \cdots, t_p, u_1, \cdots, u_q$ and $s', t'_1, \cdots, t'_{p'}, u'_1, \cdots, u'_{q'}$, where p, q, p', q' are positive integers, and operators $\mathscr{T}_j, \mathscr{L}_j, \mathscr{U}_k, \mathscr{R}_k, \mathscr{V}$ as well as $\mathscr{T}'_{j'}, \mathscr{L}'_{j'}, \mathscr{U}'_{k'}, \mathscr{R}'_{k'}, \mathscr{V}'$ defined in terms of these variables by expressions (1.16). The resulting $2(p + p' + q + q' + 1)$ operators form a basis for the Lie algebra $\mathscr{G}_{p,q} \oplus \mathscr{G}_{p',q'}$.

We try to characterize a generating function H for products of G-functions by requiring that H be a simultaneous eigenfunction of $p + p' + q + q'$ independent operators constructed from the elements of $\mathscr{G}_{p,q} \oplus \mathscr{G}_{p',q'}$ and also that $\mathscr{C}^{m,n}_{p,q}H = \mathscr{C}'^{m',n'}_{p',q'}H = 0$. An interesting case is

$$
\begin{aligned}
&\mathscr{T}_jH = a_jH, \qquad 1 \leqq j \leqq p, \qquad \mathscr{T}'_jH = a'_jH, \qquad 1 \leqq j \leqq p',\\
&(\mathscr{U}_1 + \mathscr{U}_2)H = \alpha H, \qquad (\mathscr{U}'_1 + \mathscr{U}'_2)H = \beta H,\\
&(\mathscr{U}_1 - \mathscr{U}'_1)H = \gamma H, \qquad \{\mathscr{R}_1\mathscr{R}'_1(\mathscr{U}'_2 - \delta) - \mathscr{R}_2\mathscr{R}'_2\mathscr{U}'_2\}H = 0,\\
&\mathscr{U}_kH = b_kH, \qquad 3 \leqq k \leqq q, \qquad \mathscr{U}'_kH = b'_kH, \qquad 3 \leqq k \leqq q',\\
&\mathscr{C}^{m,n}_{p,q}H = \mathscr{C}'^{m',n'}_{p',q'}H = 0.
\end{aligned}
\tag{2.23}
$$

For convenience, we set $p' = q' = 2$, $m' = n' = 1$.

It is straightforward to derive the solution

$$
\begin{aligned}
H &= B^{m,n}_{p,q}\left(s \,\middle|\, \begin{matrix} a_p \\ \beta+\gamma, b_3, \cdots, b_q, \alpha-\beta-\gamma \end{matrix}\right) B^{1,1}_{2,2}\left(s' \,\middle|\, \begin{matrix} a'_1 a'_2 \\ \beta, 0 \end{matrix}\right) \\
&\quad \cdot {}_3F_2\left(\begin{matrix} \beta+\gamma-s, \beta-s', \delta \\ s-\alpha+\beta+\gamma+1, s'+1 \end{matrix} \,\middle|\, \tau\right) u_1^{\beta+\gamma} u_1'^{\beta} u_2^{\alpha-\beta-\gamma} \\
&= \sum_{h=0}^{\infty} \frac{\tau^h}{h!} \frac{\Gamma(\delta+h)}{\Gamma(\delta)} B^{m,n}_{p,q}\left(s \,\middle|\, \begin{matrix} a_p \\ \beta+\gamma+h, b_3, \cdots, b_q, \alpha-\beta-\gamma-h \end{matrix}\right) \\
&\quad \cdot B^{1,1}_{2,2}\left(s' \,\middle|\, \begin{matrix} a'_1, a'_2 \\ \beta+h, -h \end{matrix}\right) u_1^{\beta+\gamma} u_1'^{\beta} u_2^{\alpha-\beta-\gamma},
\end{aligned} \tag{2.24}
$$

where $\tau = (u_1 u'_1)/(u_2 u'_2)$. The problem is now to take the double inverse Mellin transform of this expression. In the special case $\tau = 1$, $\delta = \beta$, $\alpha = \beta + \gamma$ the ${}_3F_2$ is well-poised and we can apply Dixon's theorem [5, p. 104] to express the ${}_3F_2$ in terms of gamma functions if $\mathrm{Re}(2s + 2s' - 2\gamma - 3\beta) > -2$. Furthermore, if $a'_2 = \beta$, $a'_1 = \beta/2$ we can easily compute the required inverse Mellin transforms and obtain after simplication and setting $\gamma = 0$,

$$
\begin{aligned}
G^{m,n}_{p,q}\left(\omega z \,\middle|\, \begin{matrix} a_p \\ 0, b_3, \cdots, b_q, -\beta/2 \end{matrix}\right) &= \sum_{h=0}^{\infty} \frac{\Gamma(\beta+h)}{h!} \frac{(\beta+2h)}{\Gamma(1+\beta/2)} \\
&\quad \cdot G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_p \\ h, b_3, \cdots, b_q, -h-\beta \end{matrix}\right) \\
&\quad \cdot {}_2F_1\left(\begin{matrix} -h, \beta+h \\ 1+\beta/2 \end{matrix} \,\middle|\, \omega\right).
\end{aligned} \tag{2.25}
$$

After the usual augmentation of indices (or equivalently, after solving equations (2.23) for more general p', q') we arrive at a result identical to the identity of Luke [6, p. 15]. For $\gamma \neq 0$ we get an extension of this identity.

Finally we note that the symbolic method of Burchnall and Chaundy [11], [12] has much in common with our approach although these authors did not use Lie algebraic techniques.

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ON A QUADRATIC EIGENVALUE PROBLEM*

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Abstract. It is shown that the eigenvalue problem $u'' + Bu = (\lambda^2 + \lambda p)u$; $u(0) = u(1) = 0$ (where p is a positive function and B an arbitrary bounded operator on $L^2[0, 1]$) possesses in general two different sets of eigenfunctions, each of which is an unconditional basis for L^2 and other spaces. The method of proof, which is applicable to more general problems, uses the ordinary quadratic formula to find a factored operator-polynomial which is "close" to the original problem; perturbation techniques are then applied to derive the desired spectral information.

Introduction. The eigenvalue problem

$$u'' = (\lambda^2 + \lambda p)u; \qquad u(0) = u(1) = 0 \tag{1}$$

(where p is a positive function on $[0, 1]$) has been studied by Deavours [1] in conjunction with a damped-wave problem. Using complex variable techniques he found asymptotic expressions for the eigenvalues and eigenfunctions and proved an expansion theorem. In this paper we shall consider a somewhat more general problem than (1), using operator-theoretic methods to obtain sharpened expansion theorems, and to bring out a property of these problems which is not readily apparent from Deavours' analysis, namely that there are in general two different complete sets of eigenfunctions, each corresponding to a particular subset of the eigenvalues. The method used here involves finding an "approximate factorization" of the quadratic operator-polynomial, and using perturbation techniques due originally to Schwartz [6]. We remark that if the operator A below is replaced by an arbitrary second order differential operator L (possibly with different boundary conditions) satisfying

(i) L is spectral,

(ii) the eigenvalues of L lie in the left half-plane, have bounded imaginary parts, and only finitely many have multidimensional eigenspaces,

then the analysis given here is changed only in superficial details (in particular it may be necessary to assume boundary conditions on p sufficient to guarantee that multiplication by p preserves $\mathscr{D}(L)$). We treat here only the restricted problem (3) in order to make the exposition more transparent.

1. Preliminaries. Let L^2 be $L^2([0, 1])$. Let H^n be the space of functions on $[0, 1]$ having $n - 1$ absolutely continuous derivatives and n L^2-derivatives, with the usual inner product [2, p. 1296]. Let H_0^n be the subspace of H^n consisting of those functions which vanish, together with their first $n - 1$ derivatives, at the endpoints of the interval. Let l_+^2 and l^2 be the spaces of square-summable sequences on index sets, respectively the natural numbers and the integers. We shall identify l^2 with $l_+^2 \oplus C \oplus l_+^2$, where C denotes a one-dimensional space. Let $\mathscr{D}(A) = H^2 \cap H_0^1$, and if $f \in \mathscr{D}(A)$, let $Af = f''$. The operator A is negative definite and self-adjoint. Let $p \in C^2([0, 1])$, $p \geqq 0$. The problem (1) can be rephrased as

$$Au = (\lambda^2 + \lambda p)u. \tag{2}$$

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If B is an arbitrary, but fixed, bounded operator on L^2, let $A_1 = A + B$. We then have the more general problem

$$A_1 u = (\lambda^2 + \lambda p)u. \tag{3}$$

We use the symbol "p" to denote either the function p or the corresponding multiplication operator on L^2. Note that this operator takes H^2, H^1, and H_0^1 into themselves. If A_1 is not invertible, choose $\lambda_0 < 0$ so that $\lambda_0^2 > \|B\| + |\lambda_0|\,\|p\|$. Then $A_1 - \lambda_0^2 + \lambda_0 p$ is invertible, and $A_1 u = (\lambda^2 + \lambda p)u$ if and only if

$$\begin{aligned}(A_1 - \lambda_0^2 + \lambda_0 p)u &= (\lambda^2 - \lambda p)u - (\lambda_0^2 - \lambda_0 p)u \\ &= [(\lambda + \lambda_0)^2 + (\lambda + \lambda_0)(p - 2\lambda_0)]u.\end{aligned} \tag{4}$$

So letting

$$\begin{aligned}\tilde{A}_1 &= A_1 - \lambda_0^2 + \lambda_0 p, \\ \tilde{\lambda} &= \lambda + \lambda_0, \\ \tilde{p} &= p - 2\lambda_0,\end{aligned}$$

the problem (3) is equivalent to

$$\tilde{A}_1 u = [\tilde{\lambda}^2 + \tilde{\lambda}\tilde{p}]u.$$

It follows that we may assume A_1 is invertible. With the convention that the square-root function takes the negative real axis to the positive imaginary axis, we define the following operators:

$$\begin{aligned}T &= A^{1/2}, \\ S &= (p^2 + 4A)^{1/2}, \\ Q &= Sp - pS, \\ K^{\pm} &= (1/2)(-p \pm S), \\ P(\lambda) &= \lambda^2 I + \lambda p - A, \\ P_1(\lambda) &= \lambda^2 I + \lambda p - A_1.\end{aligned}$$

The domains of T, S, Q, and $K^{\pm}$ are H_0^1, and those of $P(\lambda)$ and $P_1(\lambda)$ are $\mathscr{D}(A)$. We first show that the operator Q in fact extends to a bounded operator on L^2.

LEMMA 1. *Let V be an unbounded spectral operator with compact inverse, all but finitely many of whose eigenvalues have one-dimensional eigenspaces, and whose numerical range is contained in a sector of angle less than π. Let C be a bounded operator such that $(VC - CV)V^{-1/2}$ and $V^{-1/2}(VC - CV)$ are both bounded operators. Then $V^{1/2}C - CV^{1/2}$ is a bounded operator.*

Proof. We may assume without loss of generality that all eigenvalues $\{\lambda_n\}$ of V have one-dimensional eigenspaces and that the sector has vertex at the origin and is centered on the positive axis. With respect to the basis $\{\phi_n\}$ of eigenvectors of V, the operator $VC - CV$ has the matrix whose entries are:

$$L_{mn} = (\bar{\lambda}_m - \lambda_n)(C\phi_n, \phi_m),$$

and $V^{1/2}C - CV^{1/2}$ has

$$K_{mn} = ((\bar{\lambda}_m)^{1/2} - \lambda_n^{1/2})(C\phi_n, \phi_m).$$

By a standard interpolation theorem [7, Lemma 3], the operator $V^{-1/4}(VC - CV)V^{-1/4}$ is bounded, so the matrix whose entries are

$$M_{mn} = (\bar{\lambda}_m - \lambda_n)\lambda_n^{-1/4}(\bar{\lambda}_m)^{-1/4}(C\phi_n, \phi_m)$$

corresponds to a bounded operator. Now let $\mu_n = \lambda_n^{1/2}$. Then

$$K_{mn} = (\bar{\mu}_m - \mu_n)(C\phi_n, \phi_m) = \mu_n^{1/2}(\bar{\mu}_m)^{1/2}(\mu_n + \bar{\mu}_m)^{-1}M_{mn}.$$

Let $g_n(x) = \mu_n^{1/2}x^{\mu_n}$, and $h_m(x) = \mu_m^{1/2}x^{\mu_m - 1}$. Then

$$(g_n, h_m) = \mu_n^{1/2}(\bar{\mu}_m)^{1/2}\int_0^1 x^{\mu_n + \bar{\mu}_m - 1}\,dx = \mu_n^{1/2}(\bar{\mu}_m)^{1/2}(\mu_n + \bar{\mu}_m)^{-1}.$$

We have

$$\|g_n\| = |\mu_n|^{1/2}\left|\frac{1}{2\operatorname{Re}\mu_n + 1}\right|^{1/2},$$

$$\|h_m\| = |\mu_m|^{1/2}\left|\frac{1}{2\operatorname{Re}\mu_m - 1}\right|^{1/2},$$

both of which are bounded independently of m and n, and so by a theorem of Schur [3, p. 222, #300] the matrix K_{mn} corresponds to a bounded operator.

COROLLARY. *The operator Q is bounded on L^2.*

Proof. If $f \in \mathscr{D}(A)$, then $(Ap - pA)f = 2p'f' + p''f$. Since $S^2 = 4A + p^2$, we have $S^2p - pS^2 = 4[Ap - pA]$. If S is invertible, it follows from the closed graph theorem and [2, p. 1296] that $S^{-1}[Ap - pA]$ and $[Ap - pA]S^{-1}$ are bounded, and from Lemma 1 that $Sp - pS$ is bounded. If S is not invertible, then 0 is an eigenvalue; denoting by E the orthogonal projection onto the null space of S, we apply the above argument to $S + \varepsilon E$, then take limits as $\varepsilon \to 0$.

Let ρ be the set of all complex λ such that $P_1(\lambda)$ has a bounded inverse. Then ρ is open and nonempty, $P_1(\lambda)$ is analytic on ρ, and the complement of ρ consists of the eigenvalues of (3). It is easy to see that the eigenvalues of (2) all have one-dimensional eigenspaces. If f_n is a normalized eigenfunction of (2) belonging to λ_n, then

$$\lambda_n^2 + \lambda_n(pf_n, f_n) - (Af_n, f_n) = 0,$$

or

$$\lambda_n = (1/2)[-(pf_n, f_n) \pm ((pf_n, f_n)^2 + 4(Af_n, f_n))^{1/2}]. \tag{5}$$

It follows that if λ_n is real, then $-\max p \leqq \lambda_n \leqq 0$, and if λ_n is not real, then $-1/2 \max p \leqq \operatorname{Re}\lambda_n \leqq 0$. By similar reasoning the numerical range of K^+ lies in the half-strip

$$S = \{\operatorname{Im} z \geqq 0; -\max p \leqq \operatorname{Re} z \leqq 0\}.$$

LEMMA 2. *The operators $K^{\pm}$ are spectral.*

Proof. We remark first that if $\max p$ is sufficiently small, the lemma follows from a result of Turner [8]. For general p, we proceed as follows: let $\{\alpha_n\}$ and $\{h_n\}$ denote the eigenvalues and eigenfunctions of S. It follows from known results [4, Chap. II] and [6], that α_n has the asymptotic expression

$$\alpha_n = 2\pi ni + \alpha + O(n^{-1}), \quad \text{where } \alpha \text{ is constant},$$

and the functions $\{h_n\}$ can be chosen so that $\|h_n - \phi_n\| = O(n^{-1})$, where $\phi_n(x) = \sin n\pi x$. Let $h_n = \phi_n + n^{-1}g_n$; in the basis h_n, $S - p$ has the matrix whose entries are (letting $\beta_n = O(n^{-1})$)

$$(6)\qquad (2\pi ni + \alpha + \beta_n)\delta_{nm} - (\rho\phi_n, \phi_m) - n^{-1}(pg_n, \phi_m) - m^{-1}(p\phi_n, g_m) - n^{-1}m^{-1}(pg_n, g_m).$$

Now if $\psi_n(x) = \cos n\pi x$, then

$$(7)\qquad (p\phi_n, \phi_m) = (1/2)(p, \psi_{n-m}) + (2(n+m)^2\pi^2)^{-1}[a_{n+m} + (p'', \psi_{n+m})],$$

where $a_n = (-1)^k p'(1) - p'(0)$. The second term clearly represents the matrix of a Hilbert–Schmidt operator, so we find from (6) and (7) that $S - p$ differs by a Hilbert–Schmidt operator from the operator W whose matrix entries are $(2\pi ni + \alpha)\delta_{nm} - (1/2)(p, \psi_{n-m})$. Let $\sigma_k = (1/2)(p, \psi_k)$. Since $S - p + (\alpha + \sigma_0)I$ is spectral if and only if $S - p$ is, we may assume $\alpha = \sigma_0 = 0$.

Regarding W as an operator on l^2_+, let Y be the operator on $l^2 = l^2_+ \oplus C \oplus l^2_+$ defined by

$$Y = \begin{bmatrix} W^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W \end{bmatrix}.$$

Then Y differs by a Hilbert–Schmidt operator from the operator on l^2 represented by the matrix

$$(8)\qquad \operatorname{diag}\{2\pi ni\} - \sum_{n=1}^{\infty} \sigma_n(U^n + (U^*)^n),$$

where U is the bilateral right shift on l^2. Let $\omega_n(t) = e^{\pi int}$, for $t \in [-1, 1]$, n any integer. Then $\{\omega_n\}$ is an orthonormal basis of $L^2([-1, 1])$, and in this basis the matrix (8) represents the operator $Z = 2D + q$ (where D denotes differentiation), with boundary conditions $f(-1) = f(1)$, and where q is the function $q = \sum_{h=1}^{\infty} \sigma_n(\omega_n + \omega_{-n})$, which converges uniformly on $[-1, 1]$. This latter operator can be shown spectral by explicitly solving the corresponding differential equation. Its eigenvalues are $\{(1/2)\int_0^1 q(t)\,dt + 2\pi in\}$. Putting all the above together, we find that the operator

$$\begin{bmatrix} (S-p)^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S-p \end{bmatrix}$$

differs from Z by a compact operator, and so by the abovementioned theorem of Turner, it must be spectral with eigenvalues $\{\alpha_n\}$ satisfying $an \leqq |\alpha_n| \leqq bn$, for some constants a, b. It follows that $S - p = 2K^+$ has the same properties. (Note: in the case of a more general operator than A, the above proof becomes considerably more complicated, but the idea remains essentially the same.)

By well-known properties of spectral operators, we have

$$\|(\lambda - K^{\pm})^{-1}\| \leqq Md^{\pm}(\lambda)^{-1},$$

where M is constant and $d^{\pm}(\lambda)$ is the distance from λ to the spectrum of $K^{\pm}$, provided that λ stays a fixed distance away from all multiple poles of $(\lambda - K^{\pm})^{-1}$. This means, for example, that if λ is in the half-strip S, then $\|(\lambda - K^{-1})^{-1}\|$ is bounded by a multiple of $|\lambda|^{-1}$.

2. Perturbation and expansion theorems. Let $R_\lambda^{\pm} = (\lambda - K^{\pm})^{-1}$. We have the identities

$$(\lambda - K^-)(\lambda - K^+) = \lambda^2 + \lambda p - A + (1/4)Q, \tag{9}$$

$$P_1(\lambda) = (\lambda - K^-)(\lambda - K^+) - F, \quad \text{where } F = (1/4)Q - B, \tag{10}$$

$$P_1(\lambda)^{-1} = (\lambda - R_\lambda^- F - K^+)^{-1} R_\lambda^- = (I - R_\lambda^+ R_\lambda^- F)^{-1} R_\lambda^+ R_\lambda^- . \tag{11}$$

Let $\{\mu_n\}$ be the eigenvalues of K^+, numbered by increasing magnitude. The sequence satisfies $\alpha n \leqq |\mu_n| \leqq \beta n$, for some constants α, β. Let γ_n be a circle around μ_n, of radius r, chosen so that none of the circles intersect. For $\mu \in S$, let $G(\mu) = (K^+ + R_\mu^- F)$. Then

$$\begin{aligned}(\lambda - G(\mu))^{-1} &= (I - R_\lambda^+ R_\mu^- F)^{-1} R_\lambda^+ \\ &= R_\lambda^+ + (I - R_\lambda^+ R_\mu^- F)^{-1} R_\lambda^+ R_\mu^- F R_\lambda^+ .\end{aligned} \tag{12}$$

Since $\|R_\mu^-\| = O(|\mu|^{-1})$, it follows that if μ is inside γ_n, and n is large, we have

$$\frac{1}{2\pi}\left|\int_{\gamma_n} [(\lambda - G(\mu))^{-1} - (\lambda - K^+)^{-1}]\, d\lambda\right| = O(n^{-1}). \tag{13}$$

The constant in (13) may be taken independent of μ. Now two projections differing in norm by less than 1 have the same rank, so if n is sufficiently large, then for every μ inside or on γ_n there is a single simple eigenvalue of $G(\mu)$ lying inside or on γ_n; denote this eigenvalue by $\lambda(\mu)$. It is not hard to show that $\lambda(\mu)$ is an analytic function in the interior of γ_n, and continuous on the closed disk bounded by γ_n. We have $\|R_\lambda^+\| \leqq M|\lambda - \mu_n|^{-1}$, for λ inside γ_n, with M independent of n. If n is so large that $\|R_\mu^- F\| \leqq r/(2M)$ for all μ inside γ_n, then $\|R_\lambda^+ R_\mu^- F\| < 1$ whenever $|\lambda - \mu_n| > r/2$. From (12) it follows that the annulus $\{r/2 \leqq |\lambda - \mu_n| \leqq r\}$ is in the resolvent set of $G(\mu)$, and hence that $|\lambda(\mu) - \mu_n| \leqq r/2$, for all μ inside γ_n. Thus, if $\mu \in \gamma_n$,

$$|(\lambda(\mu) - \mu) - (\mu_n - \mu)| = |\lambda(\mu) - \mu_n| \leqq r/2 < r = |\mu_n - \mu| .$$

By Rouche's theorem the function $\lambda(\mu)$ has exactly one fixed point inside γ_n, which we denote by λ_n. Let f_n be the corresponding eigenvector of $G(\lambda_n)$. Since $\mathscr{R}(R_\mu^- F) \subseteq \mathscr{D}(A^{1/2})$, we have $\mathscr{D}(G(\mu)^2) = \mathscr{D}((K^+)^2) = \mathscr{D}(A)$, and hence that $f_n \in \mathscr{D}(A)$. From this follows $0 = (\lambda_n - K^-)(\lambda_n - G(\lambda_n))f_n = P_1(\lambda_n)f_n$, and so λ_n and f_n are respectively an eigenvalue and an eigenvector of (3). Now letting $\mu = \lambda$ in (12) we see that for $|\lambda|$ sufficiently large and λ lying outside the circles γ_n, the operator $(\lambda - G(\lambda))^{-1}$ exists, and hence $P_1(\lambda)^{-1} = (\lambda - G(\lambda))^{-1} R_\lambda^-$ exists. It follows that all eigenvalues of (3) in the upper half-plane lie either inside one of the $\{\gamma_n\}$ or inside a certain circle centered at the origin. The discreteness of the eigenvalues [9, p. 371] implies that only finitely many can lie in this latter circle. Summarizing the above, we have the following theorem.

THEOREM 1. *For all sufficiently large n, the circle γ_n encloses exactly one eigenvalue λ_n of* (3), *which has a one-dimensional eigenspace. Only finitely many* *eigenvalues of* (3) *in the upper half-plane lie outside the circles* $\{\gamma_n\}$. *We can choose normalized eigenvectors* f_n *of* (3), *belonging to* λ_n, *and* g_n *of* K^+, *belonging to* μ_n, *in such a way that* $\|f_n - g_n\| = O(n^{-1})$.

The last assertion of the theorem follows from (13). The same result holds, of course, if the roles of K^+ and K^- are reversed, thus locating all but finitely many of the eigenvalues of (3). We remark also that a slight refinement of the argument above shows that in fact $|\lambda_n - \mu_n| = O(n^{-1})$.

By a chain of generalized eigenvectors of (3) is meant a sequence $\{q_1, \cdots, q_n\}$ such that, for some λ,

$$P_1(\lambda)q_1 = 0,$$

$$P_1(\lambda)q_2 + P_1'(\lambda)q_1 = 0,$$

$$P_1(\lambda)q_k + P_1'(\lambda)q_{k-1} + (1/2)P_1''(\lambda)q_{k-2} = 0 \quad \text{for } k = 3, 4, \cdots, n.$$

A straightforward calculation shows that such a chain is the image under the map E of a chain of generalized eigenvectors of W, where E and W are the maps defined in the proof of Theorem 2 below. Henceforth when we refer to "the set of eigenvectors" it should be understood that finitely many generalized eigenvectors may also be included.

THEOREM 2. *The set of all eigenvectors of* (3) *generates the entire space* L^2.

Proof. (Outline): Let $V = A_1^{1/2}$ and let W be the operator on $L^2 \oplus L^2$ defined by

$$\begin{bmatrix} -p & V \\ V & 0 \end{bmatrix}$$

with domain $\mathscr{D}(T) \oplus \mathscr{D}(T)$. If $z \in L^2 \oplus L^2$, $z = \begin{bmatrix} f \\ g \end{bmatrix}$, let $Ez = f$. It is easy to see that if z is an eigenvector of W, then Ez is an eigenvector of (3); it thus suffices to show that the eigenvectors of W generate $L^2 \oplus L^2$. Define

$$Z(\lambda) = \begin{bmatrix} -P_1(\lambda)^{-1} & -P_1(\lambda)^{-1}V \\ -P_1(\lambda)^{-1}V & -P_1(\lambda)^{-1}(p + \lambda) \end{bmatrix}.$$

Then $Z(\lambda)$ is a Hilbert–Schmidt operator, since both $P_1(\lambda)^{-1}$ and $P_1(\lambda)^{-1}V$ are Hilbert–Schmidt.

$$Z(\lambda)(W - I) = I + \begin{bmatrix} 0 & 0 \\ P_1(\lambda)^{-1}(Vp - pV) & 0 \end{bmatrix} = I + Y(\lambda).$$

Using (11), and Lemma 1, one shows that $\|P_1(\lambda)^{-1}\| = O(d(\lambda)^{-2})$, that $\|P_1(\lambda)^{-}V\| = O(d(\lambda)^{-1})$, and $\|Y(\lambda)\| = O(d(\lambda)^{-2})$, where $d(\lambda)$ is the distance from λ to the strip $\{-\max p \leqq \operatorname{Re} z \leqq 0\}$. Thus $(W - \lambda)^{-1} = (I + Y(\lambda))^{-1}Z(\lambda)$, and hence $\|(W - \lambda)^{-1}\| = O(\|Z(z)\|)$ as $|\lambda| \to \infty$. It also follows that $(W - \lambda)^{-1}$ is Hilbert–Schmidt. Thus along the rays from the origin at angles $\{0, (2\pi)/5, (4\pi)/5, (6\pi)/5, (8\pi)/5\}$, $\|Z(\lambda)\| = O(d(\lambda)^{-1}) = O(|\lambda|^{-1})$. The conclusion of the theorem now follows from [2, p. 1042].

The eigenvalues of (3) in the upper half-plane are called upper eigenvalues. Those in the lower half-plane are lower eigenvalues; real eigenvalues are both upper and lower.

The eigenvectors $\{g_n\}$ of K^+ (together with at most finitely many generalized eigenvectors) form a basis (i.e., a bicontinuous image of an orthonormal basis) for L^2. From the last assertion of Theorem 1 it follows that there is an N such that

$$\sum_N^\infty \|f_n - g_n\|^2 = \theta_N < 1.$$

The spectral projections of K^+ onto $\{g_n\}$ have the form $E_n f = (f, h_n)g_n$; where h_n is a sequence of elements such that $(g_n, h_m) = \delta_{nm}$. The following is a modification of an argument of Sz.-Nagy [5, p. 208].

Define the operator K_N by $K_N f = \sum_N^\infty (f, h_n)(f_n - g_n)$. It is simple to see that K_N is bounded by a multiple of θ_N, and so $I + K_N$ is invertible; if $n \geqq N$, then $(K_N + I)g_n = f_n$. Thus $\overline{sp}\{f_n | n \geqq N\} = H_N$ has codimension $N - 1$, and so by Theorem 2 there are linearly independent eigenvectors $\{f_1, \cdots, f_{N-1}\}$ with $sp\{f_1, \cdots, f_{N-1}\} \cap H_N = \{0\}$, such that the system $\{f_n\}$ spans L^2. If we define

$$K_0 g_n = \begin{cases} f_n - g_n & \text{for } n < N, \\ 0 & \text{for } n \geqq N, \end{cases}$$

then K_0 has finite rank, and thus $I + K_N + K_0$ is invertible if and only if it is one-to-one. But if $(I + K_N + K_0)f = 0$, then $\sum_1^N (f, h_n) f_n \in H_N$, so $f = 0$. Thus $I + K_N + K_0$ is bicontinuous and takes g_n to f_n, so $\{f_n\}$ forms a basis. We have proved the following theorem.

THEOREM 3. *The eigenvalues of* (3) *can be divided into two sets* $\mathscr{A}$ and $\mathscr{B}$, *whose intersection is at most finite, with* $\mathscr{A}$ (*respectively*, $\mathscr{B}$) *containing at most finitely many lower* (*resp. upper*) *eigenvalues, and such that the eigenvectors corresponding to elements of* $\mathscr{A}$ (*resp.* $\mathscr{B}$) *form an unconditional basis of* L^2.

We thus have two "complete sets of eigenvectors," which coincide in the trivial case that p is identically constant.

Corresponding to either of the sets of eigenvectors given by Theorem 3, there is, for each $f \in L^2$, an eigenvector series which converges to f in the topology of L^2. With further restrictions on f we can strengthen the convergence: if $\{f_n\}$ is as above, then

$$A_1 f_n = \lambda_n^2 (f_n + \lambda_n^{-1} p f_n) = \lambda_n^2 k_n.$$

LEMMA 3. *The vectors* $\{k_n\}$ *form a basis.*

Proof. $\|f_n - k_n\| = \|\lambda_n^{-1} p f_n\| = O(n^{-1})$. Thus $\sum_1^\infty \|f_n - k_n\|^2 < \infty$. It follows just as in the proof of Theorem 3 that $\{k_n\}$ will be a basis if we can show that for all large N,

(a) $\{k_1, \cdots, k_N\}$ are linearly independent;

(b) $sp\{k_1, \cdots, k_N\} \cap \overline{sp}\{k_n | n > N\} = \{0\}$.

To show (a), we note that $\{k_1, \cdots, k_n\}$ are the images under a one-to-one map (namely A_1) of $\{\lambda_1^{-2} f_1, \cdots, \lambda_N^{-2} f_N\}$, which are linearly independent. To show (b), suppose

$$f \in sp\{k_1, \cdots, k_N\} \cap \overline{sp}\{k_{N+1}, k_{N+2}, \cdots\}.$$

Then $A_1^{-1}f \in sp\{f_1, \cdots, f_N\} \cap \overline{sp}\{f_{N+1}, f_{N+2}, \cdots\}$. So $A_1^{-1}f = 0$, and hence $f = 0$.

LEMMA 4. $\mathscr{D}(A_1) = \mathscr{D}(A) = \{f = \sum_1^\infty \alpha_n f_n | \{\alpha_n \lambda_n^2\} \in l_+^2\}$.

Proof. Let $f \in \mathscr{D}(A_1)$, $A_1 f = g$. Then $g = \sum \beta_n k_n$, for some $\{\beta_n\} \in l_+^2$, by Lemma 3. Now $A_1^{-1}g = \sum \beta_n \lambda_n^{-2} f_n$; so let $\alpha_n = \lambda_n^{-2}\beta_n$. Conversely, if $f = \sum \alpha_n f_n$, with $\{\lambda_n^2 \alpha_n\} \in l_+^2$, then $g = \sum \lambda_n^2 \alpha_n k_n$ converges. Since A_1 is closed it follows that $f \in \mathscr{D}(A_1)$ and $A_1 f = g$.

THEOREM 4. *If $f \in \mathscr{D}(A)$, then the eigenvector series for f converges to f in the topology of $\mathscr{D}(A)$.*

Proof. Suppose $f = \sum \alpha_n f_n$. Then $A_1 f = \sum \lambda_n^2 \alpha_n k_n$, which converges in L^2. So we have

$$\sum_1^N \alpha_n f_n \to f \quad \text{as } N \to \infty$$

and

$$A_1 \sum_1^N \alpha_n f_n \to A_1 f \quad \text{as } N \to \infty$$

which implies convergence in the topology of $\mathscr{D}(A)$.

Since $\mathscr{D}(A_1) = H^2 \cap H_0^1$, Theorem 4 says that if $f \in \mathscr{D}(A)$, then its eigenvector series converges to f uniformly, with uniform convergence of derivatives and L^2-convergence of second derivatives. All the foregoing, of course, applies to both of the eigenvector bases described by Theorem 3. By interpolation one can show that Theorem 4 remains true if $\mathscr{D}(A)$ is replaced by $\mathscr{D}(A^\theta)$, with $0 \leqq \theta \leqq 1$, which implies, for example, that if $f \in H_0^1$, the series converges to f uniformly, with L^2-convergence of derivatives.

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LINEAR EVOLUTION EQUATIONS THAT INVOLVE PRODUCTS OF COMMUTATIVE OPERATORS*

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Abstract. Let $P(\xi_1, \cdots, \xi_n)$ be a polynomial of degree $2k$ in $\xi_1, \cdots, \xi_n$ with real constant coefficients. Let X be a function space and let $A_1, \cdots, A_n$ denote operators in X such that $A_i \cdot A_j = A_j \cdot A_i$ for all i, j. This paper treats the problem of representing solutions of the Cauchy problem $u'(t) = P(A_1, \cdots, A_n)u(t)$, $t > 0$; $u(0+) = \varphi$ in the following situations: (i) X is a Banach space and the A_i are infinitesimal generators of C_0 groups in X and (ii) X is a space of entire functions and the A_i are derivative operators in X. The results are motivated by elementary operational formulas and applications are given to both well-posed and ill-posed problems.

1. Introduction. Let X be some function space and let $A_1, A_2, \cdots, A_n$ denote operators in X with $A_i \cdot A_j = A_j \cdot A_i$ for all i, j. Let $P(\xi_1, \xi_2, \cdots, \xi_n)$ denote a polynomial in $\xi_1, \cdots, \xi_n$ with real coefficients. We shall be concerned with abstract Cauchy problems of the form

$$u'(t) = P(A_1, A_2, \cdots, A_n)u(t), \qquad t > 0, \tag{1.1a}$$

$$\lim_{t \to 0+} u(t) = \varphi \quad \text{(in some sense)}, \tag{1.1b}$$

where φ is restricted to an appropriate subspace of X.

If X is a Banach space, the A_i are generators of C_0 groups $T_{A_i}(t)$ in X, and $P(A_1, \cdots, A_n)$ is the infinitesimal generator of a C_0 semigroup in X, then (1.1) is well-posed if $\varphi \in \mathscr{D}(P)$, the domain of P. The limit condition in (1.1b) can then be taken in the strong sense and the Cauchy problem (1.1) has the solution $u(t) = U_P(t) \cdot \varphi$ where $U_P(t)$ is the semigroup generated by P. From this $u(t)$, one can then construct solutions for other types of abstract Cauchy or Dirichlet problems involving the operator P by the application of integral transformations arising in related equations ([1]–[4], [8]). In certain other function spaces, there are important examples of ill-posed problems of the form (1.1) that have solutions for restricted choices of φ. Suppose we denote a solution operator of (1.1) by e^{tP} for either the well-posed or else the solvable but ill-posed problem (1.1) (and then make the identification $e^{tP} = U_P(t)$ if P is a semigroup generator in a Banach space). Of interest in this paper is a method for constructing solutions of (1.1) based upon or motivated by properties of groups generated by the corresponding A_i in the Banach space case. A similar question has been considered by R. Hersh [10] for a problem involving a more general equation than that in (1.1). However, his treatment is restricted to well-posed problems.

The suggestive basis for our method is the operational formula

$$\exp[tP(a_1, \cdots, a_n)] = \exp[tP(D_1, \cdots, D_n)] \cdot \exp\left(\sum_{i=1}^{n} a_i \xi_i\right)\Bigg|_{\xi_1 = \cdots = \xi_n = 0}, \tag{1.2}$$

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where $D_i = \partial/\partial\xi_i$ and in which the a_i are constants. An alternative form of this can also be given. Suppose that

$$b_i = \sum_{j=1}^{n} \alpha_{ij} \cdot a_j, \qquad i = 1, \cdots, n, \tag{1.3}$$

where the α_{ij} denote real or complex scalars, and suppose that $Q(b_1, \cdots, b_n) \equiv P(a_1, \cdots, a_n)$. Then

$$\begin{aligned} &\exp[tP(a_1, \cdots, a_n)] \\ &\quad = \exp[tQ(D_1, \cdots, D_n)] \exp\left\{ \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} \alpha_{ij} a_j \right) \right\} \Bigg|_{\xi_1 = \cdots = \xi_n = 0}. \end{aligned} \tag{1.4}$$

The idea will be to show that if either $P(D_1, \cdots, D_n)$ or $Q(D_1, \cdots, D_n)$ is an elliptic operator of order $2k$ with appropriate principal part (see § 2), then the formula (1.2) or (1.4) can be given a meaningful interpretation if the constants a_i are replaced by operators. The formulas (1.3) and (1.4) are of special interest since they can be used to construct solutions of (1.1) particularly when $P(\xi_1, \cdots, \xi_n)$ is a homogeneous quadratic in the ξ_i. The operator $Q(D_1, \cdots, D_n)$ can then be taken to be the Laplacian.

In § 2, we show that if $P(D_1, \cdots, D_n)$ is such an elliptic operator, then (1.2) has the interpretation that the solution operator $\exp[tP(A_1, \cdots, A_n)]$, in a Banach space, is representable in terms of the groups $T_{A_i}(t)$ by means of an integral involving a Green's function. This leads to results on solutions which agree with those obtained by Hersh. If $P(D_1, \cdots, D_n)$ fails to be an appropriate type of operator, a problem of the form (1.1) may still have a solution. By replacing $P(D_1, \cdots, D_n)$ by an appropriate operator $Q(D_1, \cdots, D_n)$, certain of the α_{ij} in (1.3) are complex. We then consider the alternatives: (i) X is a complex Banach space and the A_i (or certain of them) are bounded operators in X or (ii) X is a space of entire functions and the A_i are derivative operators D_{x_i}. Finally, we use these ideas in § 4 to construct solutions to a number of problems of the form (1.1). Among the illustrations we include an application connected with the backward heat equation, an equation with variable coefficients, and the construction of a Green's function for representing e^{tP} by using a Green's function associated with a simpler operator Q.

2. Generalization of (1.2). Let $P(D_1, \cdots, D_n)$ be an elliptic operator of order $2k$ with real constant coefficients. Then P is of the *appropriate type* if its principal part (highest order term) has the form $(-1)^{k-1}P_0(D_1, \cdots, D_n)$ in which $P_0(\xi_1, \cdots, \xi_n)$ is a positive definite homogeneous polynomial of degree $2k$. In the following, let $\xi = (\xi_1, \cdots, \xi_n)$, $\sigma = (\sigma_1, \cdots, \sigma_n)$, $\xi - \sigma = (\xi_1 - \sigma_1, \cdots, \xi_n - \sigma_n)$, and $d\sigma = d\sigma_1 \cdots d\sigma_n$. By using Fourier methods, P. C. Rosenbloom [12] has shown that if P is such an operator and $f(\xi)$ is continuous with restricted growth, then the Cauchy problem

$$\begin{aligned} &v_t(\xi, t) = P(D_1, \cdots, D_n)v(\xi, t), \qquad t > 0, \\ &v(\xi, 0+) = f(\xi) \end{aligned} \tag{2.1}$$

has the solution

$$v(\xi, t) = \int_{E_n} K(\xi - \sigma, t) f(\sigma)\, d\sigma \tag{2.2}$$

in which K is a Green's function. Bounds on K are given by

$$|K(\xi, t)| \leqq M_1 t^{-n/k} \exp\{M_2 t - M_3 |||\xi|||^{2k/(2k-1)} t^{-(2k-1)}\} \tag{2.3}$$

with $t > 0$, $|||\xi|||^2 = \sum_{i=1}^{n} \xi_i^2$, and M_1, M_2, M_3 positive constants. Symbolically we can write (2.2) as

$$v(\xi, t) = \exp[tP(D_1, \cdots, D_n)] f(\xi). \tag{2.4}$$

Now, take X to be a Banach space, let the A_i, $i = 1, \cdots, n$, be infinitesimal generators of C_0 groups $T_{A_i}(t)$ in X, and let $\varphi \in \bigcap_{i=1}^{n} \mathscr{D}(A_i^{2k})$, where $\mathscr{D}(A_i^{2k})$ denotes the domain of A_i^{2k}. We have

$$\| T_{A_i}(t) \| \leqq N_i \exp(\omega_i |t|)$$

with $N_i \geqq 0$, $\omega_i \geqq 0$ for $-\infty < t < \infty$. Set $f(\xi) = T_{A_1}(\xi_1) \cdots T_{A_n}(\xi_n)\varphi$ (we are replacing $e^{a_i \xi_i}$ in (1.2) by $T_{A_i}(\xi_i)$). From our assumptions, f is continuous and

$$\| f(\xi) \| \leqq M_4 \exp\left(\sum_{i=1}^{n} \omega_i |\xi_i| \right) \| \varphi \|, \tag{2.5}$$

where $M_4 \geqq 0$. With this, we see by (2.4) and (2.2) that (1.2) has the formal generalization

$$\begin{aligned} u(t) &= \exp[tP(A_1, \cdots, A_n)]\varphi \\ &= \int_{E_n} K(-\sigma, t) T_{A_1}(\sigma_1) \cdots T_{A_n}(\sigma_n)\varphi\, d\sigma. \end{aligned} \tag{2.6}$$

In this, we take the integral to be defined in the strong Riemann sense (see [7, Appendix]). This gives the following theorem.

THEOREM 2.1. *Let X be a Banach space and let $A_1, A_2, \cdots, A_n$ be infinitesimal generators of C_0 groups in X such that $A_i \cdot A_j = A_j \cdot A_i$ for all i, j. If $P(D_1, \cdots, D_n)$ is an elliptic operator of order $2k$ and is of appropriate type and $\varphi \in \bigcap_{i=1}^{n} \mathscr{D}(A_i^{2k})$, then the integral in* (2.6) *gives a solution to the abstract Cauchy problem* (1.1).

Proof. The existence of the integral in (2.6) follows by using the estimates (2.3) and (2.5). That (2.6) satisfies the equation in (1.1) can be easily checked by using an argument similar to the one used by Hersh [10]. We follow a slightly different approach in checking condition (1.1b). Set $h(\sigma) = \| T_{A_1}(\sigma_1) \cdots T_{A_n}(\sigma_n)\varphi - \varphi \|$. Then

$$\| u(t) - \varphi \| \leqq \int_{E_n} K(\xi - \sigma, t) h(\sigma)\, d\sigma |_{\xi_1 = \cdots = \xi_n = 0}. \tag{2.7}$$

But as $t \to 0$,

$$\int_{E_n} K(\xi - \sigma, t) h(\sigma)\, d\sigma \to h(\xi)$$

since $h(\xi)$ is continuous. Finally, since $h(0) = 0$, we see that $\| u(t) - \varphi \| \xrightarrow[t \to 0+]{} 0$ in (2.7).

A comparison of Theorem 2.1 with Theorem 2 of [10] shows that our result (2.6) agrees with Hersh's formula (1) when his equation takes the form (1.1a). We note that if $n = 1$, $A_1 = A$, and $P(D_1) = D_1^2$, then (2.6) with K the heat kernel gives the basic result

$$e^{tA^2}\varphi = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-\sigma^2/4t} T_A(\sigma)\varphi \, d\sigma. \tag{2.8}$$

This has been used to relate solutions of abstract wave problems with solutions of abstract heat problems [2]–[4]. It was developed probabilistically in [9].

We observe that if the coefficients α_{ij} in the transformations (1.3) are real and $P(D_1, \cdots, D_n)$ is elliptic of appropriate type, then the operator $Q(D_1, \cdots, D_n)$ is also elliptic and of the appropriate type. The relation (1.4) can then be generalized to give

$$\begin{aligned} &\exp\,[tP(A_1, \cdots, A_n)]\varphi \\ &\quad = \int_{E_n} K^*(-\sigma, t) T_{A_1}\left(\sum_{i=1}^{n} \alpha_{i1}\sigma_i\right) \cdots T_{A_n}\left(\sum_{i=1}^{n} \alpha_{in}\sigma_i\right)\varphi \, d\sigma \end{aligned} \tag{2.9}$$

in which K^* is the Green's function associated with a problem (1.1) involving the operator Q. Uniqueness shows that this agrees with the integral in (2.6). From a practical standpoint, it may be easier to construct K^* corresponding to Q than to construct K corresponding to P. We shall demonstrate that the complex analogue of the integral (2.9) satisfies (1.1a) in § 3.

3. Complex transformations. As was noted earlier, the operator $P(D_1, \cdots, D_n)$ may fail to be elliptic of appropriate type. The construction of a corresponding operator $Q(D_1, \cdots, D_n)$ by means of (1.3) and (1.4) then leads to certain of the α_{ij} being complex in (1.3). An examination of (2.9) shows that a number, perhaps all, of the group operators $T_{A_j}(\sum_{i=1}^n \alpha_{ij}\sigma_j)$ must then involve a complex or purely imaginary argument. This shows that if we wish to extend the applicability of (2.9), we must require the corresponding A_j to generate groups with analytic extensions to the entire complex plane. In general, such extensions are not possible if A_j is unbounded [11, p. 278]. The difficulty is tied to the problem of obtaining dense subspaces. However, we do have the following result.

THEOREM 3.1. *Let X be a complex Banach space and let $\mathscr{E}(X)$ be the space of bounded linear transformations on X. If $A_i \in \mathscr{E}(X)$, $i = 1, \cdots, n$, and $A_i \cdot A_j = A_j \cdot A_i$, then the integral in* (2.9) *defines a solution of* (1.1) *for $\varphi \in X$.*

Proof. The boundedness of the operators A_j clearly implies the existence of the integral in (2.9). We omit checking that this satisfies (1.1a) since a similar argument is needed in the proof of Theorem 3.2.

There is an important situation where we can formulate a meaningful result for unbounded operators. Let $\mathfrak{A}$ denote the set of entire functions φ of $(x_1, \cdots, x_n)$. For $z_1, \cdots, z_n$ complex variables, we assume the series representations

$$\varphi(z_1, \cdots, z_n) = \sum_{j_1, \cdots, j_n} b_{j_1 \cdots j_n} z_1^{j_1} \cdots z_n^{j_n}$$

satisfy the condition

$$\sum_{j_1,\cdots,j_n} |b_{j_1\cdots j_n}| \cdot |z_1|^{j_1} \cdots |z_n|^{j_n} \leqq M_5 \exp\{\lambda(\textstyle\sum|z_i|^2)^{\nu/2}\} \tag{3.1}$$

with $M_5 \geqq 0$, $\lambda \geqq 0$ and $0 \leqq \nu \leqq 2k/(2k-1)$. Now take $A_j = \partial/\partial x_j$ and interpret $T_{A_j}(z)$ in (2.9) as a translation operator. Then we have the following result.

THEOREM 3.2. *In* (1.1) *let* $A_i = \partial/\partial x_i$, $i = 1, \cdots, n$, *and let* $\varphi \in \mathfrak{A}$. *Then a solution of* (1.1) *is given by*

$$u(t) = \int_{E_n} K^*(-\sigma, t)\varphi\left(x_1 + \sum_{i=1}^n \alpha_{i1}\sigma_i, \cdots, x_n + \sum_{i=1}^n \alpha_{in}\sigma_i\right) d\sigma \tag{3.2}$$

at least in the interval $0 < t < (M_3/\lambda)^{1/(2k-1)}$. *If* $\nu > 2k/(2k-1)$, *then the integral* (3.2) *defines a solution of* (1.1) *for all* $t > 0$.

Proof. From the estimate (2.3) and the growth condition (3.1), it is clear that the integral in (3.2) converges for $0 < t < (M_3/\lambda)^{1/(2k-1)}$ if $\nu = 2k/(2k-1)$. If $\nu < 2k/(2k-1)$, then the integral exists for all $t > 0$. The condition (1.1b) can easily be checked as in the classical case (here, we replace $\|\cdot\|$ by absolute value). We need only show that (1.1a) is satisfied. Now

$$\begin{aligned} u_t(t) &= \int_{E_n} K_t^*(-\sigma, t)\varphi\left(x_1 + \sum_{i=1}^n \alpha_{i1}\sigma_i, \cdots, x_n + \sum_{i=1}^n \alpha_{in}\sigma_i\right) d\sigma \\ &= \int_{E_n} \{Q(-D_{\sigma_1}, \cdots, D_{\sigma_n})K^*(-\sigma, t)\} \\ &\quad \cdot \varphi\left(x_1 + \sum_{i=1}^n \alpha_{i1}\sigma_i, \cdots, x_n + \sum_{i=1}^n \alpha_{in}\sigma_i\right) d\sigma \\ &= \int_{E_n} K^*(-\sigma, t) \\ &\quad \cdot \left\{Q(D_{\sigma_1}, \cdots, D_{\sigma_n})\varphi\left(x_1 + \sum_{i=1}^n \alpha_{i1}\sigma_i, \cdots, x_n + \sum_{i=1}^n \alpha_{in}\sigma_i\right)\right\} d\sigma, \end{aligned} \tag{3.3}$$

the last step following by an integration of parts. But for each j,

$$D_{\sigma_j}\varphi = \left(\sum_{\mu=1}^n \alpha_{j\mu} D_{x_\mu}\right)\varphi.$$

Then according to (1.3), $Q(D_{\sigma_1}, \cdots, D_{\sigma_n})\varphi = P(D_{x_1}, \cdots, D_{x_n})\varphi$. Using this in the last member of (3.3), we get

$$\begin{aligned} u_t(t) &= \int_{E_n} K^*(-\sigma, t)P(D_{x_1}, \cdots, D_{x_n})\varphi\, d\sigma \\ &= P(D_{x_1}, \cdots, D_{x_n})\int_{E_n} K^*(-\sigma, t)\varphi\, d\sigma = P(D_{x_1}, \cdots, D_{x_n})u(t). \end{aligned}$$

Remark 1. Gel'fand and Shilov [15] have also obtained representations of solutions of ill-posed problems of the form $u_t(x, t) = P(iD_x)u(x, t)$, $u(x, 0+) = \varphi(x)$ in one space variable by Fourier methods (Theorem 3, p. 163). They show that if

$\exp(tP(s))$ satisfies appropriate growth conditions, then $u(x, t) = (G(\xi, t), \varphi(x - \xi))$ where $G(\xi, t)$ is a Green's function and $\varphi(x) \in \varkappa_{p,b}$, a class of entire functions analogous to $\mathfrak{A}$. The estimates used by Rosenbloom lead to more precise bounds on $G(\xi, t)$.

Remark 2. The integral in (3.2) can exist if φ satisfies

$$|\varphi(z_1, \cdots, z_n)| \leqq M_6 \exp\{\lambda(\sum |z_i|^2)^{v/2}\}$$

with $v \leqq 2k/(2k - 1)$. However, the integral appearing in the last member of (3.3) need not exist unless we impose some stronger condition such as (3.1). If $n = 1$ and $\varphi(x) = \sum_{l=0}^{\infty} a_l x^l$, one could use the definition that $\varphi(x)$ is entire of growth (ρ, τ) if and only if

$$\limsup_{j \to \infty} (j/e\rho)|a_j|^{\rho/j} \leqq \tau.$$

Then condition (3.1) could be replaced by the condition: $\varphi(x)$ is of growth (v, λ) with $0 \leqq v \leqq 2k/(2k - 1)$.

As we shall see, Theorem 3.2 has many uses in studying the question of representing the solutions of Cauchy problems in certain types of series of functions. Finally, we note that a problem of the form (1.1) may occur in which certain of the T_{A_j} appearing in the integral in (2.9) have real arguments while others involve complex arguments. In this case, it is necessary to patch together an appropriate data and solution space. We provide an example of this below.

4. Some examples. We now show how the results of §§ 2 and 3 can be applied to specific problems. Most of these involve the construction of a second operator Q along with a transformation of the form appearing in (1.3).

Example 1. Consider the problem (1.1) with $n = 2$ and $P(A_1, A_2) = A_1^2 - A_1A_2 + A_2^2$. The operator $P(D_1, D_2) = D_1^2 - D_1D_2 + D_2^2$ is an appropriate elliptic operator. To obtain the solution $\exp\{tP(A_1, A_2)\} \cdot \varphi$, we need the Green's function K in (2.6) or K^* in (2.9) for some associated $Q(D_1, D_2)$. We choose $Q(D_1, D_2) = D_1^2 + D_2^2$ and observe in (1.4) that

$$\begin{aligned}\exp[t(a_1^2 - a_1a_2 + a_2^2)] &= \exp[t(D_1^2 + D_2^2)]\{\exp[\xi_1(a_1 + a_2)/2 + \xi_2(a_1 - a_2)\sqrt{3}/2]\}_{\xi_1 = \xi_2 = 0}\\ &= \exp[t(D_1^2 + D_2^2)]\{\exp[a_1(\xi_1 + \xi_2\sqrt{3})/2 + a_2(\xi_1 - \xi_2\sqrt{3})/2]\}_{\xi_1 = \xi_2 = 0}.\end{aligned}$$

Then, since $Q(D_1, D_2)$ is the Laplacian, we can use (2.9) with $K^*(-\sigma, t) = (4\pi t)^{-1} \exp[-(\sigma_1^2 + \sigma_2^2)/4t]$. We get

$$\exp[tP(A_1, A_2)]\varphi = \int_{E_2} K^*(-\sigma, t) T_{A_1}((\sigma_1 + \sigma_2\sqrt{3})/2) T_{A_2}((\sigma_1 - \sigma_2\sqrt{3})/2)\varphi \, d\sigma.$$

With a change of variables of integration, this integral can be written as $\int_{E_2} K(-\eta, t) T_{A_1}(\eta_1) T_{A_2}(\eta_2)\varphi \, d\eta$ with

$$K(-\eta, t) = (4\pi t)^{-1} \exp[-(\eta_1^2 + \eta_1\eta_2 + \eta_2^2)/4t].$$

Example 2. Consider the Cauchy problem

$$u_t(x, y, t) = (x^2D_x^2 + xD_x + y^{4/3}D_y^2 + \tfrac{2}{3}y^{1/3}D_y)u(x, y, t), \qquad t > 0,$$

$$u(x, y, 0+) = \varphi(x, y).$$

Assume that $\varphi(x, y) \in C^1$ in x and y and has compact support. Now $A_1 = xD_x$ is a group generator with $A_1^2 = x^2D_x^2 + xD_x$ and $T_{A_1}(t) \cdot \psi(x) = \psi(xe^t)$ if $\psi(x) \in C^1$. This can be easily seen to satisfy $u_t^*(x, t) = xD_x u^*(x, t)$, $u^*(x, 0) = \psi(x)$. Similarly $A_2 = y^{2/3}D_y$ is a group generator with $A_2^2 = y^{4/3}D_y^2 + \frac{2}{3}y^{1/3}D_y$ and

$$T_{A_2}(t) \cdot \eta(y) = \eta(y + y^{2/3}t + \tfrac{1}{3}y^{1/3}t^2 + \tfrac{1}{27}t^3)$$

if $\eta \in C^1$. If we use (2.6) with K the heat kernel, we get

$$u(x, y, t)$$

$$= (4\pi t)^{-1} \int_{E_2} \exp\left[-(\sigma^2 + \xi^2)/4t\right]\varphi(xe^\sigma, y + y^{2/3}\xi + \tfrac{1}{3}y^{1/3}\xi^2 + \tfrac{1}{27}\xi^3)\, d\sigma\, d\xi.$$

By combining the method of Example 1 with that of Example 2, we can treat a variety of problems with variable coefficients.

Example 3. The Cauchy problem for the backward heat equation is a familiar example of an ill-posed problem. There are instances where certain solutions to this problem assume an important role. As an example, Widder and Rosenbloom [13] have considered the problem of representing solutions of the heat equation in series involving the heat polynomials $v_n(x, t)$ (with $v_n(x, 0) = x^n$). Certain of their theorems pertain to the convergence of these series in a strip $-\mu < t < \mu$. They prove convergence for $0 \leqq t < \mu$ and then use a result of Täcklind [14] to obtain convergence throughout the strip. We indicate how Theorem 3.2 can be used to obtain an elementary direct proof of convergence in the negative part of the time strip.

From (1.3) and (1.4), we see that $e^{tD_1^2} e^{ia\xi_1}|_{\xi_1 = 0} = e^{-ta^2}$. According to Theorem 3.2, we have as a solution to the backward heat problem

$$u_t(x, t) = -D_x^2 u(x, t), \qquad u(x, 0+) = \varphi(x),$$

$$u(x, t) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-\sigma^2/4t}\varphi(x + i\sigma)\, d\sigma \tag{4.1}$$

with $\varphi \in \mathfrak{A}$ and $v \leqq 2$ in (3.1). We call λ in (3.1) the type of $\varphi(x)$ and say that $\varphi(x)$ has grown $(2, \lambda)$ (Widder and Rosenbloom use the definition given in the remark following Theorem 3.2). The choice $\varphi(x) = x^m$ in (4.1) leads to $u(x, t) = v_m(x, -t)$, the mth heat polynomial with the sign of t reversed. By taking $\varphi(x) = e^{ax}$, (4.1) gives $u(x, t) = e^{ax - a^2t}$, a generating function for the $v_m(x, -t)$. Expanding this in powers of a, we get $v_m(x, -t) = 4^m t^{m/2} H_m(x/(2\sqrt{t}))$, where $H_m(y)$ is the Hermite polynomial of mth degree in y. With this, we now prove: *Let* $\varphi(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathfrak{A}$ *with growth* $(2, \lambda)$. *Then the series* $\sum_{n=0}^{\infty} a_n v_n(x, -t)$ *converges to a solution of the backward heat equation for* $0 < t < 1/(4\lambda)$ *and* $\sum_{n=0}^{\infty} a_n v_n(x, 0) = \varphi(x)$.

Proof. Select $R > 0$ and take $|x| \leqq R$. Now

$$\sum_{n=0}^{\infty} a_n v_n(x, -t) = \sum_{n=0}^{\infty} (4\pi t)^{-1/2} \cdot a_n \int_{-\infty}^{\infty} e^{-\sigma^2/4t}(x + i\sigma)^n\, d\sigma$$

$$= (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-\sigma^2/4t}\left(\sum_{n=0}^{\infty} a_n(x + i\sigma)^n\right) d\sigma.$$

This interchange of summation and integration is valid if

$$\int_{-\infty}^{\infty} e^{-\sigma^2/4t}\left(\sum_{n=0}^{\infty} |a_n| \cdot |x + i\sigma|^n\right) d\sigma$$

converges. But since $\varphi(x) \in \mathfrak{A}$ with growth $(2, \lambda)$, the estimate (3.1) shows that this converges if $0 < t < 1/(4\lambda)$ and uniformly for $0 < t_0 \leqq t \leqq \tau < 1/(4\lambda)$. Since

$$(4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-\sigma^2/4t} \varphi(x + i\sigma)\, d\sigma$$

satisfies the backward heat equation, the series $\sum_{n=0}^{\infty} a_n v_n(x, -t)$ must converge uniformly to this solution in these compact sets and hence for $0 < t < 1/(4\lambda)$. Since R was arbitrary, the result follows. The convergence at $t = 0$ follows readily.

One can use analogous arguments in treating polynomial representations of solutions of, say, the problem

$$u_t(x, y, t) + (D_x^4 + D_y^4)u(x, y, t) = 0, \qquad u(x, y, 0+) = \varphi(x, y).$$

Example 4. Consider the Cauchy problem

$$\begin{aligned} &u_t(x, y, t) = D_x D_y u(x, y, t), \qquad t > 0, \\ &u(x, y, 0+) = \varphi(x, y). \end{aligned} \tag{4.2}$$

We note that

$$\exp(ta_1 a_2) = \exp[t(D_1^2 + D_2^2)] \exp[a_1(\xi_1 + i\xi_2)/2 + a_2(\xi_1 - i\xi_2)/2]|_{\xi_1 = \xi_2 = 0}.$$

According to Theorem 3.2, we have

$$\begin{aligned} u(x, y, t) = (4\pi t)^{-1} &\int_{E_2} \exp[-(\sigma_1^2 + \sigma_2^2)/4t] \\ &\cdot \varphi(x + (\sigma_1 + i\sigma_2)/2, y + (\sigma_1 - i\sigma_2)/2)\, d\sigma \end{aligned} \tag{4.3}$$

if $\varphi(x, y) \in \mathfrak{A}$ with $v \leqq 2$ in (3.1).

To treat the amended Cauchy problem

$$\begin{aligned} &u_t(x, y, z, t) = (D_x \cdot D_y + D_z^2)u(x, y, z, t), \qquad t > 0, \\ &u(x, y, z, 0+) = \varphi(x, y, z), \end{aligned} \tag{4.4}$$

we require the use of both Theorems 2.1 and 3.2. For

$$\begin{aligned} &\exp[t(a_1 a_2 + a_3^2)] \\ &= \exp[t(D_1^2 + D_2^2 + D_3^2)] \cdot \exp[a_1(\xi_1 + i\xi_2)/2 + a_2(\xi_1 - i\xi_2)/2 + a_3\xi_3]|_{\xi_1 = \xi_2 = \xi_3 = 0}. \end{aligned}$$

The term $a_3\xi_3$ suggests a translation operator in Theorem 2.1 while the other terms have the interpretation in (4.3). Then (4.4) has a solution of the form

$$\begin{aligned} &u(x, y, z, t) \\ &= (4\pi t)^{-3/2} \int_{E_3} \exp[-(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)/4t] \\ &\qquad \cdot \varphi(x + (\sigma_1 + i\sigma_2)/2, y + (\sigma_1 - i\sigma_2)/2, z + \sigma_3)\, d\sigma \end{aligned} \tag{4.5}$$

if we select $\varphi(x, y, z)$ to be C^2 in z for all x, y and in class $\mathfrak{A}$ in x and y for all z.

One can use series representations of solutions of (4.4) to discuss representations of the related ultrahyperbolic problem

$$v_{tt}(x, y, z, t) = (D_x D_y + D_z^2)v(x, y, z, t),$$
$$v(x, y, z, 0) = \varphi(x, y, z), \qquad v_t(x, y, z, 0) = 0.$$

Representation theorems for similar problems have been treated in [6].

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ON SPACES OF TYPE H_μ AND THEIR HANKEL TRANSFORMATIONS*

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Abstract. The topological and algebraic properties of the spaces of type H_μ, that is, $H_{\mu,\alpha}$, H_μ^β, and $H_{\mu,\alpha}^\beta$, are investigated in this paper, where μ is any real number, α and β are nonnegative real numbers. The conventional Hankel transformation $\mathscr{h}_\mu$ for $\mu \geqq -1/2$ is a continuous linear mapping from each of the spaces of type H_μ into certain other spaces of type H_μ. This assertion is extended to any real number μ and to the generalized Hankel transformation $\mathscr{h}'_\mu$. The nontriviality of the spaces of type H_μ, the relation of certain entire functions with a space of type H_μ, and the relations between the spaces of type S and type H_μ are proved in the Appendices.

1. Spaces of type H_μ: $H_{\mu,\alpha}$, H_μ^β and $H_{\mu,\alpha}^\beta$. $\varphi \in H_{\mu,\alpha}$, $\alpha \geqq 0$, if and only if φ is a smooth function on $0 < x < \infty$ and

$$\gamma_{k,q}^\mu(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_q A^k k^{k\alpha}, \qquad k, q = 0, 1, \cdots,$$

where the constants A and C_q depend on the testing function φ. For $k = 0$, we set $k^{k\alpha} \triangleq 1$.

$\varphi \in H_\mu^\beta$, $\beta \geqq 0$ if and only if φ is a smooth function on $0 < x < \infty$ and

$$\gamma_{k,q}^\mu(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_k B^q q^{q\beta}, \qquad k, q = 0, 1, \cdots,$$

where the constants C_k and B depend on the testing function φ. As before we set $q^{q\beta} \triangleq 1$ for $q = 0$.

$\varphi \in H_{\mu,\alpha}^\beta$, $\alpha, \beta \geqq 0$, if and only if φ is a smooth function on $0 < x < \infty$ and

$$\gamma_{k,q}^\mu(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq CA^k B^q k^{k\alpha} q^{q\beta}, \qquad k, q = 0, 1, \cdots,$$

where the constants A, B and C depend on the testing function φ.

The topology for each of these spaces is generated by the seminorms $\{\gamma_{k,q}^\mu\}_{k,q=0}^\infty$. It is clear that each of these spaces is a Fréchet space. Moreover it is perfect; every bounded set is relatively compact.

Note that the spaces $H_{\mu,\alpha}$, H_μ^β and H_μ may be considered to be the limiting cases of the space $H_{\mu,\alpha}^\beta$;

$$H_{\mu,\alpha} = H_{\mu,\alpha}^\infty, \quad H_\mu^\beta = H_{\mu,\infty}^\beta \quad \text{and} \quad H_\mu = H_{\mu,\infty}^\infty,$$

where the right-hand sides are understood to be the countable-union spaces such that

$$H_{\mu,\alpha}^\infty = \bigcup_{\beta_i=1}^\infty H_{\mu,\alpha}^{\beta_i}, \quad H_{\mu,\infty}^\beta = \bigcup_{\alpha_i=1}^\infty H_{\mu,\alpha_i}^\beta \quad \text{and} \quad H_{\mu,\infty}^\infty = \bigcup_{\alpha_i,\beta_i=1}^\infty H_{\mu,\alpha_i}^{\beta_i}.$$

For the definition of the space H_μ see [16, pp. 129–130] or [17, pp. 562–563].

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2. Different ways of defining the spaces of type H_μ.

2.1. The space $H_{\mu,\alpha}$. It is readily seen from the definition of $H_{\mu,\alpha}$ that if $\alpha = 0$, then $B_{\mu,A} \subset H_{\mu,0}$, where according to [15, pp. 679–680], $B_{\mu,A}$ is the space of smooth functions φ such that $\varphi(x) = 0$ for $x > A$ and

$$\gamma_q^\mu(\varphi) \triangleq \sup_{0<x<\infty} |(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| < \infty, \qquad q = 0, 1, \cdots.$$

But using the definition of $H_{\mu,\alpha,A}$ in § 3.1, we can easily show that $B_{\mu,A} = H_{\mu,0,A}$ algebraically and topologically. Now suppose $\alpha > 0$. Then the following two theorems are not hard to prove.

THEOREM 2.1.1. *Let $\varphi \in H_{\mu,\alpha}$, where $\alpha > 0$. Then*

$$|D^q(x^{-\mu-1/2}\varphi(x))| \leqq C'_q \exp(-a'x^{1/\alpha}), \qquad q = 0, 1, \cdots, \tag{2.1.1}$$

where $a' = (\alpha/e)A'^{1/\alpha}$ is a constant less than $a = (\alpha/e)A^{1/\alpha}$.

THEOREM 2.1.2. *Let $\varphi \in H_\mu$ satisfy the inequality* (2.1.1). *Then*

$$\gamma_{k,q}^\mu(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C''_q A''^k k^{k\alpha}, \tag{2.1.2}$$

where $a'' = (\alpha/e)A''^{1/\alpha}$ is a constant less than $a' = (\alpha/e)A'^{1/\alpha}$.

Remark. The inequality (2.1.2) shows that $\varphi \in H_{\mu,\alpha,A}$, and therefore, $\varphi \in H_{\mu,\alpha}$. Consequently, Theorems 2.1.1 and 2.1.2 together imply that $\varphi \in H_{\mu,\alpha,A}$ if and only if $\varphi \in H_\mu$ and

$$|D^q(x^{-\mu-1/2}\varphi(x))| \leqq C_q \exp\{-(a-\delta)x^{1/\alpha}\}, \qquad \delta > 0.$$

2.2. The space H_μ^β. From the definition of the space H_μ^β, it is obvious that as β diminishes, the constraints on differentiation become more strict. We state the following theorem without proof.

THEOREM 2.2.1. *Let $\varphi \in H_\mu^\beta$. Then*

$$\sup_{0<x<\infty} |x^k D^q(x^{-\mu-1/2}\varphi(x))| \leqq C_{k,q}B^q q^{q\beta}, \qquad k, q = 0, 1, \cdots,$$

where the constants $C_{k,q}$ depend on φ, k and q as well.

From Theorem 2.2.1, we see that the space H_μ^β behaves quite differently from the space S^β defined by Gel'fand and Shilov [2, p. 167], namely the analytic continuation under suitable conditions on β is possible in the latter case, but not in the former case.

2.3. The space $H_{\mu,\alpha}^\beta$. By definition the space $H_{\mu,\alpha}^\beta$ consists of the smooth functions φ on $0 < x < \infty$ satisfying the inequality

$$\gamma_{k,q}^\mu(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq CA^kB^qk^{k\alpha}q^{q\beta}, \qquad k, q = 0, 1, \cdots. \tag{2.3.1}$$

Here restraints are imposed on the decrease of φ as $x \to \infty$ and on the growth of the derivatives of φ. Thus (2.3.1) shows that $H_{\mu,\alpha}^\beta$ is contained in the intersection of the spaces $H_{\mu,\alpha}$ and H_μ^β algebraically and topologically. From Theorem 2.1.1, we obtain

$$|D^q(x^{-\mu-1/2}\varphi(x))| \leqq C_qB^q q^{q\beta} \exp\{-(a'-\delta)x^{1/\alpha}\},$$

where a' is a constant less than $a = (\alpha/e)A^{1/\alpha}$ and $\delta > 0$. Note that in this case the analytic continuation of $\psi(x) \triangleq x^{-\mu-1/2}\varphi(x)$ to the complex plane $z = x + iy$ may not be possible because of the fact that $A^k k^{k\alpha} \to \infty$ as $k \to \infty$.

3. Topological properties of the spaces of type H_μ.

3.1. The space $H_{\mu,\alpha}$ as the union of countably normed spaces. Let $H_{\mu,\alpha A}$ be the space of testing functions φ in $H_{\mu,\alpha}$ such that

$$\gamma_{k,q}^{\mu}(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_{q\delta}(A+)^k k^{k\alpha} \tag{3.1.1}$$

for any $\delta > 0$. According to Theorem 2.1.1, $\varphi \in H_{\mu,\alpha,A}$ implies

$$|D^q(x^{-\mu-1/2}\varphi(x))| \leqq C'_{q\delta} \exp\{-(a' - \delta)x^{1/\alpha}\},$$

where a' is a constant less than $a = (\alpha/e)A^{1/\alpha}$ and any $\delta > 0$. By virtue of [2, pp. 86–94], $H_{\mu,\alpha,A}$ is perfect, that is, every bounded set is relatively compact. If we define

$$\rho_p^{\mu}(\varphi) \triangleq \max_{0 \leqq q \leqq p} \sup_{0<x<\infty} M_p(x)|D^q(x^{-\mu-1/2}\varphi(x))|, \tag{3.1.2}$$

where $M_p(x) \triangleq \exp\{a'(1 - 1/p)x^{1/\alpha}\}$, $p = 2, 3, \cdots$, then ρ_p^{μ} is a norm for the space $H_{\mu,\alpha,A}$. Thus $\varphi \in H_{\mu,\alpha,A}$ if and only if $\rho_p^{\mu}(\varphi)$ is finite for each $p = 2, 3, \cdots$. If we set

$$\rho_{q\delta}^{\mu}(\varphi) \triangleq \sup_k \sup_{0<x<\infty} \frac{|x^k D^q(x^{-\mu-1/2}\varphi(x))|}{(A + \delta)^k k^{k\alpha}}, \tag{3.1.3}$$

then $\rho_{q\delta}^{\mu}$ is another norm for the space $H_{\mu,\alpha,A}$ and (3.1.2) and (3.1.3) are two equivalent norms [2, pp. 176–178].

Evidently $H_{\mu,\alpha,A_1} \subset H_{\mu,\alpha,A_2}$ if $A_1 < A_2$ and the topology for H_{μ,α,A_1} is stronger than the topology induced by that of H_{μ,α,A_2}. From the definition, $H_{\mu,\alpha} = \bigcup_{A=1}^{\infty} H_{\mu,\alpha,A}$ algebraically and we treat it as a countable-union space [16, pp. 14–16]. A sequence $\{\varphi_\nu\}$ in $H_{\mu,\alpha}$ converges to zero if and only if $\{\varphi_\nu\}$ belongs to some $H_{\mu,\alpha,A}$ and converges to zero in this space [16, pp. 14–16]. But this is the case if and only if the sequence $\{\varphi_\nu\}$ converges to zero (that is, for each $q = 0, 1, 2, \cdots$, $\{\varphi_\nu^{(q)}\}$ converges uniformly to zero in any bounded interval $(0, x]$) and the norms $\rho_p^{\mu}(\varphi_\nu)$ are bounded for all p and ν (see [2, pp. 91–94]).

3.2. The space H_μ^β as the union of countably normed spaces. Let $H_\mu^{\beta,B}$ be the space of testing functions φ in H_μ^β satisfying the inequality

$$\gamma_{k,q}^{\mu}(\varphi) \triangleq \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_{k\rho}(B + \rho)^q q^{q\beta}, \quad k, q = 0, 1, \cdots, \tag{3.2.1}$$

for any $\rho > 0$. From Theorem 2.2.1 and (3.2.1), any φ in $H_\mu^{\beta,B}$ satisfies the inequality

$$\sup_{0<x<\infty} |x^k(x^{-\mu-1/2}\varphi(x))| \leqq C_{kq\rho}(B + \rho)^q q^{q\beta}, \qquad k, q = 0, 1, \cdots,$$

for any $\rho > 0$.

Note that the space $H_\mu^{\beta,B}$ does not belong to the class $K\{M_p\}$ as does the space $H_{\mu,\alpha,A}$ because no M_p exists in this case. We introduce the norms $\|\cdot\|_{k\rho}^{\mu}$ in $H_\mu^{\beta,B}$

as follows:

$$\|\varphi\|^{\mu}_{k\rho} \triangleq \sup_{q} \sup_{0<x<\infty} \frac{|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))|}{(B+\rho)^q q^{q\beta}}, \tag{3.2.2}$$
$$k = 0, 1, \cdots, \quad \rho = 1, \tfrac{1}{2}, \cdots.$$

DEFINITION 3.2.1. Let $\hat{H}^{\beta,B}_\mu$ be the space of φ's in $H^{\beta,B}_\mu$ satisfying the condition

$$\sup_{0\leqq r\leqq q} C_{k+r,\rho} = C'_{k\rho}, \qquad k = 0, 1, \cdots,$$

where $C_{k\rho}$ are constants restraining the φ's in $H^{\beta,B}_\mu$. The topology for the space $\hat{H}^{\beta,B}_\mu$ is the one induced by $H^{\beta,B}_\mu$.

For instance, $\varphi(x) = x^{\mu+1/2}\cdot \exp\{-\frac{1}{2}(B+\rho)x^2\}$, where ρ is chosen such that $B+\rho \geqq 1$, belongs to $\hat{H}^{\beta,B}_\mu$ for $\beta \geqq 1$. Now let $\varphi \in \hat{H}^{\beta,B}_\mu$. An easy computation shows that

$$\begin{aligned} &|x^kD^q(x^{-\mu-1/2}\varphi(x))| \\ &\quad \leqq C_{k\rho}\left\{1 + \frac{b_{q,q-1}(q-1)^{(q-1)\beta}}{(B+\rho)q^{q\beta}} + \cdots + \frac{b_{q,(q+I_q)/2}(q+I_q)/2^{(q+I_q)\beta/2}}{(B+\rho)^{(q-I_q)/2}q^{q\beta}}\right\} \\ &\qquad \cdot (B+\rho)^q q^{q\beta}, \end{aligned} \tag{3.2.3}$$

where $I_q = 0$ or 1 according to whether q equals an even number or an odd number respectively. Consider the inequality

$$\frac{b_{q,q-j}(q-j)^{(q-j)\beta}}{(B+\rho)^j q^{q\beta}} \leqq q^2, \tag{3.2.4}$$
$$q = 1, 2, \cdots, \quad j = 1, 2, \cdots, (q-I_q)/2.$$

The inequality (3.2.4) holds if and only if

$$\beta \geqq \frac{\log b_{q,q-j} - 2\log q - j\cdot\log(B+\rho)}{q\cdot\log q - (q-j)\log(q-j)}, \qquad j = 1, 2, \cdots, q-1.$$

Define

$$\beta_0 \triangleq \sup_{q} \sup_{1\leqq j\leqq (q-I_q)/2} \frac{\log b_{q,q-j} - 2\log q - j\cdot\log(B+\rho)}{q\cdot\log q - (q-j)\log(q-j)}. \tag{3.2.5}$$

An inductive argument on q shows that β_0 is finite. For $\beta \geqq \beta_0$, we get from (3.2.3) and (3.2.4),

$$\begin{aligned} |x^kD^q(x^{-\mu-1/2}\varphi(x))| &\leqq C_{k\rho}(B+\rho)^q q^3 q^{q\beta} \leqq C_{k\rho\varepsilon}(B+\rho)^q(1+\varepsilon)^q q^{q\beta} \\ &\leqq C_{k\rho}(B+2\rho)^q q^{q\beta}, \end{aligned}$$

since ε is arbitrary constant. Hence we have the following theorem.

THEOREM 3.2.1. *Let $\beta \geqq \beta_0$, where β_0 is defined by* (3.2.5). *Then $\varphi \in \hat{H}^{\beta,B}_\mu$ satisfies the inequality*

$$\sup_{0<x<\infty} |x^kD^q(x^{-\mu-1/2}\varphi(x))| \leqq C_{k\rho}(B+\rho)^q q^{q\beta}. \tag{3.2.6}$$

Conversely let $\varphi \in H_\mu$ satisfy the inequality (3.2.6). Since

$$\begin{aligned}|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq |x^{k-q}D^q(x^{-\mu-1/2}\varphi(x))| + |d_{q,q-1}| \\ \cdot |x^{k-q-1}D^{q-1}(x^{-\mu-1/2}\varphi(x))| \\ + \cdots + |d_{q,1}x^{k-2q+1}D(x^{-\mu-1/2}\varphi(x))|,\end{aligned} \tag{3.2.7}$$

where $d_{q,j}$ are constants ($j = 1, \cdots, q$) and $d_{q,q} = 1$, and $d_{q,q-1} = -q(q-1)/2$, we have

$$\begin{aligned}&|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \\ &\leqq C_k\left\{1 + \frac{|d_{q,q-1}|(q-1)^{(q-1)\beta}}{(B+\rho)^q q^{q\beta}} + \cdots + \frac{|d_{q,1}|}{(B+\rho)^{q-1}q^{q\beta}}\right\}(B+\rho)^q q^{q\beta}.\end{aligned} \tag{3.2.8}$$

Consider the inequality

$$\frac{|d_{q,q-j}|(q-j)^{(q-j)\beta}}{(B+\rho)^j q^{q\beta}} \leqq q^2. \tag{3.2.9}$$

Clearly this holds if and only if

$$\beta \geqq \frac{\log|d_{q,q-j}| - 2\cdot\log q - j\cdot\log(B+\rho)}{q\cdot\log q - (q-j)\cdot\log(q-j)}, \qquad j = 1, \cdots, q-1.$$

Define

$$\beta_1 \triangleq \sup_q \sup_{1 \leqq j \leqq q-1} \frac{\log|d_{q,q-j}| - 2\cdot\log q - j\cdot\log(B+\rho)}{q\cdot\log q - (q-j)\log(q-j)}. \tag{3.2.10}$$

It follows from (3.2.8) that if $\beta \geqq \beta_1$,

$$\begin{aligned}|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| &\leqq C_{k\rho}(B+\rho)^q q^3 q^{q\beta} \leqq C_{k\rho\varepsilon}(B+\rho)^q(1+\varepsilon)^q q^{q\beta} \\ &\leqq C_{k\rho}(B+2\rho)^q q^{q\beta},\end{aligned}$$

since ε is arbitrary. Hence we have obtained the following theorem.

THEOREM 3.2.2. *Let $\beta \geqq \beta_1$, where β_1 is defined by* (3.2.10). *Then $\varphi \in H_\mu^{\beta,B}$ if $\varphi \in H_\mu$ satisfies the inequality* (3.2.6).

Remark. It is easy to confirm by induction on q that β_0 and β_1 defined by (3.2.5) and (3.2.10) are finite.

From the definition, $H_\mu^\beta = \bigcup_{B_1=1}^\infty H_\mu^{\beta,B_1}$ is a countable-union space and hence every sequence $\{\varphi_v\}$ in H_μ^β converges to zero if and only if $\{\varphi_v\}$ belongs to some $H_\mu^{\beta,B}$ and converges to zero in its topology; see [16, pp. 14–16].

3.3. The space $H_{\mu,\alpha}^\beta$ as the union of countably normed spaces. Let $H_{\mu,\alpha,A}^{\beta,B}$ be the space of testing functions φ in $H_{\mu,\alpha}^\beta$ satisfying the inequality

$$\sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_{\delta\rho}(A+\delta)^k k^{k\alpha} q^{q\beta}, \qquad k, q = 0, 1, \cdots,$$

for any $\delta, \rho > 0$. We introduce the norms $H_{\mu,\alpha,A}^{\beta,B}$ as follows:

$$\|\varphi\|_{\delta\rho}^\mu \triangleq \sup_{k,q} \sup_{0<x<\infty} \frac{|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))|}{(A+\delta)^k(B+\rho)^q k^{k\alpha} q^{q\beta}}, \qquad \delta, \rho = 1, \tfrac{1}{2}, \cdots.$$

The topology for the space $H^{\beta,B}_{\mu,\alpha,A}$ is generated by the norms $\{\|\cdot\|^{\mu}_{\delta\rho}\}$. With this topology, $H^{\beta,B}_{\mu,\alpha,A}$ is a countably normed complete perfect topological linear space. Clearly $H^{\beta}_{\mu,\alpha} = \bigcup_{A,B=1}^{\infty} H^{\beta,B}_{\mu,\alpha,A}$ algebraically and we treat it as a countable-union space. Hence a sequence $\{\varphi_\nu\}$ converges to zero in $H^{\beta}_{\mu,\alpha}$ if and only if $\{\varphi_\nu\}$ belongs to some $H^{\beta,B}_{\mu,\alpha,A}$ and converges to zero in the space; this is the case if and only if the sequence $\{\varphi_\nu\}$ converges correctly to zero and the inequality

$$|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi_\nu(x))| \leqq C(A+\delta)^k(B+\rho)^q k^{k\alpha} q^{q\beta}$$

is satisfied where the constants A, B and C are independent of ν.

4. The Hankel transformation with application.

4.1. The conventional Hankel transformation. According to [16, pp. 130–131], the conventional Hankel transformation h_μ for $\mu \geqq -\frac{1}{2}$ defined by

$$\Phi(y) \triangleq (h_\mu\varphi)(x) \triangleq \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)\,dx, \qquad 0 < x < \infty,$$

exists for each φ in $H_{\mu,\alpha,A}$, $\hat{H}^{\beta,B}_{\mu}$ or $H^{\beta,B}_{\mu,\alpha,A}$. Also from [5, p. 134] or [10, p. 355], $\sqrt{xy}J_\mu(xy) = O(x^{\mu+1/2})$ as $x \to 0$, and $\sqrt{xy}J_\mu(xy) = O(1)$ as $x \to \infty$. Following [15, p. 679] we define

$$N_\mu\varphi(x) \triangleq x^{\mu+1/2}D(x^{-\mu-1/2}\varphi(x)),$$
$$M_\mu\varphi(x) \triangleq x^{-\mu-1/2}D(x^{\mu+1/2}\varphi(x)),$$
$$N_\mu^{-1}\varphi(x) \triangleq x^{\mu+1/2}\int_\infty^x t^{-\mu-1/2}\varphi(t)\,dt.$$

THEOREM 4.1.1. *For $\mu \geqq -\frac{1}{2}$, the conventional Hankel transformation h_μ is a continuous linear mapping from the space $H_{\mu,\alpha,A}$ into the space $H_\mu^{2\alpha,(2e)^{2\alpha}A^2}$.*

Proof. Let K be a bounded set in $H_{\mu,\alpha,A}$. Then every φ in K satisfies the inequality

$$|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_{q\delta}(A+\delta)^k k^{k\alpha}, \qquad k,q = 0,1,\cdots, \quad \delta > 0, \tag{4.1.1}$$

where the constants $C_{q\delta}$ are independent of the choice of φ in K. Let $\Phi(y) = h_\mu\varphi(x)$. For any pair of nonnegative integers k and q, from [16, p. 139], and with the aid of the equality

$$N_{\mu+q+k-1}\cdots N_{\mu+q}x^q\varphi(x) = x^q N_{\mu+k-1}\cdots N_\mu\varphi(x)$$

we have

$$(-y)^k N_{\mu+q-1}\cdots N_\mu\Phi(y) = h_{\mu+k+q}\{(-x)^q N_{\mu+k-1}\cdots N_\mu\varphi(x)\}$$
$$= \int_0^\infty (-x)^q\{N_{\mu+k-1}\cdots N_\mu\varphi(x)\}\sqrt{xy}J_{\mu+k+q}(xy)\,dx. \tag{4.1.2}$$

Since by induction on k and q

$$N_{\mu+q-1}\cdots N_\mu\Phi(y) = y^{\mu+q+1/2}(y^{-1}D)^q(y^{-\mu-1/2}\Phi(y)),$$
$$N_{\mu+k-1}\cdots N_\mu\varphi(x) = x^{\mu+k+1/2}(x^{-1}D)^k(x^{-\mu-1/2}\varphi(x)),$$

it follows from (4.1.2) that

$$(4.1.3)\qquad \begin{aligned}&(-1)^{k+q}y^k(y^{-1}D)^q(y^{-\mu-1/2}\Phi(y))\\ &= \int_0^\infty x^{2\mu+1+k+2q}\{(x^{-1}D)^k(x^{-\mu-1/2}\varphi(x))\}\frac{J_{\mu+k+q}(xy)}{(xy)^{\mu+q}}\,dx.\end{aligned}$$

From [4, pp. 111–112], [5, p. 134] and upon taking v as a positive integer greater than $2\mu + 1$, (4.1.3) implies

$$(4.1.4)\qquad \begin{aligned}&|y^k(y^{-1}D)^q(y^{-\mu-1/2}\varphi(y))|\\ &\leqq C_{k\delta}\{1 + (A + \delta)^{(v+2+k+2q)}(v + 2 + k + 2q)^{(v+2+k+2q)\alpha}\}\\ &\leqq C_{k\delta\varepsilon}(A^2 + \delta')^q(1 + \varepsilon')^q(2eq)^{2\alpha q} \leqq C_{k\delta}\{(2e)^{2\alpha}A^2 + \delta''\}^q q^{2\alpha q}.\end{aligned}$$

Here the constant $C_{k\delta}$ in (4.1.4) is independent of the choice of φ in K. Consequently $\Phi = h_\mu\varphi \in H_\mu^{2\alpha,(2e)^{2\alpha}A^2}$ and from (4.1.4) the mapping $\varphi \mapsto h_\mu\varphi$ maps a bounded set in $H_{\mu,\alpha,A}$ into a bounded set in $H_\mu^{2\alpha,(2e)^{2\alpha}A^2}$ and hence is continuous. This completes the proof.

A proof similar to that of Theorem 4.1.1 shows the following theorem.

THEOREM 4.1.2. *For $\mu \geqq -\frac{1}{2}$, the conventional Hankel transformation h_μ is a continuous linear mapping from the space $\hat{H}_\mu^{\beta,B}(H_{\mu,\alpha,A}^{\beta,B})$ into the space $H_{\mu,\beta,B}(H_{\mu,\alpha+\beta,e^\alpha AB}^{2\alpha,(2e)^{2\alpha}A^2})$.*

4.2. The generalized Hankel transformation. Throughout this section, $-\frac{1}{2} \leqq \mu < \infty$. We now define the generalized Hankel transformation h'_μ on each of the dual spaces $H'_{\mu,\alpha,A}$, $(\hat{H}_\mu^{\beta,B})'$ and $(H_{\mu,\alpha,A}^{\beta,B})'$ as follows:

$$\langle F, \Phi\rangle \triangleq \langle f, \varphi\rangle,$$

where $\Phi \triangleq h_\mu\varphi$, $F \triangleq h_\mu f$, φ belongs to $H_{\mu,\alpha,A}$, $\hat{H}_\mu^{\beta,B}$ or $H_{\mu,\alpha,A}^{\beta,B}$, f belongs to the corresponding space. Since $h_\mu = h_\mu^{-1}$ for $\mu \geqq -\frac{1}{2}$, from Theorems 4.1.1, 4.1.2, and [16, pp. 141–142], we have the following theorem.

THEOREM 4.2.1. *For $\mu \geqq -\frac{1}{2}$, the generalized Hankel transformation h'_μ is a continuous linear mapping from the dual space $(H_\mu^{2\alpha,(2e)^{2\alpha}A^2})'$ $\{(H_{\mu,\beta,B})'$ and $(H_{\mu,\alpha+\beta,e^\alpha AB}^{2\alpha,(2e)^{2\alpha}A^2})'$ respectively$\}$ into the dual space $(H_{\mu,\alpha,A})'$ $\{(\hat{H}_\mu^{\beta,B})'$ and $(H_{\mu,\alpha,A}^{\beta,B})'$ respectively$\}$.*

Note that if $\mu \geqq -\frac{1}{2}$, the conventional Hankel transformation h_μ acting on $f \in L_1(0, \infty)$ is a special case of the generalized Hankel transformation, that is, the generalized Hankel transform $F \triangleq h'_\mu f$ is the regular generalized function corresponding to the conventional Hankel transform $F_c = h'_\mu f$; see [16, pp. 142–143].

4.3. Hankel transformation of arbitrary order. Following [16], we extend the Hankel transformation of the spaces $H_{\mu,\alpha,A}$, $\hat{H}_\mu^{\beta,B}$, and $H_{\mu,\alpha,A}^{\beta,B}$, and their dual spaces for an arbitrary real number μ in such a way that the following properties hold:

(i) The direct and inverse Hankel transformations exist whatever be the choice of μ.

(ii) The direct Hankel transformation and its inverse are defined on the dual spaces $H'_{\mu,\alpha,A}$, $(\hat{H}_\mu^{\beta,B})'$ and $(H_{\mu,\alpha,A}^{\beta,B})'$.

(iii) For $\mu \geqq -\frac{1}{2}$, the extended direct Hankel transformation and its inverse coincide with the previous ones discussed in § 4.1.

Let μ be any real number, and let m be a positive integer such that $\mu + m \geqq -\frac{1}{2}$. According to [16, pp. 163–164], we define the extended Hankel transformation $h_{\mu,m}$ by

$$\Phi(y) \triangleq h_{\mu,m}[\varphi(x)] \triangleq (-1)^m y^{-m} h_{\mu+m} N_{\mu+m-1} \cdots N_\mu \varphi(x), \tag{4.3.1}$$

$$\varphi(x) \triangleq h_{\mu,m}^{-1}[\Phi(y)] \triangleq (-1)^m N_\mu^{-1} \cdots N_{\mu+m-1}^{-1} y^m \Phi(y). \tag{4.3.2}$$

Then we have the following theorem from Theorems 4.1.1 and 4.1.2.

THEOREM 4.3.1. *For any real number μ and a positive integer m such that $\mu + m \geqq -\frac{1}{2}$, the extended Hankel transformation $h_{\mu,m}$ defined by* (4.3.1) *is a continuous linear mapping from $H_{\mu,\alpha,A}$ ($\hat{H}_\mu^{\beta,B}$ and $H_{\mu,\alpha,A}^{\beta,B}$ respectively) into $H_\mu^{2\alpha,(2e)^{2\alpha}A^2}$ ($H_{\mu,\beta,B}$ and $H_{\mu,\alpha+\beta,e^\alpha AB}^{2\alpha,(2e)^{2\alpha}A^2}$ respectively). Its inverse is defined by* (4.3.2). *Moreover, $h_{\mu,m}$ coincides with h_μ whenever $\mu \geqq -\frac{1}{2}$.*

It is evident that in Theorem 4.3.1, $h_{\mu,m}$ and $h_{\mu,m}^{-1}$ are independent of the choice of m. If $\mu \geqq -\frac{1}{2}$, $h_{\mu,m} = h_\mu = h_\mu^{-1}$. Now for any real number μ, the generalized Hankel transformation h'_μ on each of the dual spaces $H'_{\mu,\alpha,A}$, $(\hat{H}_\mu^{\beta,B})'$ or $(H_{\mu,\alpha,A}^{\beta,B})'$ is defined as follows; let φ belong to $H_{\mu,\alpha A}$, $\hat{H}_\mu^{\beta,B}$ or $H_{\mu,\alpha,A}^{\beta,B}$, f to its corresponding dual space; then

$$\langle F, \Phi \rangle \triangleq \langle f, \varphi \rangle, \tag{4.3.3}$$

where $F \triangleq h'_\mu f$, $\Phi \triangleq h_{\mu,m}\varphi$ and m is taken as before. Using [16, p. 29], we have the following theorem.

THEOREM 4.3.2. *For any real number μ, the generalized Hankel transformation h'_μ defined by* (4.3.3) *is a continuous linear mapping from the dual space $(H_\mu^{2\alpha,(2e)^{2\alpha}A^2})'$ $\{(H_{\mu,\beta,B})'$ and $(H_{\mu,\alpha+\beta,e^\alpha A^2}^{2\alpha,(2e)^{2\alpha}A^2})'$ respectively$\}$ into the dual space $(H_{\mu,\alpha,A})'$ $\{(\hat{H}_\mu^{\beta,B})'$ and $(H_{\mu,\alpha,A}^{\beta,B})'$ respectively$\}$.*

In view of (4.3.3), the generalized inverse Hankel transformation $(h'_\mu)^{-1}$ is defined by

$$\langle (h'_\mu)^{-1} F, h_{\mu,m}^{-1}\varphi \rangle = \langle F, \Phi \rangle. \tag{4.3.4}$$

Then from [16, p. 29], we get the following theorem.

THEOREM 4.3.3. *For any real number μ, the generalized inverse Hankel transformation $(h'_\mu)^{-1}$ defined by* (4.3.4) *is a continuous linear mapping from the dual space $H'_{\mu,\alpha,A}$ $\{\hat{H}_\mu^{\beta,B})'$ and $(H_{\mu,\alpha,A}^{\beta,B})'$ respectively$\}$ into the dual space $(H_\mu^{2\alpha,(2e)^{2\alpha}A^2})'$ $(\{(H_{\mu,\beta,B})'$ and $(H_{\mu,\alpha+\beta,e^\alpha AB}^{2\alpha,(2e)^{2\alpha}A^2})'\}$.*

4.4. An application to a Dirichlet problem in cylindrical coordinates. Let us find a conventional function $v(r, z)$ on the domain $\{(r, z): 0 < r < \infty, 0 < z < 1\}$ which satisfies Laplace's equation in a cylindrical coordinate system without a θ variation:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = 0. \tag{4.4.1}$$

We impose the following generalized boundary conditions on $v(r, z)$:

(i) $\sqrt{r}v(r, z)$ converges in the dual space $(H_{0,\alpha,A})'$, $(\hat{H}_0^{\beta,B})'$ or $(H_{0,\alpha,A}^{\beta,B})'$ to $f(r) \in \mathscr{E}'(0, \infty)$ as $r \to 0+$.

(ii) $\sqrt{r}v(r,z)$ converges in the dual space $(H_{0,\alpha,A})'$, $(\hat{H}_0^{\beta,B})'$ or $(H_{0,\alpha,A}^{\beta,B})'$ to $g(r) \in \mathscr{E}'(0, \infty)$ as $z \to 1-$.

(iii) $v(r, z)$ converges to zero pointwise on $0 < z < 1$ as $r \to \infty$.

(iv) $v(r, z)$ remains finite at each point of the interval $0 < z < 1$ as $r \to 0+$.

Note that since $\mathscr{E}(0, \infty) \supset H_\mu$ and since H_μ contains $H_{0,\alpha,A}$, $\hat{H}_0^{\beta,B}$ and $H_{0,\alpha,A}^{\beta,B}$, $\mathscr{E}'(0, \infty)$ is contained in the dual spaces $(H_{0,\alpha,A})'$, $(\hat{H}_0^{\beta,B})'$ and $(H_{0,\alpha,A}^{\beta,B})'$. Set $u(r, z) \triangleq \sqrt{r}v(r, z)$,

$$M_0 N_0 u \triangleq r^{-1/2}(\partial/\partial r)r(\partial/\partial r)r^{-1/2}u(r, z).$$

Then (4.4.1) reduces to

$$M_0 N_0 u + \partial^2 u/\partial z^2 = 0. \tag{4.4.2}$$

The conventional Hankel transformation $\hbar_0$ of (4.4.2) yields

$$\frac{\partial^2}{\partial z^2} U(\rho, z) - \rho^2 U(\rho, z) = 0,$$

where $U(\rho, z) \triangleq \hbar_0 u(r, z)$. An easy computation shows

$$\begin{aligned} U(\rho, z) = {} & \frac{1}{e^\rho - e^{-\rho}}\{e^\rho \langle f(\xi), \sqrt{\rho\xi}J_0(\rho\xi)\rangle - \langle g(\xi), \sqrt{\rho\xi})J_0(\rho\xi)\rangle\}\, e^{-\rho z} \\ & + \frac{1}{e^\rho - e^{-\rho}}\{\langle g(\xi), \sqrt{\rho\xi}J_0(\rho\xi)\rangle - e^{-\rho}\langle f(\xi), \sqrt{\rho\xi}J_0(\rho\xi)\rangle\}\, e^{\rho z}. \end{aligned} \tag{4.4.3}$$

The inverse Hankel transform $\hbar_0^{-1}$ of (4.4.3) yields

$$u(r, z) = \int_0^\infty U(\rho, z)\sqrt{\rho r}J_0(\rho r)\, d\rho, \tag{4.4.4}$$

where $U(\rho, z)$ is given by the right-hand side of (4.4.3). Hence we have the solution

$$v(r, z) = r^{-1/2} \cdot u(r, z), \tag{4.4.5}$$

where $u(r, z)$ is given by (4.4.4). It is not hard to show that the solution (4.4.5) satisfies all the boundary conditions (i)–(iv).

Appendix A. On the nontriviality of the spaces of type H_μ. A space V is said to be nontrivial if V contains at least one function which is not identically zero. We first prove that the spaces $H_{\mu,\alpha}$ and H_μ^β are nontrivial for any nonnegative real numbers α and β. Throughout this section μ is any real number.

THEOREM A.1. *For any real number $\alpha \geqq 0$, the space $H_{\mu,\alpha}$ is nontrivial.*

Proof. At first let $\alpha > 0$. Let φ be a smooth function having compact support in $(0, \infty)$ whose Taylor expansion near the origin is of a form

$$x^{\mu+1/2}[a_0 + a_2 x^2 + \cdots + a_{2q}x^{2q} + R_{2q}(x)], \qquad q = 0, 1, \cdots, \tag{A.1}$$

where

$$a_{2q} = \lim_{x \to 0+} \frac{(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))}{2^q \cdot q!} \quad \text{and} \quad R_{2q}(x) = O(x^{2q+2}).$$

Let $L \triangleq \sup\{x : x \in \operatorname{supp} \varphi\}$. Then from (A.1) we obtain

$$|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq C_q\left(\frac{L}{Ak^\alpha}\right)^k A^k k^{k\alpha}, \tag{A.2}$$

where $C_q \triangleq \sup_{0 \leqq x \leqq L}|(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))|$. It is clear that $L/(Ak^\alpha) \leqq 1$ if and only if $k \geqq (L/A)^{1/\alpha}$. Define $k_0 = [(L/A)^{1/\alpha}] + 1$, where $[x]$ denotes the Gaussian symbol, that is, the greatest integer not exceeding x. Setting $C \triangleq \max\{L/A, (L/(A2^\alpha))^2, \cdots, (L/(Ak_0^\alpha))^{k_0}\}$, we get from (A.2) that

$$|x^k(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \leqq CC_qA^kk^{k\alpha}. \tag{A.3}$$

It follows from (A.3) that $\varphi \in H_{\mu,\alpha}$. Consequently $\mathscr{D}(0, \infty) \subset H_{\mu,\alpha}$ algebraically and topologically. Now suppose $\alpha = 0$. As we have noticed in § 2.1, $H_{\mu,0,A} = B_{\mu,A}$ in this case and since $B_{\mu,A}$ is nontrivial (see [16, pp. 168–169]), $H_{\mu,0} = \bigcup_{A=1}^\infty H_{\mu,0,A}$ is also nontrivial. This completes the proof.

THEOREM A.2. *For any real number $\beta \geqq 0$, the space H_μ^β is nontrivial.*

Proof. Suppose first $\mu \geqq -\frac{1}{2}$. Then, since the conventional Hankel transformation $\mathscr{h}_\mu$ maps the space $H_{\mu,\alpha,A}$ into the space $H_\mu^{2\alpha,(2e)^{2\alpha}A^2}$ from Theorem 4.1.1 and since $\mathscr{h}_\mu$ is one-to-one for $\mu \geqq -\frac{1}{2}$, the nontriviality of $H_{\mu,\alpha}$ implies the nontriviality of H_μ^β. If $\mu < -\frac{1}{2}$, by applying $\mathscr{h}_{\mu,m}$ instead of $\mathscr{h}_\mu$, where $\mathscr{h}_{\mu,m}$ is defined by (4.3.1) and m is any positive integer greater than $-\mu - \frac{1}{2}$, we obtain the same conclusion as before. This proves the theorem.

THEOREM A.3. *If $\alpha = \beta = 0$, the space $H_{\mu,0}^0$ is trivial, that is, the only testing function φ in $H_{\mu,0}^0$ is $\varphi \equiv 0$.*

Proof. Since $H_{\mu,0}^0 = \bigcup_{A,B=1}^\infty H_{\mu,0,A}^{\beta,B}$ and since $H_{\mu,0,A_1}^{0,B_1} \subset H_{\mu,0,A_2}^{0,B_2}$ if $A_1 \leqq A_2$ and $B_1 \leqq B_2$, it is enough to show the nontriviality of $H_{\mu,0,A+\delta}^{0,B+\rho}$ for any δ and ρ such that $A + \delta \geqq 1$, $B + \rho \geqq 1$. Assume there exists a testing function φ in $H_{\mu,0,A+\delta}^{0,B+\rho}$ which is not identically zero. Define $\psi(x) \triangleq x^{-\mu-1/2}\varphi(x)$. Then $\operatorname{supp}\varphi = \operatorname{supp}\psi$ and $\varphi \not\equiv 0$ if and only if $\psi \not\equiv 0$. Suppose first $\operatorname{supp}\varphi \subset [0, A + \delta]$. From the definition of $H_{\mu,0,A+\delta}^{0,B+\rho}$, we get

$$|x^k(x^{-1}D)^q\psi(x)| \leqq C_{\delta\rho}(A + \delta)^k(B + \rho)^q, \qquad k, q = 0, 1, \cdots. \tag{A.4}$$

From (3.2.3) and (A.4), an easy computation shows that

$$\begin{aligned} |D^q\psi(x)| &\leqq C_{\delta\rho}\{(A + \delta)^q(B + \rho)^q + b_{q,q-1}(A + \delta)^{q-2}(B + \rho)^{q-1} + \cdots \\ &\qquad + b_{q,(q+I_q)/2}(A + \delta)^{I_q}(B + \rho)^{(q+I_q)/2}\} \\ &\leqq C_{\delta\rho}q!(A + \delta)^q(B + \rho)^q, \end{aligned} \tag{A.5}$$

where $I_q = 0$ or 1 according to whether q is an even integer or an odd integer respectively. The last step of (A.5) is easily justified by induction on q. Since the Taylor expansion of $\psi(x)$ near the point $x = A + \delta$ is

$$\psi(x) = \sum_{r=0}^{q-1}\frac{(x - A - \delta)^r}{r!}\psi^{(r)}(A + \delta) + \frac{(x - A - \delta)^q}{q!}\psi^{(q)}(A + \delta + \varepsilon(x - A - \delta)), \qquad 0 < \varepsilon < 1, \tag{A.6}$$

and since $\psi^{(r)}(A+\delta)=0$ $(r=0,1,\cdots,q-1)$, the inequalities (A.5) and (A.6) yield

$$|\psi(x)| \leqq \frac{|x-A-\delta|^q}{q!}|\psi^{(q)}(A+\delta+\varepsilon(x-A-\delta))|$$
$$< C_{\delta\rho}|x-(A+\delta)|^q(A+\delta)^q(B+\rho)^q \to 0 \quad \text{as } q\to\infty,$$

because the magnitude $|x-(A+\delta)|$ can be taken arbitrarily small. This contradicts the assumption on ψ. Consequently $\psi(x)\equiv 0$ and therefore $\varphi(x)\equiv 0$ for all $x\in(0,A+\delta]$. Finally assume that $\operatorname{supp}\psi \not\subset [0,A+\delta]$. Then for each $x\notin[0,A+\delta]$ such that $\psi(x)\neq 0$, we obtain from (A.4) that for $q=0$,

$$|\psi(x)| \leqq C_{\delta\rho}\left(\frac{A+\delta}{x}\right)^k \to 0 \quad \text{as } k\to\infty.$$

It follows that $\operatorname{supp}\psi\subset[0,A+\delta]$ which contradicts the assumption $\operatorname{supp}\psi \not\subset [0,A+\delta]$. Hence applying the first argument we have $\psi(x)\equiv 0$ and so $\varphi(x)\equiv 0$ for all $x\in(0,\infty)$. This completes the proof.

Example. Consider the function $\varphi(x)=x^{\mu+1/2}\exp[-\frac{1}{2}(B+\rho)x^2]$ where ρ is chosen such that $(A+\delta)^2(B+\rho)\geqq 1$ for each $\delta>0$. Then $\varphi(x)\in H^{0,B}_{\mu,1/2,A} \subset H^0_{\mu,1/2}$.

Appendix B. On certain classes of entire functions and their relations with the spaces of type H_μ. I. M. Gel'fand and G. E. Shilov [2, pp. 221–222] have proved the following theorem.

THEOREM B.1. *If an entire function f satisfies for each $k=0,1,\cdots$ the inequality*

$$|x^k f(x+iy)| \leqq C_k \exp(b|y|^\gamma), \qquad \gamma>1, \tag{B.1}$$

then for any $q=0,1,\cdots$,

$$|x^k f^{(q)}(x)| \leqq C_k B^q q^{q(1-1/\gamma)},$$

where $B=(1/e)(b'e)^{1/\gamma}$, b' is any constant greater than b.

In this section we answer the following question: Under what conditions will the entire function f satisfying the condition (B.1) belong to a space of type H_μ?

Now let β_1 be defined by (3.2.10). Invoking Theorem 3.2.2, we obtain the next theorem from Theorem B.1.

THEOREM B.2. *Let $z^{-\mu-1/2}f(z)$ be an entire function, and let the restriction of $f(z)$ to $0<x<\infty$ belong to the space H_μ. If $F(z)=z^{-\mu-1/2}f(z)$ satisfies the condition* (B.1), *where $z^{-\mu-1/2}$ is understood to be the principal value, then for $\beta\geqq\beta_1$, the restriction of $f(x)$ to $0<x<\infty$ belongs to the space $H^{\beta,B}_\mu$, where $B=(1/e)(b'e)^{1/\gamma}$, b' any constant greater than b.*

Remark. It is easy to confirm by induction on q that β_1 defined by (3.2.10) is greater than 1 for large q and so $\beta_1\geqq 1-1/\gamma$, since $\gamma>1$.

Appendix C. Relations between the spaces S and H_μ, and the spaces of type H_μ and type S. The following theorem is not hard to prove.

THEOREM C.1. *$\varphi \in H_\mu$ if and only if the even extension of $x^{-\mu-1/2}\varphi(x)$ belongs to S and the Taylor expansion near the origin is of the form*

$$x^{\mu-1/2}\{a_0 + a_2x^2 + \cdots + a_{2q}x^{2q} + R_{2q}(x)\}, \tag{C.1}$$

where

$$a_{2q} = \frac{1}{2^q q} \lim_{x\to 0+} (x^{-1}D)^q\{x^{-\mu-1/2}\varphi(x)\} \quad \textit{and} \quad R_{2q}(x) = O(x^{2q+2}).$$

Remark. Consider the function $\varphi(x) = x^{\mu+1/2}e^{-|x|}$. Then $x^{-\mu-1/2}\varphi(x) \in S$. On the other hand, since

$$\lim_{x\to 0+} |(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))| \to \infty, \qquad q = 1, 2, \cdots,$$

the restriction of $\varphi(x)$ to $0 < x < \infty$ does not belong to H_μ. Therefore the condition (C.1) in Theorem C.1 is necessary.

Let β_0 and β_1 be defined by (3.2.5) and (3.2.10) respectively. Define $\beta_2 \triangleq \max\{\beta_0, \beta_1\}$. Then Theorems 3.2.1 and 3.2.2 together yield the following theorem.

THEOREM C.2. *Let $\beta \geqq \beta_2$. Then φ belongs to the space $\hat{H}_\mu^{\beta,B}$ if and only if the even extension of $x^{-\mu-1/2}\varphi(x)$ belongs to $S^{\beta,B}$ and the Taylor expansion of φ near the origin is of the form* (C.1).

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ADDENDUM: QUANTITATIVE ESTIMATES FOR A NONLINEAR SYSTEM OF INTEGRODIFFERENTIAL EQUATIONS ARISING IN REACTOR DYNAMICS*

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A recent question by D. F. Shea prompted me to obtain the following improvement of the results of this paper (both Theorems 1 and 2). Condition (2.18), p. 570, can be weakened to:

There exists a constant $\Gamma > 0$ such that $|\sigma(x)| \leqq \Gamma(1 + S(x))$, $-\infty < x < \infty$.

(2.18*)

In the statement of Theorems 1 and 2 one replaces (2.18) by (2.18*) and all conclusions remain valid.

The proofs require only the following changes in the proofs of Lemmas 2 and 3. To prove the first part of Lemma 2 replace the Lyapunov function $W(x, z)$, p. 577, by

$$U(x, z, t) = (1 + W(x, z)) \exp\left(- \Gamma \int_0^t |e_N(\tau)| \, d\tau \right),$$

where $W(x, z)$ is defined on p. 577. Differentiation of U with respect to the system (3.12) and use of (3.13) yields the estimate

$$W(x(t), z(t)) \leqq U(x(t), z(t), t) \leqq (1 + W_0) \exp\left(\frac{\Gamma k}{\lambda_0} \right), \qquad 0 \leqq t < \infty,$$

for any solution $x(t)$, $z(t)$ of the system (3.12), where W_0, k, λ_0 are constants independent of N defined previously and Γ is defined in (2.18*). This a priori estimate implies global existence and boundedness of solutions of the system (3.12). The remainder of the proof of Lemma 2 is unaffected by the change of (2.18) to (2.18*).

A similar change in the argument is required to obtain global existence and boundedness of solutions of the system (3.19) in the first part of the proof of Lemma 3. Here one replaces $W(x, y, z)$ (p. 579) by

$$U(x, y, z, t) = (1 + W(x, y, z)) \exp\left(- \Gamma \int_0^t |e_N(\tau)| \, d\tau \right).$$

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KRAWTCHOUK POLYNOMIALS AND THE SYMMETRIZATION OF HYPERGROUPS*

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Abstract. This paper introduces the method of symmetrization of the measure algebra of a compact P_*-hypergroup. This method is used to form a measure algebra whose characters are Krawtchouk polynomials (these are the finite sets of polynomials orthogonal with respect to the binomial distribution on $\{0, 1, \cdots, N\}$). As a further application, one derives a theorem about nonnegative expansions of one family of Krawtchouk polynomials in terms of another family.

Introduction. The role of orthogonal polynomials in harmonic analysis has long been of interest. For example, certain sets of orthogonal polynomials appear as sets of spherical functions of compact homogeneous spaces. In this paper we introduce the method of symmetrization of a measure algebra and use it to form a measure algebra whose characters are Krawtchouk polynomials (these are the finite sets of polynomials orthogonal with respect to the binomial distribution on $\{0, 1, \cdots, N\}$). This method also makes it possible to prove a theorem about the expansion of Krawtchouk polynomials of one family in series with positive coefficients of the polynomials from another family.

The underlying structure is that of a compact P_*-hypergroup. This is a compact space on which the space of finite regular Borel measures has a convolution structure preserving the probability measures. Suppose further that a compact group of automorphisms acts on the hypergroup; then the set of measures invariant under the action of the automorphism group forms a subalgebra, which itself has the structure of the measure algebra of a hypergroup.

In the first section of the paper we give the definitions and basic theorems about compact P_*-hypergroups, taken from Dunkl [2]. In § 2 we discuss the action of a compact group of automorphisms on a compact P_*-hypergroup, and show the existence of a symmetrization operator on the continuous functions on the hypergroup. In §§ 3 and 4 we show that the algebra of measures invariant under the automorphism group is the measure algebra of a P_*-hypergroup. Some key formulas are also determined.

Section 5 contains the harmonic analysis structure of the Krawtchouk polynomials. The idea is to take the N-fold Cartesian product of a two-point P_*-hypergroup and let the permutation group on N letters act on the product by permutation of the coordinates. The characters of the symmetrized product are the Krawtchouk polynomials. To illustrate the techniques of §§ 3 and 4 we compute the product theorems for these polynomials.

Section 6 discusses the two kinds of homomorphisms of P_*-hypergroups and uses the symmetrization technique to give nonnegative expansions of one family of Krawtchouk polynomials in terms of another family.

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1. Basic definitions and facts about hypergroups.

DEFINITION 1.1. A locally compact space H is called a *hypergroup* if there exists a map $\lambda: H \times H \to M_p(H)$ (the space of probability measures), with the following properties:

(i) $\lambda(x, y) = \lambda(y, x)$, $x, y \in H$ (so H is a commutative hypergroup).

(ii) For each $f \in C_c(H)$ (the space of continuous functions on H with compact support), the map $(x, y) \mapsto \int_H f\, d\lambda(x, y)$ is in $C^B(H \times H)$ (the space of continuous bounded functions on $H \times H$), and the map $x \mapsto \int_H f\, d\lambda(x, y)$ is in $C_c(H)$ for each $y \in H$.

(iii) The convolution on $M(H)$ defined implicitly by

$$\int_H f\, d\mu * v = \int_H d\mu(x) \int_H dv(y) \int_H f\, d\lambda(x, y),$$

μ, $v \in M(H)$, $f \in C_0(H)$ (the space of continuous functions vanishing at infinity) is associative.

(iv) There exists a unique point $e \in H$ such that $\lambda(x, e) = \delta_x$, $x \in H$.

Remark 1.2. A compact space H is a hypergroup if and only if the space $M(H)$ of regular Borel measures on H is a commutative Banach algebra and the space $M_p(H)$ of probability measures on H is a compact commutative topological semigroup with unit in the weak* topology [2].

DEFINITION 1.3. For the hypergroup H with $f \in C_c(H)$, $x \in H$, and $\mu \in M(H)$, define $R(x)f \in C_c(H)$ by $R(x)f(y) = \int_H f\, d\lambda(x, y)$, $y \in H$; and define the function $R(\mu)f \in C_0(H)$ by $R(\mu)f(y) = \int_H R(z)f(y)\, d\mu(z)$, $y \in H$. (That $R(\mu)f \in C_0(H)$ is shown in [2, Thm. 1.10].)

DEFINITION 1.4. An *invariant measure* m on the hypergroup H is a positive nonzero regular Borel measure on H which is finite on compact sets and such that $\int_H f\, dm = \int_H R(x)f\, dm$, $x \in H$, $f \in C_c(H)$.

DEFINITION 1.5. If a hypergroup H has an invariant measure m, and a continuous involution $x \mapsto x'$, $x \in H$, such that $\int_H (R(x)f)\bar{g}\, dm = \int_H f\{R(x')g\}^-\, dm$, $f, g \in C(H)$, $x \in H$, and such that $e \in \operatorname{spt} \lambda(x, x')$, $x \in H$ and spt denoting the support of a measure, then H is called a *$*$-hypergroup.*

Remark 1.6. The invariant measure m of a $*$-hypergroup is unique up to a constant [2, Prop. 3.2]. If H is compact, we denote the normalized invariant measure of H by m_H, $m_H(H) = 1$.

DEFINITION 1.7. A nonzero function $\phi \in C^B(H)$ is called a *character* of the hypergroup H if the following formula holds:

$$\phi(x)\phi(y) = \int_H \phi\, d\lambda(x, y), \qquad x, y \in H.$$

The set of all characters is denoted by $\hat{H}$. (For $\phi \in \hat{H}$, $|\phi(x)| \leqq \phi(e) = 1$, $x \in H$, [2, Prop. 2.2].)

Remark 1.8. For compact $*$-hypergroups, $\hat{H}$ is an orthogonal basis for $L^2(H)$, and $\hat{H}$ is discrete in the weak* topology from $L^2(H)$, [2, Thm. 3.5].

DEFINITION 1.9. If a compact $*$-hypergroup H has the further property that $\hat{H}\hat{H} \subset \operatorname{co}(\hat{H})$ (the convex hull of $\hat{H}$), then H is called a *compact P_*-hypergroup.*

Remark 1.10. If H is a compact P_*-hypergroup, then the linear span of $\hat{H}$ is sup-norm dense in $C(H)$ (see [2, Prop. 3.8]). For $\phi, \psi \in \hat{H}$, we write

$$\phi(x)\psi(x) = \sum_{\omega \in \hat{H}} n(\phi, \psi; \omega)\omega(x), \qquad x \in H.$$

DEFINITION 1.11. For H a compact P_*-hypergroup, and $\phi \in \hat{H}$, let

$$c(\phi) = \left(\int_H |\phi|^2 \, dm_H\right)^{-1} \geqq 1.$$

Then $\hat{H}$ (with the measure c and conjugation as the involution) is a $*$-hypergroup [2, Thm. 3.9].

THEOREM 1.12. *Let H be a compact P_*-hypergroup. For $x \in H$,*

$$S_x = \sum_{\phi \in \hat{H}} c(\phi)|\phi(x)|^2$$

is finite if and only if $m_H(\{x\}) > 0$, and in this case $S_x = 1/m_H(\{x\})$.

Proof. First note that $\hat{\delta}_x(\phi) = \overline{\phi(x)}$, $x \in H$. By the Plancherel theorem for hypergroups [2, Thm. 3.5], $\hat{\delta}_x \in L^2(H)\hat{} = l^2(\hat{H})$ if and only if $S_x < \infty$. Also $\hat{\delta}_x \in L^2(H)\hat{}$ if and only if $m_H(\{x\}) > 0$. Finally, in this case,

$$\frac{1}{m_H(\{x\})} = \sum_{\phi \in \hat{H}} c(\phi)|\phi(x)|^2.$$

COROLLARY 1.13. *If H is a compact P_*-hypergroup, then $m_H(\{e\}) \leqq m_H(\{x\})$, $x \in H$.*

Proof. $S_x \leqq \sum_{\phi \in \hat{H}} c(\phi) = S_e$.

COROLLARY 1.14. *If H is a denumerable compact P_*-hypergroup, then the identity e is a cluster point.*

Remark 1.15. The authors know of an example of a denumerable compact P_*-hypergroup, and will discuss it in a future paper. The hypergroup comes from having the group of units of Δ_p (the p-adic integers) act on Δ_p.

2. Automorphisms on hypergroups. In this section H will be a compact P_*-hypergroup.

DEFINITION 2.1. Let W be a compact *group of homeomorphisms* on the compact space X. The topology on W is the pointwise topology from X, and the map $(x, \tau) \mapsto \tau(x)$ of $X \times W \to X$ is separately continuous.

For $\tau \in W$ and $f \in C(X)$, define $\tau_1 f \in C(X)$ by $\tau_1 f(x) = f(\tau x)$, $x \in X$. Let τ_1^* be the (weak* continuous) adjoint of τ_1, that is, $\int_X f \, d\tau_1^* \mu = \int_X f \circ \tau \, d\mu$, $f \in C(X)$, $\mu \in M(X)$.

DEFINITION 2.2. An *automorphism* τ on the compact P_*-hypergroup H is a homeomorphism such that $\tau_1^* \lambda(x, y) = \lambda(\tau x, \tau y)$, $x, y \in H$. Thus for $\phi \in \hat{H}$, $\phi \circ \tau \in \hat{H}$,

$$\begin{aligned} \phi \circ \tau(x) \phi \circ \tau(y) &= \int_H \phi \, d\lambda(\tau x, \tau y) = \int_H \phi \, d\tau_1^* \lambda(x, y) \\ &= \int_H \phi \circ \tau \, d\lambda(x, y), \qquad x, y \in H. \end{aligned}$$

Also $\tau(x)' = \tau(x')$, $x \in H$: for $\phi \in \hat{H}$,

$$\phi \circ \tau(x') = \overline{\phi \circ \tau}(x) = \overline{\phi}(\tau x) = \phi((\tau x)').$$

THEOREM 2.3. *Let W be a compact group of automorphisms on the compact P_*-hypergroup H. Then the set $O(\phi) = \{\phi \circ \tau \in \hat{H} : \tau \in W\}$, $\phi \in \hat{H}$, called the orbit of ϕ, is a finite subset of $\hat{H}$.*

Proof. The set $O(\phi)$ is compact in the pointwise topology from H, and hence in the weak topology (as a subset of $C(H)$). Recall from Remark 1.8, $\hat{H}$ is discrete in the weak* topology (as a subset of $L^2(H)$), and thus in the weak topology (as a subset of $C(H)$); and so $O(\phi)$ is a finite subset of $\hat{H}$.

THEOREM 2.4. *Let W be a compact group of automorphisms on the compact P_*-hypergroup H. Then the space W is totally disconnected.*

Proof. For $\phi \in \hat{H}$, let $A_\phi = \{\tau \in W : \phi \circ \tau = \phi\}$. The set A_ϕ is an open neighborhood of the identity e_W in W; and for $\tau_1, \tau_2 \in A_\phi$, $\tau_1\tau_2 \in A_\phi : (\phi \circ \tau_1\tau_2)(x) = \phi \circ \tau_2(x) = \phi(x)$, $x \in H$. Also A_ϕ is inverse closed: for $\tau \in A_\phi$ and $x \in H$ with $\tau(x) = y$, then $\phi(y) = \phi \circ \tau(x) = \phi(x) = \phi \circ \tau^{-1}(y)$. Thus A_ϕ is an open (and closed) subgroup of W, and $\bigcap\{A_\phi : \phi \in \hat{H}\} = \{e_W\}$. This implies that W is totally disconnected.

3. Symmetrization of hypergroups. In this section H will be a compact P_*-hypergroup and W will be a compact group of automorphisms on H.

DEFINITION 3.1. Define the *symmetrization operator* σ_1 on $C(H)$ by

$$\sigma_1 f(x) = \int_W f(\tau x)\, dm_W(\tau), \qquad f \in C(H), \quad x \in H,$$

where m_W denotes the Haar measure on W. The function $\sigma_1 f$ is in $C(H)$ (by the Grothendieck theorem that the pointwise and weak topologies are equivalent on compact subsets of $C(H)$; see also Glicksberg [5]). We let $\sigma : M(H) \to M(H)$ be the (weak* continuous) adjoint of σ_1. Note that σ, σ_1 are projections.

Example 3.2. Let T denote the unit circle, and let $W = \{e_W, \tau\}$ where $\tau(x) = \bar{x}$, $x \in T$. Then for $f \in C(T)$, $\sigma_1 f(x) = \frac{1}{2}(f(x) + f(\bar{x}))$, $x \in T$.

DEFINITION 3.3. Let H be a compact P_*-hypergroup and W a compact group of automorphisms on H. We define the compact space H_W by identifying the points of H which are in the same orbit; that is, $H_W = H/\sim$ where $x \sim y$ if and only if there exists $\tau \in W$ such that $\tau x = y$.

Let $C_W(H)$ denote the space

$$\{f \in C(H) : f \circ \tau = f, \text{ all } \tau \in W\}.$$

THEOREM 3.4. $\sigma_1 C(H) = C_W(H)$.

Proof. Let $f \in \sigma_1 C(H)$, $x \in H$; then $f = \sigma_1 g$, some $g \in C(H)$, and $f(x) = \int_W g(\tau x)\, dm_W(\tau)$. Thus by the translation invariance of the Haar measure on W, $f \circ \rho(x) = \int_W g(\tau\rho x)\, dm_W(\tau) = \int_W g(\tau x)\, dm_W(\tau) = f(x)$, $\rho \in W$; and so $\sigma_1 C(H) \subset C_W(H)$.

Conversely, if $f \in C_W(H)$, $\sigma_1 f = f$ since $\sigma_1 f(x) = \int_W f(\tau x)\, dm_W(\tau) = f(x) \cdot \int_W dm_W = f(x)$, $x \in H$. Thus $C_W(H) \subset \sigma_1 C(H)$.

DEFINITION 3.5. Let $M_W(H) = \{\mu \in M(H) : \int_H f\, d\mu = \int_H f \circ \tau\, d\mu$, all $\tau \in W$, $f \in C(H)\}$.

THEOREM 3.6. *Let H be a compact P_*-hypergroup and W a compact group of automorphisms on H. Then $C_W(H) \cong C(H_W)$ and*

$$\sigma M(H) = M_W(H) \cong M(H_W).$$

Proof. That $C_W(H) \cong C(H_W)$ is immediate from the definition of H_W. That $M_W(H) \cong M(H_W)$ follows from Theorem 3.4; that is,

$$M(H_W) = C_W(H)^* \cong (\sigma_1 C(H))^* = M_W(X).$$

Remark 3.7. For convenience, we often identify $C(H_W)$, $M(H_W)$ with $C_W(H)$, $M_W(H)$ respectively.

THEOREM 3.8. *For μ, $\nu \in M(H)$, H a compact hypergroup, and W a compact group of automorphisms, $\sigma(\mu * \sigma\nu) = \sigma\mu * \sigma\nu$.*

Proof. For $\lambda \in M(H)$, $\sigma\lambda$ is defined by $\int_H f\, d\sigma\lambda = \int_H \sigma_1 f\, d\lambda$, $f \in C(H)$. Since σ is weak* continuous, it will suffice to let $\mu = \delta_x$, $x \in H$.

Now for $f \in C(H)$,

$$\begin{aligned}
\int_H f\, d\sigma\delta_x * \sigma\nu &= \int_H d\sigma\nu(v) \int_H d\sigma\delta_x(u) \int_H f\, d\lambda(u, v) \\
&= \int_H d\sigma\nu(v) \int_H \int_W \int_H f\, d\lambda(\tau u, v)\, dm_W(\tau)\, d\delta_x(u) \\
&= \int_H d\sigma\nu(v) \int_W \int_H f\, d\lambda(\tau x, v)\, dm_W(\tau) \\
&= \int_H d\sigma\nu(v) \int_W \int_H f \circ \tau\, d\lambda(x, \tau^{-1} v)\, dm_W(\tau) \\
&= \int_H d\sigma\nu(v) \int_W R(x)(f \circ \tau)(\tau^{-1} v)\, dm_W(\tau) \\
&= \int_W \int_H R(x)(f \circ \tau)(\tau^{-1} v)\, d\sigma\nu(v)\, dm_W(\tau) \\
&= \int_W \int_H R(x)(f \circ \tau)\, d(\tau^{-1})^*(\sigma\nu)\, dm_W(\tau) \\
&= \int_W \int_H R(x) f \circ \tau\, d\sigma\nu\, dm_W(\tau) \\
&= \int_W \int_H f \circ \tau\, d(\delta_x * \sigma\nu)\, dm_W(\tau) \\
&= \int_H f\, d\sigma(\delta_x * \sigma\nu).
\end{aligned}$$

COROLLARY 3.9. *The space $\sigma M(H) = M_W(H)$ is a commutative Banach algebra; and the space $M_{W_p}(H)$ of probability measures in $M_W(H)$ is isomorphic to $M_p(H_W)$, and it is a commutative topological semigroup with unit (and thus H_W is a hypergroup).*

COROLLARY 3.10. *$\sigma m_H = m_H$ and under the isomorphism of $M_W(H)$ and $M(H_W)$, m_H is identified with m_{H_W}.*

Proof. Since H_W is a compact hypergroup there exists an invariant measure on H_W which is unique [2, Thm. 1.12].

Let $\mu \in \sigma M(H)_p$; then $\sigma m_H * \sigma\mu = \sigma(m_H * \sigma\mu) = \sigma(m_H * \mu) = \sigma m_H$, and so σm_H is an invariant measure on H_W.

Further, for $\tau \in W$, $\tau^* m_H = m_H$, since $\tau^* m_H$ is an invariant probability measure on H.

COROLLARY 3.11. *If H is a finite P_*-hypergroup, then*

$$m_{H_W}(\{\sigma x\}) = \sum_{y \in O(x)} m_H(\{y\}),$$

where σx denotes the element of H_W $(= H/\sim)$ which contains x, and $O(x) = \{y \in H:$ there exists $\tau \in W$ with $\tau y = x\}$, the orbit of x.

Proof. Let χ_A denote the characteristic function of the set A, $A \subset H$. Then

$$\begin{aligned} m_{H_W}(\{\sigma x\}) &= \int_{H_W} \chi_{\sigma x}\, dm_{H_W} \\ &= \int_H \chi_{O(x)}\, d\sigma m_H \\ &= \int_H \chi_{O(x)}\, dm_H = \sum_{y \in O(x)} m_H(\{y\}). \end{aligned}$$

4. Duals of symmetrized hypergroups. In this section H is a compact P_*-hypergroup and W is a compact group of automorphisms on H. Let σ_1, σ be as in § 3.

PROPOSITION 4.1. *For $\phi \in \hat{H}$, $\sigma_1\phi \in \hat{H}_W$.*

Proof. A continuous function g on H_W is a character if g defines a nonzero multiplicative functional on $M(H_W)$ (see [2, Prop. 2.3]).

For $\mu \in \sigma M(H) \cong M(H_W)$,

$$\int_H \overline{\sigma_1\phi}\, d\mu = \int_H \overline{\phi}\, d\sigma\mu = \int_H \overline{\phi}\, d\mu = \hat{\mu}(\phi);$$

and so $\sigma_1\phi \in \hat{H}_W$.

DEFINITION 4.2. Define $\hat{\sigma}: \hat{H} \to \hat{H}_W$ by $\hat{\sigma}\phi = \sigma_1\phi$.

THEOREM 4.3. *Let $\phi, \psi \in \hat{H}$; then $(\hat{\sigma}\phi)\cdot(\hat{\sigma}\psi) = \sigma_1(\phi \cdot \hat{\sigma}\psi)$.*

Proof. Let $x \in H$; then

$$\begin{aligned} \sigma_1(\phi \cdot \hat{\sigma}\psi)(x) &= \int_W \phi(\tau x)\hat{\sigma}\psi(\tau x)\, dm_W(\tau) \\ &= \hat{\sigma}\psi(x) \int_W \phi(\tau x)\, dm_W(\tau) \\ &= \hat{\sigma}\psi(x)\hat{\sigma}\phi(x). \end{aligned}$$

THEOREM 4.4. *Let W be a compact group of automorphisms on the P_*-hypergroup H. Then H_W is a compact $*$-hypergroup.*

Proof. That H_W is a compact hypergroup is given in Corollary 3.9. The invariant measure on H_W (as a hypergroup) is m_H by Corollary 3.10.

The involution in H_W is defined by $(\sigma x)' = \sigma(x')$, $x \in H$. This is well-defined since $(\tau x)' = \tau(x')$, $\tau \in W$. This involution is continuous since the map $\sigma : M(H) \to \sigma M(H)$ is weak* continuous. Let $f, g \in C_W(H)$, $x \in H$; then

$$\begin{aligned}\int_H (R(\sigma x) f)\bar{g}\, dm_H &= \int_H (R(\sigma x)\sigma_1 f)\overline{\sigma_1 g}\, dm_H \\ &= \int_H (R(x)\sigma_1 f)\overline{\sigma_1 g}\, dm_H = \int_H \sigma_1 f\, \overline{R(x')\sigma_1 g}\, dm_H \\ &= \int_H \sigma_1 f\, \overline{R(\sigma x')\sigma_1 g}\, dm_H = \int_{H_W} f(\overline{R(\sigma x)' g})\, dm_{H_W}.\end{aligned}$$

Since $e \in \operatorname{spt} \lambda(x, x')$, $x \in H$, then $e \in \operatorname{spt} \lambda(\sigma x, \sigma x')$. We have thus shown that H_W is a compact $*$-hypergroup.

THEOREM 4.5. *Let W be a compact group of automorphisms on the compact P_*-hypergroup H. Then H_W is a compact P_*-hypergroup, and $\hat{H}_W = \hat{\sigma}\hat{H}$.*

Proof. We show for $\hat{\sigma}\phi \neq \hat{\sigma}\psi$, $\phi, \psi \in \hat{H}$, that $\int_{H_W} \hat{\sigma}\phi \cdot \overline{\hat{\sigma}\psi}\, dm_{H_W} = 0$:

$$\begin{aligned}\int_{H_W} \hat{\sigma}\phi \cdot \overline{\hat{\sigma}\psi}\, dm_{H_W} &= \int_{H_W} \sigma_1(\phi \cdot \overline{\hat{\sigma}\psi})\, dm_{H_W} \\ &= \int_H \phi \cdot \overline{\sigma_1 \psi}\, dm_H = \int_H \phi \int_W \overline{\psi \circ \tau}\, dm_W(\tau)\, dm_H \\ &= \int_W \int_H \phi \cdot \overline{\psi \circ \tau}\, dm_H\, dm_W(\tau) = 0\end{aligned}$$

since distinct characters are orthogonal on H.

Now let $\phi, \psi \in \hat{H}$ with $\hat{\sigma}\phi = \hat{\sigma}\psi$. Thus

$$\begin{aligned}\int_{H_W} \sigma\phi\overline{\sigma\psi}\, dm_{H_W} &= \int_W \int_W \int_H \phi(\tau x)\overline{\psi(\tau' x)}\, dm_H(x)\, dm_W(\tau)\, dm_W(\tau') \\ &= \int_W \int_W \int_H \phi(\tau(\tau')^{-1} x)\overline{\psi(x)}\, dm_H(x)\, dm_W(\tau)\, dm_W(\tau') \\ &= \int_W \int_H \phi(\tau x)\overline{\psi(x)}\, dm_H(x)\, dm_W(\tau) \\ &= m_W(\{\tau \in W : \phi \circ \tau = \phi\}) \int_H |\phi|^2\, dm_H.\end{aligned}$$

We next show that $\hat{\sigma}\hat{H} \cdot \hat{\sigma}\hat{H} \subset \hat{\sigma}\hat{H}$. Let $\phi, \psi \in \hat{H}$; then

$$\hat{\sigma}\phi \cdot \hat{\sigma}\psi = \sigma_1(\phi\hat{\sigma}\psi) = \sigma_1\left(\phi \cdot \frac{1}{|O(\phi)|} \sum_{\omega \in O(\psi)} \omega\right) \in \sigma_1(\hat{H} \cdot \operatorname{co} \hat{H}) \subset \sigma_1(\operatorname{co} \hat{H}) \subset \operatorname{co}(\hat{\sigma}\hat{H}).$$

(Note that for $\tau \in W$, $n(\phi, \psi; \omega) = n(\phi \circ \tau, \psi \circ \tau; \omega \circ \tau)$, $\phi, \psi, \omega \in \hat{H}$ since $n(\phi, \psi; \omega) = c(\omega) \int_H \phi\psi\bar{\omega}\, dm_H$.) Thus $\mathrm{Sp}\,(\hat{\sigma}\hat{H})$ is dense in $C_W(H)$.

We have thus shown that $\hat{\sigma}\hat{H}$ is a complete orthogonal basis for $L^2(H_W)$, and so $\hat{H}_W = \hat{\sigma}\hat{H}$. It follows that H_W is a compact P_*-hypergroup.

COROLLARY 4.6. *For* $\phi \in \hat{H}$, $c(\hat{\sigma}\phi) = c(\phi)|O(\phi)|$, *where* $|O(\phi)|$ *denotes the cardinality of the set* $O(\phi)$.

We now derive the functional equation for symmetrized characters.

THEOREM 4.7. *Let* $\phi \in C(H)$. *Then* $\phi \in \hat{H}_W$ *if and only if the following condition holds for all* $x, y \in H$:

$$\phi(x)\phi(y) = \int_W \int_H \phi\, d\lambda(\tau x, y)\, d\tau. \tag{1}$$

Proof. Assume $\phi \in \hat{H}_W$. Then

$$\begin{aligned}
\int_W \int_H \phi\, d\lambda(\tau x, y)\, dm_W(\tau) &= \int_W \int_H R(y)\phi\, d\delta_{\tau x}\, dm_W(\tau) \\
&= \int_H \int_W R(y)\phi\, d\delta_{\tau x}\, dm_W(\tau) \\
&= \int_H R(y)\phi\, d\sigma\delta_x = \int_H \phi\, d(\sigma\delta_x) * \delta_y \\
&= \int_H \sigma_1\phi\, d\sigma\delta_x * \delta_y \\
&= \int_H \phi\, d\sigma(\sigma\delta_x * \delta_y) = \int_H \phi\, d\sigma\delta_x * \sigma\delta_y \\
&= \phi(x)\phi(y).
\end{aligned}$$

Conversely, assume condition (1). Let $x_1 \sim x_2$. Then

$$\begin{aligned}
\phi(x_1)\phi(y) &= \int_W \int_H \phi\, d\lambda(\tau x_1, y)\, dm_W(\tau) \\
&= \int_H R(y)\phi\, d\sigma\delta_{x_1} = \int_H R(y)\phi\, d\sigma\delta_{x_2} \\
&= \int_W \int_H \phi\, d\lambda(\tau x_2, y)\, dm_H(\tau) = \phi(x_2)\phi(y), \qquad y \in H.
\end{aligned}$$

Therefore $\phi(x_1) = \phi(x_2)$, and $\phi \in C_W(H)$. Furthermore,

$$\begin{aligned}
\int_H \phi \, d\lambda(\sigma x, \sigma y) &= \int_H \phi \, d\sigma\delta_x * \sigma\delta_y \\
&= \int_H \phi \, d\sigma(\sigma\delta_x * \delta_y) \\
&= \int_H \sigma_1\phi \, d(\sigma\delta_x * \delta_y) \\
&= \int_H \phi \, d(\sigma\delta_x * \delta_y) = \int_H R(y)\, \phi \, d\sigma\delta_x \\
&= \int_H \int_W R(y)\phi \, d\delta_{\tau x} \, dm_W(\tau) \\
&= \int_W \int_H R(y)\phi \, d\delta_{\tau x} \, dm_W(\tau) \\
&= \int_W \int_H \phi \, d\lambda(\tau x, y) \, dm_W(\tau) \\
&= \phi(x)\phi(y).
\end{aligned}$$

And so $\phi \in \hat{H}_W$.

Remark 4.8. As an application of Theorem 4.7, one can derive the functional equation for the characters of a compact group. That is, for $f \in C(G), f(x) = \chi_\alpha/n_\alpha$ if and only if

$$f(x)f(y) = \int_G f(zxz^{-1}y) \, dm_G(z)$$

for all x, $y \in G$ (see Weil [8, p. 87]). The result is obtained by symmetrizing the noncommutative hypergroup G by the compact group of inner automorphisms of G.

5. Krawtchouk polynomials. The Krawtchouk polynomials k_n, $0 \leqq n \leqq N$, are an orthogonal set of polynomials on the discrete set $\{0, 1, \cdots, N\}$, where $0 < p < 1$, N is a positive integer, and the weight function is $\binom{N}{x} p^x(1 - p)^{N-x}$, $x = 0, 1, \cdots, N$. The polynomials are given in terms of the hypergeometric functions by

$$\begin{aligned}
k_n(x; p, N) &= (1 - p)^n \binom{x}{n} F[-n, x - N; x - n + 1; p/(p - 1)] \\
&= \frac{p^n(-N)_n}{n!} F[-n, -x; -N; 1/p]
\end{aligned}$$

(see [1, § 6]). We normalize the Krawtchouk polynomials by

$$K_n(x; p, N) = \frac{k_n(x; p, N)}{k_n(0; p, N)}.$$

Szegö [6, p. 36] gives the explicit formula

$$\begin{aligned} K_n(x; p, N) &= \sum_{j=0}^{\min(x,n)} (-1)^j \left(\frac{1-p}{p}\right)^j \frac{(N-x)!(N-n)!n!x!}{(n-j)!(x-j)!(N+j-x-n)!N!j!} \\ &= \sum_{j=0}^{\min(x,n)} \left(\frac{p-1}{p}\right)^j \binom{N-x}{n-j}\binom{x}{j} \Big/ \binom{N}{n}. \end{aligned}$$

We now show how the symmetrized characters of the product of a two-point hypergroup yields the Krawtchouk polynomials.

Let $-1 \leqq a < 0$ and define H_a to be the two-point hypergroup with points $\{\mathbf{1}, \mathbf{a}\}$ such that $\lambda(\mathbf{1}, \mathbf{1}) = \delta_{\mathbf{1}}, \lambda(\mathbf{1}, \mathbf{a}) = \lambda(\mathbf{a}, \mathbf{1}) = \delta_{\mathbf{a}}$, and $\lambda(\mathbf{a}, \mathbf{a}) = -a\delta_{\mathbf{1}} + (a+1)\delta_{\mathbf{a}}$. The identity e is $\mathbf{1}$. The hypergroup H_a has two characters χ_0, χ_1 defined by $\chi_0 = 1$, and $\chi_1(\mathbf{1}) = 1$ and $\chi_1(\mathbf{a}) = a$. Furthermore, H_a is a compact P_*-hypergroup. For notational convenience, we write

$$H_a \cong \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix},$$

where the points of H_a are the rows and the characters are the columns. Let m denote the invariant measure on H_a, defined by $m(\{\mathbf{1}\}) = 1 - p$ and $m(\{\mathbf{a}\}) = p$. Then $a = (p-1)/p$ and $\frac{1}{2} \leqq p < 1$. Also $c(\chi_0) = 1$ and $c(\chi_1) = p/(1-p) = -1/a$.

Let N be a positive integer, and let H_a^N denote the hypergroup $H_a \times H_a \times \cdots \times H_a$, ($N$ times). The elements of H_a^N are N-tuples of $\mathbf{1}$'s and $\mathbf{a}$'s, and the elements of $(H_n^N)\hat{}$ are N-tuples of χ_0's and χ_1's.

The permutation group P_N on N letters acts naturally on the hypergroup H_a^N as a compact group W of automorphisms. We let σ_1, σ respectively denote the symmetrization operators on $C(H_a^N)$, $M(H_a^N)$ respectively.

For $x \in H_a^N$, let $[x] \in \{0, 1, \cdots, N\}$ be the number of times $\mathbf{a}$ appears in the N-tuple x. Let $\sigma x \in (H_a^N)_W$ be represented by $(\mathbf{1}, \mathbf{1}, \cdots, \mathbf{1}, \mathbf{a}, \mathbf{a}, \cdots, \mathbf{a})$, $N - [x]\,\mathbf{1}$'s and $[x]$ $\mathbf{a}$'s. For $0 \leqq n \leqq N$, let $\chi_n \in (H_a^N)_W\hat{}$ be represented by $(\chi_0, \chi_0, \cdots, \chi_0, \chi_1, \chi_1, \cdots, \chi_1)$, $N - n$ χ_0's and n χ_1's.

For $\varepsilon \in \prod_{k=1}^N \{0, 1\} \cong (H_a^N)\hat{}$, let χ_ε denote the element of $(H_a^N)\hat{}$ associated with ε; that is, $\chi_\varepsilon = \chi_{\varepsilon_1}\chi_{\varepsilon_2}\cdots\chi_{\varepsilon_N}$. For $0 \leqq n \leqq N$, choose $\varepsilon \in \prod_{k=1}^N \{0, 1\}$ with $N - n$ 0's and n 1's. We compute the symmetrized character $\chi_n \in (H_a^N)_W\hat{}$. Recall that for $x \in H_a^N$,

$$\chi_n(\sigma x) = \sigma_1 \chi_\varepsilon(x).$$

To compute $\sigma_1\chi_\varepsilon$ we note that χ_ε has n χ_1's and only the χ_1's need be evaluated. Indeed to get the jth power of a, j of the $[x]$ $\mathbf{a}$'s must be paired with j of the χ_1's, and $n - j$ of the $N - [x]$ $\mathbf{1}$'s must be paired with the remaining $n - j$ of the χ_1's.

The number of times this occurs is $\binom{[x]}{j}\binom{N-[x]}{n-j}$ as the permutation group P_N

acts on x. Since characters assume the value 1 at the identity, it follows that

$$\chi_n(\sigma x) = \sum_{j=0}^{\min([x],n)} a^j \binom{[x]}{j}\binom{N-[x]}{n-j} \Big/ \binom{N}{n},$$

$0 \leqq n \leqq N$, $x \in H_a^N$. We have thus shown the following result.

THEOREM 5.1. *Let H_a^N be the product of N copies of the finite P_*-hypergroup $H_a = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$, $-1 \leqq a < 0$; and let W be the finite group of automorphisms of H_a^N given by the permutation group on N letters. Then the symmetrized characters of $(H_a^N)_W$ are given by*

$$\chi_n(\sigma x) = K_n([x]; p, N),$$

$x \in H_a^N$, $0 \leqq n \leqq N$, $p = 1/(1-a)$.

We now derive a product theorem for Krawtchouk polynomials.

Askey and Gasper [1], using a method of Eagleson [3], have shown that the Krawtchouk polynomials satisfy the following linearization theorem:

$$K_n(x; p, N)K_n(y; p, N) = \sum_{z=0}^{N} I(z, y, x; p, N)K_n(z; p, N),$$

$0 \leqq x, y, n \leqq N$, $0 < p < 1$, where

$$I(z, y, x; p, N) = \frac{N!}{p^{x+y}(1-p)^z \binom{N}{x}\binom{N}{y}} \cdot \sum_{j \geqq 0} \frac{p^j(1-p)^j(2p-1)^{z+y+x-2j}}{(j-z)!(j-y)!(j-x)!(z+y+x-2j)!(N-j)!}.$$

Therefore $I(z, y, x; p, N) \geqq 0$ for $\frac{1}{2} \leqq p < 1$.

Since the Krawtchouk polynomials are the symmetrized characters of the hypergroup H_a^N, $a = (p-1)/p$, $\frac{1}{2} \leqq p < 1$, we will be able to obtain this result for $\frac{1}{2} \leqq p < 1$ from the fact that characters ϕ satisfy the functional equation

$$\phi(x)\phi(y) = \int_H \phi \, d\delta_x * \delta_y$$

($x, y \in H$ a hypergroup) once a formula for $\delta_x * \delta_y$ is obtained.

Recall $-1 \leqq a < 0$, $a = (p-1)/p$, $H_a \cong \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$, $N = 1, 2, \cdots$, and σ_1, σ are the symmetrization projections on $C(H_a^N)$, $M(H_a^N)$ respectively. For $x \in H_a^N$, $[x]$ denotes the number of times $\mathbf{a}$ appears in the N-tuple x. The convolution structure of $M(H_a)$ is given by $\delta_1 * \delta_1 = \delta_1$, $\delta_1 * \delta_{\mathbf{a}} = \delta_{\mathbf{a}}$ and $\delta_{\mathbf{a}} * \delta_{\mathbf{a}} = -a\delta_1 + (1+a)\delta_{\mathbf{a}}$. For $u, v \in M(H_a^N)$, $\delta_u * \delta_v = (\delta_{u_1} * \delta_{v_1}) \times \cdots \times (\delta_{u_N} * \delta_{v_N})$.

We wish to compute $\sigma\delta_x * \sigma\delta_y$ for $x, y \in H_a^N$. For $z \in H_a^N$, we define $\langle z \rangle \in M_W(H_a^N)$ by $\langle z \rangle = \sigma(\delta_1 \times \delta_1 \times \cdots \times \delta_1 \times \delta_{\mathbf{a}} \times \delta_{\mathbf{a}} \times \cdots \times \delta_{\mathbf{a}}) = \sigma\delta_z$, ($N - [z]$ δ_1's and $[z]$ $\delta_{\mathbf{a}}$'s). Now $\sigma\delta_x * \sigma\delta_y = \sigma(\langle x \rangle * \delta_y)$, (Theorem 3.8). For $\tau \in W$, let $j\tau$ denote the number of coordinates i where both $x_i = \mathbf{a}$ and $(\tau y)_i = \mathbf{a}$. Hence for

$\tau \in W$, $\delta_x * \delta_{\tau y}$ is a Cartesian product in which $-a\delta_{\mathbf{1}} + (1+a)\delta_{\mathbf{a}}$ appears $j\tau$ times, δ_a appears $([x] - j\tau) + ([y] - j\tau)$ times, and $\delta_{\mathbf{1}}$ appears $N + j\tau - [x] - [y]$ times. By applying the appropriate $\tau' \in W$ to $\delta_x * \delta_{\tau y}$ we may assume $\delta_x * \delta_{\tau y} = (\delta_{\mathbf{1}})^{N+j\tau-[x]-[y]} \times (\delta_{\mathbf{a}})^{[x]+[y]-2j\tau} \times (-a\delta_{\mathbf{1}} + (1+a)\delta_{\mathbf{a}})^{j\tau}$ (where the exponent denotes the number of successive appearances in the product). The symmetrization of the last $j\tau$ factors is the same as that of $\sum_{k=0}^{j\tau} \binom{j\tau}{k} (-a)^{j\tau-k}(1+a)^k \delta_{\mathbf{1}}^{j\tau-k}\delta_{\mathbf{a}}^k$. Since the functions in $C_W(H_a^N)$ are constant on the equivalence classes in $H_a^N/\sim$, we have that

$$\sigma(\langle x\rangle * \delta_y) = \frac{1}{N!}\sum_{\tau\in W}\sum_{k=0}^{j\tau} \langle [x] - j\tau + [y] - j\tau + k\rangle \binom{j\tau}{k}(-a)^{j\tau-k}(1+a)^k$$

$$= \frac{1}{N!}\sum_{\tau\in W}\sum_{l=0}^{N} \langle l\rangle \binom{j\tau}{l - [x] - [y] + 2j}$$

$$\cdot (-a)^{[x]+[y]-j\tau-l}(1+a)^{l-[x]-[y]+2j\tau}$$

$$(\text{let } l = [x] + [y] - 2j\tau + k)$$

$$= \frac{1}{\binom{N}{[y]}}\sum_{l=0}^{N} \langle l\rangle \sum_{j=0}^{N} \binom{[x]}{j}\binom{N-[x]}{[y]-j}\binom{j}{l-[x]-[y]+2j}$$

$$\cdot (-a)^{[x]+[y]-j-l}(1+a)^{l-[x]-[y]+2j},$$

since $\binom{[x]}{j}\binom{N-[x]}{[y]-j}[y]!(N-[y])!$ is the number of $\tau \in W$ for which $j\tau = j$.

We have thus shown the following result.

THEOREM 5.2. *Let $-1 \leqq a < 0, \frac{1}{2} \leqq p < 1$, and $x, y \in H_a^N$. Then*

$$\sigma x * \sigma y = \sum_{l=0}^{N} J(l, [y], [x]; p, N)\langle l\rangle,$$

where

$$J(l, [y], [x]; p, N) = \frac{1}{\binom{N}{[y]}}\sum_{j=0}^{N} \binom{[x]}{j}\binom{N-[x]}{[y]-j}\binom{j}{l-[x]-[y]+2j}$$

$$\cdot (-a)^{[x]+[y]-j-l}(1+a)^{l-[x]-[y]+2j},$$

and where $\langle l\rangle$ denotes $\sigma(\delta_{\mathbf{1}} \times \cdots \times \delta_{\mathbf{1}} \times \delta_{\mathbf{a}} \times \cdots \times \delta_{\mathbf{a}})$, $N - l$ $\delta_{\mathbf{1}}$'s and l $\delta_{\mathbf{a}}$'s. Thus for $0 \leqq n \leqq N$,

$$K_n(x; p, N)K_n(y; p, N) = \sum_{l=0}^{N} J(l, y, x; p, N)K_n(l; p, N),$$

$$0 \leqq x, y \leqq N.$$

PROPOSITION 5.3. *$J(l, y, x; p, N) = I(l, y, x; p, N), 0 \leqq l, y, x \leqq N, 0 < p < 1$.*

Proof.

$$J(l, y, x; p, N) = \frac{y!(N-y)!}{N!} \sum_{j=0}^{N} \frac{x!(N-x)!j!}{j!(x-j)!(y-j)!(N-x-y+j)!(l-x-y+2j)!}$$
$$\cdot \frac{(1-p)^{x+y-j-l} p^{-x-y+j+l-l+x+y-2j}}{(x+y-l-j)!} \cdot (2p-1)^{l-x-y+2j}$$
$$\text{(since } 1 + a = (2p-1)/p\text{)}$$
$$= \frac{N!}{\binom{N}{y}\binom{N}{x}} \sum_{k=0}^{N} \frac{(1-p)^{k-l}(2p-1)^{l+x+y-2k} p^{k-x-y}}{(k-y)!(k-x)!(N-k)!(l-2k+x+y)!(k-l)!}$$
$$\text{(let } k = x + y - j\text{)}$$
$$= \frac{N!}{\binom{N}{y}\binom{N}{x}} p^{-x-y}(2p-1)^{x+y}$$
$$\cdot \sum_{k=0}^{N} \frac{p^k(1-p)^{k-l}(2p-1)^{l-2k}}{(k-y)!(k-x)!(N-k)!(l-2k+x+y)!(k-l)!}$$
$$= I(l, y, x; p, N).$$

THEOREM 5.4. *Let* $\frac{1}{2} \leqq p < 1, 0 \leqq x, n \leqq N$. *Then*

$$\sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} |K_n(x; p, N)|^2 = \frac{1}{\binom{N}{n}} \left(\frac{1-p}{p}\right)^n.$$

Proof. The left-hand side is

$$(c(K_n(\cdot\,; p, N)))^{-1} = \int_{H_a^N} |K_n([x]; p, N)|^2 \, d\sigma m_{H_a^N}(x),$$
$$a = (p-1)/p.$$

By Corollary 4.6, $c(\hat{\sigma}\phi) = c(\phi)\,|O(\phi)|$, ϕ a character on a compact P_*-hypergroup and σ a symmetrization projection. Let $\chi_\varepsilon = (\chi_0, \cdots, \chi_0, \chi_1, \cdots, \chi_1)$, $N - n$ χ_0's and n χ_1's, be a character on H_a^N. Then $|O(\chi_\varepsilon)| = \binom{N}{n}$, and

$$(c(\chi_\varepsilon))^{-1} = \int_{H_a^N} |\chi_\varepsilon(x)|^2 \, dm_{H_a^N}(x)$$
$$= \left(-\frac{1}{a}\right)^n = \left(\frac{p}{1-p}\right)^n.$$

Finally, recall that $\hat{\sigma}\chi_\varepsilon(x) = K_n([x]; p, N)$.

6. Homomorphisms between hypergroups. Let $-1 \leqq a \leqq b < 0$ and let $H_a = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$, $H_b = \begin{pmatrix} 1 & 1 \\ 1 & b \end{pmatrix}$ be the two-point P_*-hypergroups associated with a, b respectively (as in § 5). Define $\rho_1 : C(H_b) \to C(H_a)$ by $\rho_1 \chi_0^{(b)} = \chi_0^{(a)}$ and

$$\rho_1 \chi_1^{(b)} = \frac{b-a}{1-a} \chi_0^{(a)} + \frac{1-b}{1-a} \chi_1^{(a)}.$$

The map ρ_1 is the induced adjoint map of the point map $\rho : H_a \to H_b$ defined by $\rho \mathbf{1} = \mathbf{1}$ and $\rho \mathbf{a} = \mathbf{b}$.

We compute now the action of ρ_1^N relative to the symmetrized hypergroups $(H_a^N)_W, (H_b^N)_W$, where W is the permutation group on N letters, $N = 1, 2, \cdots$.

PROPOSITION 6.1. $\rho_1^N : C_W(H_b^N) \to C_W(H_a^N)$.

Proof. That $\rho_1^N C_W(H_b^N) \subset C_W(H_a^N)$ follows since there exists a one-to-one map α of $W^{(a)}$ into $W^{(b)}$ such that the following diagram commutes:

$$\begin{tikzcd} H_a^N \arrow[r, "\rho^N"] \arrow[d, "\tau^{(a)}"'] & H_b^N \arrow[d, "\alpha\tau^{(a)}"] \\ H_a^N \arrow[r, "\rho^N"] & H_b^N \end{tikzcd}$$

To see this, let $f \in C(H_b^N)$ with $f \circ \tau^{(b)} = f$ for all $\tau^{(b)} \in W^{(b)}$; then for $x \in H_a^N$ and $\tau^{(a)} \in W^{(a)}$, $(\rho_1^N f) \circ \tau^{(a)}(x) = f \circ \rho^N(\tau^{(a)} x) = f(\alpha\tau^{(a)} \circ \rho^N)(x) = f(\rho^N x) = (\rho_1^N f)(x)$, and so $\rho_1^N f \in C_W(H_b^N)$, which completes the proof.

For $0 \leqq n \leqq N$, define $\chi_n^{(a)}, \chi_n^{(b)}$ to be the symmetrized characters on H_a^N, H_b^N, respectively, associated with $N - n\chi_0^{(a)}$'s, $\chi_0^{(b)}$'s, respectively, and $n\chi_1^{(a)}$'s, $\chi_1^{(b)}$'s, respectively. Let $x \in H_a^N$ and so x has $[x]$ $\mathbf{a}$'s and $N - [x]$ $\mathbf{1}$'s.

We write $\chi_n^{(b)} = \sigma_1(\chi_0^{(b)}, \cdots, \chi_0^{(b)}, \chi_1^{(b)}, \cdots, \chi_1^{(b)})$. Similarly for $\chi_n^{(a)}$. Thus

$$\begin{aligned} \rho_1^N \chi_n^{(b)}(x) = \chi_n^{(b)}(\rho^N x) &= \sigma_1 \prod_{j=N-n+1}^{N} \chi_1^{(b)}((\rho^N x)_j) \\ &= \sigma_1 \prod_{j=N-n+1}^{N} \left(\frac{b-a}{1-a} \chi_0^{(a)}(x_j) + \frac{1-b}{1-a} \chi_1^{(a)}(x_j) \right) \\ &= \sum_{j=0}^{n} \binom{N}{j} \left(\frac{b-a}{1-a} \right)^{n-j} \left(\frac{1-b}{1-a} \right)^{j} \chi_j^{(a)}(x). \end{aligned}$$

For $\frac{1}{2} \leqq p_1 \leqq p_2 < 1$, let $b = (p_2 - 1)/p_2$ and $a = (p_1 - 1)/p_1$. Then $-1 \leqq a \leqq b < 0$, and the above remarks yield an expansion in Krawtchouk polynomials.

THEOREM 6.2. *Let* $\frac{1}{2} \leqq p_1 \leqq p_2 < 1, 0 \leqq n \leqq N$. *Then*

$$K_n(x; p_2, N) = \sum_{j=0}^{n} \binom{n}{j} \frac{(p_2 - p_1)^{n-j} p_1^j}{p_2^n} K_j(x; p_1, N), \qquad \textit{all } x.$$

Proof. Theorem 5.1 implies that

$$K_n(x; p_2, N) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{b-a}{1-a}\right)^{n-j} \left(\frac{1-b}{1-a}\right)^{j} K_j(x; p_1, N),$$

and the right-hand side is equal to

$$\begin{aligned} &\sum_{j=0}^{n} \binom{n}{j} \left(\frac{(p_2-1)/p_2 - (p_1-1)/p_1}{1-(p_1-1)/p_1}\right)^{n-j} \left(\frac{1-(p_2-1)/p_2}{1-(p_1-1)/p_1}\right)^{j} K_j(x; p_1, N) \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{p_2-p_1}{p_2}\right)^{n-j} \left(\frac{p_1}{p_2}\right)^{j} K_j(x; p_1, N) \\ &= \sum_{j=0}^{n} \frac{(p_2-p_1)^{n-j} p_1^j}{p_2^n} K_j(x; p_1, N). \end{aligned}$$

COROLLARY 6.3. *Let $x, n = 0, 1, \cdots, N$, and $\frac{1}{2} \leqq p_1 \leqq p_2 < 1$. Then*

$$K_n(x; p_2, N) = \sum_{y=0}^{N} \binom{x}{y} \frac{(p_2-p_1)^{x-y} p_1^y}{p_2^x} K_n(y; p_1, N).$$

Proof. The Krawtchouk polynomials are symmetric in x and n; that is,

$$K_n(x; p, N) = K_x(n; p, N).$$

Remark 6.4. Theorem 6.2 shows that the Krawtchouk polynomials $K_n(x; p_2, N)$ can be expanded in a series of $\{K_j(x; p_1, N)\}_{j=0}^{n}$ with *positive* coefficients, $\frac{1}{2} \leqq p_1 \leqq p_2 < 1$. Corollary 6.3 shows each $x = 0, 1, \cdots, N$ can be represented as a positive measure μ_x on $\{0, 1, \cdots, N\}$ such that

$$K_n(x; p_2, N) = \int K_n(y; p_1, N)\, d\mu_x(y),$$

$n = 0, 1, \cdots, N; \frac{1}{2} \leqq p_1 \leqq p_2 < 1$.

Remark 6.5. Gasper [4, (5.13)] determined the expansion of $K_n(\cdot\,; p, N)$ restricted to $\{0, 1, \cdots, M\}$ (for $M < N$) in terms of the set $\{K_j(\cdot\,; p, M)\}_{j=0}^{M}$ and showed that the coefficients were nonnegative. We can obtain this result with our techniques (one step at a time, that is, for $M = N - 1$). The idea is to first symmetrize H_a^N over P_{N-1}, the group of permutations of the first $N - 1$ coordinates, thus obtaining an expression involving $K_j(\cdot\,; p, N-1)$ (summing over P_N can be done by summing over P_{N-1} and then summing over the two two-sided cosets of P_{N-1} in P_N). One obtains

$$K_n(m; p, N) = \frac{N-n}{N} K_n(m; p, N-1) + \frac{n}{N} K_{n-1}(m; p, N-1)$$

$$\text{for } m = 0, 1, \cdots, N-1, \quad n = 0, 1, \cdots, N,$$

and

$$K_n(m; p, N) = \frac{N-n}{N} K_n(m-1; p, N-1) + \frac{an}{N} K_{n-1}(m-1; p, N-1)$$

$$\text{for } m = 1, \cdots, N, \qquad a = (p-1)/p, \quad n = 0, 1, 2, \cdots, N.$$

By combining the two formulas one can derive the difference formula:

$$K_n(m;p,N) - K_n(m-1;p,N) = \frac{n}{N}(a-1)K_{n-1}(m-1;p,N-1)$$

$$\text{for } n = 0, 1, \cdots, N, \quad m = 1, \cdots, N.$$

Historical remarks 6.6. Vere-Jones [7, p. 268] identified the Krawtchouk polynomials (in the symmetric case $p = q = 1/2$) with the spherical zonal functions associated with a certain finite group and subgroup. He asked for which values of p, $0 < p < 1$, is there such a group interpretation.

Let G be a compact group and K a closed subgroup such that there exists precisely two two-sided cosets K and KxK, $x \in G$. Let $H = \{K, KxK\}$. Then H has the structure of a finite P_*-hypergroup (see [2, Ex. 4.2]). The invariant measure m_H of H is positive on each element of H [2, Prop. 3.2]; indeed, since KxK is closed K is open. Using our previous symbolism, write

$$H = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix},$$

where $-1 \leqq a < 0$. Now $m_H(KxK) = km_H(K)$, where k is the number of distinct left K-cosets in KxK; and so k is a positive integer.

By the orthogonality of χ_0 and χ_1, we have the equations

$$m_H(K) + am_H(KxK) = 0, \qquad m_H(K) + akm_H(K) = 0,$$

and so

$$a = -1/k.$$

It follows that the only values of p, $0 < p < 1$, for which there exists such a group interpretation is for $p = k/(k+1)$, $k = 1, 2, \cdots$, for recall $a = (p-1)/p$; and indeed, let $G = P_{k+1}$ and $K = P_k$, $k = 1, 2, \cdots$.

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AN OPERATOR RELATED TO THE INVERSE LAPLACE TRANSFORM*

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Abstract. An operator T is defined and shown to be closely related to the inverse n-tuple Laplace transform. As T does not involve integration, it partially simplifies and generalizes the Laplace transform technique.

1. Introduction. Some properties of Laplace transform technique can be simplified and generalized in terms of an operator T introduced in § 2. The theory of Laplace transforms, also referred to as operational calculus is useful in solving some boundary value problems where the solution is determined by inversion of the operational solution which can be obtained from the operational problem. The inversion of the operational solution may be a difficult task. However, it is known [1], [3] that in some cases, the use of Laplace transform technique and a suitable contour in the complex plane reduce the complexity of the problem to a point where integration is not required. Actually, this method, called Cagniard's method, can be regarded as a technique in which the integral symbol, appearing in the Laplace transform definition is cancelled by the integral symbol in the inversion formula and as a result, integration is not required in obtaining the solution.

This suggests that a modification of the Laplace transforms, which includes the deletion of the integral symbol can be used instead of Laplace transforms when Cagniard's method is applicable. A simple example illustrating Cagniard's method is given in [3] but the reader need not be familiar with Cagniard's method in order to read this paper.

This suggestion led to the definition of the operator T which is closely related to the inverse L-tuple Laplace transform. Due to its simplicity and applicability, the case $L = 1$ is considered separately in §§ 3 and 4, although it is included in the general case of § 5. In using the operator T instead of Laplace transforms, when it is possible, one avoids not only integrations as in Cagniard's method but also the transformations of complex contour integrals involved in his method.

2. Formulation of the main result. Let

$$\begin{aligned} \mu_1 &= \mu_1^*(x, \lambda_1, \lambda_2, \cdots, \lambda_L), \\ &\vdots \\ \mu_L &= \mu_L^*(x, \lambda_1, \lambda_2, \cdots, \lambda_L), \end{aligned} \tag{2.1}$$

where $x = (x_1, x_2, \cdots, x_N)$, be a system of L equations, μ_i^* being analytic over a region D of the $(N + L)$-dimensional complex space C^{N+L} of the complex variables

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x and λ_i, $i = 1, 2, \cdots, L$. The Jacobian

$$\dot{\mu}^* = \frac{\partial(\mu_1^*, \mu_2^*, \cdots, \mu_L^*)}{\partial(\lambda_1, \lambda_2, \cdots, \lambda_L)}$$

is assumed to be different from zero in D such that the "inverse" (2.2) of (2.1) exists:

$$\lambda_1 = \lambda_1^*(x, \mu_1, \mu_2, \cdots, \mu_L),$$

$$\vdots \tag{2.2}$$

$$\lambda_L = \lambda_L^*(x, \mu_1, \mu_2, \cdots, \mu_L),$$

where λ_i^*, $i = 1, 2, \cdots, L$, are L analytic functions in a region R, $R \subset C^{N+L}$ of the complex variables x and μ_i. The Jacobian

$$\dot{\lambda}^* = \frac{\partial(\lambda_1^*, \lambda_2^*, \cdots, \lambda_L^*)}{\partial(\mu_1, \mu_2, \cdots, \mu_L)}$$

is different from zero in R and upon substitution of (2.2) in (2.1), (2.1) reduces to a system of identities.

For an "index exponent" $p = (p_1, p_2, \cdots, p_N)$ of order N, whose components are nonnegative integers, $|p| = p_1 + p_2 + \cdots + p_N$. For any $f = (f_1, f_2, \cdots, f_N)$,

$$f^p = f_1^{p_1} f_2^{p_2} \cdots f_N^{p_N}.$$

This notation will be used with the differentiation operator

$$\frac{\partial^p}{\partial x^p} = \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_N^{p_N}}$$

as well as with the function

$$f^p(z) = f_1^{p_1}(z) f_2^{p_2}(z) \cdots f_N^{p_N}(z).$$

Lastly, an operator T will be defined by

$$T\left\{ \sum_{q \in Q} s^q F_q(x, \lambda) \exp\left(-s_i \mu_i^*(x, \lambda)\right) \right\} = \sum_{q \in Q} \frac{\partial^q}{\partial \mu^q} \{F_q(x, \lambda^*) \dot{\lambda}^*\} \tag{2.3}$$

for arbitrary analytic functions $F_q(x, \lambda)$ and $F_q(x, \lambda^*)$ in the regions D and R respectively. Q is a finite set of index exponents of order L, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_L)$ and $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_L^*)$. Here we adopt the summation convention that whenever an index (but not an index exponent) is repeated in a given term, a sum is taken over the index from 1 to L, or otherwise the summation symbol $\sum$ is used. $s = (s_1, s_2 \cdots, s_L)$ are L complex variables.

The main result of the present work is the following commutative relation.

THEOREM 1.

$$\frac{\partial^p}{\partial x^p} T = T \frac{\partial^p}{\partial x^p}.$$

Theorem 1 still holds when some of the variables are real, provided that the functions under consideration possess an appropriate number of derivatives which are all continuous in the region. The proof of Theorem 1 will be given in § 5.

3. Single equation, $L = 1$. It is useful to prove Theorem 1 for the simple case of $L = 1$ before proving the general one as the proof of this particular case is simple and applications can be easily outlined.

For $L = 1$, a single equation is given

$$\mu = \mu^*(x, \lambda) \tag{3.1}$$

satisfying $\dot{\mu}^* = \partial\mu^*/\partial\lambda \neq 0$ in a region D of C^{N+1}. The inverse λ^* of μ^* is

$$\lambda = \lambda^*(x, \mu) \tag{3.2}$$

with $\dot{\lambda}^* = \partial\lambda^*/\partial\mu \neq 0$ in a region R of C^{N+1}. Upon substitution of (3.2) into (3.1), (3.1) reduces to the identity

$$\mu \equiv \mu^*(x, \lambda^*(x, \mu)). \tag{3.3}$$

For this case, Theorem 1 has the following form.

THEOREM 2.

$$\frac{\partial^p}{\partial x^p} T\left\{ \sum_{n\in I} s^n F_n(x, \lambda) \exp(-s\mu^*(x, \lambda))\right\} = T\frac{\partial^p}{\partial x^p}\left\{ \sum_{n\in I} s^n F_n(x, \lambda) \exp(-s\mu^*(x, \lambda))\right\},$$

where I is a finite set of nonnegative integers.

The operator T in Theorem 2 is given by

$$T\left\{\sum_{n\in I} s^n F_n(x, \lambda) \exp(-s\mu^*(x, \lambda))\right\} = \sum_{n\in I} \frac{\partial^n}{\partial\mu^n}\{F_n(x, \lambda^*)\dot{\lambda}^*\}.$$

Replacing T, formally, by $A\int_0^\infty d\lambda$, we have

$$\begin{aligned} F(x, \lambda^*)\dot{\lambda}^* &= T\{F(x, \lambda) \exp(-s\mu^*(x, \lambda))\} \\ &= A\int_0^\infty F(x, \lambda) \exp(-s\mu^*(x, \lambda))\, d\lambda \\ &= A\int_0^\infty F(x, \lambda^*)\dot{\lambda}^* e^{-s\mu}\, d\mu \end{aligned}$$

provided that the integrals exist and the appropriate path transformation in the complex plane, resulting from the change of the variable of integration, is justified. According to the definition of Laplace transforms,

$$h(x; \mu) = F(x, \lambda^*)\dot{\lambda}^*$$

is the inverse Laplace transform of

$$g(x; s) = \int_0^\infty F(x, \lambda^*)\dot{\lambda}^* e^{-s\mu}\, d\mu,$$

where s is the transformed variable of the original variable μ. Therefore, under the above assumptions, the symbol A represents the inverse Laplace transform operator L^{-1} and T is related to it by

$$T = L^{-1}\int_0^\infty d\lambda.$$

The validity of this relation depends on the operand, and generally it is not valid as the integrals involved may not exist or the above path transformation may not be justified. However, if this relation holds and differentiation commutes with integration, obviously, Theorem 2 is true. Using Cagniard's method, one proves the commutativity in Theorem 2 by proving that $T = L^{-1}\int_0^\infty d\lambda$ when applied to the operand under consideration. Although the proof (or its converse) may be extremely difficult, as in the case of Cagniard's method applied to problems of wave propagation in horizontally layered media, Theorem 2 has not been conjectured hitherto.

In order to prove Theorem 2, the following lemma is needed.

LEMMA 1.

$$\frac{\partial}{\partial x_i}\{F(x, \lambda^*)\dot{\lambda}^*\} = \frac{\delta F(x, \lambda^*)}{\delta x_i}\dot{\lambda}^* - \frac{\partial}{\partial \mu}\left\{\frac{\delta \mu^*(x, \lambda^*)}{\delta x_i}F(x, \lambda^*)\dot{\lambda}^*\right\},$$

$$i = 1, 2, \cdots, N,$$

where δ denotes differentiation with respect to an explicit variable.

Proof of Lemma 1.

$$\begin{aligned}\frac{\partial}{\partial x_i}\{F(x, \lambda^*)\dot{\lambda}^*\} &= \frac{\partial F}{\partial \lambda^*}\frac{\partial \lambda^*}{\partial x_i}\dot{\lambda}^* + F(x, \lambda^*)\frac{\partial \dot{\lambda}^*}{\partial x_i} + \frac{\delta F}{\delta x_i}\dot{\lambda}^* \\ &= -\frac{\partial F}{\partial \lambda^*}\frac{\delta \mu^*}{\delta x_i}\dot{\lambda}^{*2} - F(x, \lambda^*)\left[\frac{\partial}{\partial \lambda^*}\frac{\delta \mu^*}{\delta x_i}\dot{\lambda}^{*2} + \frac{\delta \mu^*}{\delta x_i}\ddot{\lambda}^*\right] + \frac{\delta F}{\delta x_i}\dot{\lambda}^*\end{aligned}$$

and

$$\frac{\delta}{\partial \mu}\left\{F(x, \lambda^*)\frac{\delta \mu^*(x, \lambda^*)}{\delta x_i}\dot{\lambda}^*\right\} = \frac{\partial F}{\partial \lambda^*}\frac{\delta \mu^*}{\delta x_i}\dot{\lambda}^{*2} + F(x, \lambda^*)\frac{\partial}{\partial \lambda^*}\frac{\delta \mu^*}{\delta x_i}\dot{\lambda}^{*2} + F(x, \lambda^*)\frac{\delta \mu^*}{\delta x_i}\ddot{\lambda}^*$$

and hence the result.

Proof of Theorem 2.

$$\begin{aligned}&T\frac{\partial}{\partial x_i}\{s^n F(x, \lambda)\exp(-s\mu^*(x, \lambda))\} \\ &\quad = Ts^n\left[\frac{\delta F(x, \lambda)}{\delta x_i} - sF(x, \lambda)\frac{\delta \mu^*(x, \lambda)}{\delta x_i}\right]\exp(-s\mu^*(x, \lambda)) \\ &\quad = \frac{\partial^n}{\partial \mu^n}\left\{\frac{\delta F(x, \lambda^*)}{\delta x_i}\dot{\lambda}^* - \frac{\partial}{\partial \mu}\left[F(x, \lambda^*)\frac{\delta \mu^*(x, \lambda^*)}{\delta x_i}\dot{\lambda}^*\right]\right\}.\end{aligned}$$

Here $\delta F(x, \lambda)/\delta x_i = \partial F(x, \lambda)/\partial x_i$ as x_i appears only explicitly in $F(x, \lambda)$, but obviously, they are not equal when λ is replaced by λ^*.

On the other hand,

$$\frac{\partial}{\partial x_i}T\{s^n F(x, \lambda)\exp(-s\mu^*(x, \lambda))\} = \frac{\partial}{\partial x_i}\frac{\partial^n}{\partial \mu^n}\{F(x, \lambda^*)\dot{\lambda}^*\} = \frac{\partial^n}{\partial \mu^n}\frac{\partial}{\partial x_i}\{F(x, \lambda^*)\dot{\lambda}^*\}.$$

Hence, by Lemma 1, $T(\partial/\partial x_i) = (\partial/\partial x_i)T$ for $i = 1, 2, 3, \cdots, N$, and the theorem is true for p such that $|p| = 1$.

The proof of the theorem is by induction on $|p|$. Assuming that the theorem is true for all p' such that $|p'| = |p|$, then we have

$$T\frac{\partial}{\partial x_i}\frac{\partial^{p'}}{\partial x^{p'}} = \frac{\partial}{\partial x_i}T\frac{\partial^{p'}}{\partial x^{p'}} = \frac{\partial}{\partial x_i}\frac{\partial^{p'}}{\partial x^{p'}}T$$

for $i = 1, 2, 3, \cdots, N$. Thus the theorem is true also for all p'' such that $|p''| = |p| + 1$, and the proof is complete.

4. Applications. As an application to Theorem 2, consider the particular case of

$$F(x, \lambda) = F(\lambda)$$

and

$$\mu^*(x, \lambda) = \sum_{i=0}^{N} x_i f_i(\lambda), \qquad \frac{\partial \mu^*}{\partial \lambda} \neq 0 \tag{4.1}$$

in a region where $x_0 \equiv 1$ and $f_i(\lambda)$ are analytic functions of λ in the region.

THEOREM 3. *Let $\lambda^*(x, \mu)$ be defined implicitly by* (4.1) *in a region R and let F be an arbitrary function such that $F(\lambda^*(x, \mu))$ is analytic in R. Then*

$$\frac{\partial^p}{\partial x^p}\{F(\lambda^*)\dot{\lambda}^*\} = (-1)^{|p|}\frac{\partial^{|p|}}{\partial \mu^{|p|}}\{f^p(\lambda^*)F(\lambda^*)\dot{\lambda}^*\}$$

in R.

Proof of Theorem 3. By the definition of T,

$$\frac{\partial^p}{\partial x^p}T\left\{F(\lambda)\exp\left(-s\sum_{i=0}^{N} x_i f_i(\lambda)\right)\right\} = \frac{\partial^p}{\partial x^p}\{F(\lambda^*)\dot{\lambda}^*\}.$$

On the other hand,

$$\begin{aligned} T\frac{\partial^p}{\partial x^p}\left\{F(\lambda)\exp\left(-s\sum_{i=0}^{N} x_i f_i(\lambda)\right)\right\} &= T\left\{(-s)^{|p|}f^p(\lambda)F(\lambda)\exp\left(-s\sum_{i=0}^{N} x_i f_i(\lambda)\right)\right\} \\ &= (-1)^{|p|}\frac{\partial^{|p|}}{\partial \mu^{|p|}}\{f^p(\lambda^*)F(\lambda^*)\dot{\lambda}^*\}. \end{aligned}$$

But, by Theorem 2, $(\partial^p/\partial x^p)T = T(\partial^p/\partial x^p)$, and hence the result.

Theorem 3 is useful for finding progressing waves (see [2] for definition) and solving some boundary value problems in terms of them as Theorem 3 enables one to replace differentiation with respect to the variables x by differentiation with respect to the single "parameter" μ. An illustrative example follows below.

Example. For $N = 4$ denote (x_1, x_2, x_3, x_4) by (x, y, z, t) and restrict x, y, z, t to real values. Let $\lambda^*(x, y, z, t; \mu)$ be defined implicitly by

$$\mu = \mu^*(x, y, z, t; \lambda) = x \sinh\alpha \sinh\lambda - iy \sinh\alpha \cosh\lambda + z\cosh\alpha - t, \qquad \alpha > 0.$$

Then λ^* is given explicitly by

$$\lambda^*(x, y, z, t; \mu) = \sinh^{-1}\left(\frac{t + \mu - z\cosh\alpha}{r\sinh\alpha}\right) + i\theta$$

and

$$\dot{\lambda}^* = \frac{\partial \lambda^*}{\partial \mu} = [(t + \mu - z \cosh \alpha)^2 + r^2 \sinh^2 \alpha]^{-1/2},$$

where r, θ, z are cylindrical coordinates satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Let $\square$ be the operator

$$\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2};$$

then, by Theorem 3,

$$\begin{aligned}\square\{F(\lambda^*)\dot{\lambda}^*\} &= \frac{\partial^2}{\partial \mu^2}\{(\sinh^2\alpha \sinh^2\lambda^* - \sinh^2\alpha\cosh^2\lambda^* + \cosh^2\alpha - 1)F(\lambda^*)\dot{\lambda}^*\} \\ &= 0,\end{aligned}$$

and hence, $F(\lambda^*)\dot{\lambda}^*$ is a progressing wave satisfying the wave equation

$$\square\phi = 0. \tag{4.2}$$

μ appears in $\phi = F(\lambda^*)\dot{\lambda}^*$ as a parameter and one can put $\mu = 0$ for convenience.

It should be noted that $F(\lambda^*)$ in this example is also a progressive wave. A necessary and sufficient condition for λ^* to be a phase (see [2] for definition) of an undistorted progressing wave satisfying (4.2) is that λ^* satisfies both

$$(\nabla\lambda^*)^2 = \left(\frac{\partial \lambda^*}{\partial t}\right)^2$$

and

$$\square\lambda^* = 0,$$

where ∇ is the nabla operator $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. However, the progressing wave $F(\lambda^*)$ is not applicable for Theorem 3.

Applications of Theorems 2 and 3 to the theory of wave propagation in horizontally layered media are introduced in [4]–[6].

5. The system of L equations. In order to prove Theorem 1 the following lemma is needed.

LEMMA 2.

$$\frac{\partial}{\partial x_i}\{F(x, \lambda^*)\dot{\lambda}^*\} = \frac{\delta F(x, \lambda^*)}{\delta x_i}\dot{\lambda}^* - \frac{\partial}{\partial \mu_n}\left\{\frac{\delta \mu_n^*(x, \lambda^*)}{\delta x_i}F(x, \lambda^*)\dot{\lambda}^*\right\}, \quad i = 1, 2, \cdots, N.$$

Proof of Lemma 2. For the system (2.1),

$$I_j = \frac{\partial \mu^*}{\partial \lambda_k^*}\frac{\partial \lambda_k^*}{\partial \mu_j}, \tag{5.1}$$

where I_j is a vector whose L components are zero but the jth component is unity and

$$-\frac{\delta \mu^*}{\delta x_i} = \frac{\partial \mu^*}{\partial \lambda_k^*}\frac{\partial \lambda_k^*}{\partial x_i}. \tag{5.2}$$

Hence, by (5.1)

$$\frac{\delta\mu^*}{\delta x_i} = \frac{\delta\mu_n^*}{\delta x_i} I_n = \frac{\partial\mu^*}{\partial\lambda_k^*}\frac{\delta\mu_n^*}{\delta x_i}\frac{\partial\lambda_k^*}{\partial\mu_n}. \tag{5.3}$$

On comparing (5.2) with (5.3) and taking into account that the matrix $\dot{\mu}^* = (\partial\mu_n^*/\partial\lambda_k^*)$ is invertible in the region under consideration, we have

$$\frac{\partial\lambda_k^*}{\partial x_i} = -\frac{\delta\mu_n^*}{\delta x_i}\frac{\partial\lambda_k}{\partial\mu_n}, \quad i = 1, 2, \cdots, N, \quad k = 1, 2, \cdots, L. \tag{5.4}$$

Differentiation of (5.4) with respect to μ_j yields

$$\frac{\partial^2\lambda_k^*}{\partial x_i\,\partial\mu_j} = -\frac{\partial\delta\mu_n^*}{\partial\lambda_m^*\,\delta x_i}\frac{\partial\lambda_m^*}{\partial\mu_j}\frac{\partial\lambda_k^*}{\partial\mu_n} - \frac{\delta\mu_n^*}{\delta x_i}\frac{\partial^2\lambda_k^*}{\partial\mu_n\,\partial\mu_j}, \quad j = 1, 2, \cdots, L. \tag{5.5}$$

Consider the Jacobian

$$\dot{\lambda}^* = \frac{\partial(\lambda_1^*, \lambda_2^*, \cdots, \lambda_L^*)}{\partial(\mu_1, \mu_2, \cdots, \mu_L)} = \varepsilon_{ijk\cdots n}\frac{\partial\lambda_1^*}{\partial\mu_i}\frac{\partial\lambda_2^*}{\partial\mu_j}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial\lambda_L^*}{\partial\mu_n},$$

where $ijk\cdots n$ are L indices and $\varepsilon_{ijk\cdots n}$ is the Levi–Civita symbol that is antisymmetric in all the indices and has the value 1 for $\varepsilon_{123\cdots L}$.

By means of (5.5), $\partial\dot{\lambda}^*/\partial x_\omega$ is given by

$$\begin{aligned}
\frac{\partial\dot{\lambda}^*}{\partial x_\omega} = &-\frac{\partial\delta\mu_r^*}{\partial\lambda_m^*\,\delta x_\omega}\frac{\partial\lambda_1^*}{\partial\mu_r}\varepsilon_{ijk\cdots n}\frac{\partial\lambda_m^*}{\partial\mu_i}\frac{\partial\lambda_2^*}{\partial\mu_j}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial\lambda_L^*}{\partial\mu_n}\\
&-\frac{\partial\delta\mu_r^*}{\partial\lambda_m^*\,\delta x_\omega}\frac{\partial\lambda_2^*}{\partial\mu_r}\varepsilon_{ijk\cdots n}\frac{\partial\lambda_1^*}{\partial\mu_i}\frac{\partial\lambda_m^*}{\partial\mu_j}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial\lambda_L^*}{\partial\mu_n}\\
&-\cdots\\
&-\frac{\partial\delta\mu_r^*}{\partial\lambda_m^*\,\delta x_\omega}\frac{\partial\lambda_L^*}{\partial\mu_r}\varepsilon_{ijk\cdots n}\frac{\partial\lambda_1^*}{\partial\mu_i}\frac{\partial\lambda_2^*}{\partial\mu_j}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial\lambda_m^*}{\partial\mu_n}\\
&-\frac{\delta\mu_r^*}{\delta x_\omega}\bigg[\varepsilon_{ijk\cdots n}\frac{\partial^2\lambda_1^*}{\partial\mu_i\,\partial\mu_r}\frac{\partial\lambda_2^*}{\partial\mu_j}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial\lambda_L^*}{\partial\mu_n}\\
&\qquad+\varepsilon_{ijk\cdots n}\frac{\partial\lambda_1^*}{\partial\mu_i}\frac{\partial^2\lambda_2^*}{\partial\mu_j\,\partial\mu_r}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial\lambda_L^*}{\partial\mu_n}\\
&\qquad+\cdots\\
&\qquad+\varepsilon_{ijk\cdots n}\frac{\partial\lambda_1^*}{\partial\mu_i}\frac{\partial\lambda_2^*}{\partial\mu_j}\frac{\partial\lambda_3^*}{\partial\mu_k}\cdots\frac{\partial^2\lambda_L^*}{\partial\mu_n\,\partial\mu_r}\bigg].
\end{aligned} \tag{5.6}$$

The expressions

$$\varepsilon_{i_1\cdots i_r\cdots i_L}\frac{\partial\lambda_1^*}{\partial\mu_{i_1}}\cdots\frac{\partial\lambda_m^*}{\partial\mu_{i_r}}\cdots\frac{\partial\lambda_L^*}{\partial\mu_{i_L}}, \qquad r = 1, 2, \cdots, L,$$

appearing in (5.6) are zero for $m \neq r$ since they represent determinants with two equal rows. The expression in the brackets in (5.6) is equal to $\partial\dot{\lambda}^*/\partial\mu_r$. Therefore,

$$\frac{\partial\dot{\lambda}^*}{\partial x_\omega} = -\frac{\partial\delta\mu_r^*}{\partial\lambda_m^*\,\delta x_\omega}\frac{\partial\lambda_m^*}{\partial\mu_r}\dot{\lambda}^* - \frac{\delta\mu_r^*}{\delta x_\omega}\frac{\partial\dot{\lambda}^*}{\partial\mu_r}.$$

Now both sides of the equation in Lemma 2 can be evaluated and shown to be equal.

Proof of Theorem 1.

$T(\partial/\partial x_i)\{s^q F(x,\lambda)\exp(-s_i\mu_i^*(x,\lambda))\}$

$$= Ts^q\left\{\frac{\delta F(x,\lambda)}{\delta x_i} - s_n F(x,\lambda)\frac{\delta\mu_n^*(x,\lambda)}{\delta x_i}\right\}\exp(-s_i\mu_i^*(x,\lambda))$$

$$= \frac{\partial^q}{\partial\mu^q}\left[\frac{\delta F(x,\lambda^*)}{\delta x_i}\dot{\lambda}^* - \frac{\partial}{\partial\mu_n}\left\{F(x,\lambda^*)\frac{\delta\mu_n^*(x,\lambda^*)}{\delta x_i}\dot{\lambda}^*\right\}\right].$$

Further,

$$\frac{\partial}{\partial x_i}T\{s^q F(x,\lambda)\exp(-s_i\mu_i^*(x,\lambda))\}$$

$$= \frac{\partial}{\partial x_i}\frac{\partial^q}{\partial\mu^q}\{F(x,\lambda^*)\dot{\lambda}^*\} = \frac{\partial^q}{\partial\mu^q}\frac{\partial}{\partial x_i}\{F(x,\lambda^*)\dot{\lambda}^*\}.$$

Therefore, by Lemma 2, $T(\partial/\partial x_i) = (\partial/\partial x_i)T$ for $i = 1, 2, \cdots N$, and the theorem is true for all p such that $|p| = 1$.

The proof of the theorem is by induction on $|p|$. Assuming that the theorem is true for all p' such that $|p'| = |p|$, then we have

$$T\frac{\partial}{\partial x_i}\frac{\partial^{p'}}{\partial x^{p'}} = \frac{\partial}{\partial x_i}T\frac{\partial^{p'}}{\partial x^{p'}} = \frac{\partial}{\partial x_i}\frac{\partial^{p'}}{\partial x^{p'}}T, \qquad i = 1, 2, 3, \cdots, N.$$

Hence the theorem is true also for all p'' such that $|p''| = |p| + 1$ and the proof is complete.

If A_k has the form $s^q F(x,\lambda)\exp(-s_i\mu_i^*(x,\lambda))$ where F and μ^* depend on the integer k and if $A = \sum_{k=0}^K A_k$, then T is defined on A by

$$TA = \sum_{k=0}^{K} TA_k,$$

and obviously, Theorem 1 still holds. For an infinite series $A_\infty = \sum_{k=0}^\infty A_k$, T is also defined by

$$TA_\infty = \sum_{k=0}^{\infty} TA_k$$

provided that the series converges and Theorem 1 is satisfied. The application of T to infinite series will be considered in a subsequent paper.

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ON A PROBLEM OF E. L. DE FOREST IN ITERATED SMOOTHING*

Dedicated to the Late Hugh H. Wolfenden on his 80th Birthday

T. N. E. GREVILLE†

Abstract. Schoenberg's results on the limiting behavior of the normalized coefficients of the n-fold iterate of a symmetrical linear smoothing formula are extended to the unsymmetrical case. When the formula is exact to an odd degree, the family of limiting functions obtained is the same as that deduced by Schoenberg. When it is exact to an even degree (possible only for an unsymmetrical formula), the limiting functions belong to a different family defined by very slowly converging Fourier integrals and including the Airy function as a particular case.

A critique is made of a certain theoretical objection to even-degree smoothing formulas, and as a curious by-product, a certain unsymmetrical 5-term formula exact to degree two is shown to be a better smoothing agent than the corresponding 5-term symmetrical formula.

1. Introduction. We shall consider linear adjustment formulas of the form

$$v_l = \sum_{j=p}^{q} c_j y_{l-j}, \tag{1.1}$$

where y_l is an observed, or "crude" value, v_l is the corresponding adjusted value, and the coefficients c_j are real and satisfy

$$\sum_{j=p}^{q} c_j = 1. \tag{1.2}$$

As we shall see, under certain conditions, such a formula can appropriately be called a "smoothing" formula. Formula (1.1) is called *exact for the degree r* when the coefficients c_j are such that, whenever there exists a polynomial $p(x)$ of degree r or less such that

$$y_{l-j} = p(l-j), \qquad j = p, p+1, \cdots, q,$$

then

$$v_l = p(l),$$

but there is a polynomial of degree $r + 1$ for which the corresponding relation does not hold.

If we subject the sequence $\{y_l\}$ n times in succession to the same transformation (1.1), we obtain a linear transformation

$$v_l^{(n)} = \sum_{j=np}^{nq} c_j^{(n)} y_{l-j}, \tag{1.3}$$

which is the n-fold iterate of (1.1).

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E. L. De Forest in 1878 [5] proposed the problem of finding the limiting form, if any, of the (suitably normalized) coefficients $c_j^{(n)}$ of (1.3). He quickly realized that the degree r for which the formula (1.1) is exact plays an essential role, and he obtained the correct answers for the cases of $r = 1, 2$ and 3. For $r = 1$ (as shown also by G. B. Dantzig [3]), the limiting curve is the normal probability function. De Forest's methods do not meet modern standards of rigor. He showed that the limiting function, if it exists, must satisfy the differential equation

$$d^r y/dx^r = k\,xy \tag{1.4}$$

for a suitable constant k, subject to the constraints

$$\int_{-\infty}^{\infty} x^j y\,dx = \delta_{0j}, \qquad j = 0, 1, \cdots, r,$$

and he obtained power series expansions for functions satisfying these conditions. He does not seem to have appreciated that the cases of odd r and even r are fundamentally different. (When expressed as power series, the limiting functions for the two cases look much alike.)

He probably understood that convergence to a limiting function does not occur for all formulas (1.1) with coefficients satisfying (1.2), and that some further conditions on these coefficients are needed. However, he does not discuss necessary or sufficient conditions for convergence. Such conditions would have been difficult to formulate before the introduction of the characteristic function by Schoenberg [21] in 1946.

The characteristic function of (1.1) is defined by

$$\phi(t) = \sum_{j=p}^{q} c_j e^{ijt}. \tag{1.5}$$

Evidently this is a periodic function of period 2π, and has the properties

$$\phi(0) = 1$$

and

$$\phi(-t) = \overline{\phi(t)}. \tag{1.6}$$

Schoenberg showed [21]–[23] that the condition

$$|\phi(t)| < 1, \qquad 0 < t < 2\pi \tag{1.7}$$

(sometimes called the von Neumann condition [29]), plays a central role in consideration of questions of convergence of the normalized coefficients $c_j^{(n)}$ to a limiting function. When the sign of inequality is reversed in (1.7) for any $t \in (0, 2\pi)$ such convergence does not occur, and it is doubtful when inequality is merely replaced by equality for some t in the open interval. For the case of odd r, he showed in 1948 in [22] that when (1.7) holds convergence does in fact occur, and he obtained expressions for the limiting functions in the form of Fourier integrals.

The case of even r has been less studied, partly for a technical reason. Most formulas (1.1) used in practice are *symmetrical*, i.e., $p = -q$ and

$$c_{-l} = c_l, \qquad l = 1, 2, \cdots, q. \tag{1.8}$$

It is easily shown that (1.8) implies that (1.1) is exact for an odd degree. Thus, a formula (1.1) exact for an even degree is necessarily unsymmetrical. Except for a few general observations, Schoenberg confines his attention to symmetrical formulas. De Forest, on the contrary, was especially intrigued by the unsymmetrical formulas [6], and his work on the subject has motivated me to pursue this question. The main purpose of this paper is to show that in the case of even r convergence of the normalized coefficients $c_j^{(n)}$ to a limiting function occurs under essentially the same conditions as in the case of odd r.

I have been investigating this problem off and on for many years, and have benefited from conversations and correspondence with numerous persons. My indebtedness to De Forest and Schoenberg is very obvious, and very extensive. As Schoenberg pointed out long ago [23], transformations of the type (1.1) are utilized in connection with difference methods for numerical solution of partial differential equations, and the stability of such methods involves questions analogous to those raised by De Forest. In this connection I am especially indebted to G. W. Hedstrom, W. G. Strang and V. Thomée. While the normalization of the coefficients $c_j^{(n)}$ introduces complications that make it difficult to apply their results directly, some mathematical techniques developed by these authors [11], [28], [29] have been utilized in the proofs. A special insight that has made it possible, after many unsuccessful attempts, to complete the proof of the main theorem, is due to J. Barkley Rosser, to whom I am therefore especially grateful. In other connections, I am indebted to R. Hersh and to the late H. H. Wolfenden, who almost fifty years ago [35] rescued from obscurity the mathematical work of E. L. De Forest. I am grateful also to W. F. Trench, whose careful reading of the manuscript has eliminated some mathematical errors.

2. Normalization and the limiting functions. The Maclaurin expansion of the characteristic function (1.5) is of the form

$$\phi(t) = 1 + b(it)^{r+1} + \cdots,$$

where r is the degree to which the formula is exact and

$$b = \frac{1}{(r+1)!} \sum_{j=p}^{q} c_j j^{r+1} \neq 0.$$

There are two distinct cases according as r is odd or even, and the coefficient of t^{r+1} is consequently real or imaginary. First, we proceed heuristically to derive what may appear to be the limiting functions approached by the normalized coefficients.

In the case of odd r, we write

$$\phi(t) = 1 - at^{r+1} + \cdots, \tag{2.1}$$

where $a = -bi^{r+1}$. In a sufficiently small neighborhood of the origin, we have

$$\ln \phi(t) = -at^{r+1} + \cdots,$$

and for the n-fold iterate of (1.1),

$$\ln \phi^n(t) = -ant^{r+1} + \cdots.$$

Following Schoenberg [22], we normalize by means of the substitution

$$t = u(an)^{-1/(r+1)}, \tag{2.2}$$

and obtain

$$\ln \psi_n(u) = -u^{r+1} + O(n^{-1/(r+1)}), \tag{2.3}$$

where $\psi_n(u) = \phi^n(t)$. (If $h \geqq 0$, then $h^{1/(r+1)}$ will denote the nonnegative $(r+1)$th root of h; if $h < 0$ and $r+1$ is odd, it will denote the negative $(r+1)$th root of h.) Now,

$$\begin{aligned} c_j^{(n)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ijt}\phi^n(t)\,dt = (an)^{-1/(r+1)}\frac{1}{2\pi}\int_{-\alpha_n}^{\alpha_n} e^{-ixu}\psi_n(u)\,du \\ &= (an)^{-1/(r+1)}\frac{1}{\pi}\int_0^{\alpha_n}(\cos xu \operatorname{Re}\psi_n(u) + \sin xu \operatorname{Im}\psi_n(u))\,du, \end{aligned} \tag{2.4}$$

in view of (1.6), where

$$\alpha_n = \pi(an)^{1/(r+1)} \tag{2.5}$$

and

$$x = j(an)^{-1/(r+1)}. \tag{2.6}$$

A geometrical description may help the reader to understand what is going on. Think of the array of coefficients c_j as represented by a histogram with bars (in general, both positive and negative) of unit width. The substitution (2.6) changes the width of the bars to $(an)^{-1/(r+1)}$, and in order to preserve area we compensate by multiplying the height $c_j^{(n)}$ by $(an)^{1/(r+1)}$. As n tends to infinity, the width of the bars approaches zero, and we would like to know if the heights converge to the ordinates of a continuous curve. Equations (2.3) and (2.4) suggest that it is a plausible conjecture that under appropriate conditions the limiting function is

$$G_{r+1}(x) = \frac{1}{\pi}\int_0^{\infty} \cos xu \exp(-u^{r+1})\,du. \tag{2.7}$$

For $r = 1$, (2.7) is the normal probability curve, which, of course, is everywhere positive and decreases monotonically toward zero with increasing $|x|$. For $r = 3, 5, \cdots$, the corresponding function $G_{r+1}(x)$ is again an even function, but takes on some negative values. It has the general appearance of a probability density function in the middle portion and that of a rapidly damped sine curve in the tails. As Schoenberg [22] has pointed out, these functions were studied by F. Bernstein [2] (for $r + 1 = 4$) and Pólya [18].

As we shall state more precisely in § 5, Schoenberg [22] has shown that the normalized coefficients do in fact converge to this limiting function when r is odd and (1.7) is satisfied.

When r is even, we have

$$\phi(t) = 1 + ait^{r+1} + \cdots, \tag{2.8}$$

where $a = bi^r$, and

$$\ln \psi_n(u) = iu^{r+1} + O(n^{-1/(r+1)}).$$

Note that in the odd case a must be positive for (1.7) to hold, but in the even case there is no such restriction, and accordingly (an) must be replaced by $|an|$ in (2.4) and (2.5). In this case, the conjectured limiting function (see [8], [9]) is

$$H_{r+1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixu} \exp(iu^{r+1})\, du = \frac{1}{\pi}\int_{0}^{\infty} \cos(u^{r+1} - xu)\, du. \tag{2.9}$$

Superficially, these functions have some resemblance to the functions (2.7). However, they are no longer even functions, and damping is very much slower for positive x than for negative x. Here again the smallest value of r ($r = 2$) is a special case, in which for negative x, $H_3(x)$ is everywhere positive and decreases monotonically toward zero as $x \to -\infty$. $H_3(x)$ is, in fact, a form of the well-known Airy function of geometrical optics: more specifically,

$$H_3(x) = 3^{-1/3}\,\mathrm{Ai}(-3^{-1/3}x).$$

Both families of limiting functions, (2.7) and (2.9), have been derived independently by Hersh [13], and functions related to (2.9) have been studied by Watson [33] and Titchmarsh [30]. De Forest was unacquainted with Fourier integrals and gave [6] only power series expansions for the cases of $r = 2$ and 3.

3. The De Forest differential equation. For $r = 2$, the differential equation (1.4) is the well-known Airy equation (of which the Airy function is a solution). On the other hand, (1.4) is itself a particular case of the more general equation

$$\frac{d^r y}{dx^r} = x^\nu y$$

(where ν is a natural number), which has been extensively studied [10], [17], [19], [32] in connection with asymptotic solutions of ordinary differential equations.[1] In particular, Molins [17], about the same time that De Forest developed his equation, obtained by an elegant method power series for that set of solutions (of the more general equation) which is principal at the origin. Heading [10] wrote the equation in the form

$$\frac{d^n u}{dz^n} = (-1)^n z^m u,$$

and noted that "there is a fundamental difference in the character of the solutions for n even and n odd."

The following demonstration that the limiting functions, if they exist, must satisfy the differential equation (1.4) is essentially that given by De Forest [6] (though he used generating functions rather than characteristic functions).

[1] I became aware of these references only at the time of acceptance of this paper for publication, when they were brought to my attention by F. W. J. Olver, Managing Editor of this Journal. There is a certain irony in the fact that I never discussed the iterated smoothing problem with R. E. Langer, who was Director of the Mathematics Research Center when I first became a member, because I was unaware that it was related to his interests. He was an eminent authority on asymptotic solutions of differential equations and is mentioned in some of the references.

For any characteristic function

$$\Phi(t) = \sum_j d_j \; e^{ijt}$$

note that

$$\Phi'(t) = \sum_j jd_j \; e^{ijt}$$

may be regarded as i times the characteristic function of the sequence $\{jd_j\}$. Evidently also,

$$\frac{d}{dt}\Phi^{n+1}(t) = (n+1)\Phi'(t)\Phi^n(t). \tag{3.1}$$

Equating coefficients of e^{ilt} on both sides of (3.1) gives

$$lc_l^{(n+1)} = (n+1)\sum_{j=p}^{q} jc_j c_{l-j}^{(n)}. \tag{3.2}$$

By Gauss' "forward" formula,

$$c_{l-j}^{(n)} = c_l^{(n)} - j\Delta c_l^{(n)} + \binom{-j}{2}\Delta^2 c_{l-1}^{(n)} + \binom{-j+1}{3}\Delta^3 c_{l-1}^{(n)} + \cdots, \tag{3.3}$$

terminating after a finite number of terms. Now, we substitute (3.3) in (3.2), noting that (1.1) is exact for the degree r, and that the successive polynomials of degree $1, 2, \cdots, r$ obtained by multiplying by j the coefficients of the first r terms of the right member of (3.3) all vanish for $j = 0$. We obtain, therefore,

$$lc_l^{(n+1)} = (n+1)(d_r\Delta^r c_{l-s}^{(n)} + d_{r+1}\Delta^{r+1}c_{l-t} + \cdots), \tag{3.4}$$

where d_r and d_{r+1} are coefficients independent of l, s is the largest integer not exceeding $r/2$, and t is either s or $s+1$, depending on the parity of r.

If we now think of the coefficients $c_l^{(n)}$ as being spaced at intervals of $h = |an|^{-1/(r+1)}$, (3.4) can be written in the form

$$\begin{aligned} &l(a(n+1))^{-1/(r+1)}|a(n+1)|^{1/(r+1)}c_l^{(n+1)} \\ &\qquad = \frac{n+1}{an}\left(d_r\frac{\Delta^r h^{-1}c_{l-s}^{(n)}}{h^r} + hd_{r+1}\frac{\Delta^{r+1}h^{-1}c_{l-t}^{(n)}}{h^{r+1}} + \cdots\right). \end{aligned} \tag{3.5}$$

If we postulate that $|an|^{1/(r+1)}c_l^{(n)}$, regarded as a function of $x = l(an)^{-1/(r+1)}$, approaches an infinitely differentiable function $f(x)$, as $n \to \infty$ and therefore $h \to 0$, then the arguments x corresponding to l and $l - s$ approach equality, and the limit of (3.5) is

$$xf(x) = \frac{d_r}{a}f^{(r)}(x),$$

which is equivalent to (1.4).

De Forest's procedure was to seek a power series solution of (1.4), subject to the conditions for exactness to the degree r. It is interesting to note that he obtained

[6] an expression for the Airy function of the form

$$\mathrm{Ai}(x) = d_1 f(x) = d_2 g(x),$$

where

$$f(x) = 1 + \frac{1}{3!}x^3 + \frac{1 \cdot 4}{6!}x^6 + \frac{1 \cdot 4 \cdot 7}{9!}x^9 + \cdots,$$

$$g(x) = x + \frac{2}{4!}x^4 + \frac{2 \cdot 5}{7!}x^7 + \frac{2 \cdot 5 \cdot 8}{10!}x^{10} + \cdots,$$

$$d_1 = .36, \qquad d_2 = .262444.$$

The well-known handbook [1] gives the identical expression but with

$$d_1 = .35503, \qquad d_2 = .25882.$$

Thus, De Forest's values of the two constants were correct to two decimal places. As it is unlikely that he had any mechanical calculating aid at his disposal, I consider this a remarkable achievement.

In the case of odd r, it is easily seen that the function $G_{r+1}(x)$ satisfies the differential equation. Since the integral (2.7) and its derivatives are absolutely and uniformly convergent, one can merely differentiate under the integral sign.

For even r, it is less obvious that the function $H_{r+1}(x)$ satisfies (1.4). For the following argument, I am indebted to R. Hersch. If we regard $\exp{(it)^{r+1}}$ and its Fourier transform as tempered Schwartz distributions, then the limit function $H_{r+1}(x)$ satisfies the differential equation, at least in the sense of generalized functions [7], [14]. However, since $H_{r+1}(x)$ is an entire function (as shown by Hersh in [13]), differentiable in the classical sense, it is therefore a classical solution of the differential equation.

4. Nonconvergence when the stability condition is reversed. In the symmetrical case, in which $\phi(t)$ assumes only real values for real t, Schoenberg [22] has proved that $|c_0^{(n)}|$ diverges to infinity with increasing n if the inequality (1.7) is reversed anywhere. In the general (not necessarily symmetrical) case, the corresponding statement is no longer true. However, we can prove the following related theorem, portions of the proof being closely analogous to Schoenberg's proof. This theorem is also an obvious corollary of the main theorem of Thomée's [29].

THEOREM 1. *For some t_0, let*

$$|\phi(t_0)| > 1.$$

Then, to every positive M, there corresponds a positive integer N_M, such that for all $n > N_M$,

$$\max_j |c_j^{(n)}| > M.$$

Proof. Since $\phi(t)$ is analytic, there is an interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ in which

$$|\phi(t)| > \gamma > 1.$$

Define

$$S_n = \sum_{j=np}^{nq} (c_j^{(n)})^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^n(t)\overline{\phi^n(t)}\,dt. \tag{4.1}$$

Since the integrand is everywhere nonnegative

$$S_n > \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} |\phi^n(t)|^2\,dt > \frac{\varepsilon\gamma^{2n}}{\pi}.$$

For a given S_n, $\max_j |c_j^{(n)}|$ is smallest when the terms of the summation in (4.1) are all equal. Thus

$$\max_j |c_j^{(n)}| \geqq \left(\frac{S_n}{n(q-p)+1}\right)^{1/2} > \left(\frac{\varepsilon}{\pi(n(q-p)+1)}\right)^{1/2} \gamma^n.$$

Therefore,

$$\max_j |c_j^{(n)}| > \left(\frac{\varepsilon}{\pi(q-p+1)}\right)^{1/2} n^{-1/2}\gamma^n,$$

from which the desired result follows easily.

Some ambiguity remains about the situation in which the inequality is never reversed in (1.7), but equality holds for some values of t not integral multiples of 2π. Schoenberg has shown by an example that in such a case $c_0^{(n)}$ may fail to approach a single limiting value. His example is the adjustment formula

$$v_l = \tfrac{1}{2}(y_{l-1} + y_{l+1}), \tag{4.2}$$

for which

$$\phi(t) = \cos t.$$

Evidently,

$$|\phi(t)| < 1, \qquad 0 < |t| < \pi,$$

but

$$\phi(\pi) = \phi(-\pi) = -1.$$

He shows that if n is restricted to even integers $2m$,

$$\lim_{m\to\infty} m^{1/2}c_0^{(2m)} = G_1(0) = \tfrac{1}{2}\sqrt{\pi},$$

as required by the theory. However, $c_0^{(n)}$ vanishes for all odd n.

Nevertheless, it is my conjecture, without anything very solid to base it on, that this is not typical behavior, and that convergence to the expected limiting function normally occurs in the situation being considered. I would point out that (4.2) belongs to a rather special class of formulas, those having all coefficients c_j equal to 0 for even j. For every such formula, it is easily seen that $c_j^{(n)} = 0$ always for odd n and even j, while $\phi(\pi) = -1$. (An analogous situation exists in the more general case in which nonzero coefficients c_j occur only for indices j spaced at intervals of k and not including $j = 0$.)

5. Convergence in the case of odd r. Schoenberg in [22] has essentially solved the problem of convergence of the normalized coefficients to a limiting function when r is odd. However, he limits himself to the case of symmetrical formulas (1.1), for which the characteristic function (1.5) assumes only real values for real t, and can be written in the form

$$\phi(t) = c_0 + 2 \sum_{j=1}^{q} c_j \cos jt.$$

The modifications required to extend his result to the unsymmetrical case, in which $\phi(t)$ may assume complex values for real t, are not quite trivial and enter into the proof in a way that is most easily explained by giving the entire proof. Therefore, we state and prove the following more general theorem, emphasizing that the proof is largely due to Schoenberg [22].

THEOREM 2. *Let* (1.1) *be an adjustment formula exact for odd r* ($r = 1, 3, 5, \cdots$) *with a characteristic function $\phi(t)$ having the properties* (1.7) *and* (2.1). *Then, to every positive ε there corresponds a positive integer N_ε, such that*

$$|(an)^{1/(r+1)} c_j^{(n)} - G_{r+1}(j(an)^{-1/(r+1)})| < \varepsilon$$

for all j if $n > N_\varepsilon$. In other words, for $n \to \infty$,

$$c_j^{(n)} = (an)^{-1/(r-1)} G_{r+1}(j(an)^{-1/(r+1)}) + o(n^{-1/(r+1)}),$$

uniformly for all integral values of j.

Proof. Note that (1.7) requires $a > 0$. Following Schoenberg [22], we define

$$F_n(x) = \frac{1}{2\pi} \int_{-\alpha_n}^{\alpha_n} e^{-iux} \psi_n(u)\, du; \tag{5.1}$$

then it follows from (2.4) that

$$(an)^{1/(r+1)} c_j^{(n)} = F_n(j(an)^{-1/(r+1)}),$$

and it is therefore sufficient to prove that

$$\lim_{n \to \infty} F_n(x) = G_{r+1}(x)$$

uniformly in x.

We write

$$\psi_n(u) = \rho_n(u) \exp i\theta_n(u),$$

with $\rho_n(u)$ and $\theta_n(u)$ real functions. Then

$$\rho_n(u) = |\phi(u(an)^{-1/(r+1)})|^n, \tag{5.2}$$

$$\theta_n(u) = n \arg \phi(u(an)^{-1/(r+1)}). \tag{5.3}$$

In view of (1.6), we have

$$\begin{aligned} F_n(x) - G_{r+1}(x) = {} & \frac{1}{\pi} \int_0^{\beta\alpha_n} [\rho_n(u) \cos(\theta_n(u) - xu) - \exp(-u^{r+1}) \cos xu]\, du \\ & + \frac{1}{\pi} \int_{\beta\alpha_n}^{\alpha_n} \rho_n(u) \cos(\theta_n(u) - xu)\, du - \frac{1}{\pi} \int_{\beta\alpha_n}^{\infty} \exp(-u^{r+1}) \cos xu\, du \\ = {} & I_1 + I_2 - I_3, \end{aligned}$$

where β is an element of (0, 1) on which we shall impose conditions later.

Again following Schoenberg [22], we let $\gamma = \max |\phi(t)|$ for $t \in [\beta\pi, \pi]$. Then $\gamma < 1$, and $\rho_n(u) \leqq \gamma^n$ for $u \in [\beta\alpha_n, \alpha_n]$. Therefore,

$$|I_2| \leqq (1 - \beta)(an)^{1/(r+1)}\gamma^n < \varepsilon/4 \tag{5.4}$$

for sufficiently large n. Also,

$$|I_3| < \frac{1}{\pi}\int_{\beta\alpha_n}^{\infty} \exp(-u^{r+1})\, du < \frac{\varepsilon}{4}$$

for sufficiently large n. We have

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^{\beta\alpha_n} (\rho_n(u) - \exp(-u^{r+1}))\cos xu\, du \\ &\quad - \frac{1}{\pi}\int_0^{\beta\alpha_n} \rho_n(u)(\cos xu - \cos(\theta_n(u) - xu))\, du \\ &= I_4 - I_5. \end{aligned}$$

Evidently

$$|I_4| \leqq \frac{1}{\pi}\int_0^{\beta\alpha_n} |\rho_n(u) - \exp(-u^{r+1})|\, du. \tag{5.5}$$

Employing the same ingenious device utilized by Schoenberg [22], we let

$$0 < a_1 < a < a_2,$$

where we shall later impose further conditions on a_1 and a_2. Then,

$$\exp\left(-\frac{a_2}{a}u^{r+1}\right) < \exp(-u^{r+1}) < \left(-\frac{a_1}{a}u^{r+1}\right) \tag{5.6}$$

for $u > 0$. In view of (2.1), $|\phi(t)|$ has a Maclaurin expansion of the form

$$|\phi(t)| = 1 - at^{r+1} + \cdots.$$

Hence, we can choose β so that

$$\exp(-a_2 t^{r+1}) < |\phi(t)| < \exp(-a_1 t^{r+1})$$

for $t \in (0, \beta]$. Then it follows from (5.2) that

$$\exp\left(-\frac{a_2}{a}u^{r+1}\right) < \rho_n(u) < \exp\left(-\frac{a_1}{a}u^{r+1}\right) \tag{5.7}$$

for $u \in (0, \beta\alpha_n]$.

From (5.6) and (5.7) it follows that

$$|\rho_n(u) - \exp(-u^{r+1})| < \exp\left(-\frac{a_1}{a}u^{r+1}\right) - \exp\left(-\frac{a_2}{a}u^{r+1}\right).$$

for $u \in (0, \beta\alpha_n]$, and substitution of this result in (5.5) gives

$$(5.8) \qquad |I_4| < \frac{1}{\pi}\left[\left(\frac{a}{a_1}\right)^{1/(r+1)} - \left(\frac{a}{a_2}\right)^{1/(r+1)}\right] \int_0^\infty \exp(-u^{r+1})\, du.$$

The expression in square brackets can be made as small as we please by choosing a_1 and a_2 sufficiently close together, and we then choose β so that (5.7) holds. As the integral in (5.8) is finite, $|I|_4$ can be made less than $\varepsilon/4$, uniformly in x for sufficiently large n.

Since $\theta_n(u) \equiv 0$ in the symmetrical case considered by Schoenberg, I_5 vanishes in that case. In order to extend the result to unsymmetrical formulas with odd r, we write

$$(5.9) \qquad I_5 = \frac{2}{\pi}\int_0^{\beta\alpha_n} \rho_n(u) \sin\frac{1}{2}\theta_n(u) \sin\left(\frac{1}{2}\theta_n(u) - xu\right) du.$$

In a sufficiently small neighborhood of the origin, $\arg\phi(t)$ has a convergent Maclaurin expansion of the form

$$\arg\phi(t) = ct^{r+h} + \cdots,$$

where h is a positive even integer and $c \neq 0$. Thus (5.3) gives

$$|\theta_n(u)| = |nct^{r+h} + \cdots| < 2n|c|t^{r+h} = \frac{2|c|}{a}u^{r+h}(an)^{-(h-1)/(r+1)}$$

for $0 \leqq u \leqq \beta\alpha_n$ if β is sufficiently small (in addition to the condition previously imposed upon it). Thus, we have

$$|\sin \tfrac{1}{2}\theta_n(u)| < Lu^{r+h}(an)^{-(h-1)/(r+1)}$$

for some L for $u \in (0, \beta\alpha_n)$, since $|\sin y| < |y|$ for all $y \neq 0$. Thus, (5.9) and (5.7) give

$$|I_5| < \frac{2}{\pi}L(an)^{-(h-1)/(r+1)} \int_0^\infty u^{r+h} \exp\left(-\frac{a_1}{a}u^{r+1}\right) du.$$

Since the integral converges to a finite constant, $|I_5| < \varepsilon/4$ for sufficiently large n.

Therefore, there is some N_ε such that for $n > N_\varepsilon$,

$$|F_n(x) - G_{r+1}(x)| < |I_2| + |I_3| + |I_4| + |I_5| < \varepsilon,$$

uniformly in x.

6. Convergence in the case of even *r*. The main result of this paper is the proof that the normalized coefficients $c_j^{(n)}$ converge to the postulated limiting function in the case of even r. As convergence is much slower than in the case of odd r, the proof is correspondingly more difficult. Nevertheless, pointwise convergence is not difficult to prove; it is the requirement to prove uniform convergence that introduces complications. We shall need to utilize several lemmas, which grew out of suggestions made by J. Barkley Rosser.

LEMMA 1. *Let $f(u) \in C'$ be such that $f'(u)$ is strictly positive or strictly negative for $u \in [u_1, u_3]$, let $g(u)$ be integrable and such that $|g(u)/f'(u)|$ is monotonic decreasing* *for* $u \in [u_1, u_3]$, *and let* u_1, u_2, u_3 *be three consecutive zeros of* $\cos f(u)$. *Then*

$$\left| \int_{u_1}^{u_2} g(u) \cos f(u)\, du \right| > \left| \int_{u_2}^{u_3} g(u) \cos f(u)\, du \right|. \tag{6.1}$$

Proof. Let $w = f(u)$. Since $f'(u)$ is strictly positive or strictly negative for $u \in [u_1, u_3]$, u is a single-valued monotonic function of w, say $h(w)$, for w in the interval between $f(u_1)$ and $f(u_3)$, and

$$\int_{u_j}^{u_{j+1}} g(u) \cos f(u)\, du = \int_{(v+1/2)\pi \pm j\pi}^{(v+1/2)\pi \pm (j+1)\pi} \frac{g(h(w)) \cos w}{f'(h(w))}\, dw$$

for $j = 1, 2$ for some integer v, where we use the plus or minus signs according as $f'(u)$ is positive or negative. For $j = 2$, $\cos w$ goes through the same absolute values as for $j = 1$, but with the opposite sign. Since $|g(u)/f'(u)|$ is monotonic decreasing for $u \in [u_1, u_3]$, the coefficients of corresponding values of $\cos w$ have the same sign and are smaller in absolute value for $j = 2$ than for $j = 1$. Inequality (6.1) therefore follows.

LEMMA 2. *Let* $g(u)$ *be integrable on* (u_0, u_1), *and let* $f(u) \in C''$ *be such that* $f'(u)$ *does not change sign and* $f''(u) > 0$ *in* $[u_0, u_1]$, *where* u_1 *is the smallest argument greater than* u_0 *such that* $\cos f(u_1) = 0$. *Let* $u_0 < b \leqq u_1$. *Also let*

$$\eta = \min_{u \in [u_0, u_1]} f''(u)$$

and

$$\xi = \max_{u \in [u_0, u_1]} |g(u)|.$$

Then

$$\left| \int_{u_0}^{b} g(u) \cos f(u)\, du \right| \leqq \xi \sqrt{\frac{2\pi}{\eta}}.$$

Proof. First, consider the case in which $f'(u) \geqq 0$ in $[u_0, u_1]$. We have

$$f(u_1) = f(u_0) + (u_1 - u_0) f'(u_0) + \int_{u_0}^{u_1} (u_1 - u) f''(u)\, du$$

and

$$f(u_1) - f(u_0) \leqq \pi.$$

Therefore,

$$\pi \geqq (u_1 - u_0) f'(u_0) + \int_{u_0}^{u_1} (u_1 - u) f''(u)\, du \geqq \eta \int_{u_0}^{u_1} (u_1 - u)\, du = \tfrac{1}{2}\eta (u_1 - u_0)^2.$$

Hence,

$$u_1 - u_0 \leqq \sqrt{2\pi/\eta},$$

and, consequently,

$$\left| \int_{u_0}^{b} g(u) \cos f(u)\, du \right| \leqq \int_{u_0}^{u_1} |g(u)|\, du \leqq \xi \sqrt{\frac{2\pi}{\eta}}.$$

Note that equality holds only if $\xi = 0$.

If $f'(u) \leqq 0$ in $[u_0, u_1]$ we employ the substitution $u = -v$ and consider the resulting integral with respect to v.

LEMMA 3. *Let $f(u) \in C''$ be such that $f'(u)$ does not change sign and $f''(u) > 0$ for $u \in [a, b]$, and let $g(u)$ be integrable with no change of sign on (a, b), and be such that $|g(u)/f'(u)|$ is monotonic on $[a, b]$. Moreover, let*

$$\eta = \min_{u \in [a,b]} f''(u)$$

and

$$\xi = \max_{u \in [a,b]} |g(u)|.$$

Then

$$\left| \int_a^b g(u) \cos f(u)\, du \right| \leqq \xi \sqrt{\frac{2\pi}{\eta}}. \tag{6.2}$$

Proof. If $\cos f(u)$ has no zero in (a, b), the result follows from Lemma 2. Otherwise, let $u_1, u_2, \cdots, u_m$ be the successive arguments u in (a, b) such that $\cos f(u) = 0$, and consider first the case in which $|g(u)/f'(u)|$ is monotonic decreasing on $[a, b]$. Then

$$\int_{u_1}^b g(u) \cos f(u)\, du = \sum_{j=1}^{m-1} \int_{u_j}^{u_{j+1}} g(u) \cos f(u)\, du + \int_{u_m}^b g(u) \cos f(u)\, du,$$

where the successive terms of the right member are of alternating sign, and, by Lemma 1, of decreasing absolute value. Therefore, the left member has the same sign as

$$I_1 = \int_{u_1}^h g(u) \cos f(u)\, du,$$

and is less than or equal to I_1 in absolute value, where $h = b$ when $m = 1$ and $h = u_2$ when $m > 1$. Moreover,

$$I_2 = \int_a^{u_1} g(u) \cos f(u)\, du$$

has the opposite sign. Therefore,

$$\left| \int_a^b g(u) \cos f(u)\, du \right| < \max(|I_1|, |I_2|).$$

Since the minimum of $f''(u)$ on $[a, b]$ is less than or equal to its minima on $[a, u_1]$ and $[u_1, h]$ and an analogous remark applies to the maximum of $|g(u)|$, (6.2) now follows from Lemma 2.

In case $|g(u)/f'(u)|$ is monotonic increasing on $[a, b]$, we employ the substitution $u = -v$ and consider the equivalent integral with respect to v.

LEMMA 4. *The integral in (2.9) converges uniformly in x.*

Proof. We apply Lemma 3 to the integral

$$I_M = \int_M^\infty \cos(u^{r+1} - xu)\, du,$$

taking $f(u) = u^{r+1} - xu$ and $g(u) \equiv 1$. Let

$$u_x = \left(\frac{x}{r+1}\right)^{1/r},$$

which is the only real zero of $f'(u)$. If $u_x \leqq M$, the hypotheses of Lemma 3 are fulfilled, and we have

$$|I_M| < \left(\frac{2\pi}{r(r+1)M^{r-1}}\right)^{1/2}. \tag{6.3}$$

We shall denote by K_M the right member of (6.3). If $u_x > M$, we must consider separately the intervals (M, u_x) and (u_x, ∞). On each subinterval considered separately, the hypotheses of Lemma 3 are fulfilled, and we have, therefore, for all x,

$$|I_M| < 2K_M.$$

It is clear that, if M is sufficiently large, $|I_M|$ is less than an arbitrary positive ε.

With the help of Lemmas 3 and 4, we can now prove the main theorem.

THEOREM 3. *Let* (1.1) *be an adjustment formula exact for even* r ($r = 2, 4, 6, \cdots$) *with a characteristic function* $\phi(t)$ *having the properties* (1.7) *and* (2.8). *Then, to every positive* ε *there corresponds a positive integer* N_ε *such that*

$$\left|\, |an|^{1/(r+1)} c_j^{(n)} - H_{r+1}(j(an)^{-1/(r+1)})\right| < \varepsilon$$

for all j *if* $n > N_\varepsilon$. *In other words, for* $n \to \infty$,

$$c_j^{(n)} = |an|^{-1/(r+1)} H_{r+1}(j(an)^{-1/(r+1)}) + o(n^{-1/(r+1)}),$$

uniformly for all integral values of j.

Proof. We use the same notations (5.1)–(5.3) as in the proof of Theorem 2, but take

$$\alpha_n = \pi |an|^{1/(r+1)},$$

since the coefficient a is no longer restricted as to sign. It is then sufficient to show that

$$\lim_{n\to\infty} F_n(x) = H_{r+1}(x)$$

uniformly in x, and we write

$$\begin{aligned} F_n(x) - H_{r+1}(x) = &\frac{1}{\pi}\int_0^M [\rho_n(u)\cos(\theta_n(u) - xu) - \cos(u^{r+1} - xu)]\, du \\ &+ \frac{1}{\pi}\int_M^{\beta\alpha_n} \rho_n(u)\cos(\theta_n(u) - xu)\, du \\ &+ \frac{1}{\pi}\int_{\beta\alpha_n}^{\alpha_n} \rho_n(u)\cos(\theta_n(u) - xu)\, du \\ &- \frac{1}{\pi}\int_M^{\infty} \cos(u^{r+1} - xu)\, du \\ = &\ I_1 + I_2 + I_3 - I_4. \end{aligned}$$

Here we shall choose M sufficiently large so that

$$|I_4| < \varepsilon/4$$

by Lemma 4. Conditions on β and further conditions on M will be imposed later. By the reasoning of Schoenberg already employed in the proof of Theorem 2, we show that for sufficiently large n,

$$|I_3| < \varepsilon/4.$$

Since

$$\lim_{n\to\infty} [\phi(u(an)^{-1/(r+1)})]^n = e^{iu^{r+1}}$$

uniformly in u over $[0, M]$, multiplying by e^{-iux} and equating real parts gives

$$\lim_{n\to\infty} \rho_n(u)\cos(\theta_n(u) - xu) = \cos(u^{r+1} - xu)$$

uniformly in u over $[0, M]$. Thus, for n sufficiently large, the integrand of I_1 is less than $\pi\varepsilon/4M$ for all u in $[0, M]$, and therefore

$$|I_1| < \varepsilon/4.$$

Finally, we consider I_2. We observe that $\phi(t)$ has a Maclaurin expansion of the form

$$\phi(t) = 1 + ct^{r+h} + \cdots + i(at^{r+1} + dt^{r+k} + \cdots),$$

where h is a positive even integer and k an odd integer not less than 3. By the definition of the absolute value, we obtain for $|\phi(t)|$ an expansion of the form

$$|\phi(t)| = 1 - Bt^{r+w} + \cdots, \tag{6.4}$$

where w is a positive integer. We note that the hypothesis that $\phi(t)$ satisfies (1.7) requires w even and $B > 0$ (since otherwise (1.7) would be violated for values of t sufficiently close to zero on one side or the other). We have also the expansion

$$\arg\phi(t) = at^{r+1} + \cdots,$$

and, consequently,

$$\frac{d}{dt}(\arg\phi(t)) = a(r+1)t^r + \cdots, \tag{6.5}$$

$$\frac{d^2}{dt^2}(\arg\phi(t)) = ar(r+1)t^{r-1} + \cdots. \tag{6.6}$$

We are interested in the behavior of these expressions in the interval between 0 and $\beta\pi \operatorname{sgn} a$. It follows from (6.4)–(6.6) that we can choose β sufficiently small so that in this interval $|\phi(t)|$ is monotonic decreasing with increasing $|t|$ and both $(d^2/dt^2)(\arg\phi(t))$ and $(\operatorname{sgn} a)(d/dt)(\arg\phi(t))$ are positive and monotonic increasing with increasing $|t|$, and so that, moreover,

$$\frac{d^2}{dt^2}(\arg\phi(t)) \geqq K|t|^{r-1} \tag{6.7}$$

in the interval under consideration, for some positive K.

Now with

$$t = u(an)^{-1(r+1)},$$

equations (5.2) and (5.3) give

$$\begin{aligned} \rho_n(u) &= |\phi(t)|^n, \\ \theta'_n(u) &= n(an)^{-1/(r+1)}\frac{d}{dt}(\arg\phi(t)), \qquad (6.8) \\ \theta''_n(u) &= n(an)^{-2/(r+1)}\frac{d^2}{dt^2}(\arg\phi(t)). \end{aligned}$$

Therefore, if β is chosen as indicated, $\rho_n(u)$ is monotonic decreasing and $\theta'_n(u)$ and $\theta''_n(u)$ are positive and monotonic increasing for $u \in [0, \beta\alpha_n]$.

We turn now to the appraisal of I_2. Different cases must be considered, depending on the value of x. First if $x \leqq \theta'_n(M)$, we apply Lemma 3 with $f(u) = \theta_n(u) - xu$ and $g(u) = \rho_n(u)$. We have then

$$\left|\frac{g(u)}{f'(u)}\right| = \frac{\rho_n(u)}{\theta'_n(u) - x}.$$

Since the numerator is monotonic decreasing and the denominator is monotonic increasing, this ratio is monotonic decreasing for $u \in [M, \beta\alpha_n]$. Therefore, the hypotheses of Lemma 3 are satisfied. Moreover, by (6.8) and (6.7),

$$\eta = \theta''_n(M) = n(an)^{-2/(r+1)}\frac{d^2}{dt^2}(\arg\phi(M(an)^{-1/(r+1)})) \geqq \frac{KM^{r-1}}{|a|},$$

while $\xi < 1$. Therefore, by Lemma 3,

$$|I_2| \leqq \left(\frac{2|a|\pi}{KM^{r-1}}\right)^{1/2}. \qquad (6.9)$$

We shall denote by L_M the right member of (6.9).

If $x \geqq \theta'_n(\beta\alpha_n)$, we write

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_M^{\beta\alpha_n}\cos(\theta_n(u) - xu)\,du - \frac{1}{\pi}\int_M^{\beta\alpha_n}(1 - \rho_n(u))\cos(\theta_n(u) - xu)\,du \\ &= I_5 - I_6. \end{aligned}$$

The first integral satisfies the hypotheses of Lemma 3 with $f(u) = \theta_n(u) - xu$ and $g(u) = 1$, and we have

$$\left|\frac{g(u)}{f'(u)}\right| = \frac{1}{x - \theta'_n(u)},$$

which is monotonic increasing. Consequently

$$|I_5| \leqq L_M.$$

Similarly, in the case of I_6, we take $f(u) = \theta_n(u) - xu$ and $g(u) = 1 - \rho_n(u)$, so that

$$\left|\frac{g(u)}{f'(u)}\right| = \frac{1 - \rho_n(u)}{x - \theta_n'(u)}.$$

Since the numerator is monotonic increasing and the denominator monotonic decreasing, this ratio is monotonic increasing, and Lemma 3 gives

$$|I_6| \leqq L_M.$$

Therefore,

$$|I_2| \leqq |I_5| + |I_6| \leqq 2L_M.$$

Finally, when $\theta_n'(M) < x < \theta_n'(\beta\alpha_n)$, it is necessary to consider separately the two subintervals (M, x) and $(x, \beta\alpha_n)$ and to apply the methods of the two preceding cases to the respective subintervals. This gives

$$|I_2| \leqq 2L_M + L_M = 3L_M.$$

Therefore, for all x,

$$|I_2| \leqq 3L_M.$$

From the definition of L_M as the right member of (6.9), it follows that M can be chosen sufficiently large so that

$$|I_2| < \varepsilon/4.$$

As this proof is rather complicated, it may be well to recapitulate the order in which the various constants are chosen. First we must choose β sufficiently small so that the required positivity and monotonicity conditions on $|\phi(t)|$ and the derivatives of $\arg \phi(t)$ are satisfied in the interval between 0 and $\beta\pi \operatorname{sgn} a$. The choice of β determines the positive constant K of (6.7) and the positive constant γ of (5.4). Next we take M sufficiently large so that both $3L_M$ and $2K_M/\pi$ (where K_M was defined as the right member of (6.3)) are less than $\varepsilon/4$. This ensures that $|I_2|$ and $|I_4|$ are less than $\varepsilon/4$. Finally, n must be taken large enough to satisfy three conditions: (i) we must have $\beta\alpha_n \geqq M$; (ii) the integrand of I_1 must be less than $\pi\varepsilon/4M$ for all $u \in [0, M]$, and (iii) the second inequality of (5.4) (with (an) replaced by $|an|$) must be satisfied. It then follows that $|I_1|$ and $|I_3|$ are less than $\varepsilon/4$, and therefore

$$|F_n(x) - H_{r+1}(x)| \leqq |I_1| + |I_2| + |I_3| + |I_4| < \varepsilon.$$

7. The classical theory of smoothing formulas. There is a fairly extensive literature on adjustment formulas of the form (1.1) intended for application to equally spaced observational data with the object of replacing irregular data by "smoother" adjusted values. Such replacement is sometimes called "graduation". Though this literature includes such eminent names as G. V. Schiaparelli [20], W. F. Sheppard [24]–[27], E. T. Whittaker [34], and I. J. Schoenberg [21]–[23], it appears to be relatively little known among contemporary mathematicians. In the past, the chief applications of such formulas have been in actuarial science and time-series analysis [15], but in recent years interest has been shown by workers concerned with other kinds of observational data. For a more detailed account of this classical theory, the interested reader may consult references [12], [16], [35]. For the present purpose, it will be sufficient to mention that one approach emphasizes numerical quantities R_m, $m = 0, 1, 2, \cdots$, associated with every such formula, sometimes called the "smoothing coefficient of order m" of the particular formula [35], [8].

If, for notational convenience, we extend the limits of summation in (1.1) to $-\infty$ and $+\infty$, with the understanding that the additional coefficients c_j vanish, the quantities R_m are defined by

$$R_m^2 = \sum_{j=-\infty}^{\infty} (\Delta^m c_j)^2 \Big/ \binom{2m}{m}.$$

It has been customary to judge smoothness on the basis of the numerical magnitude of the mth finite differences of the graduated data (if these are numerically small, the data are "smooth"), for some chosen m (frequently 3), and it has been shown that a small value of R_m tends to be associated with numerically small mth differences.

Schoenberg has shown [22] that R_m can be expressed in terms of the characteristic function by

$$R_m^2 = \int_{-\pi}^{\pi} (2 \sin \tfrac{1}{2} t)^{2m} \phi(t)\overline{\phi(t)}\, dt \Big/ \int_{-\pi}^{\pi} (2 \sin \tfrac{1}{2} t)^{2m}\, dt. \tag{7.1}$$

Greville has pointed out [8] that, consequently,

$$R_\infty = \lim_{m \to \infty} R_m = |\phi(\pi)|. \tag{7.2}$$

Schoenberg's definition of a smoothing formula, as a formula (1.1) such that (1.7) holds, is eminently consistent with the classical theory. Indeed, it follows at once from (7.1) and (7.2) that if (1.7) holds, then $R_m < 1$ for all m, including ∞. The converse is not true, as shown by the example

$$v_l = \tfrac{1}{22}(6y_{l-2} + 9y_{l-1} - 8y_l + 9y_{l+1} + 6y_{l+2}),$$

which is exact for degree 1. Here,

$$\phi(t) = \tfrac{1}{22}(-8 + 18 \cos t + 12 \cos 2t),$$

and it is easily verified that

$$\phi(\arccos(-\tfrac{3}{8})) = -\tfrac{17}{16},$$

so that (1.7) fails. Yet, it can be shown that

$$R_m^2 = \frac{149}{242} + \frac{18}{121}\frac{m}{m+1} - \frac{15}{242}\frac{m(m-1)}{(m+1)(m+2)} - \frac{54}{121}\frac{m(m-1)(m-2)}{(m+1)(m+2)(m+3)} + \frac{18}{121}\frac{m(m-1)(m-2)(m-3)}{(m+1)(m+2)(m+3)(m+4)},$$

and consequently $R_m < 1$ for all m.

On the other hand, formulas (1.1) that minimize R_m for some m, subject to certain constraints, have been much studied [4], [16], [35] and Greville [8] has shown that an important class of such formulas do, in fact, have characteristic functions satisfying (1.7). Trench [31] has extended Greville's results.

8. Critique of formulas exact to an even degree. As stated in the introductory section, iteration of a transformation of the form (1.1) is utilized as a numerical procedure in certain difference schemes for numerical solution of partial differential equations. Some results obtained in the study of such procedures have an important bearing on the problem considered in this paper. In particular, it is an immediate corollary of a theorem of Thomée [29, Theorem 1] that, for every formula (1.1) satisfying (1.2) and (1.7), the quantity

$$C_n = \sum_{j=-\infty}^{\infty} |c_j^{(n)}|$$

is bounded for all n if and only if the formula is exact to an odd degree. This finding is remarkable and interesting, but, I think, not surprising, since the limiting function $H_{r+1}(x)$ for the case of even r is not absolutely integrable over the real line.

It has been suggested that, because C_n is unbounded, any formula (1.1) that is exact to an even degree should be regarded with suspicion. It is pointed out that if y and v are infinite vectors of "crude" and "graduated" values, respectively, (1.1) defines a transformation T, so that

$$v = Ty,$$

and C_n is actually the supremum norm of T^n. It is evident, in fact, that

$$\|T^n\|_\infty \leqq C_n,$$

while if we take $y^{(n)}$ as a vector such that $y_j^{(n)} = \operatorname{sgn} c_j^{(n)}$, we have

$$v_0 = C_n = C_n \|y^{(n)}\|_\infty .$$

I am indebted to V. Thomée for the suggestion that by placing the nonzero portions of the vectors $y^{(n)}$ end to end for $n = 1, 2, \cdots$, one can even construct a fixed vector y such that $\|T^n y\|_\infty$ is unbounded as n tends to infinity. Indeed, Thomée has pointed out that the existence of such a fixed vector follows from general considerations of functional analysis.

It should be pointed out, however, that there are two basic differences between classical smoothing and difference schemes for partial differential equations. In the latter case, iteration of a transformation (1.1) is utilized as an actual numerical procedure. In the former, such iteration is only a mental construct, motivated in De Forest's case solely by intellectual curiosity (and possibly the hope of discovering new probability density functions, a hope not realized because the new functions are found to assume negative values), and in Schoenberg's case additionally by the desire for further validation of his definition of a smoothing formula as one having the property (1.7). I have never heard of anyone iterating a classical smoothing formula upon numerical data.

Secondly, in classical smoothing one is not seeking to approximate a well-defined mathematical function. At most one can postulate an "underlying trend" in the observational data, on which errors have been superimposed.

A critic strongly oriented toward difference schemes for partial differential equations has argued that application of an adjustment formula having an unbounded norm in the sense described will "make the data rougher rather than smoother". In my opinion, this argument is based on a misunderstanding. Every adjustment formula having some negative coefficients c_j (as it must have if it is exact for a degree greater than 1) evidently has a supremum norm $\|T\|_\infty$ greater than unity. It is therefore possible to construct a pathological data vector such that application of the formula will make the data rougher. This is true *whether or not* the norm of the nth iterate is bounded as n tends to infinity. The difference between the two cases is that if a linear adjustment formula is applied repeatedly to pathological data, in the one case there is a finite upper bound to how rough the adjusted data can become, and in the other case there is not. This does not seem to me an important practical distinction when the formula is intended to be applied only once in the hope of increasing smoothness.

However, since this question has been raised, it is of some interest to examine more carefully the smoothing properties of formulas exact to an even degree. First, for the benefit of the uninitiated reader, it may be well to establish that we are not talking about an empty class. Consider the formula

$$v_l = y_l - a\,\Delta^3 y_{l-1}.$$

If $a \neq 0$, this is exact to the degree 2. It is easily verified that

$$|\phi(t)|^2 = 1 - 16a\sin^4 \tfrac{1}{2}t + 64a^2 \sin^6 \tfrac{1}{2}t,$$

and (1.7) is satisfied if and only if $0 < a < \frac{1}{4}$.

Since a formula exact for degree 2 must have at least four terms, we must go to six terms in order to have a meaningful comparison with a symmetrical formula. For comparison, we first note that the minimum-R_m formula employing the values y_{l-2} to y_{l+2}, inclusive, and exact for degree at least 2 is

$$v_l = y_l - \frac{(m+3)(m+4)}{4(2m+5)(2m+7)}\Delta^4 y_{l-2}. \tag{8.1}$$

Since this is a symmetrical formula, it is actually exact for degree 3, and therefore not subject to the criticism that the operator T^n is unbounded.

Let us now broaden the class of formulas considered, by including y_{l+3} (but not y_{l-3}) in the summation (1.1). In the resulting class of 6-term formulas exact for degree 2, we seek the one for which R_m is smallest. The result is most surprising, and I think not previously noted in the literature. The minimum-R_m formula of this class is

$$v_l = y_l - \frac{m+4}{2(2m+5)}\Delta^3 y_{l-1} - \frac{(m+4)(m+5)}{4(2m+5)(2m+7)}\Delta^4 y_{l-1}. \tag{8.2}$$

Note that this is again a 5-term formula, as the coefficient of y_{l-2} is zero! The formula is even more unsymmetrical than we required it to be, involving three arguments greater than l and only one less. Since the class of 6-term formulas considered includes (8.1), this means that the unsymmetrical 5-term formula (8.2) has a smaller R_m than the symmetrical 5-term formula (8.1).

This result was so surprising that it seemed desirable to make a numerical experiment. For the customary $m = 3$, (8.1) and (8.2) become

$$v_l = y_l - \tfrac{21}{286}\Delta^4 y_{l-2} \tag{8.3}$$

and

$$v_l = y_l - \tfrac{7}{22}\Delta^3 y_{l-1} - \tfrac{14}{143}\Delta^4 y_{l-1}. \tag{8.4}$$

R_3 is .274 for (8.3) and .153 for (8.4).

The "crude" data in the second column of Table 1 are observed mortality rates from a life insurance experience taken from [16]. These are the first and only numerical data on which I have tested the two formulas. It should be pointed out that, as a practical matter, a formula with as short a range as 5 terms would not be used to graduate data as irregular as these. It is evident that the third differences of the graduated rates obtained by (8.4) are numerically smaller than those derived from (8.3), whatever norm is used to compare them. It is clear also that (8.4) has been more successful in attempting to smooth out the accidental fluctuations in the data; the remaining undulations in the graduated rates are of smaller amplitude than when (8.3) is used. On the basis of either the classical theory or common sense, (8.4) is a better smoothing formula than (8.3).

TABLE 1

Comparison of graduations by symmetrical and unsymmetrical 5-term formulas

Age	Crude mortality rate	Graduated rate		Third differences of graduated rates	
		by (8.3)	by (8.4)	by (8.3)	by (8.4)
35	.00215				
36	.00212		.00185		−.00071
37	.00169	.00174	.00191	−.00093	.00080
38	.00192	.00218	.00231	.00188	−.00013
39	.00320	.00274	.00234	−.00087	−.00066
40	.00238	.00249	.00280	−.00163	−.00037
41	.00259	.00331	.00356	.00132	.00109
42	.00553	.00433	.00396	.00188	.00086
43	.00311	.00392	.00363	−.00145	−.00131
44	.00365	.00340	.00366	−.00100	.00012
45	.00446	.00465	.00491	.00068	−.00040
46	.00632	.00622	.00607	−.00038	−.00053
47	.00741	.00711	.00726	−.00075	.00095
48	.00726	.00800	.00808	.00203	.00029
49	.00945	.00851	.00800	.00022	.00113
50	.00749	.00789	.00797	−.00060	
51	.00763	.00817	.00828		
52	.01064	.00957	.01006		
53	.00999	.01149			
54	.01378				
55	.00967				

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Note added in proof. It has been brought to my attention by J. M. Hoem that K. Weichselberger (*Über eine Theorie der gleitenden Durchshnitte und verschiedene Anwendungen dieser Theorie*, Metrika, 8 (1964), pp. 185–230) gives (8.1) for the case of $m = 0$. He points out that R_0 is less for this formula than for the corresponding symmetrical formula, but does not mention that the minimum-R_0 6-term formula with range $l - 2$ to $l + 3$ degenerates to this 5-term formula.

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CONJUGATE INEQUALITIES FOR FUNCTIONS AND THEIR DERIVATIVES*

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Abstract. We consider finding the best possible constants $C = C(n, \alpha, \beta, p, q)$ such that $\|f\|_p \leqq C(b - a)^{n+1/p-1/q}\|f^{(n)}\|_q$ under several different sets of boundary conditions. Specifically in one case, f is required to have α zeros at a and β zeros at b where $n \leqq \alpha + \beta \leqq 2n$, $\alpha \leqq n$ and $\beta \leqq n$. In the other case, f has α zeros at a, β zeros at b and $\int_a^b f(x)x^r(x - a)^\alpha(x - \beta)^\beta\,dx = 0$, $r = 0, \cdots, n - 1 - \alpha - \beta$, where $0 \leqq \alpha$, $0 \leqq \beta$ and $\alpha + \beta < n$. The various problems are related and the numbers C are calculated exactly in some cases. Finally, we show how some of these can be applied to disconjugacy problems in differential equations.

1. Introduction. We shall consider inequalities between functions and their derivatives when the function has a certain distribution of zeros. The case when the number of zeros is the same as the order of the derivative considered has been discussed by Brink [1]. Some of these inequalities have also been considered by Boyd [2]. We are primarily interested in the case when the number of zeros exceeds the order of the derivative.

Specifically, let $\|f\|_p \equiv (\int_a^b|f|^p)^{1/p}$ for $1 \leqq p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}|f(t)|$. Then we will consider inequalities of the form

$$\|f\|_p \leqq C\|f^{(n)}\|_q(b - a)^{n+1/p-1/q} \tag{1}$$

when either

(2) (α, β): f has a zero of order α at a and a zero of order β at b with $n \leqq \alpha + \beta \leqq 2n$, $\alpha \leqq n$ and $\beta \leqq n$;

or

(3) (α, β): f has a zero of order α at a and a zero of order β at b with $\alpha + \beta < n$ and $\int_a^b f(x)(x - a)^\alpha(b - x)^\beta\, q(x)\,dx = 0$ for all polynomials q of degree $\leqq n - 1 - \alpha - \beta$.

We have chosen the exponent of $(b - a)$ in (1) so that C is independent of translation or change of scale. We thus take $a = 0$ and $b = 1$ for the remainder of the discussion. We also assume that f has $n - 1$ continuous derivatives and that $f^{(n-1)}$ is absolutely continuous so that $f^{(n)}$ exists almost everywhere and is integrable in the appropriate sense.

Our discussion includes a novel way of deriving the variational equations, that is, the equations satisfied by the extremals of the problem. Our study was, in part, motivated by an attempt to get inequalities between the derivatives of different orders of a given function. A discussion of this problem and its applications to differential equations is indicated in the last part of this paper.

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2. The case $\alpha + \beta = n$. The consideration of the boundary conditions (2) (α, β) when $\alpha + \beta = n$ is relatively simple. We show in this section, by way of introduction to the more complicated case, $\alpha + \beta \neq n$, how to derive certain variational equations.

We will have occasion to use the well-known fact that there is a Green's function $G(x, t)$ so that when f satisfies (2) (α, β), then the representation

$$f(x) = \int_0^1 G(x,t) f^{(n)}(t)\,dt \tag{4}$$

holds with G being continuous as a function of two variables. Furthermore, $G(x, t)$ will be of constant sign. Indeed if $f^{(n)}(t) \geqq 0$, then f is nonzero on $(0, 1)$. In the contrary case, if f has a zero on $(0, 1)$, then $f^{(n-1)}$ has two zeros on $[0, 1]$ and thus $f^{(n)}$ must change sign. An easy argument using $f^{(n)}$ with small support completes the proof that G does not change sign on $[0, 1] \times [0, 1]$. In fact $(-1)^{\beta}G(x, t) \geqq 0$. This is easy to see for

$$g(x) = \int_0^1 G(x,t)1\,dt = \frac{x^{\alpha}(1-x)^{\beta}(-1)^{\beta}}{n!}.$$

As a standard assumption for the remainder of the paper, we write $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$ with the usual assumptions when one of these numbers is 1 or ∞.

Using the representation (4), it is easy to see that there are numbers C which satisfy (1) under the boundary conditions (2) (α, β). In fact the constant

$$\left(\int_0^1 \left(\int_0^1 |G(x,t)|^{q'}\,dt \right)^{p/q'} dx \right)^{1/p}$$

is one. We are interested in the best possible constants so we let $C(n, \alpha, p, q)$ be the infimum over all constants for which (1) is valid.

THEOREM 1. *For* $1 \leqq p \leqq \infty$ *and* $1 \leqq q \leqq \infty$,

$$C(n,\alpha,p,q) = C(n,\alpha,q',p').$$

Proof. As a preliminary fact, we note that $C(n, \alpha, p, q) = C(n, \beta, p, q)$ by consideration of the mapping of the independent variable $x \to 1 - x$. Now we let f satisfy the $(2)(\alpha, \beta)$ conditions and define h by $h^{(n)} = |f|^{p-1}\operatorname{sgn}(f)$ and h is to satisfy the boundary conditions $(2)(\beta, \alpha)$. The existence of h follows from the representation (4) with the appropriate Green's function. Then $\|f\|_p^p = \int_0^1 f h^{(n)} = (-1)^n \int_0^1 f^{(n)} h$ by use of integration by parts and the boundary conditions. Thus

$$\begin{aligned}\|f\|_p^p &\leqq \|f^{(n)}\|_q \|h\|_{q'} \leqq \|f^{(n)}\|_q C(n,\beta,q',p')\|h^{(n)}\|p' \\ &= \|f^{(n)}\|^q C(n,\alpha,q',p')\|f\|_p^{p-1}.\end{aligned}$$

Now $\|f\|_p \leqq C(n, \alpha, q', p')\|f^{(n)}\|_q$ and by the minimality of $C(n, \alpha, p, q)$ we have $C(n, \alpha, p, q) \leqq C(n, \alpha, q', p')$. By symmetry the reverse inequality is true. If $p = \infty$, the inequality follows by taking limits.

The two equivalent inequalities of Theorem 1 are called conjugate inequalities for obvious reasons. If there is an f for which the inequality (1) is an equality, then all the inequalities in the above proof must be equalities. Assuming now that

$1 \leqq p < \infty$ and $1 < q < \infty$ we then must have

$$|f^{(n)}|^q = \lambda^2 |h|^{q'} \quad \text{and} \quad h^{(n)} = |f|^{p-1} \operatorname{sgn}(f) \tag{5}$$

such that f satisfies (2)(α, β) and h satisfies (2)(β, α), for some constant λ. Note that both f and h must give equality, so that both are extremals or neither. The equations (5) give candidates for extremals, the existence of which is the content of the next two lemmas.

LEMMA 1. *If $1 \leqq p \leqq \infty$ and $1 < q < \infty$, then there is a function f for which $\|f\|_p = C(n, \alpha, p, q)\|f^{(n)}\|_q$ and equations* (5) *hold if $1 \leqq p < \infty$ and $1 < q < \infty$.*

Proof. By the homogeneity of the inequality (1) we may assume that $\|f^{(n)}\|_q = 1$ so that we need to show that max $\|f\|_p$ is obtained. Let $\|g\|_q = 1$ and define $f(x) = \int_0^1 G(x, t)g(t)\,dt$ for the Green's function G for (2)(α, β). As g varies over the unit sphere in L_q, the family F so defined is uniformly bounded and equicontinuous. If f_k is a sequence so that $\|f_k\|_p \to \sup \|f\|_p = C(n, \alpha, p, q)$, then we may take subsequences, which we do not relabel, so that $f_k \to f_0$ uniformly on $[0, 1]$ and $f_k^{(n)} \to g_0$ weakly. The first is by the Ascoli theorem and the second by the weak compactness of the unit sphere in L_q. Thus for each x,

$$f_0(x) = \lim f_k(x) = \lim \int_0^1 G(x, t) f_k^{(n)}(t)\,dt = \int_0^1 G(x, t) g_0(t)\,dt.$$

Now f^0 has $n - 1$ continuous derivatives, $f_0^{(n)} = g_0$, a.e., and f_0 satisfies the boundary conditions. Clearly $\|f_0\|_p = C(n, \alpha, p, q)$.

LEMMA 2. *Extremals exist unless* $(p, q) = (1, 1)$ *or* (∞, ∞).

Proof. Lemma 1 gives extremals for $1 \leqq p \leqq \infty$ and $1 < q < \infty$. By Theorem 1 this also holds for $1 \leqq q' \leqq \infty$ and $1 < p' < \infty$ or $1 < p < \infty$ and $1 \leqq q \leqq \infty$. Only $p = q = 1$ or $p = q = \infty$ remain.

In order to give one result with numbers, we give a result proved by Brink [1] by geometric methods. One method is exceedingly simple and can be generalized.

LEMMA 3. *For* $1 < q < \infty$,

$$C(n, \alpha, 1, q) = \left\| \frac{x^\alpha (1 - x)^\beta}{n!} \right\| = \frac{1}{n!} \frac{\Gamma(\alpha q' + 1)\Gamma(\beta q' + 1)^{1/q'}}{\Gamma(nq' + 1)},$$

while $C(n, \alpha, 1, 1) \leqq \alpha^\alpha \beta^\beta/(n! n^n)$.

Proof. Since $G(x, t)$ does not change sign, we note that if $g(t) = \int_0^1 G(x, t) \cdot |f^{(n)}(t)|\,dt$, then $|f(t)| \leqq |g(t)|$ so $\|f\|_1 \leqq \|g\|_1$. We may thus restrict ourselves to functions for which $f^{(n)}$ does not change sign. It now follows that f does not change sign so we assume $f \geqq 0$. Then $\|f\|_1 = \int_0^1 f = \int_0^1 f g^{(n)}$, where

$$g(x) = \frac{x^\beta (1 - x)^\alpha}{n!} (-1)^\alpha.$$

Then an integration by parts leads to $\|f\|_1 = (-1)^n \int_0^1 f^{(n)} g \leqq \|f^{(n)}\|_q \|g\|_{q'}$ with equality $(1 < q < \infty)$ for $|f^{(n)}|^q = \lambda^2 |g|^{q'}$ for some λ.

Once we had shown $f \geqq 0$ we could have appealed directly to (5) but we preferred to show that one does not need to prove that extremals exist; one merely exhibits it by the proof. The constancy of sign of f and $f^{(n)}$ and the inequality

$(-1)^\beta G(x,t) \geqq 0$ allows one to rewrite the extremal equations as

$$f^{(n)} = (-1)^\beta \lambda^{2/q} h^{q'-1} \quad \text{for } h \text{ satisfying } (2)(\beta,\alpha);$$

$$(6) \qquad h^{(n)} = (-1)^\alpha f^{p-1} \quad \text{for } f \text{ satisfying } (2)(\alpha,\beta);$$

$$f \geqq 0, \quad h \geqq 0.$$

Note that these are linear if $p = q = 2$ and then f and h satisfy a $2n$-order equation but with peculiar type boundary conditions. One might expect some sort of uniqueness of extremals. If this were the case, then $h(x) = cf(1-x)$ certainly satisfies the boundary conditions. This leads to $p = q'$ when h is eliminated from (6). For $p = q = 2$ and $n = 1$ this is indeed the case since one can verify that $f(x) = \sin \pi x/2 = (2/\pi)h(1-x)$ satisfies (6) with $\lambda = \pi^2/4$. It follows that $C(1,1,2,2) = 2/\pi$. Even the general first order case seems hard to solve. We do these computations in a later section.

We can simplify the extremal equations when $p = q = 2$. We use the theory of positive operators. Note that any solution of the extremal equations has a continuous nth derivative so we can work in the space of continuous functions. Let $G(t,s)$ be the Green's function operator for the operator $y^{(n)}$ with boundary conditions $(2)(\alpha,\beta)$ with $0 < \alpha < n$ and $\alpha + \beta = n$.

Define the operator K by $Kf(t) = \int_0^1 (-1)^\beta G(t, 1-s) f(s)\, ds$. Note that $(-1)^\beta G(t,1-s) \geqq 0$ so K is a positive operator over the cone of functions which are nonnegative. We first observe that for $f(t) \equiv 1$, $Kf(t) = t^\alpha (1-t)^\beta / n!$ Next, we note that Brink [1] has observed that Rolle's theorem is more appropriately stated that f' must change sign if f changes sign twice. Iterating this version, one notes that if f has $n+1$ zeros, counting multiplicaties, then $f^{(n)}$ actually must change sign. This means that the range of K as $f \geqq 0$ consists of functions with exactly n zeros which are then at the endpoints as specified by the Green's function. This shows that for any $f \geqq 0$, $g(t) = Kf(t)/(t^\alpha(1-t)^\beta) > 0$ on $[0,1]$ and continuous there since $\lim_{t \to 0^+} g(t) \neq 0$; similarly, at 1. Hence there are positive constants δ_1 and δ_2 so that $\delta_1 t^\alpha (1-t)^\beta \leqq Kf(t) \leqq \delta_2 t^\alpha (1-t)^\beta$. In the terminology of Krasnoselski [3], K is a $U_0 = t^\alpha(1-t)^\beta$ positive operator. The conclusion of the theory of U_0 positive operators is that there is a unique positive eigenfunction and the corresponding positive eigenvalue has the maximum modulus of all eigenvalues. We conclude that there is a function f_1 so that ($\lambda^{-1/2}$ is the eigenvalue)

$$\lambda^{-1/2} f_1(t) = \int_0^1 (-1)^\beta G(t, 1-s) f(s)\, ds = (-1)^\beta \int_0^1 G(t,s) f_1(1-s)\, ds.$$

Let $h(x) = \lambda^{1/2} f_1(1-x)$. Then $f_1^{(n)}(x) = (-1)^\beta \lambda^{1/2} f_1(1-x) = (-1)^\beta h(x)$, f_1 satisfies the boundary conditions $(2)(\alpha,\beta)$, and h satisfies $(2)(\beta,\alpha)$. Then $h^{(n)}(x) = (-1)^n \lambda^{1/2} f^{(n)}(1-x) = (-1)^\alpha \lambda f_1(x)$. That is, f_1 and h form a solution of (6) with $p = q = 2$. Furthermore $C(n,\alpha,p,q) = \|f\|_2/\|h\|_2 = \lambda^{-1/2}$ so C is the eigenvalue of K.

Conversely suppose the pair (f,h) forms a solution of (6) with $f \geqq 0$. Then $(-1)^\beta f^{(n)} \geqq 0$ so $h \geqq 0$ and $(-1)^\alpha h^{(n)} \geqq 0$. Then $\lambda^{-1} f(x) = (-1)^n \int_0^1 (\int_0^1 G_1(x,t) \cdot G_2(t,s)\, dt) f(s)\, ds$, where G_1 and G_2 are the Green's functions for the boundary conditions $(2)(\alpha,\beta)$ and $(2)(\beta,\alpha)$ respectively. As before, the operator in question is a U_0 positive operator and so only one eigenfunction is positive. Thus the one constructed above is it. We may thus state the following theorem.

THEOREM 2. *If* $0 < \alpha < n$ *then* $C(n, \alpha, 2, 2)$ *is the largest eigenvalue of the operator* $Kf(t) = \int_0^1 (-1)^\beta G(t, 1-s) f(s)\, ds$ *and* f *is the positive eigenfunction. In particular if* $2\alpha = n$, *then* $Kf(t) = \int_0^1 (-1)^\beta G(t,s) f(s)\, ds$ *is also correct. Furthermore, the differential equations equivalent to the integral equations are* $f^{(n)}(x) = \lambda(-1)^\beta \cdot f(1-x)$ *and* $f^{(n)}(x) = \lambda(-1)^\alpha f(x)$ *respectively.*

We have proved this theorem above except for the case $2\alpha = n$. But then $h(x) = f_1(x)$ satisfies the correct boundary condition and we proceed as before. As an example, to compute $C(4, 2, 2, 2)$ we look at $f^{(4)} = \lambda f$, $f(0) = f(1) = f'(0) = f'(1) = 0$. Writing $\lambda = r^4$ with $r > 0$ then $f(0) = f'(0) = 0$ implies that $f(x) = a(\cosh(xr) - \cos(xr)) + b(\sinh(xr) - \sin(xr))$. Writing the conditions at 1 requires that $1 - \cos r \cosh r = 0$ and $C(4, 2, 2, 2) = \lambda^{-1/2} = r^{-2}$ with r the smallest positive root of $\cos r \cosh r = 1$.

The above argument for Theorem 2 does not work if $\alpha = n$ since we have used the fact that the range of K consists of functions with exactly n zeros. This is not true for $\alpha = n$ since $Kf(x) = \int_0^x ((x-t)^{n-1}/(n-1)!) f(t)\, dt$ and for t^m this gives t^{m+n} so this condition no longer holds.

3. The case $\alpha + \beta \neq n$. The extra complexity here is two-fold. First, there is no well-known Green's function representation and the orthogonality relations in $(3)(\alpha, \beta)$ are hard to verify. We begin by showing how one might find solutions of differential equations of the form $y^{(n)} = g$ with the boundary conditions $(2)(\alpha, \beta)$ or $(3)(\alpha, \beta)$.

LEMMA 4. *Let* g *be integrable. Then there exists a unique function* h *such that* $h^{(n)} = g$, *a.e. and* h *satisfies the conditions* $(3)(\alpha, \beta)$.

Proof. Let h_0 be a solution of the differential equation $h^{(n)} = g$ that satisfies the zero conditions at 0 and 1. Since $\alpha + \beta < n$ this is possible. The general solution of $h^{(n)} = g$ that satisfies the zero conditions is then given by $h_0(x) + ax^\alpha(1-x)^\beta q(x)$, where $q(x)$ is an arbitrary monic polynomial of degree $\leqq n - 1 - \alpha - \beta$, and a is a real number. This collection of functions is a closed finite-dimensional convex set in L_2. It therefore has an element of minimum norm. We assume the notation has been chosen so that h_0 is this function. Writing out the inequality $\|h_0(x) + ax^\alpha(1-x)^\beta q(x)\|_2^2 \geqq \|h_0\|_2^2$ gives

$$2a \int_0^1 h_0(x) x^\alpha (1-x)^\beta q(x)\, dx + a^2 \int_0^1 x^{2\alpha}(1-x)^{2\beta} q^2(x)\, dx \geqq 0$$

for all real a. This requires that $\int_0^1 h_0(x) x^\alpha (1-x)^\beta q(x)\, dx = 0$; thus h_0 satisfies $(3)(\alpha, \beta)$.

If h_1 is a second function satisfying $h^{(n)} = g$ and $(3)(\alpha, \beta)$, then $h_1(x) = h_0(x) + x^\alpha(1-x)^\beta q_1(x)$. Multiplying by $x^\alpha(1-x)^\beta q_1(x)$ and integrating yields $\int_0^1 x^{2\alpha} \cdot (1-x)^{2\beta} q_1^2(x)\, dx = 0$ and thus $q_1(x) \equiv 0$.

There is an integral representation for h in terms of $h^{(n)}$ with the boundary conditions $(3)(\alpha, \beta)$. See Reid [4, Chap. III].

LEMMA 5. *Suppose* $n < \alpha + \beta \leqq 2n$, $\alpha \leqq n$, $\beta \leqq n$ *and let* g *satisfy* $(3)(n - \alpha, n - \beta)$. *There exists a unique function* h *so that* $h^{(n)} = g$ *and* h *satisfies* $(2)(\alpha, \beta)$.

(If $n = \alpha$ or $n = \beta$ then delete the appropriate factor in (3)$(n - \alpha, n - \beta)$ *and require no zeros at* 0 *or* 1 *respectively.)*

Proof. If $\alpha = n$ then take $h(x) = (1/(n-1)!)\int_0^x (x-t)^{n-1}g(t)\,dt$. It follows that $h^{(k)}(0) = 0$ for $k = 0, \cdots, n-1$ by the representation. To get the required zeros at 1 we need $\int_0^1 (1-t)^{n-1-k}g(t)\,dt = 0$ for $k = 0, 1, \cdots, \beta - 1$. The integral condition in (3) has become $\int_0^1 (1-t)^{n-\beta}q(t)g(t)\,dt = 0$ if degree $q(t) \leqq \alpha + \beta - n - 1$. Take for $q(t)$ the polynomial $(1-t)^j, j = 0, \cdots, \alpha + \beta - n - 1$. This gives the required condition. By the change of variable $x \to 1 - x$ we also have the lemma when $\beta = n$, and any α. We thus assume that $\alpha < n$ and $\beta < n$. Let $\delta = n - \beta$ and let $h(x) = \int_0^1 G(x,t)g(t)\,dt$, where G is the Green's function for the boundary conditions (2)(δ, β). Let $p(x) = x^{n-\alpha}(1-x)^{n-\beta}q(x)$, where q is a polynomial of degree $\leqq \alpha + \beta - n - 1$. Then we know that $\int_0^1 p(x)g(x)\,dx = 0$. We write this as $\int_0^1 p(x)h^{(n)}(x)\,dx = 0$ and integrate by parts n times to conclude that

$$\sum_{k=0}^{n-1} (-1)^{k+1} h^{(n-1-k)}(x)p^{(k)}(x)\Big|_0^1 = 0.$$

We now use the boundary conditions $p^{(k)}(1) = 0$ for $k = 0, \cdots, n - \beta - 1$; $h^{(n-1-k)}(1) = 0$ for $k = n - \beta, \cdots, n-1$; $p^{(k)}(0) = 0$ for $k = 0, \cdots, n - \alpha - 1$; and $h^{(n-1-k)}(0) = 0$ for $k = \beta, \cdots, n-1$. Then the above becomes

$$\sum_{k=n-\alpha}^{n-\delta-1} (-1)^k h^{(n-1-k)}(0)p^k(0) = 0.$$

Take for $p(x)$ the choices $p_j(x) = x^{n-\alpha+j}(1-x)^{n-\beta}$ for $j = 0, \cdots, \alpha + \beta - n - 1$ and note that $p_j^{(k)}(0) = 0$ for $k = 0, \cdots, n - \alpha + j - 1$ and $p_j^{(n-\alpha+j)}(0) \neq 0$. The equations become

$$\sum_{k=n-\alpha+j}^{n-\delta-1} (-1)^k h^{(n-1-k)}(0)p_j^{(k)}(0) = 0, \qquad j = 0, \cdots, \alpha + \beta - n - 1.$$

This is a system of $\alpha - \delta$ equations in the $\alpha - \delta$ unknowns $h^{(k)}(0)$, $k = \delta, \cdots, \alpha - 1$. The determinant of coefficients is $\pm\prod_{j=0}^{\alpha-\delta-1} p_j^{(n-\alpha+j)}(0) = 0$. Thus $h^{(k)}(0) = 0$ for $k = \delta, \cdots, \alpha - 1$. This gives h the correct number of zeros. Uniqueness follows since the difference of two solutions is a polynomial that vanishes at $\alpha + \beta > n$ points and so is zero.

Note that in all cases of Lemma 5, h may be given by a Green's function. It can be shown that the Green's function for $2(\delta, \gamma)$ boundary conditions will work so long as $\delta \leqq \alpha$ and $\gamma \leqq \beta$ and $\gamma + \delta = n$.

We now seek best possible constants for which

$$\begin{aligned} &\|f\|_p \leqq D(n, \alpha, \beta, p, q)\|f^{(n)}\|_q \\ &\text{when } f \text{ satisfies } (2)(\alpha, \beta),\ n < \alpha + \beta,\ \alpha \leqq n,\ \beta \leqq n; \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\|f\|_p \leqq E(n, \alpha, \beta, p, q)\|f^{(n)}\|_q \\ &\text{when } |f|^{p-1}\operatorname{sgn}(f) \text{ satisfies } (3)(n-\alpha, n-\beta),\ n < \alpha + \beta,\ \alpha \leqq n,\ \beta \leqq n. \end{aligned} \tag{8}$$

It is easy to see that the numbers D exist since $D(n, \alpha, \beta, p, q) \leqq C(n, \delta, p, q)$ for any δ such that $\delta \leqq \alpha$ and $n - \delta \leqq \beta$. We now show that the numbers D and E are conjugately related.

THEOREM 3. *For all p and q*, $D(n, \alpha, \beta, p, q) = E(n, \alpha, \beta, q', p')$.

Proof. Let $|f|^{p-1} \operatorname{sgn}(f)$ satisfy $(3)(n - \alpha, n - \beta)$, and $h^{(n)} = |f|^{p-1} \operatorname{sgn}(f)$ with h satisfying $(2)(\alpha, \beta)$. Then

$$\|f\|_p^p = \int_0^1 fh^{(n)} = (-1)^n \int_0^1 f^{(n)}h \leqq \|f^{(n)}\|_q \|h\|_{q'}$$
$$\leqq D(n, \alpha, \beta, q', p')\|f^{(n)}\|_q \|h^{(n)}\|_{p'} = D(n, \alpha, \beta, q', p')\|f^{(n)}\|_q \|f\|_p^{p-1}.$$

In particular, $E(n, \alpha, \beta, p, q)$ exists and is $\leqq D(n, \alpha, \beta, q', p')$.

For the reverse inequality, let f satisfy $(2)(\alpha, \beta)$ and select $h^{(n)} = |f|^{p-1} \operatorname{sgn}(f)$ with h satisfying $(3)(n - \alpha, n - \beta)$. As before,

$$\|f\|_p^p = (-1)^n \int_0^1 f^{(n)}h \leqq \|f^{(n)}\|_q \|h\|_{q'} \leqq \|f^{(n)}\|_q E(n, \alpha, \beta, q', p')\|f\|_p^{p-1},$$

so $D(n, \alpha, \beta, p, q) \leqq E(n, \alpha, \beta, q', p')$.

Again we can extract candidates for the extremals from the proof of Theorem 3. They are

$$\begin{aligned} &h^{(n)} = |f|^{p-1}\operatorname{sgn}(f) && \text{for } |f|^{p-1} \operatorname{sgn}(f) \text{ satisfying } (3)(n - \alpha, n - \beta); \\ &|f^{(n)}| = \lambda^2 |h|^q && \text{for } h \text{ satisfying } (2)(\alpha, \beta). \end{aligned} \tag{9}$$

Here $1 \leqq p < \infty$ and $1 < q < \infty$.

Note again that f is an extremal for $(3)(n - \alpha, n - \beta)$ and h for $(2)(\alpha, \beta)$ so extremals exist for both problems or neither.

LEMMA 6. *For* $1 \leqq p \leqq \infty$ *and* $1 < q < \infty$, *there exist functions f and h so that* $\|f\|_p = D(n, n, n, p, q)\|f^{(n)}\|_q$ *and* $\|h\|_{q'} = E(n, n, n, q', p')\|h^{(n)}\|_{p'}$ *and the pair* (f, h) *satisfies* (9).

Proof. We use the fact that if f exists which gives D, then h must exist and the pair satisfies (9). To establish the existence of f we note that the proof of Lemma 5 shows that $f(x) = \int_0^1 G(x, t)f^{(n)}(t)\, dt$ for G the Green's function for $(2)(n - \beta, \beta)$. We can thus repeat the argument of Lemma 1. Note that weak convergence of $f_k^{(n)} \to g_0$ loses any zero conditions, so $\alpha = \beta = n$ is necessary for this argument.

Further simplifications of the extremal equation is difficult. We cannot have $h^{(n)}$ of constant sign, for h has too many zeros, and we cannot have f of constant sign if the orthogonal conditions in $(3)(n - \alpha, n - \beta)$ are to be satisfied. As a result, $p = q = 2$ does not result in a linear equation, for the conditions become $(|f|^{p-1} \operatorname{sgn}(f) = f)$

$$\begin{aligned} &h^{(n)} = f; \quad f^{(n)} = \lambda h(x) \operatorname{sgn}(h) \operatorname{sgn}(f^{(n)}); \quad \text{with} \\ &f \text{ satisfying } (3)(n - \alpha, n - \beta) \text{ and } h \text{ satisfying } (2)(\alpha, \beta). \end{aligned} \tag{10}$$

Note also that the case $p = 1$ leads to $h^{(n)} = \operatorname{sgn}(f)$ so that h is a spline function. Also for $n = 1$ we have only one problem, that is, $\alpha = \beta = 1$ is the only choice. We can show that we need only consider extremals h for which $h \geqq 0$. In any case λ in (9) is to be minimized, for using $|h^{(n)}|^{p'} = |f|^p$ and $|f^{(n)}|^q = \lambda^2 |h|^{q'}$ and the normalization $\|f^{(n)}\|_q = 1$, then $\|h\|_{q'} = \lambda^{-2/q'}$ and since both h and f are extremals for the same constant we have

$$\|f\|_p = \frac{\|h\|_{q'}}{\|h^{(n)}\|_{p'}} = \frac{\lambda^{-2/q'}}{\|f\|_p^{p-1}} \quad \text{or} \quad \lambda = \|f\|_p^{-pq'/2}.$$

Since we are trying to maximize $\|f\|_p$ we must minimize λ.

LEMMA 7. *For* $n = 1$, $h \geqq 0$ *characterizes the extremal which obtains* $D(1, 1, 1, p, q)$.

Proof. Suppose (f, h) is a pair of extremals and that $h(c) = 0$ for $0 < c < 1$. Let c be the smallest such number. Assume this exists, for the zeros of h are not dense, so there is some interval which we can translate to $(0, c)$ if necessary. Assume $h > 0$ on $(0, c)$. Then define $f_1(x) = f(xc)$ and $h_1(x) = c^{-1}h(xc)$. Then $h_1'(x) = |f_1(x)|^{p-1}f_1(x)$ and $|f_1'(x)| = \lambda^{2/q}c|h_1(x)|^{q-1}$. Furthermore $h_1(0) = h_1(1) = 0$ and

$$\begin{aligned}\int_0^1 |f_1|^{p-1} \operatorname{sgn} f_1\, dx &= \int_0^1 |f(xc)|^{p-1} \operatorname{sgn} f(xc)\, dx \\ &= c^{-1}\int_0^c |f(t)|^{p-1} \operatorname{sgn} f(t)\, dt \\ &= c^{-1}\int_0^c h'(t)\, dt = c^{-1}[h(c) - h(0)] = 0.\end{aligned}$$

Now $\lambda^{2/q}c < \lambda^{2/q}$, and therefore is not the minimum $D(1, 1, 1, p, q)$.

For example, when $p = q = 2$ it is easy to verify that $h(x) = \sin \pi x$ and $f(x) = \pi \cos \pi x$ satisfy (9) with $\lambda = \pi^2$. Thus $C(1, 1, 1, 2, 2) = \pi^{-1}$. We compute the general case in the next section.

4. Computations for $n = 1$. We first give a method for computing $C(1, 1, p, q)$ and this will lead to a similar computation for $D(1, 1, 1, p, q) = E(1, 1, 1, q', p')$. For $p = 1$ it is easier to give a direct proof rather than use the extremal equation. Recall that we may assume $f \geqq 0$ and $f' \geqq 0$.

$$\|f\|_1 = \int_0^1 f = \int_0^1 (1 - x)f' \leqq \|f'\|_q\left(\int_0^1 (1 - x)^{q'}\right)^{1/q'}$$

with equality when $|f'|^q = \lambda^2(1 - x)^{q'}$ or $f(x) = \lambda^{2/q}(1 - x)^{q'-1} - \lambda^{2/q}$. Thus $C(1, 1, 1, q) = (1 + q')^{-1/q'}$ for $1 < q < \infty$, and $C(1, 1, 1, \infty) \leqq \frac{1}{2}$. Consideration of $f(x) = x$ shows that $C(1, 1, 1, \infty) = \frac{1}{2}$. For $q = \infty$ we note that $C(1, 1, p, \infty) = C(1, 1, 1, p') = (1 + p)^{-1/p}$ if $1 < p' \leqq \infty$, that is, $1 \leqq p < \infty$. To compute $C(1, 1, \infty, \infty)$ note that $f(x) = \int_0^x f'$ so $\|f\|_\infty \leqq \|f'\|_\infty$ and equality holds if $f' = 1$. Thus $C(1, 1, 1, 1) = C(1, 1, \infty, \infty) = 1$. We thus may assume that $p \neq 1$ and $q' \neq 1$. Recall $(a = \lambda^{2/q})$ that we have $h' = -f^{p-1}$, $f' = ah^{q'-1}$, $f(0) = h(1) = 0$ and consequently $h'(0) = f'(1) = 0$. Also $f \geqq 0$ implies that $h' \leqq 0$ so $h \geqq 0$. We have both f and h twice differentiable and

$$f'' = a(q' - 1)h^{q'-2}h' = a(1 - q')f^{p-1}\left(\frac{f'}{a}\right)^{(q'-2)/(q'-1)}.$$

Multiply both sides of this equation by $(f')^{q/q'}$ and integrate to get

$$(f')^q = q\frac{1-q'}{p}a^{q-1}f^p + c \tag{11}$$

and

$$\int_0^1 (f')^q = q\frac{1-q'}{p}a^{q-1}\int_0^1 f^p + c. \tag{12}$$

We now take the qth root of both sides of (11), divide both sides by the right-hand side and integrate from 0 to 1. The result is

$$1 = \int_0^1 \left(q\frac{1-q'}{p}a^{q-1}f^p(x) + c\right)^{-1/q} f'(x)\,dx.$$

Let $u = (q[q'-1)/p]a^{q-1}c^{-1})^{1/p}f(x)$ to get

$$\int_0^1 \frac{du}{(1-u^p)^{1/q}} = [f(1)]^{-1}f'(0). \tag{13}$$

We have used the identities $q[(q'-1)/p]a^{q-1}f^p(1) = c$ and $(f'(0))^q = c$ which follow from (11). By a similar analysis one gets

$$\int_0^1 \frac{du}{(1-u^{q'})^{1/p'}} = -h'(1)(h(0))^{-1}. \tag{14}$$

We now normalize f by assuming that $f'(0) = 1$. Then $f(1)$ is determined by (13). By the extremal equation $-h'(1) = f^{p-1}(1)$ is thus determined. Now (14) determines $h(0)$ but $h(0)^{q'-1}a = f'(0) = 1$. Eliminating $f(1)$ from all these equations we find that

$$a = \left(\int_0^1 \frac{du}{(1-u^{q'})^{1/p'}}\right)^{q'-1}\left(\int_0^1 \frac{du}{(1-u^p)^{1/q}}\right)^{(p-1)(q'-1)}. \tag{15}$$

We now use the identities $|h'|^{p'} = |f|^p$, $|f'|^q = a|h|^{q'}$, $C(1,1,p,q) = \|f\|_p\|f'\|_q^{-1} = \|h\|_{q'}\|h'\|_{p'}^{-1}$ and (11) and (12). Combining the first four yields

$$\begin{aligned} C(1,1,p,q) &= \|h\|_{q'}\|h'\|_{p'}^{-1} = a^{-1/q'}\|f'\|_q^{q-1}\|f\|_p^{1-p} \\ &= a^{-1/q'}\|f'\|_q^{q-1}C(1,1,p,q)^{1-p}\|f'\|_q^{1-p} \end{aligned}$$

or

$$C(1,1,p,q)^p = a^{-1/q'}\|f'\|_q^{q-p}.$$

The constant c in (12) is 1 since in (11), $f(0) = 0$, $f'(0) = 1$. We then use (12) to eliminate $\|f'\|_q^{q-p}$ from the formula for $C(1,1,p,q)$. We have first for $q = p$ that $C(1,1,p,q) = a^{-1/pq'}$. If $q \neq p$ then

$$\|f'\|_q^q = C(1,1,p,q)^{pq/(q-p)}a^{(q-1)/(q-p)} = q\frac{1-q'}{p}a^{q-1}C(1,1,p,q)^p\|f'\|_q^p + 1.$$

Thus

$$C^{pq/(q-p)}\left(a^{(q-1)/(q-p)} + \frac{q'}{p}a^{(q-1)+p/((q-p)q')}\right) = 1, \tag{16}$$

with a given by (15). To compute $D(1, 1, 1, p, q)$ we use the similarity of the extremal equations (6) and (9).

LEMMA 8. *For* $1 < p < \infty$ *and* $1 < q < \infty$,

$$D(1,1,1,p,q) = (\tfrac{1}{2})^{1+1/p-1/q'}.$$

Proof. Since $C(1, 1, p, q) = C(1, 1, q', p')$ we let (f, h) be extremals for this problem, that is,

$$h' = -f^{q'-1}, \quad f(0) = 0, \quad (f')^{p'} = \lambda^2(h)^p, \quad h(1) = 0$$

and $h'(0) = 0, f'(1) = 0, h \geqq 0, f \geqq 0, h' \leqq 0, f' \geqq 0$. Define $k(x) = (\lambda/2)^{-2/p'}f(2x)$, $0 \leqq x \leqq \frac{1}{2}$, and $k(x) = k(1-x)$. Similarly, take $g(x) = h(2x)$ on $0 \leqq x \leqq \frac{1}{2}$ with $g(x) = -g(1-x)$. It follows that g and k are differentiable on $(0, 1)$, $k \geqq 0$, $k(0) = k(1) = 0$, $g(x) \geqq 0$ on $(0, \frac{1}{2})$ and $g(x) \leqq 0$ on $(\frac{1}{2}, 1)$. Also

$$|g'(x)|^q = 2^q|h'(2x)|^q = 2^q|f(2x)|^{q'} = 2^{q+q'}\lambda\frac{2q'}{p'}|k(x)|^{q'}$$

on $0 \leqq x \leqq \frac{1}{2}$ and a similar equation on $\frac{1}{2} \leqq x \leqq 1$. It is easy to see that $\int_0^1 |g|^{p-1} \cdot \operatorname{sgn}(g) = 0$. Finally,

$$h' = \lambda^{-2/p'}f'(2x) = \lambda^{-2/p'}\lambda^{2/p'}h(2x)^{p-1} = g(x)^{p-1} = g(x)^{p-1}\operatorname{sgn}(g)$$

on $0 \leqq x \leqq \frac{1}{2}$ and a similar computation on $(\frac{1}{2}, 1)$. Thus the pair (g, k) satisfies the extremal equations (9) and $g \geqq 0$. According to Lemma 7, this characterizes g as giving $D(1, 1, 1, p, q)$. Thus

$$D(1,1,1,p,q) = \frac{\|g\|_p}{\|g'\|_q} = \frac{(2\int_0^{1/2}|h(2x)|^p)^{1/p}}{(2\int_0^{1/2}|h'(2x)|^q 2^q)^{1/q}} = \frac{C(1,1,p,q)}{2}.$$

Again for $p = 1$ there is a direct proof, for we may assume that $f \geqq 0$. Then $\int_0^1 f = \int_0^1 f'(x)(c - x)\,dx$ for any c. Thus $\|f\|_1 \leqq \|f'\|_q\|(c-x)\|_{q'}$ and we pick c to minimize this. However, the choice $c = \frac{1}{2}$ and $f' = |\frac{1}{2} - x|^{q-1}\lambda^{2/q}$, $0 \leqq x \leqq \frac{1}{2}$, and $f' = -|\frac{1}{2} - x|^{q-1}\lambda^{2/q}, 0 \leqq x \leqq \frac{1}{2}$, and $f' = -|\frac{1}{2} - x|^{q-1}\lambda^{2/q}$ shows that there is an f for which equality holds. Thus

$$D(1,1,1,1,q) = \left(\int_0^1\left|\frac{1}{2} - x\right|^{q'}\right)^{1/q'} = \frac{1}{2}(1+q')^{-1/q'}, \qquad D(1,1,1,1,\infty) \leqq 1/4,$$

and $D(1,1,1,1,1) \leqq \frac{1}{2}$. The choice $f = |x - 1/2| - 1/2$ shows $D(1,1,1,1,\infty) = 1/4$. More generally the last part of the argument of Lemma 8 leads to $D(n,n,n,p,q) \leqq C(n,n,p,q)/2^{n+1/p-1/q}$.

5. Computations for $n \geqq 2$. Recall that extremals for $p = q = 2 = n$ satisfy $f'' = (-1)^\beta\lambda h$, $h'' = (-1)^\alpha f$ if $\alpha + \beta = 2$ and for $\alpha + \beta \neq 2$, $h'' = f$, $f'' = \lambda \cdot h(\operatorname{sgn}(h))(\operatorname{sgn}(f''))$. In either case the appropriate constant C or D was shown to be $\lambda^{-1/2}$. We first note the classical result that $C(2, 1, 2, 2) = \pi^{-2}$, which results from the choice $f(x) = \sin \pi x$, $h(x) = \pi^{-2} \sin \pi x$, $\lambda = \pi^4$. The other case when $\alpha + \beta = 2$ is $\alpha = 2$ and $\beta = 0$. We note then that $f^{(4)} - \lambda f = 0$, $f(0) = f'(0) = f''(1) = f'''(1) = 0$. Writing $\lambda = r^4$, then

$$f(x) = C_1(e^{rx} - \cos(rx) - \sin(rx)) + C_2(e^{-rx} - \cos(rx) + \sin(rx))$$

satisfies the left end conditions. A simple computation shows that the two conditions at 1 are satisfied if and only if $1 + \cos r \cosh r = 0$. Taking the smallest positive solution of this equation and $h = (f'')r^{-4}$, then (f, h) is an extremal pair and $C(2, 2, 2, 2) = r^{-2}$.

For the computation of $D(2, 2, 2, 2, 2)$, we may start as above and take $f^{(4)} - \lambda f = 0$, $\lambda = s^4$, and f as above. Note that the same choice for h, namely $h(x) = s^{-4} f''$, will satisfy (3)(0, 0) and thus will be an extremal pair. This time the computation to get the right endpoints leads to the equation $\cos(s) \cosh s = 1$ and $D(2, 2, 2, 2, 2) \leqq s^{-2}$. Is $s^2 = 2r^2$? As a final example we consider the boundary condition $\alpha = 2$, $\beta = 1$. It is not possible to solve the boundary value problem $f^{(4)} + \lambda f = 0, f(0) = f'(0) = f(1) = f''(1) = 0$ nontrivially. If we look at $f^{(4)} - \lambda f = 0$ with these boundary conditions with $\lambda = r^4$, again we start as before and get the equation $\tan r_0 = \tanh r_0$. If we define $h(x) = \lambda^{-1} f''(x)$ as we need to do to get the extremal equations, then we note that h must satisfy the orthogonal relation $\int_0^1 (1 - x)h(x)\,dx = 0$. This is satisfied. So again we have $D(2, 2, 1, 2, 2) < r_0^{-2}$.

For $n > 1$ and $\alpha + \beta \neq n$ we have not been able to show that extremals are nonnegative. However, as we can see in the next two sections, adding this hypothesis still gives some interesting applications. Meanwhile, for $p = 1$ we can see how other inequalities can be generated.

LEMMA 9. *If $f \geqq 0$ also satisfies* (2)(α, β), *then*

$$\|f\|_1 \leqq \|f^{(n)}\|_q \|h\|_{q'},$$

where $h(x) = (1/n!)x^{n-\alpha}(1 - x)^{n-\beta}(-1)^{n-\beta}p(x)$ and $p(x)$ is a monic polynomial of degree $\alpha + \beta - n$. If $n - \alpha \leqq 0$ or $n - \beta \leqq 0$, delete the appropriate factor from h.

Proof. Merely integrate by parts n times as we have done previously. Naturally we would like to minimize the norm $\|h\|_{q'}$. We have solved this problem in Lemma 4 for $q = 2$. We take $h(x)$ so that $\int_0^1 h(x)x^{n-\alpha}(1 - x)^{n-\beta} r(x)\,dx = 0$ for all polynomials r of degree $\alpha + \beta - n - 1$. Since h is a polynomial this looks easy, but seems difficult to do in the general case. For example, for $\alpha = \beta = n = 2$, h is an arbitrary quadratic with leading term $\frac{1}{2}$, and we need to min $\|x^2/2 + ax + b\|_{q'}$. One can do this indirectly, for if $h(x)$ is such that $f'' = h^{q-1}$ has a solution which is positive and has the required zeros, then this h must minimize the above norm. According to Lemma 5 this requires $\int_0^1 |h|^{q'-2} \operatorname{sgn}(h) = 0$. This required that h have a zero in (0, 1), but the choice $h(x) = \frac{1}{2}(x - \frac{1}{2})^2$ does not work. A good guess is probably $h(x) = \frac{1}{2}(x - a)(x - 1 + a)$ for $0 < a < 1$. In general a is a function of q.

As another simple example, consider the case $\alpha = n = 2$, $\beta = 1$. Then $h(x) = (x - 1)(\frac{1}{2}x + a)$ and a must be selected to minimize the q'-norm. The requirement that equality hold in the use of Hölder's inequality then is $f'' = h^{q-1}$ and $\int_0^1 |h|^{q-2} \operatorname{sgn} h(1 - x) = 0$. This then is $\int_0^1 |1 - x|^{q-1}|x - 2a|^{q-2} \operatorname{sgn}(x + 2a)dx = 0$. It is clear a exists, for example, when $q = 3$, $a = -1/8$, and when $q = 4$, then

a is a solution of

$$236a^4 + 108a^3 - 15a^2 + 20a + 15 = 0.$$

When $q = 2$ then h is an appropriate Jacobi polynomial. This leads to the formula

$$D^2(n, \alpha, \beta, 1, 2) = \frac{4^{\alpha+\beta-n}(n+\beta-\alpha)!(n+\alpha-\beta)![(n-\alpha-\beta)!]^2}{(2n+1)!(2n)!(3n-\alpha-\beta)!}.$$

As a final result in this section, we compute $C(2, 1, p, p')$. The extremal here should be symmetric about $x = \frac{1}{2}$ so we make this guess. Let

$$g(y) = \frac{1}{a}\int_0^y \frac{du}{(1-u^p)^{1/2}} \quad \text{for } 0 \leqq y \leqq 1,$$

with a chosen so that $g(1) = \frac{1}{2}$. Since $g' > 0$, g has an inverse function f, such that $f(0) = 0$ and $f(\frac{1}{2}) = 1$ and $ax = \int_0^{f(x)} du/(1-u^p)^{1/2}$. Thus $f' = a(1-f^p)^{1/2}$ and $f'(\frac{1}{2}) = 0$ and $(f')^2 = a^2(1-f^p)$. A further differentiation yields $f'' = -(pa^2/2) \cdot f^{p-1}$ on $(0, \frac{1}{2})$. We now define f on $(\frac{1}{2}, 1)$ by $f(1-x) = f(x)$. Since $f'(\frac{1}{2}) = 0$ this extended f is differentiable on $[0, 1]$ and satisfies the extremal equation (6) with $h = f$. Now $C(2, 1, p, p') = \|f\|_p/\|f''\|_{p'} = (2/a^2p)\,\|f\|_p^{2-p}$, but using the equation

$$\frac{pa^2}{2}\int_0^1 |f|^p = -\int_0^1 ff'' = \int_0^1 (f')^2 = \int_0^1 a^2(1-f^p) = a^2 - a^2\int_0^1 |f|^p$$

one computes $\|f\|^p$. Thus

$$C(2, 1, p, p') = \left(\frac{2}{p+2}\right)^{(2-p)/p}\frac{1}{2p}\left(\int_0^1 \frac{du}{(1-u^p)^{1/2}}\right)^{-2}.$$

Note that

$$\int_0^1 \frac{du}{(1-u^p)^{1/2}} = \frac{2}{p}\,\frac{\Gamma(2/p)\Gamma(1/2)}{\Gamma(2/p+1/2)}.$$

6. Applications to disconjugacy. A linear differential equation is disconjugate on an interval if no nontrivial solution has the same number of zeros on this interval as the order of the equation. All sufficiently small intervals are intervals of disconjugacy, but we may like to get lower bounds for this length. We illustrate for $n = 2$; the generalization to nth order equations is clear. Consider the equation $y'' + r(x)y = 0$. This equation is disconjugate on (a, b) only if there is a positive solution on this interval. Suppose on the contrary that there is a nontrivial solution with two zeros at c and d but positive in between. Then, computing norms on this interval

$$\|y''\|_1 = \|ry\|_1 \leqq \|r\|_{p'}\|y\|_p \leqq \|r\|_{p'}C(2, 1, p, 1)\|y''\|_1(d-c)^{1+1/p}$$

so $1 \leqq \|r\|_{p'}C(2, 1, p, 1)(d-c)^{1+1/p}$. This gives a lower bound for $(d-c)$. That is, if $1 > \|r\|_{p'}C(2, 1, p, 1)(d-c)^{1+1/p}$ then $[c, d]$ is an interval of disconjugacy.

If one wants to extend the above method to equations of the form $y'' + r_1(x)y' + r_2(x)y = 0$ then one needs to have an estimate of the form $\|y'\| \leqq K\|y''\|$. Now y' has a zero but it is unknown where it is. However, Brink [1] has shown that in general if f has n zeros on $[a, b]$ then $\|f\|_p \leqq \max_{0 \leqq \alpha \leqq n} C(n, \alpha, p, q)\|f^{(n)}\|_q$. In our case then, one would get $\|f'\|_p \leqq C(1, 1, p, q)\|f''\|_p$. What has motivated this study is the fact that this is not a very good inequality. In fact, note that $\int_c^d f' = 0$ if $f(c) = f(d) = 0$, so $\|f'\|_p \leqq E(1, 1, 1, p, q)\|f''\|_q$. But

$$E(1,1,1,p,q) = D(1,1,1,p,q) = (\tfrac{1}{2})^{1+1/p-1/q'}C(1,1,p,q)$$

according to Lemma 8. One can argue that if $y > 0$ on (a, b) but $y(a) = y(b) = 0$, then

$$\begin{aligned}\|y''\|_2 &< \|r_1\|_\infty\|y'\|_2 + \|r_2\|_\infty\|y\|_2 \\ &\leqq \|r_1\|_\infty(b-a)\frac{C(1,1,2,2)}{2}\|y''\|_2 + \|r_2\|_\infty(b-a)^2C(2,1,2,2)\|y''\|_2.\end{aligned}$$

Thus $1 \leqq \|r_1\|_\infty(b - a)(1/\pi) + \|r_2\|_\infty(b - a)^2(1/\pi^2)$. Willett [5] reports that the best previous constant pairs have been (1/4, 1/8) instead of $(1/\pi, 1/\pi^2)$.

The general problem to find best possible constants for $\|f^{(k)}\|_p \leqq C\|f^{(n)}\|_q$ when f has n zeros remains to be solved.

Other sources for inequalities related to ours should be mentioned. Beesack [6] is a long paper giving inequalities between functions and the first derivative, and his list of references is extensive. Most of his results do not overlap ours. At one point of contact we have given a slight improvement. He shows that if f has a zero and $f(-\pi/2) = f(\pi/2)$ then $\int_{\pi/2}^{\pi/2} f^2 \leqq \int_{\pi/2}^{\pi/2} (f')^2$. Putting this in our form it is $\|f\|_2 \leqq \pi^{-1}\|f'\|_2$ if f has a zero on [0, 1] and $f(0) = f(1)$. By our observations, if f has a zero on [0, 1] then $\|f\|_2 \leqq C(1, 1, 2, 2)\|f'\|_2 = \pi^{-1}\|f'\|_2$. This shows that the condition $f(0) = f(1)$ is superfluous.

Hardy, Littlewood and Pólya [7] give the result $C(1, 1, 2k, 2k) = (2k - 1)^{-1/2k} \cdot (2k/\pi)\sin(\pi/2k)$ while we have shown that for all $q \in (1, \infty)$,

$$C(1,1,q,q) = \left(\int_0^1 \frac{du}{(1-u^{q'})^{1/q'}}\right)^{-1/q^2}\left(\int_0^1 \frac{du}{(1-u^q)^{1/q}}\right)^{(1-q)/q^2}.$$

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A VOLTERRA EQUATION IN HILBERT SPACE*

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Abstract. This paper concerns the asymptotic behavior of the solution of a Volterra equation in Hilbert space. The proof uses spectral decomposition and a result of independent interest on the global dependence on a parameter of the solution of a scalar integro-differential equation.

1. Introduction. Let $\mathbf{H}$ denote a real Hilbert space. Let L be a self-adjoint linear operator with domain in $\mathbf{H}$ and such that $\langle L\varphi, \varphi \rangle \geqq \lambda_0 \langle \varphi, \varphi \rangle$, where $\lambda_0 > 0$. We consider the integral equation

$$y(t) + L \int_0^t h(t-s)y(s)\,ds = \xi + t\varkappa \tag{1.1}$$

(ξ and $\varkappa$ are prescribed elements of $\mathbf{H}$), where $h(t) = \int_0^t a(s)\,ds$ and a is a real-valued function such that

$$\begin{aligned} &a \in C^1(0,\infty) \cap L^1(0,1);\ a \text{ is nonnegative, nonincreasing, and convex;} \\ &\text{and } a(t) \not\equiv a(0). \end{aligned} \tag{1.2}$$

With certain additional assumptions, we shall prove that $y(t)$ tends to a limit in $\mathbf{H}$ as $t \to \infty$.

We study (1.1) in terms of the formula

$$y(t) = \int_{\lambda_0}^\infty u(t;\lambda)\,dE_\lambda \xi + \int_{\lambda_0}^\infty w(t;\lambda)\,dE_\lambda \varkappa, \tag{1.3}$$

where $\{E_\lambda\}$ is the spectral family associated with L and where u and w (defined below) are particular solutions of the scalar equation

$$x'(t;\lambda) + \lambda \int_0^t a(t-s)x(s;\lambda)\,ds = k, \qquad x(0) = x_0. \tag{1.4}$$

Here k and x_0 are fixed real numbers and λ is a positive parameter. (Primes denote differentiation with respect to the first variable, t in this case.)

We proved in [6] that when (1.2) holds, the solution $x(\cdot\,;\lambda)$ of (1.4) belongs to BC, the Banach space of bounded, continuous functions on $[0,\infty)$ and that

$$\lim_{t\to\infty} x(t;\lambda) = \frac{k}{\lambda \int_0^\infty a(t)\,dt}, \tag{1.5}$$

where the right-hand side is interpreted as zero if $a \notin L^1(0,\infty)$. With $R^+ = (0,\infty)$, x_0 and k arbitrary but fixed, define

$$\Phi : R^+ \to BC \quad \text{by} \quad \Phi(\lambda) = x(\cdot\,;\lambda).$$

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We denote by $u(t;\lambda)$ the solution of (1.4) with $k = 0$, $x_0 = 1$; $w(t;\lambda)$ is the solution of (1.4) with $k = 1$, $x_0 = 0$. It is easy to verify that

$$x = kw + x_0 u \quad \text{and} \quad w' = u. \tag{1.6}$$

THEOREM 1. *Suppose* (1.2) *holds. Then* Φ *is differentiable and*

$$\lambda \frac{d\Phi}{d\lambda}(t) = -kw(t;\lambda) + \int_0^t x'(t-s;\lambda)u(s;\lambda)\,ds, \tag{1.7}$$

$$0 < \lambda < \infty, \quad 0 \leqq t < \infty.$$

We apply Theorem 1 to obtain our results for (1.1).

THEOREM 2. *Assume that* (1.2) *holds and*

$$\int_0^\infty a(t)\,dt < \infty. \tag{1.8}$$

If y *is a continuous solution of* (1.1), *then*

$$\lim_{t\to\infty} \left\| y(t) - \left(\int_0^\infty a(t)\,dt \right)^{-1} L^{-1} n \right\| = 0. \tag{1.9}$$

If $n = 0$, (1.2) *alone implies that* $\|y(t)\| \to 0$ $(t \to \infty)$.

THEOREM 3. *Assume that* (1.2) *holds and* $a(t) \to 0$ *as* $t \to \infty$ *but* $\int_0^\infty a(t)\,dt = \infty$. *Assume further that* $a(0+) < \infty$, *a is twice differentiable on* $(0, \infty)$, *and* a'' *is bounded away from zero on every finite interval* $(0, T]$. *If* y *is a continuous solution of* (1.1), *then*

$$\lim_{t\to\infty} \|y(t)\| = 0. \tag{1.10}$$

The existence of y is a consequence of the proofs of Theorems 2 and 3.

If L has a compact inverse, Theorems 2 and 3 are contained in [8, Thms. 2, 3, 4 and Cor. 1]. (The conclusion of Corollary 1 in [8] is correct only if $a \notin L^1(0, \infty)$; if $a \in L^1(0, \infty)$, equation (1.9) above holds.)

We make the assumptions in the second sentence of Theorem 3 so that the uniform boundedness result [8, Thm. 4] holds; alternative conditions for the validity of that theorem are given in [8] and would suffice here.

Our proof of (1.10) also works with $a(\infty) > 0$ and hypothesis (1.2) alone. Using a Lyapunov method as in [7], one establishes that $|u(t;\lambda)| + |w(t;\lambda)|$ is bounded in $\{0 \leqq t < \infty, \lambda_0 \leqq \lambda < \infty\}$; the remainder of the proof is as below.

C. M. Dafermos [1], [2], A. Friedman [3], A. Friedman and M. Shinbrot [4], and R. C. MacCamy and J. S. W. Wong [10] have studied Volterra equations in Hilbert and Banach spaces. In particular, MacCamy and Wong consider equations including (1.1) with L a symmetric, strongly elliptic differential operator with discrete spectrum. They determine the asymptotic behavior of the solution if (1.2) holds and either $\int_0^\infty [a(t) + \int_t^\infty a(s)\,ds]\,dt < \infty$ or $a(\infty) > 0$ and $\int_0^\infty [a(t) - a(\infty)]\,dt < \infty$. Our method, based on spectral decomposition and the scalar equation with parameter, follows [3] and [4]. Dafermos and MacCamy and Wong operate directly in **H**. We shall consider variable kernels $L(t-s)$ in [9].

Concerning our proof of Theorem 1, we note that S. I. Grossman and R. K. Miller [5] have used this method (the variation of constants formula together with L^2 estimates on the transforms u^* and $(u')^*$) systematically to prove perturbation theorems for nonlinear versions of (1.4).

Remarks on piecewise linear kernels. Assume that $a(0) < \infty$ and that (1.2) holds, except that $a \notin C^1$. Instead suppose that a' is piecewise constant with jumps only at integral multiples of $t_0 = 2\pi[\lambda a(0)]^{-1/2}$; in this case we say condition $H(\lambda)$ holds.

We proved in [6] that when $H(\lambda)$ holds, $x(t;\lambda)$ is asymptotically periodic $(t \to \infty)$. In all other cases, the condition $a \in C^1$ can be replaced by $a \in C$ in (1.2) and (1.5) still holds. For a given kernel $a(t)$, $H(\lambda)$ can hold for countably many values of λ at most. It follows that Φ is not continuous at λ if $H(\lambda)$ holds.

With this exception, our proof of Theorem 1 goes through with $a \in C$ instead of $a \in C^1$ in (1.2).

2. Proof of Theorem 1. We shall use the Fourier transforms

$$u^*(\tau;\lambda) = \lim_{T\to\infty} \int_0^T e^{-i\tau t} u(t;\lambda)\,dt$$

and $a^*(\tau)$.

In [6, Lemmas 3 and 5] we established the following consequences of (1.2): $a^*(\tau)$ is continuous in $\{|\tau| > 0\}$ and tends to 0 as $|\tau| \to \infty$; $a^*(\tau)$ is continuous at 0 if $a \in L^1(0,\infty)$, while $|a^*(\tau)| \to \infty$ $(\tau \to 0)$ if $a \notin L^1(0,\infty)$; and $\operatorname{Re} a^*(\tau) > 0$ (all τ).

As in [6], one may now use Laplace transforms to establish that

$$u^*(\tau;\lambda) = [i\tau + \lambda a^*(\tau)]^{-1}. \tag{2.1}$$

LEMMA 1. *Suppose (1.2) holds. Then $u(\cdot\,;\lambda)$ and $u'(\cdot\,;\lambda)$ belong to $L^2 = L^2(0,\infty)$. The formula $\Psi(\lambda) = (u(\cdot\,;\lambda), u'(\cdot\,;\lambda))$ defines a continuous map from R^+ to $L^2 \times L^2$.*

Proof of Lemma 1. It follows from (2.1) and the remarks above that $u^*(\tau;\lambda)$ is continuous in $\{|\tau| < \infty, \lambda > 0\}$ and is $O(\tau^{-1})$ as $|\tau| \to \infty$, uniformly in compact subsets of $\{\lambda > 0\}$. In particular, $u^*(\tau;\lambda)$ is in $L^2(-\infty,\infty)$ as a function of τ and it is easy to estimate $\|u(\cdot\,;\lambda) - u(\cdot\,;\mu)\|_2 = \|u^*(\cdot\,;\lambda) - u^*(\cdot\,;\mu)\|_2$ and show that $u(\cdot\,;\lambda) \to u(\cdot\,;\mu)$ in L^2 as $\lambda \to \mu$.

Since $[u']^*(\tau;\lambda) = -\lambda a^*(\tau)[i\tau + \lambda a^*(\tau)]^{-1} = -1 + i\tau[i\tau + \lambda a^*(\tau)]^{-1}$, the same argument works for $u'(t,\lambda)$. This proves Lemma 1.

It suffices to prove (1.7) at the point $\lambda = 1$; in the general case where $\lambda = \lambda_0$ we substitute $b = \lambda a$, $\mu = \lambda/\lambda_0$ and obtain (1.7).

In the following, we write $u(t) = u(t;1)$, $x(t) = x(t;1)$, and $x_\lambda(t) = x(t;\lambda)$.

Set $z_\lambda = (\lambda - 1)^{-1}(x_\lambda - x)$ when $\lambda \neq 1$. Then by (1.4), $z_\lambda(0) = 0$ and

$$z'_\lambda(t) = -\int_0^t a(t-\tau)z_\lambda(\tau)\,d\tau + \lambda^{-1}[x'(t;\lambda) - k].$$

Variation of constants yields the formula

$$z_\lambda(t) = \frac{-1}{\lambda}kw(t) + \frac{1}{\lambda}\int_0^t x'(t-s;\lambda)u(s)\,ds. \tag{2.2}$$

To complete the proof, we must show that as $\lambda \to 1$, this function of t tends uniformly to the function given by the right-hand side of (1.7) with $\lambda = 1$, and that the latter function is bounded. Since $w(t)$ is bounded (see (1.5)). we need only consider the last term in (2.2). But by (1.6),

$$\left| \int_0^t x'(t-s;\lambda)u(s)\,ds \right| \leqq \left| k\int_0^t u(t-s;\lambda)u(s)\,ds \right| + \left| x_0 \int_0^t u'(t-s;\lambda)u(s)\,ds \right|$$
$$\leqq |k|\,\|u(\cdot\,;\lambda)\|_2\|u\|_2 + |x_0|\,\|u'(\cdot\,;\lambda)\|_2\|u\|_2,$$

and this is uniformly bounded for λ near 1, by Lemma 1. Similarly,

$$\left| \int_0^t x'(t-s;\lambda)u(s)\,ds - \int_0^t x'(t-s;1)u(s)\,ds \right|$$
$$\leqq \|u\|_2(|k|\,\|u(\cdot\,;\lambda) - u\|_2 + |x_0|\,\|u'(\cdot\,;\lambda) - u'\|_2);$$

this tends to zero as $\lambda \to 1$, by Lemma 1, and the estimate is uniform in t. This proves Theorem 1.

3. Proof of Theorems 2 and 3. Consider the integrals

$$U(t) = \int_{\lambda_0}^{\infty} u(t;\lambda)\,dE_\lambda\xi, \qquad W(t) = \int_{\lambda_0}^{\infty} w(t;\lambda)\,dE_\lambda\varkappa. \tag{3.1}$$

In [8, Thms. 2, 3, 4] we proved that under the hypotheses of the present Theorems 2 and 3, $u(t,\lambda)$ and $w(t,\lambda)$ are uniformly bounded, say $|u(t,\lambda)| + |w(t,\lambda)| \leqq B$, in $\{0 \leqq t \leqq \infty, \lambda_0 \leqq \lambda < \infty\}$. It follows that the integrals (3.1) exist. Moreover,

$$\begin{aligned}
&L^{-1}U(t) + \int_0^t h(t-s)U(s)\,ds \\
&= \int_{\lambda_0}^{\infty} \left[\lambda^{-1}u(t;\lambda) + \int_0^t h(t-s)u(s;\lambda)\,ds\right] dE_\lambda\xi \\
&= \int_{\lambda_0}^{\infty} \left\{\lambda^{-1} + \int_0^t \left[\lambda^{-1}u'(\tau,\lambda) + \int_0^\tau a(\tau-s)u(s;\lambda)\,ds\right] d\tau\right\} dE_\lambda\xi \\
&= L^{-1}\xi.
\end{aligned}$$

Similarly, $L^{-1}W(t) + \int_0^t h(t-s)W(s)\,ds = tL^{-1}\varkappa$. Then $y = U + W$ is a solution of (1.1). (This argument follows Friedman [3, Thm. 4.1].) If Y is another continuous solution of (1.1) and $Z = y - Y$, one sees that

$$\|E_\lambda Z(t)\| \leqq \lambda \int_0^t h(t-s)\|E_\lambda Z(s)\|\,ds \leqq \lambda h(t) \int_0^t \|E_\lambda Z(s)\|\,ds$$

for any fixed λ. Thus $E_\lambda Z(t) = 0$ (all t and λ) and $Y = y$.

By Theorem 1 Φ is, in particular, continuous. It follows that (1.5) holds uniformly on compact subsets of $\{\lambda > 0\}$. Thus for Theorem 2 (or 3) set $A = (\int_0^\infty a(t)\,dt)^{-1}$ (or 0) and let $\varepsilon > 0$. Choose $\mu > \lambda_0$ so that

$$(A\lambda_0^{-1} + B)(\|(I - E_\mu)\xi\| + \|(I - E_\mu)\varkappa\|) < \varepsilon/2,$$

and choose T so that

$$|u(t;\lambda)| + |w(t;\lambda) - \lambda^{-1}A| < \varepsilon/2(\|\xi\| + \|\varkappa\|), \qquad T \leqq t < \infty, \quad \lambda_0 \leqq \lambda \leqq \mu.$$

Then if $t \geqq T$,

$$\begin{aligned}
\|y(t) - AL^{-1}\varkappa\| &\leqq \|E_\mu U(t)\| + \|E_\mu[W(t) - AL^{-1}\varkappa]\| + \|(I - E_\mu)y(t)\| \\
&\quad + A\|L^{-1}(I - E_\mu)\varkappa\| \\
&\leqq \left\| \int_{\lambda_0}^{\mu} u(t, \lambda)\, dE_\lambda \xi \right\| + \left\| \int_{\lambda_0}^{\mu} [w(t, \lambda) - A\lambda^{-1}]\, dE_\lambda \varkappa \right\| + \varepsilon/2 \\
&< \varepsilon .
\end{aligned}$$

This proves Theorems 2 and 3.

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SINGULAR PERTURBATION OF AN IMPROPERLY POSED CAUCHY PROBLEM*

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Abstract. In this paper we consider the solution of an improperly posed Cauchy problem (assumed to exist) for a coupled system of two second order elliptic differential equations one of which has a small coefficient ε multiplying the highest order derivative. We compare the solution of this problem with the solution of the appropriately defined Cauchy problem for the elliptic differential equation resulting from setting ε equal to zero. We prove that if the two solutions belong to the appropriate spaces of functions, then their difference in the $\mathscr{L}^2$-norm over some appropriately defined subdomain is of order ε to some positive power which depends on the coefficients of the system and the particular subdomain considered.

1. Introduction. In an earlier paper [2], we considered the following type of singular perturbation of an improperly posed quasi-linear elliptic Cauchy problem:

$$\left.\begin{aligned} \varepsilon bLv + v &= u \\ Lu &= E(x, \varepsilon, v, u) \end{aligned}\right\} \text{ in } D$$

with appropriate Cauchy data specified on Σ a portion of the ∂D, where L denotes a symmetric strongly elliptic operator. We proved that if v and w (the solution to the corresponding appropriate unperturbed problem) were suitably restricted, then the difference of v and w in the $\mathscr{L}^2$-norm for some subdomain of D is of order ε to some positive power. The power depends on the constant b and on the size of the subdomain. These results were also extended to include the case when E depends also on any first partial derivatives of u and v.

In this paper we wish to consider the case when the operators on u and v differ. This introduces additional difficulties, but the basic results remain valid.

2. Notation and statement of the problem. Let D be an N-dimensional domain bounded by a closed surface C, and let Σ be that portion of C on which Cauchy data are prescribed. The complement of Σ with respect to C is denoted Σ' and on Σ' no data are given. For the purpose of this paper we shall assume $\bar{\Sigma}$ (the closure of Σ) is a $C^{3+\alpha}$-surface.

Let L_1 and L_2 denote the elliptic operators

$$L_1 u = (a_{ij}u_{,i})_{,j},$$

$$L_2 u = (b_{ij}u_{,i})_{,j},$$

where we have adopted the summation convention over repeated indices and the comma denotes partial differentiation. We also assume that the a_{ij}'s and b_{ij}'s are C^1-functions of the space variables $x = (x_1, x_2, \cdots, x_N)$.

Let the operators L_1 and L_2 be symmetric and strongly elliptic, that is the matrices a_{ij} and b_{ij} are symmetric and there exist positive constants a_0 and b_0 such

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that for all vectors ξ_i the inequalities

$$\frac{1}{a_0}\sum_{i=1}^{N}\xi_1^2 \geqq a_{ij}\xi_i\xi_j \geqq a_0\sum_{i=1}^{N}\xi_i^2 \tag{2.1}$$

and

$$\frac{1}{b_0}\sum_{i=1}^{N}\xi_i^2 \geqq b_{ij}\xi_i\xi_j \geqq b_0\sum_{i=1}^{N}\xi_i^2 \tag{2.2}$$

hold at every point in D.

We shall compare solutions v and w of the following set of improperly posed Cauchy problems.

Problem A:

$$\left.\begin{aligned} \varepsilon b L_2 v + v &= u \\ L_1 u &= E(x,\varepsilon,v,u) \end{aligned}\right\} \text{ in } D$$

with

$$L_2^i v = h_i(x,\varepsilon), \quad \operatorname{grad}(L_2^i v) = \vec{g}_i(x,\varepsilon) \quad \text{on } \Sigma, \qquad i = 0, 1.$$

Problem B:

$$L_1 w = E(x,0,w,w) \quad \text{in } D$$

with

$$w = h_0(x,0), \quad \operatorname{grad} w = \vec{g}_0(x,0) \quad \text{on } \Sigma,$$

where b is a constant and $\vec{g}_i(x,\varepsilon)$ denotes for each i a vector-valued function.

We assume that E satisfies a uniform Lipschitz condition in its last three arguments, that is there exist constants λ_0, λ_1, and λ_2 such that

$$|E(x,\varepsilon,v,u) - E(x,0,\bar{v},\bar{u})| \leqq \lambda_0\varepsilon + \lambda_1|v-\bar{v}| + \lambda_2|u-\bar{u}|. \tag{2.3}$$

Furthermore, we assume that

$$\int_D E^2(0)\,dx \leqq P^2, \tag{2.4}$$

where $E(0) = E(x,0,0,0)$ and P is a constant.

On Σ we require the Cauchy data $h_i(x,\varepsilon)$ and $\vec{g}_i(x,\varepsilon)$ to satisfy

$$\int_\Sigma h_i^2\,ds \leqq \pi_i^2, \quad \int_\Sigma |\vec{g}_i|^2\,ds \leqq \mu_i^2 \tag{2.5}$$

for known constants π_i and μ_i ($i = 0, 1$) independent of ε. Also we assume

$$|h_0(x,\varepsilon) - h_0(x,0)| = O(\varepsilon) \tag{2.6}$$

and

$$|\vec{g}_0(x,\varepsilon) - \vec{g}_0(x,0)| = O(\varepsilon). \tag{2.7}$$

To determine the boundary data for u and grad u, we substitute the data for v and its derivatives into the first equation of Problem A. In this way we always know that

$$\|u - v\|_\Sigma = \left(\int_\Sigma (u - v)^2 \, ds \right)^{1/2} = O(\varepsilon)$$

and

$$\|\operatorname{grad}(u - v)\|_\Sigma = O(\varepsilon).$$

Similarly

$$\|u - w\|_\Sigma = O(\varepsilon)$$

and

$$\|\operatorname{grad}(u - w)\|_\Sigma = O(\varepsilon).$$

As mentioned in [1] and [2] we can allow for small errors in the measurement of the Cauchy data without affecting our results. In fact such relaxation in the data might result in Problem A having a solution for a range of values of the parameter ε.

To assure ourselves that the solutions of our problems depend Hölder continuously on the data, we must restrict the class of admissible solutions (see John [3], Laurentiev [4], [5] and Pucci [6]). To this end we introduce a class of functions $\tilde{M}$ as follows: a function φ will be said to belong to $\tilde{M}$ if

$$\int_D \varphi^2 \, dx \leqq M^2$$

for some prescribed constant M. In addition a function ψ will be said to belong to $\tilde{M}_1$ if

$$\int_D \psi^2 \, dx + \int_D |\operatorname{grad} \psi|^2 \, dx \leqq M_1^2$$

for some prescribed constant M_1. We shall be concerned with solutions v of Problem A and w of Problem B which belong either to $\tilde{M}$ or $\tilde{M}_1$. We assume throughout that for the particular value of ε under consideration these solutions exist and belong to the appropriate spaces. We assume further that these solutions are sufficiently differentiable for carrying out the indicated operations. In each case sufficient conditions can be readily found in the literature. Note that we do not require a priori that $u \in \tilde{M}$ or $\tilde{M}_1$.

We propose to prove that if v and w belong to $\tilde{M}$, then the difference v and w in the $\mathscr{L}^2$-norm for some subdomain of D is of order ε to some positive power. The power depends on the constant b and also on the size of the subdomain.

We do not compare v and w over all of D, but only over a class of subdomains $D_\alpha \subset D$. We define these subdomains as follows.

Let $f(x) =$ const. define a set of (not necessarily closed) surfaces. This set is to be so chosen that for each α satisfying

$$0 < \alpha \leqq 1 \tag{2.8}$$

the surface $f(x) = \alpha$ intersects D and forms a closed region D_α whose boundary points consist only of points of Σ and points on the surface $f =$ const.

We shall require that $f(x)$ have continuous second derivatives in $\bar{D}_1$ and further that if f satisfies (2.8), then

$$\beta \leqq \gamma \Rightarrow D_\beta \subset D_\gamma, \qquad 0 < \beta \leqq \gamma \leqq 1. \tag{2.9}$$

We assume that the surfaces have been so chosen that for α satisfying (2.8), D_α has nonzero measure, but that D_0 has zero measure.

As in [2] we compare the solutions v and w in the following sense. We show that

$$\|v - w\|_{D_\alpha}^2 \equiv \int_{D_\alpha} (v - w)^2 \, dx = O(\varepsilon^{\gamma(\alpha)}),$$

where $\gamma(\alpha)$ is a positive function of α for $0 \leqq \alpha < \alpha_1 < 1$ and $\gamma(\alpha_1) = 0$. Thus for $0 \leqq \alpha < \alpha_1 < 1$ our inequality will show that if ε is sufficiently small, v will be arbitrarily close to w in $\mathscr{L}^2$ over D_α.

3. Results. We first show that for $0 < \alpha < 1$ the quantity $\|u - v\|_{D_\alpha}^2$ is of order ε in general and under certain conditions is of order ε^2.

We introduce the function $\tau(x)$ defined in $\bar{D}_1$ as

$$\tau(x) = \begin{cases} 1 & \text{in } D_\alpha \cup \Sigma_\alpha, \\ \dfrac{1 - f(x)}{1 - \alpha} & \text{in } \bar{D}_1 - (D_\alpha \cup \Sigma_\alpha), \end{cases} \tag{3.1}$$

where Σ_α is the portion of Σ which lies on the boundary of D_α, and S_α will denote the portion of the surface $f(x) = \alpha$ so that the entire boundary of D_α is $\Sigma_\alpha \cup S_\alpha$. Clearly $\tau(x) = 0$ on S_1 and $|\tau(x)| \leqq 1$ in $\bar{D}_1$. Since $f \in C^2(\bar{D}_1)$, $|\tau_{,i}\tau_{,i}| \leqq M_3$ and $|\tau_{,ij}\tau_{,ij}| \leqq M_4$ in $\bar{D}_1$ for constants M_3 and M_4.

Thus we have

$$\|u - v\|_{D_\alpha}^2 \equiv \int_{D_\alpha} (u - v)^2 \, dx \leqq \int_{D_1} \tau^s (u - v)^2 \, dx, \tag{3.2}$$

where s is a positive integer to be chosen so large that all subsequent integrals over S_1 vanish.

We begin by proving the analogues of Lemmas 3.1 and 3.3 of [2].

LEMMA 3.1. *If $v \in \tilde{M}$ is a solution of Problem* A *with $b < 0$, or if $v \in \tilde{M}_1$ is a solution of Problem* A *with $b > 0$, then*

$$\int_{D_1} \tau^s (u - v)^2 \, dx \leqq O(\varepsilon) + K_0 \varepsilon \int_{D_1} \tau^{s-2} (u - v)^2 \, dx \tag{3.3}$$

for a computable constant K_0.

Proof. From the equations of Problem A and Green's identity, we have

$$
\begin{aligned}
\int_{D_1} \tau^s(u-v)^2\,dx &= b\varepsilon \int_{D_1} \tau^s(u-v)L_2 v\,dx \\
&= b\varepsilon \int_{\Sigma_1} \tau^s(u-v)\frac{\partial v}{\partial \nu_2}\,ds \\
&\quad - b\varepsilon \int_{D_1} \tau^s b_{ij}(u-v),_j v,_i\,dx \\
&\quad - b\varepsilon \int_{D_1} \tau^s,_j b_{ij}(u-v)v,_i\,dx,
\end{aligned} \tag{3.4}
$$

where $\partial/\partial\nu_2$ is the conormal derivative $b_{ij}n_j(\partial/\partial x_i)$ on the boundary Σ_1.

Also since we can bound the conormal derivative in terms of the normal and tangential derivatives, the boundary integral in (3.4) involves data terms and is $O(\varepsilon^2)$.

We shall in this paper make frequent use of the arithmetic–geometric mean inequality (henceforth abbreviated A–G inequality). Unless we specifically need the constants which enter, we shall use the letters γ_j for the coefficients of the terms we shall subsequently wish to make small and k_j as coefficients of the other terms which are computable and may be large (but do not depend on ε).

Thus using the A–G inequality and the bounds for derivatives of τ we find

$$
\begin{aligned}
\int_{D_1} \tau^s(u-v)^2\,dx &\leqq O(\varepsilon^2) + b\varepsilon\int_{D_1}\tau^s b_{ij}v,_i v,_j\,dx - b\varepsilon\int_{D_1}\tau^s b_{ij}u,_j v,_i\,dx \\
&\quad + k_1\varepsilon\int_{D_1}\tau^{s-2}(u-v)^2\,dx + \gamma_1\varepsilon\int_{D_1}\tau^s v,_i v,_i\,dx \\
&\leqq O(\varepsilon^2) + b\varepsilon\int_{D_1}\tau^s b_{ij}v,_i v,_j\,dx + [\gamma_1+\gamma_2]\varepsilon\int_{D_1}\tau^s v,_i v,_i\,dx \\
&\quad + k_1\varepsilon\int_{D_1}\tau^{s-2}(u-v)^2\,dx + k_2\varepsilon\int_{D_1}\tau^s u,_i u,_i\,dx.
\end{aligned} \tag{3.5}
$$

By the ellipticity of L_1, we have for the last term on the right-hand side (R.H.S.) of (3.5),

$$
k_2\varepsilon\int_{D_1}\tau^s u,_i u,_i\,dx \leqq \frac{k_2}{a_0}\varepsilon\int_{D_1}\tau^s a_{ij}u,_i u,_j\,dx. \tag{3.6}
$$

And by Green's identity we see that

$$
\begin{aligned}
k_2\varepsilon\int_{D_1}\tau^s u,_i u,_i\,dx &\leqq \frac{k_2}{a_0}\left[\varepsilon\int_{\Sigma_1}\tau^s u\frac{\partial u}{\partial \nu_1}\,ds - \frac{\varepsilon}{2}\int_{\Sigma_1}\frac{\partial\tau^s}{\partial\nu_1}u^2\,ds\right] \\
&\quad - \frac{k_2}{a_0}\varepsilon\int_{D_1}\tau^s u L_1 u\,dx + \frac{k_2}{2a_0}\varepsilon\int_{D_1}(L_1\tau^s)u^2\,dx.
\end{aligned} \tag{3.7}
$$

Thus

$$k_2\varepsilon \int_{D_1} \tau^s u_{,i}u_{,i}\, dx \leqq O(\varepsilon) + k_3\varepsilon \int_{D_1} \tau^{s-2}v^2\, dx + k_4\varepsilon \int_{D_1} \tau^s E^2(0)\, dx + k_5\varepsilon \int_{D_1} \tau^{s-2}(u-v)^2\, dx \tag{3.8}$$

by the Lipschitz condition on E and the A–G inequality. Substitution of (3.8) into (3.5) yields

$$\begin{aligned}\int_{D_1} \tau^s(u-v)^2\, dx \leqq O(\varepsilon) &+ b\varepsilon \int_{D_1} \tau^s b_{ij}v_{,i}v_{,j}\, dx \\ &+ [\gamma_1 + \gamma_2]\varepsilon \int_{D_1} \tau^s v_{,i}v_{,i}\, dx \\ &+ k_0\varepsilon \int_{D_1} \tau^{s-2}(u-v)^2\, dx \\ &+ k_3\varepsilon \int_{D_1} \tau^{s-2}v^2\, dx + k_4\varepsilon \int_{D_1} \tau^s E^2(0)\, dx.\end{aligned} \tag{3.9}$$

It follows then that if $b < 0$, we may choose γ_1 and γ_2 sufficiently small so that the second term dominates the third term on the right of (3.9) and both terms may be dropped. If $b > 0$ and $v \in \tilde{M}_1$, we combine the second and third terms and bound them by an $O(\varepsilon)$ term. Hence with our bounds for $E(0)$ and v, we have completed the proof of Lemma 3.1.

LEMMA 3.2. *Regardless of the sign of* b, *if* $v \in \tilde{M}$ *is a solution of Problem* A, *then for any positive integer* σ *so that all integrals over* S_1 *vanish,*

$$\int_{D_1} \tau^\sigma(u-v)^2\, dx \leqq K \tag{3.10}$$

for some computable constant K.

Proof. We proceed as in the proof of Lemma 3.1 up to equation (3.5) which becomes by a slightly different use of the A–G inequality

$$\begin{aligned}\int_{D_1} \tau^\sigma(u-v)^2\, dx \leqq O(\varepsilon^2) &+ b\varepsilon \int_{D_1} \tau^\sigma b_{ij}v_{,i}v_{,j}\, dx - b\varepsilon \int_{D_1} \tau^\sigma b_{ij}u_{,j}v_{,i}\, dx \\ &+ \gamma_4 \int_{D_1} \tau^\sigma(u-v)^2\, dx + k_5\varepsilon^2 \int_{D_1} \tau^{\sigma-2}v_{,i}v_{,i}\, dx.\end{aligned} \tag{3.11}$$

Hence we may choose $\gamma_4 < 1$ and solve for the left-hand side (L.H.S.) of (3.11). We then apply the A–G inequality to the third term on the right and use the ellipticity conditions on all the grad v and grad u terms to obtain

$$\tag{3.12} \begin{aligned} \int_{D_1} \tau^\sigma (u - v)^2\, dx \leqq O(\varepsilon^2) + \gamma_5 \int_{D_1} \tau^{\sigma+2} a_{ij} u_{,i} u_{,j}\, dx \\ + k_6 \varepsilon \int_{D_1} \tau^{\sigma-2} b_{ij} v_{,i} v_{,j}\, dx. \end{aligned}$$

We use Green's identity on the last two terms on the R.H.S. of (3.12) to get

$$\tag{3.13} \begin{aligned} \int_{D_1} \tau^\sigma (u - v)^2\, dx \leqq O(\varepsilon^2) &+ \gamma_5 \int_{\Sigma_1} \tau^{\sigma+2} u \frac{\partial u}{\partial \nu_1}\, ds \\ &- \frac{\gamma_5}{2} \int_{\Sigma_1} \frac{\partial \tau^{\sigma+2}}{\partial \nu_1} u^2\, ds - \gamma_5 \int_{D_1} \tau^{\sigma+2} u L_1 u\, dx \\ &+ \frac{\gamma_5}{2} \int_{D_1} (L_1 \tau^{\sigma+2}) u^2\, dx \\ &+ k_6 \varepsilon \int_{\Sigma_1} \tau^{\sigma-2} v \frac{\partial v}{\partial \nu_2}\, ds - \frac{k_6}{2} \varepsilon \int_{\Sigma_1} \frac{\partial \tau^{\sigma-2}}{\partial \nu_2} v^2\, ds \\ &- k_6 \varepsilon \int_{D_1} \tau^{\sigma-2} v L_2 v\, dx + \frac{k_6}{2} \varepsilon \int_{D_1} (L_2 \tau^{\sigma-2}) v^2\, dx. \end{aligned}$$

We now use the equations of Problem A, the Lipschitz condition and the A–G inequality to obtain

$$\tag{3.14} \begin{aligned} \int_{D_1} \tau^\sigma (u - v)^2\, dx \leqq O(1) + \gamma_5 \int_{D_1} \tau^\sigma (u - v)^2\, dx \\ + k_{13} \int_{D_1} \tau^{\sigma-4} v^2\, dx + k_{10} \int_{D_1} \tau^{\sigma+4} E^2(0)\, dx. \end{aligned}$$

Hence we may choose γ_5 sufficiently small to allow us to solve for the term on the L.H.S. of (3.14). If we then apply our bounds for v and $E(0)$ we have the desired result.

By combining Lemmas 3.1 and 3.2 with $\sigma = s - 2$ and using the fact that

$$\int_{D_\alpha} (u - v)^2\, dx \leqq \int_{D_1} \tau^s (u - v)^2\, dx,$$

we obtain, if we choose $\sigma > 5$, the following.

THEOREM 3.3. *If $v \in \tilde{M}$ is a solution of Problem* A *with $b < 0$, or if $v \in \tilde{M}_1$ is a solution of Problem* A *with $b > 0$, then*

$$\|u - v\|_{D_\alpha}^2 = O(\varepsilon)$$

for α in the range $0 \leqq \alpha < 1$.

We now use the results of the convexity argument of § 3 of [2] with $L = L_1$ to conclude that

$$\tag{3.15} \|u - w\|_{D_\alpha}^2 = O(\varepsilon^{1-\nu(\alpha)})$$

for Problem set (A, B).

Therefore, combining Theorem 3.3 with (3.15) and using the triangle inequality we have the following theorem.

THEOREM 3.4. *If $v \in \tilde{M}$ is a solution of Problem* A *with $b < 0$ or if $v \in \tilde{M}_1$ is a solution of Problem* A *with $b > 0$ and $w \in \tilde{M}$ is a solution of Problem* B, *and u is a solution to*

$$L_1 u = E(x, \varepsilon, v, u)$$

as in Problem A, *then the difference $v - w$ satisfies the following continuous dependence inequality for α in the range $0 \leqq \alpha < \alpha_1 < 1$ and $\nu(\alpha)$ with $0 \leqq \nu(\alpha) < 1$:*

$$\|v - w\|_{D_\alpha}^2 = O(\varepsilon^{1-\nu(\alpha)}).$$

Remark. We note that the order of ε here is half of what it was for Problem (A, B) in [2] with $L = L_1 = L_2$ and E dependent only on x, ε, v and u, and $b < 0$. If we assumed that b_{ij} and a_{ij} were related by

$$b_{ij} = a_{ij} + \varepsilon C_{ij},$$

then we would again obtain the same order as in [2].

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ASYMPTOTIC EXPANSION OF LAPLACE CONVOLUTIONS FOR LARGE ARGUMENT AND TAIL DENSITIES FOR CERTAIN SUMS OF RANDOM VARIABLES*

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Abstract. An asymptotic series $\sum_{m,n=0}^{\infty} a_{mn}t^{a(m)}(\log t)^n$ with Re $[a(m)]$ either increasing or decreasing is called a Mellin series respectively near either $0+$ or $+\infty$. Let f be a complex-valued locally integrable function on $[0, +\infty)$, and let $L[f; s]$, its Laplace transform, be absolutely convergent on $\mathrm{Re}(s) > 0$ and have a Mellin series near $0+$. Then f need not have a Mellin series near $+\infty$, but if it does then this series is uniquely determined by the given series for $L[f; s]$. Two earlier theorems of Doetsch, slightly extended, give sufficient conditions on $L[f; s]$ that f have a Mellin series near $+\infty$, while a counterexample shows that these conditions are not necessary.

The class of functions f having Mellin series near $+\infty$ is closed under pointwise addition, scalar multiplication, and Laplace convolution; whence it yields a number of useful subalgebras. These results are used to calculate Mellin series near $+\infty$ for convolutions of: (i) gamma densities; (ii) auxiliary functions $h_{a,m}(t) = t^{a-1}(\log t)^m$ on $[1, +\infty)$, $= 0$ elsewhere; (iii) Pareto densities with L^1 but asymptotically small perturbations.

1. Introduction. In this paper we consider locally integrable functions on $[0, +\infty)$ with certain asymptotic expansions near $+\infty$, and we recover such expansions, here called Mellin series, from the corresponding series near $0+$ for the Laplace transforms of these functions. We show also that the set of functions with such expansions is closed under Laplace convolution, hence that the series near $+\infty$ for these convolutions are obtainable via Laplace transforms. However, the convolution of two probability densities corresponds to the sum of the associated random variables. Thus we apply these results to expand the tail density for finite sums of random variables with gamma, Pareto, and related distributions—extending various special computations through our general technique.

Let f be a complex-valued locally integrable function on $[0, +\infty)$, and let

$$f(t) \sim \sum_{m,n=0}^{\infty} a_{mn}t^{a(m)}(\log t)^n \quad \text{as } t \to +\infty, \tag{1.1}$$

where $\{n : a_{mn} \neq 0\}$ is finite for each m and $\mathrm{Re}\,[a(m)]\downarrow -\infty$ as $m \to \infty$. According to van der Corput [12, p. 366], asymptotic series of form (1.1) were first treated systematically by Mellin [33]. Hence series near $+\infty$ of this form, or near $0+$ with $\mathrm{Re}\,[a(m)]\uparrow +\infty$, will be called *Mellin series* or *expansions* in our work. Series of form (1.1) occur naturally in the expansion of many special functions, and in the solution of ordinary differential equations. Some further series with nonintegral n have been treated by Erdélyi [18].

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For any such function f, the Laplace transform

$$L[f;s] = \int_0^\infty \exp(-st)f(t)\,dt \tag{1.2}$$

is absolutely convergent and holomorphic in $\mathrm{Re}(s) > 0$, while the Mellin transform

$$M[f;z] = \int_0^\infty t^{z-1}f(t)\,dt \tag{1.3}$$

is well-defined and meromorphic in some right half-plane, either through the integral (1.3) or through analytic continuation (Handelsman and Lew [23, (4.9)]). From $M[f;z]$ in this half-plane we have shown that $L[f;s]$ near $0+$ can be expanded systematically as a Mellin series in s. Indeed we have obtained Mellin expansions in two further papers (Handelsman and Lew [22], [24]) for all integral transforms

$$H[f;s] = \int_0^\infty h(st)f(t)\,dt \tag{1.4}$$

with suitably dominated kernels $h(t)$.

In this investigation we wish to recover a Mellin series for f near $+\infty$ from a Mellin series for $L[f;s]$ near $0+$. First we show that this can be done uniquely when f is assumed a priori to have a Mellin expansion—and cannot be done at all when f is assumed only to be locally integrable. Then we slightly generalize two earlier theorems (Hull and Froese [26], Doetsch [16, pp. 144–162], Riekstiņa [38], [39]) which derive a Mellin series for f from additional hypotheses on $L[f;s]$. We recall that an Abelian theorem is a result which obtains the limiting behavior of a transform from the limiting behavior of f, whereas a Tauberian theorem is a converse result which, in its characterization by Doetsch, involves an auxiliary condition on f. Hence our first result is a simple Tauberian theorem for $L[f;s]$, whereas the other results are Abelian theorems for the inverse transform. The auxiliary condition for our first theorem is clearly necessary and sufficient, but we shall construct examples of functions f with Mellin series near $+\infty$ which cannot be recovered by these other theorems.

Naturally, to benefit from the first theorem we must establish by other means that f has a Mellin series near $+\infty$. Therefore we prove that the set of all functions which have such expansions is closed under pointwise addition, scalar multiplication, and Laplace convolution:

$$[f * g](t) = \int_0^t f(t-u)g(u)\,du. \tag{1.5}$$

Thus we produce a family of convolution algebras all of whose elements have Mellin expansions, and we need only show that a given f is some algebraic combination of suitable functions. Analogously we might express f, to prove it differentiable, in terms of functions already known to be differentiable. To obtain these closure theorems and simplify our later applications, we introduce on $[0, +\infty)$ a class of functions $h_{a,m}$ whose Laplace transforms and Laplace convolutions involve well studied functions.

These results are used to approximate finite sums of random variables with probability density functions defined on $[0, +\infty)$. Indeed if the independent random variables T_1 and T_2 have respective densities f_1 and f_2, then the random variable $T_1 + T_2$ has associated density $f_1 * f_2$. If f_1 and f_2 have expansions of the form

$$f(t) \sim \sum_{m,n=0}^{\infty} a_{mn} \exp[a(m)t]t^n \quad \text{as } t \to +\infty, \tag{1.6}$$

under the same restrictions as (1.1), then $f_1 * f_2$ has an expansion of the stated form with coefficients obtainable by earlier methods (Doetsch [16, pp. 98, 110]). (Convergent series of form (1.6) have been called *hyperdirichlet* series by Lepson [29].) However if f_1 and f_2 have Mellin series near $+\infty$ then $f_1 * f_2$ has a Mellin series near $+\infty$ which can be computed sytematically via our relations between functions and transforms. In particular the series (1.1) for f, through the derived expansion of $L[f; s]$, yields a corresponding series for f^{*n}, the nth convolution power of f. Thus we do not treat densities with expansions like (1.6), but give several examples involving Mellin series near $+\infty$ to show the range of our techniques.

In a recent talk (Handelsman and Lew [25]) we have previously sketched a preliminary version of these results. In our future work we shall also consider the corresponding developments for cumulative distribution functions. Moreover we plan to discuss applications to compound stochastic processes, as in the theory of single-server queues and the theory of collective risk. Parts of this paper anticipate the needs of these extensions.

The classical Tauberian theorems for the Laplace transform have been used primarily to approximate distributions of high eigenvalues and of large prime numbers; but the functions f which occur in such problems have sizeable jumps for arbitrarily large t, and thus have no complete expansion near $+\infty$. Accordingly, results like Karamata's theorem (Widder [47, pp. 189–197]) deduce merely a leading term ct^{a-1} for $f(t)$ from a leading term $cs^{-a}/\Gamma(a)$ for $L[f; s]$. Similar Tauberian theorems for generalized functions f have been obtained recently by Lavoine [28], but through a regularization of f which suppresses some information about its global behavior. Further Tauberian theorems of Wagner [46] which approximate $\log f(t)$ can produce asymptotic forms for $f(t)$ more singular than ct^{a-1}.

Of course we can sometimes recover $f(t)$ exactly from $L[f; s]$ by the classical inversion integral or the Post–Widder formula (Widder [47, pp. 66, 288], Doetsch [15, pp. 212, 290]). Moreover $f(t)$ can be determined exactly by a method of Goldenberg [21] when $L[f; s]$ satisfies a linear differential or difference equation with polynomial coefficients; and $f(t)$ can be obtained as a convergent power series by a more recent method of Widder [48], [49, Chap. 9] when $f(t)$ and $L[f; s]$ satisfy appropriate smoothness conditions. If $L[f; s]$ is known only for $0 < s < +\infty$ then $f(t)$ can be computed approximately by various methods (Bellman, Kalaba and Lockett [3]), but all of these demand prior assumptions about $f(t)$ near $+\infty$. If $L[f; s]$ is known for $\mathrm{Re}(s) > 0$ then $f(t)$ can be obtained numerically by the fast Fourier transform (Cooley, Lewis and Welch [11]); indeed, the convolutions $f_1 * f_2$ can be obtained directly through this algorithm. However, for large t these numerical methods become less reliable while our analytic results become more accurate, so that these algorithms complement our theorems.

In a closely related paper, Riekstiņš [40] considers two locally integrable functions with asymptotic series near $+\infty$ in related powers of t, and expands their Laplace convolution as a corresponding series near $+\infty$, with a possible factor $\log(t/2)$. He extracts a series for this integral by detailed arguments on two subintervals, but claims the validity of his method for all functions with Mellin series. However we need such direct methods to establish only the existence of Mellin series for general convolutions $f * g$, and use Laplace transforms to obtain conveniently the coefficients of such series, even for powers f^{*n}.

The limiting behavior of the density function for an algebraic combination of random variables has been widely studied by other investigators. Indeed by a standard result of probability theory (Feller [19, p. 271]) if

$$\int_t^{\infty} f_i(u)\,du \sim c_i t^{-r} L(t) \quad \text{for } i = 1, 2 \text{ as } t \to +\infty, \tag{1.7}$$

with $L(t)$ a slowly varying function, then

$$\int_t^{\infty} [f_1 * f_2](u)\,du \sim (c_1 + c_2) t^{-r} L(t) \text{ as } t \to +\infty, \tag{1.8}$$

with an immediate corollary for f^{*n}. Moreover a leading term for $f_1 * f_2$, under asymmetric hypotheses on f_1 and f_2, has been given by Muki and Sternberg [36] in a study of integral equations. However an expansion like (1.1) can be carried to any number of terms, and the polynomials in $\log t$ which occur in such series are all slowly varying functions, so that our results are sharper than these alternatives.

Through the use of integral transforms, other authors have derived general formulas for densities of sums and products (Abraham and Prasad [1], Prasad [37]) or power series for combinations of special densities (Brennan, Reed, and Sollfrey [9], Springer and Thompson [43], Blum [8]). However our results produce a Mellin expansion near $+\infty$ for any $f_1 * f_2$ from the corresponding series and moments for f_1 and f_2. Products of random variables are not discussed in this paper—because a Mellin series near $+\infty$ for the resulting density

$$\int_0^t f_1(t/u) f_2(u)\,du/u \tag{1.9}$$

can be obtained by earlier theorems (Doetsch [16, pp. 131–135], Handelsman and Lew [24, (1.9)]) from the corresponding behavior of f_1 and f_2.

If the function f has variance $< +\infty$ and Fourier transform in $L^1(-\infty, +\infty)$ then the central limit theorem for densities (Feller [19, p. 489]) states that f^{*n} for large n is approximated by a normal density. However the relative error for such approximations need in general be small only in some neighborhood of the mean whose width grows slowly with n (Feller [19, p. 517]). Thus our results hold when the central limit theorem fails, namely for n fixed and $t \to +\infty$. If the function f has a series (1.1) with exponent $a(0) > -3$ then other well-known theorems in probability (Feller [19, p. 303]) state that f lies in the domain of attraction for some nonnormal stable density. Possibly f^{*n} for large n may be approximated in a useful sense by this stable density near $+\infty$—under some further hypotheses; but our results are valid even when no limit density is relevant, that is, for all negative $a(0)$ in the expansion (1.1).

2. Convolution algebras. In this section we introduce a number of function spaces on $[0, +\infty)$ and discuss their closure under Laplace convolution. Later we distinguish those functions which have Mellin expansions near $+\infty$, but initially we consider all functions which are locally integrable on $[0, +\infty)$. We also define some simple isomorphisms among these spaces and review some basic properties of Laplace transforms. We establish our notation through this summary, and state various results for convenient reference.

For any nonnegative t let $R(t)$ denote the interval $[t, +\infty)$ and let $R'(t)$ denote the interval $[0, t)$. For any such interval I let AI denote the set of all complex-valued locally (Lebesgue-) integrable functions on $[0, +\infty)$ which vanish outside the given I, so that in particular $AR(0)$ is the set of all such functions which satisfy no such restriction. Furthermore let $AR(0+)$ denote the union of all $AR(t)$ with $t > 0$, and let $AR'(\infty-)$ denote the union of all $AR'(t)$ with $t < +\infty$. Clearly these sets of functions are all complex vector spaces under pointwise addition and scalar multiplication, with

$$AR(0) = AR(t) \oplus AR'(t) = AR(0+) + AR'(\infty-). \tag{2.1}$$

As usual in integration theory, we identify any two functions which differ only on a set of Lebesgue measure zero.

For any f and g in $AR(0)$ let $f * g$ denote their Laplace convolution, given by

$$[f * g](t) = \int_0^t f(t-u)g(u)\,du. \tag{2.2}$$

For almost all nonnegative t this convolution can be shown to exist and to define a function in $AR(0)$ (Widder [47, pp. 91–92]). We shall usually call this function simply the convolution of f and g, since we shall rarely need to discuss other convolutions (e.g., Fourier, Mellin). We observe for all $t, u \geqq 0$ that

$$\begin{aligned} AR(t) * AR(u) &\subset AR(t+u), \\ AR'(t) * AR'(u) &\subset AR'(t+u) \end{aligned} \tag{2.3}$$

by definition, so that $AR(0+)$ and $AR'(\infty-)$ are both closed under convolution.

Moreover convolution is associative, commutative, and distributive with pointwise addition, by standard integration theorems, so that $AR(0)$, $AR(0+)$, and $AR'(\infty-)$ are complex algebras under this further composition (Mikusinski [34, pp. 345–349]) . The Dirac delta "function" is not contained in $AR(0)$ since it is only a distribution on $[0, +\infty)$, but will be denoted by e since it is clearly an identity for $AR(0)$, and will be adjoined at times to generate the algebra $e + AR(0)$. If $f * g \equiv 0$ in $AR(0)$ then either $f \equiv 0$ or $g \equiv 0$ by Titchmarsh's theorem (Mikusinski [34, pp. 15–23]), so that both $AR(0)$ and $e + AR(0)$ are integral domains under these compositions.

For any f in $AR(0)$ we can define its nonnegative integral $*$-powers by

$$f^{*0} = e, \quad f^{*1} = f, \qquad (2.4)$$
$$f^{*n+1} = f * f^{*n} \quad \text{for } n = 1, 2, \cdots .$$

Following Mikusinski, we might define negative integral $*$-powers as suitable generalized functions, but we shall not need these and prefer to work within $e + AR(0)$. For any polynomial $P(z) = \sum_{m=0}^{n} p_m z^m$ with coefficients p_m in C we can define

$$P^*(f) = \sum_{m=0}^{n} p_m f^{*m}; \qquad (2.5)$$

and for any fixed f in $AR(0)$ the mapping $P(z) \to P^*(f)$ is a homomorphism from $C[z]$ into $e + AR(0)$. Indeed if $f \neq 0$ then this mapping is one-to-one, since $e + AR(0)$ is a commutative integral domain. Moreover if $P(z)$ has no constant term, that is, if $p_0 = 0$, then $P^*(f)$ is in a given subalgebra of $AR(0)$ whenever f is in this same subalgebra of $AR(0)$. Actually we can define $P^*(f)$ for any power series $P(z)$ which converges in some neighborhood of 0 (Lew [30]), but we shall not need this extension.

For any positive a, any complex c, and any f in $AR(0)$ we let

$$D(a)f(t) = af(at), \qquad (2.6)$$
$$E(c)f(t) = \exp(ct)f(t);$$

whence by direct computation we find

$$D(a)[f * g] = [D(a)f] * [D(a)g], \qquad (2.7)$$
$$E(c)[f * g] = [E(c)f] * [E(c)g].$$

Moreover $D(a)$ and $E(c)$ are linear and invertible, so that they are automorphisms of $AR(0)$, $AR(0+)$, and $AR'(\infty-)$.

For any f in $AR(0)$ and any $p \geqq 1$ we let

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p\right)^{1/p} \qquad (2.8)$$

and we allow $\|f\|_p$ the value $+\infty$ so that $\|f\|_p$ is defined for all such f. We recognize the set of all f in AI with $\|f\|_p < +\infty$ as the Banach space $L^p(I)$ imbedded naturally in $AR(0)$. Extending this notation, we thus let $E(c)L^p(I)$ be the set of all f in AI with $\|E(-c)f\|_p < +\infty$ and we let $E(c+)L^p(I)$ be the intersection of all $E(d)L^p(I)$ with $\operatorname{Re}(d - c) > 0$, or equivalently the set of all f in AI with

$$\int_0^\infty |\exp(-ct - \varepsilon t)f(t)|^p \, dt < +\infty \quad \text{for every } \varepsilon > 0. \qquad (2.9)$$

If f is in $L^1(R(t))$ and g is in $L^p(R(u))$ then $f * g$ is in $L^p(R(t+u))$ and

$$\|f * g\|_p \leqq \|f\|_1 \|g\|_p \qquad (2.10)$$

by a well-known result of harmonic analysis (e.g., Dunford and Schwartz [17,

p. 528]). Thus the following spaces, with the following norms, are Banach algebras under convolution for $0 \leqq t < +\infty$, by (2.7) and (2.10).

$$
\begin{array}{lll}
 & L^1(R(t)), & \|f\|_1, \\
 & L^1(R(t)) \cap L^p(R(t)), & \max(\|f\|_1, \|f\|_p), \\
(2.11) & E(c)L^1(R(t)), & \|E(-c)f\|_1, \\
 & E(c)[L^1(R(t)) \cap L^p(R(t))], & \max(\|E(-c)f\|_1, \|E(-c)f\|_p).
\end{array}
$$

Moreover if $1 \leqq p \leqq q$ then $L^1(R(t)) \cap L^q(R(t)) \subset L^1(R(t)) \cap L^p(R(t))$ by the Riesz convexity theorem (e.g., Dunford and Schwartz [17, p. 535]). The spaces $E(c+)L^1(R(t))$ and $E(c+)[L^1(R(t)) \cap L^p(R(t))]$, like the basic space $AR(0)$, are closed under convolution; however these spaces have natural topologies which are metrizable but not normable.

For any f in $AR(0)$ let $L[f; s]$ denote its Laplace transform, given by

$$(2.12) \qquad L[f; s] = \int_0^\infty \exp(-st)f(t)\,dt$$

whenever and wherever this integral exists. In particular if f is in $E(c)L^1(R(0))$, then $L[f; s]$ is absolutely convergent for $\operatorname{Re}(s) \geqq c$, hence continuous on this half-plane and analytic on its interior; while if f is in $E(c+)L^1(R(0))$, then $L[f; s]$ is absolutely convergent, continuous, and analytic for $\operatorname{Re}(s) > c$ (Widder [47, p. 57]). The map $f \to L[f; s]$ is a linear function on either of these spaces, and

$$(2.13) \qquad L[f * g; s] = L[f; s]L[g; s]$$

on the corresponding domain of definition (Widder [47, pp. 91–92]).

It is convenient to put $f(t) = 0$ on $(-\infty, 0)$ in discussing the inverse map, and thus extend any f in $AR(0)$ to a function on $(-\infty, +\infty)$. If this extended function has bounded variation on some (t_1, t_2), then $f(t+)$ and $f(t-)$ are well-defined in this interval; and if this f is also in $E(a)L^1(R(0))$ for some (real) a, then

$$(2.14) \qquad \tfrac{1}{2}[f(t+) + f(t-)] = (2\pi i)^{-1} \lim_{b\to\infty} \int_{a-ib}^{a+ib} \exp(ts)L[f; s]\,ds$$

for $t_1 < t < t_2$ (Widder [47, p. 66], Doetsch [15, p. 212]). If f is in either $E(c)L^1(R(0))$ or $E(c+)L^1(R(0))$, then

$$
(2.15) \qquad \begin{aligned} L[D(a)f; s] &= L[f; s/a], \\ L[E(b)f; s] &= L[f; s - b] \end{aligned}
$$

on the appropriate domain of definition. Thus problems involving these two spaces can respectively be reduced to problems involving $L^1(R(0))$ and $E(0+)L^1(R(0))$ through (2.7), (2.11), and (2.15).

If f and g are elements of $AR(0)$ which satisfy

$$(2.16) \qquad f + g = f * g,$$

then each is called a quasi-inverse of the other. It can be shown for the algebra $AR(0)$ that each element f has a unique quasi-inverse f', hence that there exists no linear map except $L \equiv 0$ which takes all of $AR(0)$ into some space of functions and which satisfies (2.13) for all f and g (e.g., Lew [30]). Thus while the Laplace transform can be defined for a large class of generalized functions (Krabbe [27]), it cannot be defined for all the functions in $AR(0)$ (but see Ditkin [14], Berg [5]).

3. Mellin series. We wish now to discuss functions f in $AR(0)$ for which there exist complex numbers a_{mn} and $a(m)$ such that

$$f(t) \sim \sum_{m,n=0}^{\infty} a_{mn} t^{a(m)} (\log t)^n \quad \text{as } t \to +\infty, \tag{3.1}$$

where $\{n : a_{mn} \neq 0\}$ is finite for each m and $\operatorname{Re}[a(m)] \downarrow -\infty$ as $m \to \infty$. Series of this form near $+\infty$, or with $\operatorname{Re}[a(m)] \uparrow +\infty$ near $0+$, are called *Mellin series* or *expansions* in our work for the historical reasons offered in § 1. If A is any set of functions on $[0, +\infty)$ then MA will denote the subset of functions with Mellin series near $+\infty$. Hence if A is a vector space then MA is clearly a vector space, but if A is a convolution algebra then MA is not obviously such an algebra.

For any function f in $MAR(0)$, by the remarks of § 2, the transform $L[f; s]$ is defined and analytic for $\operatorname{Re}(s) > 0$; and its asymptotic expansion near $0+$, through the results of our earlier work (Handelsman and Lew [23]), can be obtained systematically as a Mellin series in s. Conversely, given any function f in $AR(0)$ such that $L[f; s]$ converges for $\operatorname{Re}(s) > 0$, we should like to deduce a Mellin series for f near $+\infty$ from a Mellin series for $L[f; s]$ near $0+$. However such inverse results are not generally true without some auxiliary conditions. To see this we need only exhibit some nontrivial functions g in $AR(0)$ (see also Lew [31]) such that

$$L(g; s] = o(s^n) \quad \text{as } s \to 0+ \text{ for } n = 1, 2, \cdots, \tag{3.2}$$

since any multiple of some such g can be added to any f in $AR(0)$ without changing the expansion of $L[f; s]$.

Example 1 (Stieltjes [45, p. 105]). If we let

$$g(t) = \begin{cases} \exp(-t^{1/4}) \sin(t^{1/4}) & \text{on } [0, +\infty), \\ 0 & \text{on } (-\infty, 0), \end{cases} \tag{3.3}$$

and let $M[g; z]$ be its Mellin transform for $\operatorname{Re}(z) > 0$, then $M[g, r] = 0$ for $r = 1, 2, \cdots$ through integration by parts. However $L[g; s]$ can be expanded with these $M[g; r]$ as coefficients (Handelsman and Lew [23]), so that

$$\begin{aligned} L[g; s] &\sim \sum_{m=0}^{\infty} M[g; m+1](-s)^m/m! \\ &= o(s^n) \quad \text{as } s \to 0+ \text{ for } n = 1, 2, \cdots. \end{aligned} \tag{3.4}$$

Example 2. If for any $k > 0$ we let

$$g_k(t) = \begin{cases} t^{-1/2} \cos(kt)^{1/2} & \text{on } (0 + \infty), \\ 0 & \text{on } (-\infty, 0], \end{cases} \tag{3.5}$$

then from standard tables (Abramowitz and Stegun [2, (29.3.76)]) we find

$$L[g_k; s] = (\pi/s)^{1/2} \exp(-k/4s), \tag{3.6}$$

which satisfies (3.2) for any such k.

Example 3 (Berg [6]). We plan eventually to treat problems in which f is a Mellin-expandible probability density function; hence we wish also to give an example in which g is a slowly decaying but absolutely integrable function. Thus we define $g_k(t)$ by (3.5), we let

$$g(t) = t^{-1}[c_1 g_1(t-1) + c_2 g_2(t-1)] \quad \text{on } (-\infty, +\infty), \tag{3.7}$$

and we choose nontrivial constants c_1, c_2 such that

$$c_1 \int_0^\infty \exp(-s) L[g_1; s]\, ds + c_2 \int_0^\infty \exp(-s) L[g_2; s]\, ds = 0. \tag{3.8}$$

Then $g(t) = 0$ on $[0, 1]$ and $g(t) = O(t^{-3/2})$ near $+\infty$, so that g is in $L^1(R(0))$, while

$$\begin{aligned} L[g; s] &= \int_s^\infty \exp(-z) L[c_1 g_1 + c_2 g_2; z]\, dz \\ &= -\int_0^s \exp(-z) L[c_1 g_1 + c_2 g_2; z]\, dz, \end{aligned} \tag{3.9}$$

which satisfies (3.2) by virtue of (3.6).

The function of Example 1 has an exponentially decaying oscillation and thus has a vanishing Mellin expansion near $+\infty$, whereas the functions of Examples 2 and 3 have algebraically decaying oscillations and thus have *no* Mellin expansion near $+\infty$. To recover a series for $f(t)$ near $+\infty$ from a series for $L[f; s]$ near $0+$, we must therefore insure that $f(t)$ near $+\infty$ contains no slowly decaying oscillation, whence we shall simply require that $f(t)$ near $+\infty$ possess a Mellin expansion (3.1). This straightforward condition is clearly necessary for the result, and is shown sufficient by Theorem 1. The proof involves two preliminary lemmas, of which the second is a simple remark leading to a partial summation for Mellin series, and doubtless has much stronger forms corresponding to the related theorems for power series (Ritt [41], Carleman [10, Chap. 5], Franklin [20]).

LEMMA 1. *If f is in $MAR(0)$ with expansion* (3.1), *and if* $L[f; s] \sim \sum_{r=0}^\infty b_r s^r$ *near* $0+$ *for some complex* b_r, *then* (3.1) *is identically zero.*

Proof. Choosing any fixed $t_0 > 0$ we define

$$\begin{aligned} f_0(t) &= \sum_{-1 < \mathrm{Re}[a(m)]} a_{mn} t^{a(m)} (\log t)^n \quad \text{on } (0, +\infty), \qquad f_0(0) = 0, \\ f_1(t) &= \begin{cases} f(t) - f_0(t) & \text{on } [0, t_0), \\ 0 & \text{on } [t_0, +\infty), \end{cases} \\ f_2(t) &= f(t) - f_0(t) - f_1(t) \quad \text{on } [0, +\infty). \end{aligned} \tag{3.10}$$

The Laplace transforms of these f_i can be evaluated explicitly or expanded analytically (Handelsman and Lew [23]) as

$$
\begin{aligned}
&L[f_0;s] = \sum_{-1<\mathrm{Re}[a(m)]} a_{mn}\left[\frac{\partial}{\partial a(m)}\right]^n \Gamma(1+a(m))s^{-1-1-a(m)},\\
(3.11)\qquad &L[f_1;s] \sim \sum_{m=0}^{\infty} M[f_1;m+1](-s)^m/m! \quad \text{as } s\to 0+,\\
&L[f_2;s] \sim \sum \mathrm{Res}\,\{-M[f_2;z]\Gamma(1-z)s^{z-1}\} \quad \text{as } s\to 0+.
\end{aligned}
$$

The exponents of s in the series (3.11) have respectively negative real parts, nonnegative integer values, and nonnegative real parts; the exponents of $\log s$ in these Mellin series are always zero for $L[f_1;s]$ and nonnegative integers for the others.

By linearity the sum of all series (3.11) must equal $\sum_{r=0}^{\infty} b_r s^r$, so that $a_{mn} = 0$ whenever $\mathrm{Re}\,[a(m)] > -1$. If the expansion (3.1) is not identically zero then it has a nonzero term $a_{mn}t^a(\log t)^n$ of highest order, in which $\mathrm{Re}\,(a) \leqq -1$ by the preceding remark. However by our previous results, if a is not an integer, then in $L[f_2;s]$ this yields a sum

$$(3.12)\qquad (-1)^n a_{mn}s^{-1-a}\sum_{j=0}^{n}\binom{n}{j}(\log s)^j\left(\frac{d}{dz}\right)^{n-j}[\Gamma(z)]_{z=1+a};$$

while if a is an integer then in $L[f_2;s]$ this yields a sum

$$(3.13)\quad (-1)^n n!\,a_{mn}s^{-1-a}\sum_{j=0}^{n+1}\binom{n+1}{j}(\log s)^j\left(\frac{d}{dz}\right)^{n+1-j}[\pi z/\sin \pi z\cdot\Gamma(z-a)]_{z=0}.$$

The term of highest order in either (3.12) or (3.13) can be cancelled neither by any term of $\sum_{r=0}^{\infty} b_r s^r$ nor by any term of $L[f_1;s]$; so that $a_{mn} = 0$, a contradiction.

LEMMA 2. *If f is in $MAR(0)$ with expansion* (3.1), *then for any real p there exists a finite sum*

$$(3.14)\qquad f_p(t) = \exp(-t^{-1/2})\sum_{-p\leqq \mathrm{Re}[b(m)]} b_{mn}t^{b(m)}(\log t)^n$$

such that $f(t) - f_p(t) = o(t^{-p})$ near $+\infty$ and $L[f-f_p;s] =$ (polynomial in s) $+\, o(s^{p-1})$ near $0+$. Also for any $\theta < \pi$, f_p has a Mellin expansion near $+\infty$ and $f_p(t) = o(t^{q-1})$ near 0 for all real q, both uniformly in $|\arg t| \leqq \theta$, while $L[f_p;s]$ has a Mellin expansion near 0 and $L[f_p;s] = o(s^{-q})$ near ∞ for all real q, both uniformly in $|\arg s| \leqq \theta + \pi/2$.

Proof. By (3.11)–(3.13), logarithmic terms and nonintegral exponents in the expansion of $L[f;s]$ can arise only from terms $a_{mn}t^{a(m)}(\log t)^n$ in the expansion of $f(t)$, and indeed the set of terms involving s^{-1-a} can arise only from the corresponding set involving t^a. Taking $\{b(m)\} = \{a(m) - l/2 : l, m = 0, 1, 2, \cdots\}$ we can thus choose $\{b_{mn}\}$ by induction so that $f - f_p$ satisfies the required estimate, and we can then verify by the preceding remark that $L[f - f_p;s]$ has the required form. The remaining properties of f_p are obvious by construction, while those of $L[f_p;s]$ are obtained by superposition from the properties of $h(t) = \exp(-t^{-1/2})t^b(\log t)^n$.

Our previous work (Handelsman and Lew [23]) yields

$$
\begin{aligned}
M[h;z] &= 2\left(\frac{\partial}{\partial b}\right)^n \Gamma(-2b-2z),\\
L[h;s] &= (2\pi i)^{-1}\int_{c-i\infty}^{c+\infty} s^{z-1}\Gamma(1-z)M[h;z]\,dz
\end{aligned}
\tag{3.15}
$$

for $\operatorname{Re}(c) < b$, and Stirling's approximation implies $M[h;x+iy] = o[\exp(-\theta|y|)]$ for any $\theta < \pi$. By shifting the contour we thus get an expansion of $L[h;s]$ near either 0 or ∞, with a remainder estimate valid in $|\arg s| \leqq \theta + \pi/2$.

THEOREM 1. *If f is any element of $MAR(0)$, then the Mellin expansion for $f(t)$ near $+\infty$ is determined uniquely and linearly by the Mellin expansion for $L[f;s]$ near $0+$. If $C(\theta)$ is the contour which runs in from $\infty \exp(-i\theta)$, circles counterclockwise about the origin, and runs out to $\infty \exp(i\theta)$, then the series (3.1) for $f(t)$ can be recovered to any order by the following replacements, with $\pi/2 < \theta \leqq \pi$:*

$$
s^b \to (2\pi i)^{-1}\int_{C(\theta)} \exp(ts)s^b\,ds = t^{-b-1}/\Gamma(-b)
\tag{3.16}
$$

for all complex b (including $b = 0, 1, 2, \cdots$ for which $1/\Gamma(-b) = 0$);

$$
s^b(\log s)^n \to (2\pi i)^{-1}\int_{C(\theta)} \exp(ts)s^b(\log s)^n\,ds = \left(\frac{\partial}{\partial b}\right)^n t^{-b-1}/\Gamma(-b)
\tag{3.17}
$$

for all integers $n = 0, 1, 2, \cdots$ and all complex $b \neq 0, 1, 2, \cdots$;

$$
\begin{aligned}
s^b(\log s)^n &\to (2\pi i)^{-1}\int_{C(\theta)} \exp(ts)s^b(\log s)^n\,ds\\
&= (-1)^{b+1}\sum_{j=0}^{(n-1)/2}(-1)^j\pi^{2j}\binom{n}{2j+1}\left(\frac{\partial}{\partial b}\right)^{n-2j-1}\Gamma(b+1)t^{-b-1}
\end{aligned}
\tag{3.18}
$$

for all integers $n, b = 0, 1, 2, \cdots$.

Proof. The integral in (3.17) is evaluated by differentiation from the integral in (3.16), which is evaluated by the inversion (2.14) when $\operatorname{Re}(b) < 0$ and by deforming the contour into $C(\pi)$ when $\operatorname{Re}(b) \geqq 0$. The circle about 0 contributes nothing to the integral when $\operatorname{Re}(b) \geqq 0$. Thus the contour in (3.18) is deformed into $C(\pi)$, and the two integrals along rays are combined into

$$
\begin{aligned}
&(2\pi i)^{-1}(-1)^{b+1}\int_0^\infty \exp(-tu)u^b[(\log u + i\pi)^n - (\log u - i\pi)^n]\,du\\
&= (-1)^{b+1}\sum_{j=0}^{(n-1)/2}(-1)^j\pi^{2j}\binom{n}{2j+1}\int_0^\infty \exp(-tu)u^b(\log u)^{n-2j-1}\,du.
\end{aligned}
\tag{3.19}
$$

If g is in $MAR(0)$ then $f - g$ is in $MAR(0)$, and if $L[g;s]$ near $0+$ has the same expansion as $L[f;s]$, then $f(t) - g(t)$ near $+\infty$ has zero expansion by Lemma 1. Clearly the linearity of $f \to L[f;s]$ implies the linearity of this asymptotic inversion, and the terms involving s^{-a-1} yield the terms involving t^a. However by Lemma 2 we can choose $p > 0$ and so large that we get the same expansions, to any preassigned order, for $f(t)$ and $f_p(t)$ near $+\infty$, hence for $L[f;s]$ and $L[f_p;s]$ + (polynomial in s) near $0+$. Then we can use (2.14) to recover f_p from its transform, and deform the contour to some $C(\theta)$ with $\pi/2 < \theta \leqq \pi$. Terms in the expansion of $L[f_p; s]$ up to $O(s^{p-1})$ can be inverted by (3.16)–(3.18), and the remainder in the expansion of f_p near $+\infty$ can be estimated by Watson's lemma.

4. Other inversion theorems. Asymptotic inversion by Theorem 1 for any function f in $MAR(0)$ obtains the Mellin series for f near $+\infty$ from the Mellin series for $L[f; s]$ near $0+$. We wish now to compare this result with some earlier theorems which treat the inversion integral for the Laplace transform as an integral transform in its own right:

$$f(t) = L_c^{-1}\{F; t\} = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \exp(ts)F(s)\, ds \tag{4.1}$$

with a given function $F(s)$ which is analytic in some region. The asymptotic behavior of (4.1) is known to depend primarily on the singularity s_0 lying left of the contour but having largest real part (or, in general, on all such singularities with the same largest real part). In particular, Doetsch [16, Chap. 7] proves two Abelian theorems for this inverse transform according as the contour can be deformed into a vertical line through the point s_0, or into an angle opening to the left, with vertex s_0.

These results are valid for any s_0 but can be stated specifically for $s_0 = 0$ at no loss in generality and some gain in convenience. We shall therefore assume for $F(s)$ a general Mellin expansion

$$F(s) \sim \sum_{m,n=0}^{\infty} b_{mn} s^{b(m)} (\log s)^n \quad \text{as } s \to 0+, \tag{4.2}$$

and introduce for real r the associated finite truncations

$$F_r(s) = \sum_{\mathrm{Re}[b(m)] \leqq r} b_{mn} s^{b(m)} (\log s)^n, \tag{4.3}$$

where $\{n : b_{mn} \neq 0\}$ is finite for each m and $\mathrm{Re}\,[b(m)] \uparrow +\infty$ as $m \to \infty$. In his work Doetsch treats either a single term $s^b(\log s)^n$ (Doetsch [15, pp. 500–501]) or a series (4.2) with $r \leqq 1$, so that we sketch here the required extension of his original proofs. The explicit series for $f(t)$ is determined uniquely by Theorem 1, and need not be stated in our reformulation. For contours deformable into an angle, together with slightly stronger hypotheses, Hull and Froese [26] have considered some series of form (4.2) and some singularities of more general type, while Riekstiņa [38], [39] has discussed all Mellin series with real $b(m)$ and further singularities of other types.

THEOREM 2 (compare Doetsch [16, pp. 150–154]). *For some real r and some complex $b(m), b_{mn}$, let $F(s) = F_r(s) + G(s)$ on $0 \leqq Re(s) \leqq c$, where $F(s)$ and $G(s)$ are holomorphic on $0 < \mathrm{Re}(s) \leqq c$ and $G(s)$ is continuous on $0 \leqq \mathrm{Re}(s) \leqq c$ so that $F(s)$ is singular at $s = 0$. For $s = x + iy$ and some integer $m \geqq 0$ let $(d/dy)^{m-1}G(iy)$ be absolutely continuous on $-\infty < y < +\infty$, so that $(d/dy)^m G(iy)$ is locally integrable on $-\infty < y < +\infty$. Let $F(s) \to 0$ as $s \to \infty$ uniformly in $0 \leqq \mathrm{Re}(s) \leqq c$;*

let $(d/dy)^k F(iy) \to 0$ *as* $y \to \pm\infty$ *simultaneously for* $k = 0, \cdots, m-1$; *and let the integrals*

$$\int_{-\infty}^{-u} \text{ and } \int_{u}^{\infty} dy \exp(ity)(d/dy)^m F(iy) \quad \text{for some } u > 0 \tag{4.4}$$

converge uniformly for all $t \geqq$ *some* $t_0 > 0$. *If* $f(t)$ *is defined by* (4.1), *then* $f(t)$ *is continuous for* $t \geqq t_0$, *and has the finite Mellin expansion corresponding to* $F_r(s)$ *with remainder* $o(t^{-m})$ *as* $t \to +\infty$.

Proof. Construct f_p by Lemma 2 with a Mellin expansion near $+\infty$ which agrees to $o(t^{-m-1})$ with the finite series determined by $F_r(s)$. Then $L[f_p; s] - F_r(s)$ $=$ (polynomial in s) $+ o(s^m)$ near 0, while $L[f_p; s]$ and its derivatives vanish to all $o(s^{-q})$ near ∞; so that the assumptions of this theorem are valid with all $b_{mn} = 0$ for $F(s) - L[f_p; s]$. Thus we may shift the contour to the imaginary axis and integrate m times by parts to get

$$\begin{aligned} f(t) - f_p(t) &= L_c^{-1}\{F(s) - L[f_p; s]; t\} \\ &= (-t)^{-m} \int_{-\infty}^{\infty} dy \exp(ity)(d/ds)^m \{F(s) - L[f_p; s]\}_{s=iy}. \end{aligned} \tag{4.5}$$

The last integral for $t \geqq t_0$ is the uniform limit of continuous functions, and as $t \to +\infty$, is $o(1)$ by an extension of the Riemann–Lebesgue lemma (Doetsch [15, p. 171]).

THEOREM 3 (compare Doetsch [16, pp. 159–160]). *In any sector* $|\arg s| \leqq \theta$ *with* $\pi/2 < \theta \leqq \pi$, *let* $F(s)$ *be holomorphic on some punctured neighborhood* $0 < |s| \leqq \delta$ *and locally integrable on the rays* $\arg s = \pm\theta$ *even outside this neighborhood. Let the contour* $C(\theta)$, *in the counterclockwise sense, be the union of* $\{\delta \exp(i\phi) : |\phi| \leqq \theta\}$ and $\{\rho \exp(\pm i\theta) : \delta \leqq \rho\}$. *For all* $t \geqq$ *some* t_0 *let the integral*

$$f(t) = (2\pi i)^{-1} \int_{C(\theta)} \exp(ts)F(s)\, ds \tag{4.6}$$

be well-defined. If $F(s)$ *has a Mellin expansion near* 0 *uniformly in* $|\arg s| \leqq \theta$, *then* $f(t)$ *has the Mellin expansion near* $+\infty$ *determined uniquely by Theorem* 1.

Proof. By hypothesis $F(s) = F_r(s) + o(s^r)$ near 0, with r positive and arbitrarily large. However the terms of $F_r(s)$ yield a finite Mellin series via the integrals (3.16)–(3.18), while the remainder $o(s^r)$ yields a term $o(t^{-r-1})$ near $+\infty$ by Watson's lemma. We obtain the same series by this result and by Theorem 1, since we evaluate the same integrals in both demonstrations.

As an alternative to equations (3.16)–(3.18) in the use of Theorems 1–3, we can obtain the expansion of $f(t)$ by assuming an arbitrary form (3.1), computing the series for $L[f; s]$, and equating coefficients. If the terms which involve s^b in the expansion of $L[f; s]$ have the form $s^b Q(\log s)$, where Q is a polynomial of degree n, then the corresponding terms in the expansion of $f(t)$ have the form $t^{-b-1}P(\log t)$, where P is a polynomial of degree $n-1$ when $b = 0, 1, 2, \cdots$ and is a polynomial of degree n otherwise. In particular if $b = 0, 1, 2, \cdots$ and Q is a constant, then P is identically zero, as Lemma 1 has shown.

The existence near $+\infty$ of a Mellin series for $f(t)$ is assumed in Theorem 1 but proved in Theorems 2 and 3, so that these additional results offer sufficient conditions that f be in $MAR(0)$, given that f is in $AR(0)$. However the assumptions of Theorem 1 are patently necessary, whereas the assumptions of Theorems 2 and 3 are demonstrably not. Indeed the following construction yields many functions f with Mellin series near $+\infty$ such that Theorems 2 and 3 do not apply to the corresponding $L[f; s]$. Our further development provides other conditions under which functions f lie in $MAR(0)$.

Example 4. For an arbitrary sequence $A = (a_1, a_2, \cdots)$ of complex numbers and any increasing sequence $T = (t_1, t_2, \cdots)$ of positive numbers, we may define $A(T; s)$ to be the Dirichlet series

$$A(T; s) = \sum_{i=1}^{\infty} a_i \exp(-st_i) \tag{4.7}$$

whenever and wherever it converges. It follows from standard theorems (e.g., Widder [47, pp. 44–45]) that $A(T; s)$ has a maximal half-plane of convergence $\operatorname{Re}(s) > \sigma_c$, within which it is holomorphic, and has a maximal half-plane of analyticity $\operatorname{Re}(s) > \sigma_h$ into which it may be continued, where $-\infty \leqq \sigma_h \leqq \sigma_c \leqq +\infty$. Moreover if

$$\sum_{i=1}^{\infty} \frac{1}{t_i} < +\infty, \tag{4.8}$$

then $\operatorname{Re}(s) = \sigma_h$ is a natural boundary for $A(T; s)$ by a theorem of Bernstein ([7, pp. 139–141], Schwartz [42, p. 63]).

Now let T be any increasing sequence which satisfies (4.8), and let A be any positive sequence with $\{a_i(t_i)^r : i = 1, 2, \cdots\}$ bounded, $\{a_i \exp(rt_i) : i = 1, 2, \cdots\}$ unbounded, for all $r > 0$; for example take $a_i = \exp(-t_i^{1/2})$. Also on $-\infty < t < +\infty$ let $q(t)$ be the characteristic function of $[0, 1)$ and let

$$g(t) = \sum_{i=1}^{\infty} a_i q(t - t_i). \tag{4.9}$$

Then $g(t)$ vanishes on $(-\infty, 0]$ and converges on $(0, +\infty)$, since no bounded interval, by (4.8), can contain an infinite number of t_i. Indeed $g(t)$ is bounded on $(-\infty, +\infty)$ and $g(t) = o(t^{-r})$ near $+\infty$ for all $r > 0$, so that $M[g; z]$ converges absolutely for all complex z and $L[g; s]$ converges absolutely for $\operatorname{Re}(s) \geqq 0$. Thus $L[g; s]$ can be expanded by moments to yield

$$L[g; s] \sim \sum_{n=0}^{\infty} M[g; n+1](-s)^n/n! \quad \text{as } s \to 0+. \tag{4.10}$$

By construction $g(t)$ has a Mellin expansion near $+\infty$ which is identically zero, and thus can be recovered from (4.10) by either Lemma 1 or Theorem 1 but cannot, as we shall see, be recovered by either Theorem 2 or Theorem 3. Indeed if Theorem 2 were valid in this situation, then $g(t)$ would be continuous near $+\infty$, whereas this is false by construction. Moreover $L(g; s]$ diverges for $s < 0$ so that, by a standard theorem (Widder [47, pp. 58–59]), it has a singularity at the origin, and

$$L[g; s] = s^{-1}[\exp(s) - 1]A(T; s) \tag{4.11}$$

so that, by the cited theorem of Bernstein, it has a natural boundary on the imaginary axis. Clearly Theorem 3 cannot apply in this situation. Finally if f is in $MAR(0)$ and c is any nonzero complex number, then $f + cg$ is in $MAR(0)$ with the same Mellin series near $+\infty$; but if $L[f; s]$ can be treated by either Theorem 2 or Theorem 3, then $L[f + cg; s]$ cannot be so treated.

Remark. Let U be a region in the complex s plane which has the origin on its (piecewise smooth) boundary, and let $F(s)$ be a C^∞-function on U which has an asymptotic power series at the origin. Then $F(s)$ is uniquely determined, through some theorems on quasianalytic functions (Carleman [10, Chap. 5], Mandelbrojt [32], Davis [13]), whenever the remainders for this series, or the derivatives of $F(s)$, are bounded in a suitable norm by a related sequence of constants. Similarly one might try to find bounds on analytic functions which would uniquely determine $F(s) = L[f; s]$. However any such result would then uniquely specify f, whereas we wish not to distinguish between f and $f + g$, where g is any function in $MAR(0)$ with vanishing Mellin series near $+\infty$, for instance, the functions of Examples 1 and 4.

5. Auxiliary functions. Our next goal is to prove $MAR(0)$ closed under convolution, so that if f is a $*$-polynomial in elements of $MAR(0)$, then necessarily f is also an element of this space, and Theorem 1 can be used, even when Theorems 2 and 3 fail, to obtain the Mellin series for f near $+\infty$ from the Mellin series for $L[f; s]$ near $0+$. However we postpone to § 6 the study of general convolutions, and introduce first a set of auxiliary functions $h_{a,m}$ whose form is motivated by the expansion (3.1) and whose convolutions are described by Theorem 4. Previous authors (Brennan, Reed, and Sollfrey [9], Blum [8]) have treated some convolutions of this type, but their results apparently fail when $a = 0, -1, -2, \cdots$, so that we give an independent analysis of this problem.

Our result involves an application of Theorem 1, but the preliminaries require some identities for the incomplete beta function $B(a, b; s)$ including several well-known expressions (Abramowitz and Stegun [2, § 6.6]) and a nontrivial Barnes integral representation. However in order to use these relations over the full complex range of a, b, and s, we define $B(a, b; s)$ in terms of the hypergeometric function and sketch proofs of these identities in the next lemma. In particular, for all complex s in the set

$$C^* = C - (-\infty, 0] - [1, +\infty), \tag{5.1}$$

all complex b, and all complex $a \neq 0, -1, -2, \cdots$, we define

$$B(a, b; s) = a^{-1}s^a F(1 - b, a, 1 + a; s), \tag{5.2}$$

where $F(a, b, c; z)$ denotes the hypergeometric function. Also for any complex number a and any nonnegative integer m, we let

$$\begin{aligned} h_{a,m}(t) &= \begin{cases} 0 & \text{on } [0, 1), \\ t^{a-1}(\log t)^m & \text{on } [1, +\infty), \end{cases} \\ H_{a,m}(s) &= L[h_{a,m}; s] \quad \text{on } \operatorname{Re}(s) > 0. \end{aligned} \tag{5.3}$$

LEMMA 3. *The incomplete beta function satisfies the following identities*:

(i) $B(a, b; s) = a^{-1}s^a(1 - s)^b F(1, a + b, 1 + a; s)$;

(ii) $B(a, b; s) = \int_0^s u^{a-1}(1 - u)^{b-1}\, du$ *for* $\operatorname{Re}(a) > 0$;

(iii) $B(a, b; s) + B(b, a; 1 - s) = B(a, b; 1) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ *for* $a, b \neq 0, -1, -2, \cdots$;

(iv) $B(b, a; 1 - s) = \dfrac{i(1 - s)^b}{2\Gamma(1 - a)\Gamma(a + b)} \displaystyle\int_{-c-i\infty}^{-c+i\infty} \frac{\Gamma(-a - z)\Gamma(a + b + z)}{\sin \pi z} s^{a+z}\, dz$

for $\operatorname{Re}(-b) < \operatorname{Re}(a) < 0$, *where c is any positive number* $< 1, \operatorname{Re}(a + b)$.

Proof. We may obtain (i) from a standard identity for hypergeometric functions (Abramowitz and Stegun [2, (15.3.3)]), verify (ii) for $0 < s < 1$ by expanding $(1 - u)^{b-1}$ and integrating term by term, then extend (ii) to all other s in C^* by analytic continuation. By a simple manipulation (iii) now follows from (ii) for $\operatorname{Re}(a) > 0$, $\operatorname{Re}(b) > 0$, and by analytic continuation it extends to all other $a, b \neq 0, -1, -2, \cdots$. Thus by (i) and (iii) we may write

$$B(b, a; 1 - s) = B(a, b; 1) - a^{-1}s^a(1 - s)^b F(1, a + b, 1 + a; s) \tag{5.4}$$

for $a, b \neq 0, -1, -2, \cdots$; whence to establish (iv) for all stated values of a, b, and s we need only prove

$$\begin{aligned} &B(a, b; 1)(1 - s)^{-b} - a^{-1}s^a F(1, a + b, 1 + a; s) \\ &\quad = [-2i\Gamma(1 - a)\Gamma(a + b)]^{-1} \int_{-c-i\infty}^{-c+\infty} \Gamma(-a - z)\Gamma(a + b + z)\csc \pi z \cdot s^{a+z}\, dz \end{aligned} \tag{5.5}$$

for $0 < s < 1$ and for *nonintegral* a with $\operatorname{Re}(-b) < \operatorname{Re}(a) < 0$.

As a function of $z = x + iy$ with any fixed x, the integrand of (5.5) is $O(|y|^{b-1} e^{-2\pi|y|})$ as $y \to \pm\infty$ (Abramowitz and Stegun [2, (6.1.45)]); so that the contour of integration may be shifted rightward as far as we please, and the integral in (5.5) may be evaluated as a sum of residues. Indeed the remainder estimates go to zero, so that the resulting series converges to this integral. But under the stated restrictions on a and b the singular points of the integrand to the right of $-c$ are simple poles at $z = -a, -a + 1, -a + 2, \cdots$ and at $z = 0, 1, 2, \cdots$. By computation we find that the terms corresponding to these two sets of poles yield respectively the standard expansions of $B(a, b; 1)(1 - s)^{-b}$ and $-a^{-1}s^a F(1, a + b, 1 + a; s)$.

LEMMA 4. *For all complex numbers a, b and all nonnegative integers m, n, the function* $h_{a,m} * h_{b,n}$ *is in* $MAR(2)$.

Proof. This convolution must clearly vanish on $[0, 2)$ and be locally integrable on $[2, +\infty)$, so that we need only check the form of its asymptotic expansion. However by a simple manipulation,

$$t[h_{a,m} * h_{b,n}](t) = [h_{a+1,m} * h_{b,n}](t) + [h_{a,m} * h_{b+1,n}](t) \tag{5.6}$$

so that by induction, for $q = 0, 1, 2, \cdots$,

$$[h_{a,m} * h_{b,n}](t) = t^{-q} \sum_{p=0}^{q} \binom{q}{p} [h_{a+p,m} * h_{b+q-p,n}](t). \tag{5.7}$$

Thus, through the commutativity of $*$ and repeated use of (5.6) we can express the given $h_{a,m} * h_{b,n}$ as a linear combination of such convolutions with $\mathrm{Re}(-b) < \mathrm{Re}(a) < 0$.

However for all complex a and b, by differentiating under the integral,

$$h_{a,m} * h_{b,n} = \left(\frac{\partial}{\partial a}\right)^m \left(\frac{\partial}{\partial b}\right)^n [h_{a,0} * h_{b,0}], \tag{5.8}$$

and for $\mathrm{Re}(-b) < \mathrm{Re}(a) < 0$, with $s = t^{-1} < 1/2$,

$$\begin{aligned} [h_{a,0} * h_{b,0}](t) &= t^{a+b-1} \int_s^{1-s} (1-v)^{a-1} v^{b-1}\, dv \\ &= t^{a+b-1}[B(b,a;1-s) - B(b,a;s)]. \end{aligned} \tag{5.9}$$

Thus for all $t > 2$, by Lemma 3.4,

$$\begin{aligned} [h_{a,m} * h_{b,n}](t) = \left(\frac{\partial}{\partial a}\right)^m \left(\frac{\partial}{\partial b}\right)^n \Bigg\{ & -b^{-1} t^{a-1} F(1-a, b, 1+b; t^{-1}) \\ & + \frac{i(1-t^{-1})^b}{2\Gamma(1-a)\Gamma(a+b)} \int_{-c-i\infty}^{-c+i\infty} \frac{\Gamma(-a-z)\Gamma(a+b+z)}{\sin \pi z} t^{b-z-1}\, dz \Bigg\}. \end{aligned} \tag{5.10}$$

In the first term on the right side of (5.10) we may expand in t^{-1} and differentiate term by term, since the hypergeometric function is absolutely convergent for $|t^{-1}| < 1$. In the second term on the right side of (5.10) we may differentiate under the integral and again shift the contour rightward, since the integrand remains exponentially decreasing along vertical lines. However $(\partial/\partial a)^m t^a = t^a (\log t)^m$, $(\partial/\partial b)^n t^b = t^b (\log t)^n$, and $(\partial/\partial a)^m \Gamma(-a-z)$ is a meromorphic function with poles of order $m+1$ at $z = -a, -a+1, -a+2, \cdots$, so that the resulting series has the form (3.1) as $t \to +\infty$.

LEMMA 5. *The transform $H_{a,m}(s)$ of* (5.3) *can be analytically continued to a holomorphic function on the Riemann surface of* $\log s$. *Near* 0, *uniformly in* $|\arg s| \leqq \pi/2$,

$$\begin{aligned} H_{a,m}(s) \sim & \sum_{n=0}^{\infty} m!(-s)^n / n!(-a-n)^{m+1} \\ & + s^{-a} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{\partial}{\partial a}\right)^{m-k} \Gamma(a)(-\log s)^k \end{aligned} \tag{5.11}$$

for all $a \neq 0, -1, -2, \cdots$, *and*

$$\begin{aligned} H_{a,m}(s) \sim & \sum_{a \neq n = 0}^{\infty} m!(-s)^n / n!(-a-n)^{m+1} \\ & + s^{-a} \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{(-1)^{m+1}(\log s)^k}{(m+1)\cdot(-a)!} \left(\frac{d}{dz}\right)^{m+1-k} \left\{ \frac{\pi z \cdot \Gamma(1-a)}{\sin \pi z \cdot \Gamma(1-a+z)} \right\}_{z=0} \end{aligned} \tag{5.12}$$

for all $a = 0, -1, -2, \cdots$. *Near* ∞, *uniformly in* $|\arg s| \leqq \theta$,

$$H_{a,m}(s) \sim \exp(-s) \cdot s^{-m-1} \sum_{r=0}^{\infty} c(a,m,r) s^{-r} \tag{5.13}$$

for any $\theta < 3\pi/2$ *and some real constants* $c(a, m, r)$.

Proof. The Laplace transform of $h_{a,m}$ is continuable to any $s \neq 0$ via

$$H_{a,m}(s) = s^{-a} \int_s^{+\infty} \exp(-u) u^{a-1} (\log u - \log s)^m \, du. \tag{5.14}$$

From the Mellin transform of $h_{a,m}$ we get (5.11) and (5.12) by shifting the contour to the right in

$$H_{a,m}(s) = (2\pi i)^{-1} \Gamma(m+1) \int_{c-i\infty}^{c+\infty} s^{z-1} (1-a-z)^{-m-1} \Gamma(1-z) \, dz. \tag{5.15}$$

Indeed the remainder has form (5.15) to any order with $\Gamma(1-z) = O[e^{-\pi|y|/2}|y|^{-\mathrm{Re}(c)+1/2}]$ as $y \to \pm\infty$ (Abramowitz and Stegun [2, (6.1.45)]) whence this remainder is $O(s^{c-1})$ for $|\arg s| \leqq \pi/2$. Moreover

$$H_{a,m}(s) = \exp(-s) \int_0^{\infty} \exp(-st)(1+t)^{a-1} [\log(1+t)]^m \, dt, \tag{5.16}$$

which has the desired expansion near ∞ by the complex form of Watson's lemma (e.g., Doetsch [16, p. 48]).

THEOREM 4. *For all complex numbers a, b and all nonnegative integers m, n, the function $[h_{a,m} * h_{b,n}](t)$ has an asymptotic expansion near $+\infty$ of form*

$$t^{a+b-1} P(\log t) + t^{a-1} \sum_{j=0}^{m} f_j(t^{-1})(\log t)^j + t^{b-1} \sum_{k=0}^{n} g_k(t^{-1})(\log t)^k. \tag{5.17}$$

In (5.17) *the expressions $f_j(t^{-1})$ and $g_k(t^{-1})$ are asymptotic power series in t^{-1}, and the expression $P(\log t)$ is a polynomial in $\log t$, of degree $m+n$ when both $a, b \neq 0, -1, -2, \cdots$, and of degree $m+n+1$ otherwise.*

Proof. If $a, b \neq 0, -1, -2, \cdots$, then this follows from (5.8) and the identity

$$\begin{aligned} [h_{a,0} * h_{b,0}](t) = {} & t^{a+b-1} B(a,b,1) - a^{-1} t^{b-1} F(1-b, a, 1+a; t^{-1}) \\ & - b^{-1} t^{a-1} F(1-a, 1+b; t^{-1}) \quad \text{for } 2 < t, \end{aligned} \tag{5.18}$$

which in turn follows from (5.9) and Lemma 3.3 by analytic continuation. However for the remaining values of a and b it is tedious to argue from (5.10), whereas by Theorem 1, having just proved that $h_{a,m} * h_{b,n}$ is in $MAR(0)$, we need only show that $L[h_{a,m}; s] L[h_{b,n}; s]$ has a Mellin series near $0+$ corresponding to (5.17). However by Lemma 5,

$$L[h_{a,m}; s] \sim s^{-a} Q_a(\log s) + S_a(s) \quad \text{as } s \to 0+, \tag{5.19}$$

where $S_a(s)$ is an asymptotic power series in s, and $Q_a(s)$ is a polynomial in $\log s$, of degree m if $a \neq 0, -1, -2, \cdots$ and of degree $m+1$ otherwise. Since $L[h_{b,n}; s]$ has a similar expansion, we find

$$L[h_{a,m} * h_{b,n}; s] \sim s^{-a-b} Q_a Q_b + s^{-a} Q_a S_b + s^{-b} Q_b S_a + S_a S_b, \tag{5.20}$$

which is, for all a and b, a form corresponding to (5.17).

By more detailed study of (5.10), of the hypergeometric equation, or of some other representation for $h_{a,m} * h_{b,n}$ (Blum [8]), we can show that the series f_j and g_k converge for $|t| > 1$, but we shall not need this additional information.

6. Closure under convolution. In the preceding sections we have introduced various convolution algebras of functions defined on $[0, +\infty)$, and have distinguished the corresponding subspaces of functions with Mellin expansions near $+\infty$. Also we have introduced the auxiliary functions $h_{a,m}$ in $MAR(1)$ and proved that their convolutions lie in $MAR(2)$. We proceed now to show that the space $MAR(0)$ is closed under convolution, hence that various subspaces of $MAR(0)$ are convolution algebras. Our argument relies upon estimates of many remainders, hence begins with a lemma on finite expansions. Also we shall use without further comment the following immediate consequence of Theorem 4: if f and g are two complex-valued, locally bounded, locally integrable functions on $[0, +\infty)$, both vanishing on some nonvoid interval $[0, t_0)$, and if $f = O(h_{a,m})$, $g = O(h_{b,n})$ near $+\infty$, then $f * g$ is $O(h_{a+b,m+n+1} + h_{a,m} + h_{b,n})$ near $+\infty$. To prove this we need only change the t-scale so that $t_0 \geqq 1$, and observe that $f * g$ is dominated by some multiple of $h_{a,m} * h_{b,n}$.

In a closely related investigation, Riekstiņš [40] considers two locally integrable functions with asymptotic series near $+\infty$ in certain related powers of t, and expands their convolution in a corresponding series near $+\infty$, with a possible factor $\log(t/2)$. He calculates the leading term of his result explicitly for functions assumed of this type, but claims the validity of his approach essentially for all functions with Mellin series. His method involves splitting the integral into more tractable parts, treating each by an iterative method (e.g., Tihonov and Samarskii [44], Millar [35]), and regrouping terms into a final series. By comparison, we use such direct calculations, as in the next lemma, only to show the existence of a Mellin expansion, and invoke Laplace transforms, as in Theorems 1 and 4, to obtain systematically the successive terms of the series.

LEMMA 6. *Let F be a complex-valued function on $[0, +\infty)$ which is of bounded variation on each finite interval, is absolutely continuous on some nonvoid $[t_0, +\infty)$, and satisfies $dF/dt = O(h_{b,0})$ near $+\infty$ for some $b < 0$. Then for any complex number a and nonnegative integer m the function*

$$h(t) = \int_0^t h_{a,m}(t-u)\,dF(u) \tag{6.1}$$

is defined for $t > t_0 + 1$ and can be expanded near $+\infty$ in a finite series of form

$$h(t) = \sum_{j=0}^{r-1}\sum_{k=0}^{m} b_{jk}t^{a-j-1}(\log t)^k + R_r(t) \tag{6.2}$$

for $b + r < 0$, where R_r is $O(h_{a-r,m} + h_{b,m})$ near $+\infty$. In particular if $dF/dt = o(t^{-n})$ near $+\infty$ for $n = 1, 2, \cdots$, then $h(t)$ can be expanded in an infinite Mellin series.

Proof. For $|x| < 1$ we have the expansion

$$(1-x)^{a-1}[\log(1-x)]^j = \sum_{k=0}^{r-1} c_{jk}x^k + Q_{jr}(x), \tag{6.3}$$

where the c_{jk} are the coefficients given by Taylor's theorem; and for $0 \leqq x < 1$ we have the inequality

$$|x^{-r}Q_{jr}(x)| \leqq C_1 + C_2(1-x)^{a-1}|\log(1-x)|^j, \tag{6.4}$$

where C_1 and C_2 are some positive constants.

If $t > t_0 + 1$ then

$$h(t) = \int_0^{t_0} h_{a,m}(t-u)\,dF(u) + \int_{t_0}^{t} h_{a,m}(t-u)\frac{dF}{du}\,du, \tag{6.5}$$

and both parts of this integral are defined for almost all t. Without loss of generality we may therefore take $1 \leqq t_0 < t - 1$ and use (5.3) to get

$$\begin{aligned}
h(t) &= \int_0^{t-1} t^{a-1}(1 - t^{-1}u)^{a-1}[\log t + \log(1 - t^{-1}u)]^m\,dF(u) \\
&= \sum_{j=0}^{m}\binom{m}{j} t^{a-1}(\log t)^{m-j}\int_0^{t-1}(1 - t^{-1}u)^{a-1}[\log(1 - t^{-1}u)]^j\,dF(u) \\
&= \sum_{j=0}^{m}\binom{m}{j} t^{a-1}(\log t)^{m-j}\Bigg\{\sum_{k=0}^{r-1} c_{jk}t^{-k}\left[\int_0^{\infty} - \int_{t-1}^{\infty}\right]u^k\,dF(u) \\
&\qquad\qquad + \int_0^{t-1} Q_{jr}(t^{-1}u)\,dF(u)\Bigg\}.
\end{aligned} \tag{6.6}$$

The integrals $\int_0^\infty u^k\,dF(u)$ in (6.6) are convergent since $k + b < 0$, and thus yield a finite series as required; the integrals $\int_{t-1}^\infty u^k\,dF(u)$ and the functions $Q_{jr}(t^{-1}u)$ yield remainder terms, and must therefore be shown small enough.

Since $b < -r$, a typical remainder term of the first kind is

$$\begin{aligned}
c_{jk}\binom{m}{j} t^{a-k-1}(\log t)^{m-j}\int_{t-1}^{\infty} u^k\,dF(u) &= t^{a-k-1}(\log t)^{m-j}O\left(\int_{t-1}^{\infty} u^{b+k-1}\,du\right) \\
&= O(h_{a+b,m-j}) = O(h_{a-r,m}) \quad \text{as } t \to +\infty.
\end{aligned} \tag{6.7}$$

By (6.4) a typical remainder term of the second kind is

$$\begin{aligned}
\binom{m}{j} t^{a-1}(\log t)^{m-j}\int_0^{t-1} Q_{jr}(t^{-1}u)\,dF(u) &= t^{a-r-1}(\log t)^{m-j}O\left(\int_0^{t-1} u^r|dF(u)|\right) \\
&+ t^{a-r-1}(\log t)^{m-j}O\left(\int_0^{t-1}(1 - t^{-1}u)^{a-1}|\log(1 - t^{-1}u)|^j u^r\,|dF(u)|\right).
\end{aligned} \tag{6.8}$$

The first O-term in (6.8) is

$$\begin{aligned}
t^{a-r-1}(\log t)^{m-j}\left[\int_0^{t_0} + \int_{t_0}^{t-1}\right]u^r\,|dF(u)| &= t^{a-r-1}(\log t)^{m-j}O(1 + t^{b+r}) \\
&= O(h_{a-r,m-j}) \quad \text{as } t \to +\infty.
\end{aligned} \tag{6.9}$$

The second O-term in (6.8) may likewise be split into an integral from 0 to t_0, whose contribution is $O(h_{a-r,m-j})$, and an integral from t_0 to $t - 1$, whose contribution is dominated by

$$\begin{aligned}
&t^{-r}(\log t)^{m-j}\int_1^{t-1}(t-u)^{a-1}|\log(t-u) - \log t|^j u^{b+r-1}\,du \\
&\leqq \sum_{k=0}^{j}\binom{j}{k} t^{-r}(\log t)^{m-k}\int_1^{t-1}(t-u)^{a-1}[\log(t-u)]^k u^{b+r-1}\,du.
\end{aligned} \tag{6.10}$$

However a typical term in (6.10) is

$$\begin{aligned} &\binom{j}{k} t^{-r}(\log t)^{m-k}[h_{a,k} * h_{b+r,0}](t) \\ (6.11) \qquad &= t^{-r}(\log t)^{m-k} O(h_{a+b+r,k+1} + h_{a,k} + h_{b+r,0}) \\ &= O(h_{a-r,m} + h_{b,m-k}) \quad \text{as } t \to +\infty. \end{aligned}$$

Collecting (6.7), (6.9), and (6.11), we verify the stated estimate for $R_r(t)$.

THEOREM 5. *The following sets are commutative complex algebras (in fact, integral domains) under pointwise addition, scalar multiplication, and Laplace convolution*:

(i) $MAR(0+)$,

(ii) $MAR'(\infty-) = AR'(\infty-)$,

(iii) $MAR(t)$ *for* $0 \leqq t < +\infty$,

(iv) $ML^1(R(t))$ *for* $0 \leqq t < +\infty$,

(v) $ML^1(R(t)) \cap ML^p(R(t))$ *for* $1 \leqq p \leqq +\infty, 0 \leqq t < +\infty$.

Proof. By definition, $MAR'(\infty-) = AR'(\infty-)$ and by (2.3), $AR'(\infty-)$ is a complex algebra. We recall that the intersection of two algebras is an algebra, and that $AR(0)$, hence any subalgebra, is an integral domain; and we note that each of the other given sets is the intersection of $MAR(0)$ with the set obtained by deleting M. In § 2 we found that the latter sets are all complex algebras, whence now we need only show that $MAR(0)$ is such an algebra.

Since $MAR(0)$ is a complex vector space, we need only show that it contains h^{*2} whenever it contains h. However we can decompose $h = g + f$ with g in $AR'(\infty-)$, f locally bounded, and f in $MAR(t_0)$ for some $t_0 \geqq 1$. By (2.2) and (2.3),

$$g^{*2} \in AR'(\infty-) \subset MAR(0), \tag{6.12}$$

while $g * f = f * g$ and f^{*2} are in $AR(0)$. Hence we need only verify to any order that the last two convolutions have Mellin expansions near $+\infty$.

If the Mellin expansion of f has the general form (3.1), then

$$f(t) = \sum_{c < \mathrm{Re}[a(m)]} a_{mn} h_{a(m),n} + O(h_{c,p}(t)) \quad \text{as } t \to +\infty \tag{6.13}$$

for any real number c and some nonnegative integer p. Thus $g * f$ is a finite linear combination of terms $g * h_{a,m}$, each of which is in $MAR(1)$ by Lemma 6, plus a term $g * O(h_{c,p})$, which is $O(h_{c,p})$ by Lemma 6. Also f^{*2} is a finite linear combination of terms $h_{a,m} * h_{b,n}$, each of which is in $MAR(2)$ by Lemma 4, plus a finite linear combination of terms $h_{a,m} * O(h_{c,p})$ and $O(h_{c,p}) * O(h_{c,p})$, each of which, for any $\varepsilon > 0$, is a finite Mellin series $+ O(h_{c-\varepsilon,0})$ by Lemma 6. However c can be chosen arbitrarily near $-\infty$, so that h^{*2} is in $MAR(0)$ as required.

If the function f, for some complex c, is an element of $E(c)L^1(R(0))$, and if $P(z)$ is a complex polynomial without constant term, then (2.5) yields an element $P^*(f)$ in $E(c)L^1(R(0))$ and (2.13) shows that

$$L[P^*(f); s] = P(L[f; s]). \tag{6.14}$$

Moreover if f is in one of the algebras treated by Theorem 5 then $P^*(f)$ is in this same subalgebra of $MAR(0)$, so that $P^*(f)$ has a Mellin expansion near $+\infty$. However this expansion follows by Theorem 1 from the Mellin series for $L[P^*(f); s]$ near $0+$, which follows by (6.14) from the Mellin series for $L[f; s]$ near $0+$, which follows by our previous work (Handelsman and Lew [23]) from the Mellin series for f near $+\infty$. Thus Theorems 1 and 5 enable us to find the series for $P^*(f)$ to any order from the series for f to a corresponding order, via straightforward calculation with approximate transforms. The examples of § 3 show that such calculations are unreliable without the rigorous foundation of these theorems.

7. Applications to probability. Our results find immediate use in the theory of probability, specifically in the study of random sums. By a simple argument (Feller [19, p. 7]), if the independent random variables T_1 and T_2 have respective probability densities f_1 and f_2 on $[0, +\infty)$, then the sum $T_1 + T_2$ has density $f_1 * f_2$. Hence by induction if the independent positive random variables $T_1, \cdots, T_n$ have identical probability density f on $[0, +\infty)$, then the sum $T_1 + \cdots + T_n$ has density f^{*n}. However, f_1, f_2, and f lie in $L^1(R(0))$ so that, by Theorems 1 and 5 of this paper, we can obtain the behavior of $f_1 * f_2$ near $+\infty$ given Mellin series for f_1 and f_2 near $+\infty$, or the behavior of f^{*n} near $+\infty$ given a Mellin series for f near $+\infty$. To expand an n-fold convolution by the direct method of Riekstiņš [40] would require $n - 1$ repetitions of his detailed procedure for such integrals. We introduce some notation for standard densities, and offer some examples of our method.

Hereafter g_0 will be any function in $L^1(R(0))$ which is $o(t^{-m})$ near $+\infty$ for all $m > 0$; in particular g_0 may be the function of Example 4, which is not tractable by Theorems 2 and 3. Then the moments of g_0 are absolutely convergent, and its Laplace transform near $0+$ is given by

$$L[g_0; s] \sim \sum_{n=0}^{\infty} \mu_n(-s)^n/n! \quad \text{with } \mu_n = \int_0^{\infty} t^n g_0(t)\, dt. \tag{7.1}$$

For any a, $p > 0$ the gamma density is defined by

$$g_{a,p}(t) = \exp(-at)a^p t^{p-1}/\Gamma(p) \quad \text{on } [0, +\infty), \tag{7.2}$$

and its Laplace transform on $|\arg(s + a)| < \pi$ is found to be

$$L[g_{a,p}; s] = (1 + a^{-1}s)^{-p}. \tag{7.3}$$

For any $r > 0$ the Pareto density is defined by $h_r = rh_{-r,0}$ or, from (5.3), by

$$h_r(t) = \begin{cases} 0 & \text{on } [0, 1), \\ rt^{-r-1} & \text{on } [1, +\infty), \end{cases} \tag{7.4}$$

and its Laplace transform near $0+$ is given by Lemma 5. Moreover if $\Gamma(a, z)$ is the usual incomplete gamma function and if $M(a, b, z)$ is Kummer's confluent hypergeometric function, then (Abramowitz and Stegun [2, (6.5.3), (6.5.12)])

$$L[h_r; s] = rs^r\Gamma(-r, s) = -\Gamma(1 - r)s^r + M(-r, 1 - r, -s) \quad \text{for } r \neq 1, 2, \cdots. \tag{7.5}$$

To construct some examples amenable only to Theorem 1, we define

$$k_r(t) = h_r(t) + g_0(t), \tag{7.6}$$

and we note that k_r has only a finite number of moments but has a Mellin series near $+\infty$. Near $0+$, for $r \neq 1, 2, \cdots$,

$$\begin{aligned} L[k_r; s] &= -\Gamma(1-r)s^r + G_r(s) \sim -\Gamma(1-r)s^r + \sum_{n=0}^{\infty} c_{rn}s^n, \\ c_{rn} &= (-1)^n[\mu_n + r/(r-n)]/n! \quad \text{for } n = 0, 1, 2, \cdots, \end{aligned} \tag{7.7}$$

by Lemma 5 and equation (7.1); while near $0+$, for $r = 1, 2, \cdots$,

$$\begin{aligned} L[k_r; s] &= -(-s)^r \log s/\Gamma(r) + G_r(s) \sim -(-s)^r \log s/\Gamma(r) + \sum_{n=0}^{\infty} c_{rn}s^n, \\ c_{rn} &= \text{same as (7.7) except } c_{rr} = (-1)^r[\mu_r + r\psi(r+1)]/r!, \end{aligned} \tag{7.8}$$

with $\psi(z) = (d/dz)\log\Gamma(z)$. The transformation $D(a)$ of (2.6) preserves integrals on $[0, +\infty)$ and commutes with operations in $MAR(0)$, hence changes the scale of t and generalizes our results for k_r. Our results will be obtained for this unnormalized function k_r and must be corrected by suitable factors c_{r0}^n to be valid for the probability density k_r/c_{r0}.

Example 5. If $0 < p, q$, then $g_{a,p} * g_{a,q} = g_{a,p+q}$ by (7.3), so that $g_{a,p}^{*n} = g_{a,np}$ by induction; but if $0 < b < a$ then both $g_{a,p}$ and $g_{b,q}$ are in $E(-b)MAR(0)$ by (2.6), so that

$$\exp(bt)[g_{a,p} * g_{b,q}](t) = f(t) \in MAR(0) \tag{7.9}$$

by (2.7). Moreover by (7.3) we have

$$\begin{aligned} L[f; s] &= a^p b^q s^{-q}(s + a - b)^{-p} \\ &= a^p b^q (a-b)^{-p} \sum_{n=0}^{\infty} \binom{-p}{n} (a-b)^{-n} s^{n-q}, \end{aligned} \tag{7.10}$$

whence by Theorem 1 we can recover the Mellin expansion for f near $+\infty$. On the other hand,

$$[g_{a,p} * g_{b,q}](t) = a^p b^q e^{-bt} t^{p+q-1} M(p, p+q, bt - at)/\Gamma(p+q) \tag{7.11}$$

from an integral representation of $M(a, b, z)$, so that the Mellin series for f can be found in standard compilations (Abramowitz and Stegun [2, (13.2.1), (13.5.1)]). However the method involved, after more work, yields the expansion near $+\infty$ of any $g_{a,p} * g_{b,q} * g_{c,r}$.

Example 6. The Mellin series near $+\infty$ for $h_r * h_s$ is given by (5.13) for r, s, $\neq 1, 2, \cdots$ and described by Theorem 4 for all r, s. The Mellin series for h_r^{*n} is given by Brennan, Reed and Sollfrey [9] for $r \neq 1, 2, \cdots$ and generalized to k_r^{*n} in the next example. Here we note by Theorem 5 that $g_{a,p} * k_r$ is in $ML^1(R(0))$ and by Theorem 4 that $[g_{a,p} * g_0](t) = o(t^{-m})$ near $+\infty$ for all $m > 0$, hence directly that

$g_{a,p} * k_r$ and $g_{a,p} * h_r$ have the same Mellin series near $+\infty$. If $f = g_{a,p} * h_r$, then by (7.3) the transform $L[g_{a,p}; s]$ is analytic at the origin, whence by Lemma 1 the expansion of f near $+\infty$ is determined by the function

$$\begin{aligned} &-\Gamma(1-r)s^r(1+a^{-1}s)^{-p} && \text{for } r \neq 1, 2, \cdots, \\ &-(-s)^r(1+a^{-1}s)^{-p}\log s/\Gamma(r) && \text{for } r = 1, 2, \cdots. \end{aligned} \tag{7.12}$$

In either case we can expand near $s = 0$ and invert by Theorem 1 to get

$$f(t) \sim rt^{-r-1}\sum_{n=0}^{\infty} p\cdots(p+n-1)(r+1)\cdots(r+n)(at)^{-n}/n! \quad \text{as } t \to +\infty. \tag{7.13}$$

To understand (7.13) for $r \neq 1, 2, \cdots$, we take the integral (2.14) for $f(t)$ and deform its contour to the path $C(\pi)$ which runs from $-\infty$ to $0+$ just below the real axis, then from $0+$ to $-\infty$ just above the real axis. The integrand is the product of (7.3) and (7.5), so that

$$\begin{aligned} f(t) = (2\pi i)^{-1}\Big\{\Gamma(1-r)\int_{C(\pi)} &\exp(st)s^r(1+a^{-1}s)^{-p}\,ds \\ - \int_{C(\pi)} &\exp(st)M(-r, 1-r, -s)(1+a^{-1}s)^{-p}\,ds\Big\}; \end{aligned} \tag{7.14}$$

and $M(a, b, z)$ is an entire function, so that the second integral is $O(e^{-at}t^{p-1})$ near $+\infty$. If we split the contour $C(\pi)$ at $-a$ both below and above the real axis, and recall standard integral formulas for the confluent hypergeometric functions (Abramowitz and Stegun [2, (13.2.1), (13.2.6)]), then we can evaluate the resulting four integrals for $p < 1$ and extend the resulting identity to all other p. Hence by analytic continuation the first integral in (7.14) becomes

$$\begin{aligned} &[a^{r+1}\Gamma(1-p)/\Gamma(r-p+2)][rM(r+1, r-p+2, -at) \\ &+ \Gamma(1-r)\exp(-at)U(1-p, r-p+2, at)/\Gamma(p-r-1)]. \end{aligned} \tag{7.15}$$

As $t \to +\infty$ in (7.15), the first term yields (7.13) and the second term is $O(e^{-at}t^{p-1})$, whence the first term approximates $f(t)$ within $o(t^{-m})$ for all $m > 0$, although it is not equal to $f(t)$ on $[0, 1)$, or any other interval.

Example 7. We now consider the functions k_r^{*n}, which are in $ML^1(R(0))$ by Theorem 5, and recall the functions G_r with coefficients c_{rn} which were introduced in (7.7)–(7.8). The constants $c(r, m, j)$ are defined here for convenience by

$$G_r(s)^m \sim \sum_{j=0}^{\infty} c(r, m, j)s^j \quad \text{as } s \to 0+ \tag{7.16}$$

and can be expressed simply in terms of the c_{rn}. For example if $m \geqq 3$, then

$$\begin{aligned} c(r, m, 0) &= c_{r0}^m, \\ c(r, m, 1) &= mc_{r0}^{m-1}c_{r1}, \\ c(r, m, 2) &= mc_{r0}^{m-2}[c_{r0}c_{r2} + (m-1)c_{r1}^2/2], \\ c(r, m, 3) &= mc_{r0}^{m-3}[c_{r0}^2c_{r3} + (m-1)c_{r0}c_{r1}c_{r2} + (m-1)(m-2)c_{r1}^3/6]. \end{aligned} \tag{7.17}$$

For all $r \neq 1, 2, \cdots$, by (7.7) and (3.16),

$$L[k_r^{*n}; s] = L[k_r; s]^n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \Gamma(1-r)^m s^{mr} G_r(s)^{n-m}$$

$$\sim \sum_{m=0}^{n} \sum_{j=0}^{\infty} \binom{n}{m} (-1)^m \Gamma(1-r)^m c(r, n-m, j) s^{mr+j} \quad \text{as } s \to 0+; \tag{7.18}$$

$$k_r^{*n}(t) \sim \sum_{m=0}^{n} \sum_{j=0}^{\infty} \binom{n}{m} (-1)^m \Gamma(1-r)^m \Gamma(-mr-j)^{-1} c(r, n-m, j) t^{-mr-j-1}$$

$$\text{as } t \to +\infty.$$

Some distributions of insurance claims have recently been described by densities of type k_r with $1 \leqq r \leqq 2$ (Benktander [4]). Thus we take $r = 3/2$ as a special case, and find

$$L[k_{3/2}; s]^n = G_{3/2}(s)^n + 2n\Gamma\left(\frac{1}{2}\right) s^{3/2} G_{3/2}(s)^{n-1} + 4\pi \binom{n}{2} s^3 G_{3/2}(s)^{n-2}$$

$$+ 8\pi \binom{n}{3} \Gamma\left(\frac{1}{2}\right) s^{9/2} G_{3/2}(s)^{n-3} + \cdots,$$

$$\begin{aligned} k_{3/2}^{*n}(t) = n\{&(3/2)c(3/2, n-1, 0)t^{-5/2} - (15/4)c(3/2 \cdot n-1, 1)t^{-7/2} \\ &+ (105/8)c(3/2, n-1, 2)t^{-9/2} \\ &- (945/16)[c(3/2, n-1, 3) \\ &+ (2\pi/3)(n-1)(n-2)c(3/2, \ n-3, 0]t^{-11/2} \\ &+ O(t^{-13/2})\} \quad \text{as } t \to +\infty. \end{aligned} \tag{7.19}$$

For all $r = 1, 2, \cdots$, by (7.8) and (3.18),

$$\begin{aligned} L[k_r^{*n}; s] = L[k_r; s]^n &= \sum_{m=0}^{n} \binom{n}{m} (-1)^{mr+m} \Gamma(r)^{-m} s^{mr} (\log s)^m G_r(s)^{n-m} \\ &\sim \sum_{m=0}^{n} \sum_{j=0}^{\infty} \binom{n}{m} (-1)^{mr+m} \Gamma(r)^{-m} c(r, n-m, j) s^{mr+j} (\log s)^m \end{aligned} \tag{7.20}$$

as $s \to 0+$, and k_r^{*n} has an explicit series near $+\infty$ in powers of t^{-1} and $\log t$ which is more complicated than (7.18). Thus we give the rest of this computation for the special density k_2, which lies just within the domain of attraction for the normal distribution (Feller [19, p. 544]). In this case,

$$L[k_2; s]^n = G_2(s)^n - nG_2(s)^{n-1} s^2 \log s + \binom{n}{2} G_2(s)^{n-2} s^4 (\log s)^2$$

$$- \binom{n}{3} G_2(s)^{n-3} s^6 (\log s)^3 + \cdots,$$

$$\begin{aligned} k_2^{*n} = n\{&2c(2, n-1, 0)t^{-3} - 6c(2, n-1, 1)t^{-4} \\ &+ 24[c(2, n-1, 2) + (n-1)c(2, n-2, 0)(\log t - \psi(5))]t^{-5} \\ &- 120[c(2, n-1, 3) + (n-1)c(2, n-1, 1)(\log t - \psi(6))]t^{-6} \\ &+ O[t^{-7}(\log t)^2]\} \quad \text{as } t \to +\infty. \end{aligned} \tag{7.21}$$

For an arbitrary g_0, as in Example 4, we might conjecture these expansions from Theorems 2 and 3, but can prove them only by Theorems 1 and 5. On the other hand, for $g_0 = 0$, we can use standard results (Abramowitz and Stegun [2, (6.3.5)]) to obtain the simplification

$$\begin{aligned} h_2^{*n}(t) &= 2n^{(1)}t^{-3} + 12n^{(2)}t^{-4} + [48n^{(3)} - 14n^{(2)} + 24n^{(2)}\log t]t^{-5} \\ &\quad + [160n^{(4)} - 188n^{(3)} + 40n^{(2)} + 240n^{(3)}\log t]t^{-6} \\ &\quad + O[t^{-7}(\log t)^2] \quad \text{as } t \to +\infty, \\ n^{(m)} &= n(n-1)\cdots(n-m+1). \end{aligned} \tag{7.22}$$

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AN ESTIMATE FOR THE RATE OF CONVERGENCE OF CONVOLUTION PRODUCTS OF SEQUENCES*

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Abstract. Suppose that the series $\sum_{k=0}^{\infty} p_k x^k$ has a positive radius of convergence R and suppose that the sequence of positive numbers (a_n) satisfies the condition $a_n/a_{n+1} = \lambda + O(\delta_n)$ where $0 < \lambda \leqq R$ and (δ_n) is a sequence of positive numbers converging to zero. If $0 < \lambda < R$ and if $\delta_n/\delta_{n+1} \to 1$, then

$$\sum_{k=0}^{n} p_k \frac{a_{n-k}}{a_n} = \sum_{k=0}^{\infty} p_k \lambda^k + O(\delta_n) \quad (n \to \infty).$$

If $\lambda = R$, the same result is true, but one has to assume that $\sum_{k=1}^{\infty} k^{\alpha}|p_k|R^k < \infty$ for every $\alpha > 0$ and $n\delta_n = O(1)$, $n(\delta_n/\delta_{n-1} - 1) = O(1)\,(n \to \infty)$.

1. Let (a_n) and (p_n) be two sequences. The convolution product of (a_n) and (p_n) is the sequence (c_n) defined by

$$c_n = \sum_{k=0}^{n} p_k a_{n-k}, \qquad n = 0, 1, 2, \cdots.$$

One of the simplest results in the study of convolution products can be stated as follows.

If the series $\sum_{k=0}^{\infty} p_k$ converges absolutely and if $\lim_{n\to\infty} a_n = c$, where $0 < c < \infty$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} p_k \frac{a_{n-k}}{a_n} = \sum_{k=0}^{\infty} p_k.$$

If $\sum_{k=1}^{\infty} |p_k| k^{\eta} < \infty$ for some $\eta > 0$, the same result is true if we assume, more generally, that (a_n) is a slowly varying sequence, that is,

$$\lim_{n\to\infty} \frac{a_{[\lambda n]}}{a_n} = 1, \text{ for every } \lambda > 1.$$

This result follows from a general theorem of M. Vuilleumier (see [1, Thm. 1]).

Results of this type under much weaker hypotheses about the sequence (a_n) are also known [2, Chap. IV, prob. 178]. A typical result in this direction can be stated as follows.

Suppose that the series $\sum_{k=0}^{\infty} p_k x^k$ has a positive radius of convergence R. If (a_n) is a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lambda \tag{1.1}$$

and if $0 < \lambda < R$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} p_k \frac{a_{n-k}}{a_n} = \sum_{k=0}^{\infty} p_k \lambda^k. \tag{1.2}$$

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The aim of this paper is to give an estimate of the rate of convergence of the convolution products assuming that we have information about the rate of convergence of the sequence (a_n/a_{n+1}). We shall assume here, instead of (1.1), that

$$\frac{a_n}{a_{n+1}} = \lambda + O(\delta_n) \quad (n \to \infty),$$

where (δ_n) is a sequence of positive numbers converging to zero, satisfying the condition

$$\lim_{n\to\infty} \delta_{n+1}/\delta_n = 1. \tag{1.3}$$

Sequences like $\delta_n = n^{-\alpha}(\log n)^\beta$ $(\alpha > 0)$ or $\delta_n = n^\beta e^{-n^\gamma}$ $(0 < \gamma < 1)$ obviously satisfy these conditions.

If $0 < \lambda < R$, we obtain then, instead of (1.2), the asymptotic relation

$$\sum_{k=0}^{n} p_k \frac{a_{n-k}}{a_n} = \sum_{k=0}^{\infty} p_k \lambda^k + O(\delta_n) \quad (n \to \infty).$$

If $\lambda = R$, the same result is true under somewhat stronger hypotheses about the sequence (δ_n) and the nature of convergence of the series $\sum_{k=0}^{\infty} p_k x^k$ on the boundary of the circle of convergence. More precisely, we shall prove in the next sections the following results.

THEOREM 1. *Suppose that the series* $\sum_{k=0}^{\infty} p_k x^k$ *has a positive radius of convergence* R *and suppose that a sequence of positive numbers* (a_n) *satisfies the condition*

$$\left| \frac{a_n}{a_{n+1}} - \lambda \right| \leqq \delta_n, \qquad n = 0, 1, 2, \cdots, \tag{1.4}$$

where (δ_n) *is a sequence of positive numbers such that*

$$\delta_n \to 0 \quad \textit{and} \quad \delta_{n+1}/\delta_n \to 1 \quad (n \to \infty).$$

If $0 < \lambda < R$, *then we have*

$$\limsup_{n\to\infty} \frac{1}{\delta_n} \left| \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k \lambda^k \right| \leqq \sum_{k=1}^{\infty} k|p_k|\lambda^{k-1} < \infty. \tag{1.5}$$

THEOREM 2. *Let* R *be the radius of convergence of the series* $\sum_{k=0}^{\infty} p_k x^k$. *Suppose that*

$$\sum_{k=1}^{\infty} k^\alpha |p_k| R^k < \infty \quad \textit{for every } \alpha > 0. \tag{1.6}$$

Let (a_n) *be a sequence of positive numbers such that*

$$\left| \frac{a_{n-1}}{a_n} - R \right| \leqq \delta_{n-1}, \qquad n = 1, 2, \cdots, \tag{1.7}$$

where (δ_n) *is a sequence of positive numbers satisfying the following conditions*:

$$n\delta_n = O(1) \quad \textit{and} \quad n\left(\frac{\delta_n}{\delta_{n-1}} - 1\right) = O(1) \quad (n \to \infty). \tag{1.8}$$

Then

$$\limsup_{n\to\infty} \frac{1}{\delta_n}\left| \sum_{k=1}^{\infty} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k R^k \right| < \infty. \tag{1.9}$$

A typical series which satisfies the condition (1.6) is $\sum_{k=1}^{\infty} e^{-\sqrt{k}}x^k$. The conditions (1.8) are obviously true if $\delta_n = n^{-\alpha}$ ($\alpha \geqq 1$). The sequence $\delta_n = n^{-\alpha}$ ($0 < \alpha < 1$) does not satisfy the first of the conditions (1.8), but it satisfies the second. On the other hand, the sequence $\delta_n = e^{-\sqrt{n}}$ satisfies the first of these conditions but not the second. If we assume that $n\delta_n = o(1)$ instead of $n\delta_n = O(1)$ it is interesting to observe that the sequence (a_n) has then the representation $a_n = R^{-n}l_n$ where (l_n) is a slowly varying sequence.

The proofs of Theorems 1 and 2 will be given in §§ 2 and 3, respectively.

2. The proof of Theorem 1 is based on two lemmas. The first lemma shows essentially that a sequence (δ_n) satisfies condition (1.3) if and only if the sequence $(\rho^n\delta_n)$ is eventually increasing[1] and the sequence $(\rho^{-n}\delta_n)$ eventually decreasing for every $\rho > 1$. The second lemma establishes a basic inequality.

LEMMA 1. *Let (δ_n) be a sequence of positive numbers. Then*

$$\lim_{n\to\infty} \delta_{n+1}/\delta_n = 1 \tag{2.1}$$

if and only if the sequence $(\rho^n\delta_n)$ is eventually increasing and $(\rho^{-n}\delta_n)$ eventually decreasing for every $\rho > 1$.

Remark. If $(\rho^n\delta_n)$ is eventually increasing for every $\rho > 1$, it is easy to see that $\rho^n\delta_n \to \infty$ ($n \to \infty$) for every $\rho > 1$. If $\rho > 1$, we can find ρ_0 such that $\rho > \rho_0 > 1$ and N_{ρ_0} such that the sequence $(\rho_0^n\delta_n)$ is increasing for $n > N_{\rho_0}$. We have then

$$\rho^n\delta_n = (\rho/\rho_0)^n\rho_0^n\delta_n \geqq (\rho/\rho_0)^n\rho_0^{N_{\rho_0}}\delta_{N_{\rho_0}} \to \infty \quad (n \to \infty).$$

Proof. Suppose that the sequence $(\rho^n\delta_n)$ is eventually increasing and $(\rho^{-n}\delta_n)$ eventually decreasing for every $\rho > 1$. We have then, for $n \geqq N_\rho$,

$$\rho^n\delta_n \leqq \rho^{n+1}\delta_{n+1} \quad \text{and} \quad \rho^{-n}\delta_n \geqq \rho^{-(n+1)}\delta_{n+1} \tag{2.2}$$

or

$$\frac{1}{\rho} \leqq \frac{\delta_{n+1}}{\delta_n} \leqq \rho. \tag{2.3}$$

Hence

$$\frac{1}{\rho} \leqq \liminf_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} \leqq \limsup_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} \leqq \rho,$$

and (2.1) follows, since ρ can be chosen arbitrarily close to 1.

Next, suppose that (2.1) holds. Given $\rho > 1$, we can find N_ρ such that (2.3) holds for $n \geqq N_\rho$. But then (2.2) holds for all $n \geqq N_\rho$. Hence $(\rho^n\delta_n)$ is an eventually increasing sequence and $(\rho^{-n}\delta_n)$ is an eventually decreasing sequence.

LEMMA 2. *If*

$$\left| \frac{a_{n-1}}{a_n} - \lambda \right| \leqq \delta_{n-1}, \qquad n = 1, 2, \cdots, \tag{2.4}$$

[1] We say that a sequence (q_n) is eventually increasing if there exists N such $q_{n+1} \geqq q_n$ for all $n \geqq N$. The eventually decreasing sequence is defined in the same manner.

where $\delta_n \to 0$ *and* $\delta_{n+1}/\delta_n \to 1$ $(n \to \infty)$, *then for every* $\rho > 1$ *and* $\varepsilon > 0$ *there exists* N *such that for* $n \geqq N + l$ *and* $l = 1, 2, \cdots$ *we have*

$$\left| \frac{a_{n-k}}{a_n} - \lambda^k \right| \leqq k\delta_{n-k}\rho^{k-1}(\lambda + \varepsilon)^{k-1} \quad \textit{for } k = 1, 2, \cdots, l. \tag{2.5}$$

Proof. Suppose that we are given $\rho > 1$ and $\varepsilon > 0$. Since, by Lemma 1, $(\rho^{-n}\delta_n)$ is eventually decreasing, we can find N_ρ such that

$$\rho^{-n}\delta_n \geqq \rho^{-(n+1)}\delta_{n+1} \quad \text{for } n \geqq N_\rho. \tag{2.6}$$

Since $\delta_n \to 0$ $(n \to \infty)$, we can find N_ε such that

$$\delta_n < \varepsilon \quad \text{for } n \geqq N_\varepsilon. \tag{2.7}$$

Let $N > \max(N_\rho, N_\varepsilon)$. The statement (2.5) is clearly true if $l = 1$. Suppose that it is true for some $l = r$. We shall prove then that it is true for $l = r + 1$.

Suppose that $n \geqq N + r + 1$. Then $n > N + r$ and consequently we have, by the induction hypothesis,

$$\left| \frac{a_{n-k}}{a_n} - \lambda^k \right| \leqq k\delta_{n-k}\rho^{k-1}(\lambda + \varepsilon)^{k-1} \quad \text{for } k = 1, 2, \cdots, r.$$

Hence in order to prove that the statement (2.5) is true for $l = r + 1$, we only have to show that

$$\left| \frac{a_{n-(r+1)}}{a_n} - \lambda^{r+1} \right| \leqq (r + 1)\delta_{n-(r+1)}\rho^r(\lambda + \varepsilon)^r. \tag{2.8}$$

We have

$$\frac{a_{n-r-1}}{a_n} - \lambda^{r+1} = \left(\frac{a_{n-r-1}}{a_{n-1}} - \lambda^r\right)\left(\frac{a_{n-1}}{a_n} - \lambda\right) + \lambda\left(\frac{a_{n-r-1}}{a_{n-1}} - \lambda^r\right) + \lambda^r\left(\frac{a_{n-1}}{a_n} - \lambda\right)$$

and so

$$\left| \frac{a_{n-r-1}}{a_n} - \lambda^{r+1} \right| \leqq \left| \frac{a_{n-r-1}}{a_{n-1}} - \lambda^r \right| \left(\left| \frac{a_{n-1}}{a_n} - \lambda \right| + \lambda \right) + \lambda^r \left| \frac{a_{n-1}}{a_n} - \lambda \right|.$$

Since $n - 1 \geqq N + r$, we have, by the induction hypothesis,

$$\left| \frac{a_{n-r-1}}{a_{n-1}} - \lambda^r \right| \leqq r\delta_{n-r-1}\rho^{r-1}(\lambda + \varepsilon)^{r-1}.$$

Using this inequality and (2.4) we find that

$$\left| \frac{a_{n-r-1}}{a_n} - \lambda^{r+1} \right| \leqq r\delta_{n-r-1}\rho^{r-1}(\lambda + \varepsilon)^{r-1}(\delta_{n-1} + \lambda) + \lambda^r\delta_{n-1}.$$

Since $n - 1 \geqq N_\varepsilon$ and $n - r - 1 \geqq N_\rho$, we have by (2.6) and (2.7),

$$\delta_{n-1} < \varepsilon \quad \text{and} \quad \rho^{-(n-r-1)}\delta_{n-r-1} \geqq \rho^{-(n-1)}\delta_{n-1}.$$

Hence it follows that

$$\left| \frac{a_{n-r-1}}{a_n} - \lambda^{r+1} \right| \leqq r\delta_{n-r-1}\rho^{r-1}(\lambda + \varepsilon)^r + \lambda^r \rho^r \delta_{n-r-1}$$

$$\leqq (r + 1)\rho^r \delta_{n-r-1}(\lambda + \varepsilon)^r.$$

Thus, the inequality (2.8) is proved and the proof of Lemma 2 is completed.

Proof of Theorem 1. Let

$$S_n = \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k \lambda^k.$$

Since $\lambda < R$ we can choose $\varepsilon > 0$ such that $\lambda + \varepsilon < R$. Next, choose $\rho > 1$ such that

$$1 < \rho^2 < \frac{R}{\lambda + \varepsilon}. \tag{2.9}$$

By Lemma 2 we can find a number N such that for $n > N$ and $1 \leqq k \leqq n - N$, we have

$$\left| \frac{a_{n-k}}{a_n} - \lambda^k \right| \leqq k\delta_{n-k}\rho^{k-1}(\lambda + \varepsilon)^{k-1}.$$

Since $(\rho^n \delta_n)$ is eventually increasing we can assume also that

$$\rho^n \delta_n \leqq \rho^{n+1}\delta_{n+1} \quad \text{for all } n \geqq N. \tag{2.10}$$

If $n > N$, we have

$$\begin{aligned} |S_n| &= \left| \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k \lambda^k \right| \\ &\leqq \sum_{k=1}^{n-N} |p_k| \left| \frac{a_{n-k}}{a_n} - \lambda^k \right| + \sum_{k=n-N}^{n} |p_k| \frac{a_{n-k}}{a_n} + \sum_{k=n-N}^{\infty} |p_k| \lambda^k \\ &= I_n + J_n + K_n. \end{aligned} \tag{2.11}$$

We have first, by (2.5),

$$I_n = \sum_{k=1}^{n-N} |p_k| \left| \frac{a_{n-k}}{a_n} - \lambda^k \right|$$

$$\leqq \sum_{k=1}^{n-N} k|p_k|\rho^{k-1}(\lambda + \varepsilon)^{k-1}\delta_{n-k}.$$

Since for every $1 \leqq k \leqq n - N$, we have $n - k \geqq N$, it follows from (2.10) that $\delta_{n-k} \leqq \rho^k \delta_n$. Using this inequality we find that

$$I_n \leqq \delta_n \sum_{k=1}^{n-N} k|p_k|\rho^{2k-1}(\lambda + \varepsilon)^{k-1},$$

and it follows from (2.9) that

$$\limsup_{n\to\infty} \frac{I_n}{\delta_n} \leqq \sum_{k=1}^{\infty} k|p_k|\rho^{2k-1}(\lambda+\varepsilon)^{k-1} < \infty. \tag{2.12}$$

Next, we shall estimate J_n. We have

$$\begin{aligned} J_n &= \sum_{k=n-N}^{n} |p_k| \frac{a_{n-k}}{a_n} \\ &\leqq \left(\max_{0\leqq k\leqq N} a_k\right) \frac{1}{a_n} \sum_{k=n-N}^{n} |p_k|. \end{aligned}$$

Since $0 < \lambda < R$, we can find $\eta \in (0, (R-\lambda)/(R+\lambda))$ such that

$$\frac{1-\eta}{\lambda} > \frac{1+\eta}{R}. \tag{2.13}$$

Then from $\limsup_{n\to\infty} \sqrt[n]{|p_n|} = R^{-1}$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = \lambda^{-1}$, we can conclude that, for $n > N_\eta$, we have

$$|p_n| \leqq \left(\frac{1+\eta/2}{R}\right)^n, \qquad a_n \geqq \left(\frac{1-\eta}{\lambda}\right)^n \geqq \left(\frac{1+\eta}{R}\right)^n. \tag{2.14}$$

Hence, for $n \geqq N_\eta + N$, we have

$$J_n \leqq \left(\max_{0\leqq k\leqq N} a_k\right) \left(\frac{1+\eta/2}{1+\eta}\right)^n \sum_{k=n-N}^{n} R^{n-k},$$

and so

$$\frac{J_n}{\delta_n} \leqq \left(\max_{0\leqq k\leqq N} a_k\right) \left(\sum_{k=0}^{N} R^k\right) \left(\frac{1+\eta/2}{1+\eta}\right)^n \frac{1}{\delta_n}.$$

Since $1 + \eta/2 < 1 + \eta$, by the remark following Lemma 1, we have

$$\left(\frac{1+\eta/2}{1+\eta}\right)^n \bigg/ \delta_n \to 0 \quad (n \to \infty).$$

Hence,

$$\lim_{n\to\infty} J_n/\delta_n = 0. \tag{2.15}$$

Finally, for $k \geqq n - N$ and $n > N_\eta + N$, we have $k \geqq N_\eta$ and so by (2.14), we have

$$|p_k| \leqq \left(\frac{1+\eta}{R}\right)^k,$$

and it follows that, for $n \geqq N_\eta + N$, we have

$$K_n = \sum_{k=n-N}^{\infty} |p_k|\lambda^k \leqq \sum_{k=n-N}^{\infty} \left(\frac{1+\eta}{R}\right)^k \lambda^k.$$

Since, by (2.13), $(1 + \eta)\lambda < R$, it follows that

$$K_n \leqq \left(\left(\frac{1+\eta}{R}\right)\lambda\right)^{n-N} \frac{1}{1 - ((1+\eta)/R)\lambda}$$

and we conclude, as before, that

$$\lim_{n\to\infty} K_n/\delta_n = 0. \tag{2.16}$$

Thus, from (2.11), (2.12), (2.15) and (2.16), we find that

$$\limsup_{n\to\infty} \frac{1}{\delta_n}\left|\sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k \lambda^k\right| \leqq \sum_{k=1}^{\infty} k|p_k|(\lambda + \varepsilon)^{k-1}\rho^{2k} < \infty,$$

and the theorem follows by letting $\rho \to 1+$ and $\varepsilon \to 0+$.

3. In order to prove Theorem 2, we shall first establish an inequality similar to that of Lemma 2. Then we shall obtain some simple consequences of our hypotheses (1.8).

LEMMA 3. *If*

$$\left|\frac{a_{n-1}}{a_n} - R\right| \leqq \delta_{n-1}, \qquad n = 1, 2, \cdots, \tag{3.1}$$

where (δ_n) *satisfies*

$$n\delta_n = O(1) \quad \textit{and} \quad n\left(\frac{\delta_n}{\delta_{n-1}} - 1\right) = O(1) \quad (n \to \infty), \tag{3.2}$$

then there exist positive numbers M *and* N *such that for* $n \geqq N + l$ *and* $l = 1, 2, \cdots$ *we have*

$$\left|\frac{a_{n-k}}{a_n} - R^k\right| \leqq kR^{k-1}\delta_{n-k} \prod_{j=n-k+1}^{n} \left(1 + \frac{M}{j}\right) \quad \textit{for } k = 1, 2, \cdots, l. \tag{3.3}$$

Proof. By (3.2) there are numbers M and N such that for $n \geqq N$ we have

$$\delta_n \leqq \frac{MR}{n+1} \quad \text{and} \quad \left|\frac{\delta_n}{\delta_{n-1}} - 1\right| \leqq \frac{M}{n}. \tag{3.4}$$

The statement (3.3) is clearly true if $l = 1$. Suppose that it is true for some $l = r$. We shall then prove that it is true for $l = r + 1$.

Suppose that $n \geqq N + r + 1$. Then $n \geqq N + r$ and consequently we have, by the induction hypothesis,

$$\left|\frac{a_{n-k}}{a_n} - R^k\right| \leqq kR^{k-1}\delta_{n-k} \prod_{j=n-k+1}^{n} \left(1 + \frac{M}{j}\right), \qquad k = 1, 2, \cdots, r.$$

Hence we only have to show that

$$\left|\frac{a_{n-(r+1)}}{a_n} - R^{r+1}\right| \leqq (r+1)R^r\delta_{n-(r+1)} \prod_{j=n-r}^{n} \left(1 + \frac{M}{j}\right).$$

We have, as in the proof of Lemma 2,

$$\left|\frac{a_{n-(r+1)}}{a_n} - R^{r+1}\right| \leqq \left|\frac{a_{n-r-1}}{a_{n-1}} - R^r\right|\left(\left|\frac{a_{n-1}}{a_n} - R\right| + R\right) + R^r\left|\frac{a_{n-1}}{a_n} - R\right|.$$

Since $n - 1 \geqq N + r$, we have, by the induction hypothesis,

$$\left|\frac{a_{n-r-1}}{a_{n-1}} - R^r\right| \leqq rR^{r-1}\delta_{n-1-r}\prod_{j=n-r}^{n-1}\left(1 + \frac{M}{j}\right).$$

Using this inequality and (3.1) we find that

$$\left|\frac{a_{n-r-1}}{a_n} - R^{r+1}\right| \leqq rR^r\delta_{n-1-r}\left(1 + \frac{\delta_{n-1}}{R}\right)\prod_{j=n-r}^{n-1}\left(1 + \frac{M}{j}\right) + R^r\delta_{n-1}. \tag{3.5}$$

Since $n - r \geqq N$, from (3.4) it follows that

$$\delta_{n-1} \leqq \frac{MR}{n} \quad \text{and} \quad \delta_{n-1} \leqq \prod_{j=n-r}^{n-1}\left(1 + \frac{M}{j}\right)\delta_{n-1-r}$$

and (3.3) follows from these inequalities and (3.5).

LEMMA 4. *If $n\delta_n = O(1)$ and $n(\delta_n/\delta_{n-1} - 1) = O(1)$ $(n \to \infty)$ then there exists a number M such that*

$$\limsup_{n\to\infty}\frac{\prod_{k=1}^n(1 + \delta_k/R)}{n^M} < \infty \tag{3.6}$$

and

$$\limsup_{n\to\infty} n^{-M}/\delta_n < \infty. \tag{3.7}$$

Proof. For $n \geqq N$ we have, by (3.4),

$$\prod_{k=N}^n\left(1 + \frac{\delta_k}{R}\right) \leqq \prod_{k=N}^n\left(1 + \frac{M}{k+1}\right) \leqq \exp\left(M\sum_{k=N}^n\frac{1}{k+1}\right) \leqq (n+1)^M$$

and (3.6) follows.

Next, let N be such that $M/N < \frac{1}{2}$. For $n \geqq N$ we have, by (3.4),

$$\delta_n > \left(1 - \frac{M}{n}\right)\delta_{n-1} \tag{3.8}$$

and so

$$\delta_n \geqq \prod_{k=N+1}^n\left(1 - \frac{M}{k}\right)\delta_N.$$

On the other hand we have

$$\prod_{k=N+1}^{n}\left(1-\frac{M}{k}\right) = \exp\left(\sum_{k=N+1}^{n}\log\left(1-\frac{M}{k}\right)\right)$$
$$= \exp\left(-M\sum_{k=N+1}^{n}\frac{1}{k} + \sum_{k=N+1}^{n}\left(\frac{M}{k}+\log\left(1-\frac{M}{k}\right)\right)\right)$$
$$\geqq \exp\left(-M\log n + \sum_{k=N+1}^{n}\left(\frac{M}{k}+\log\left(1-\frac{M}{k}\right)\right)\right)$$
$$\geqq n^{-M}\exp\left(\sum_{k=N+1}^{n}\left(\frac{M}{k}+\log\left(1-\frac{M}{k}\right)\right)\right).$$

Hence,

$$n^M\delta_n \geqq \delta_N \exp\left(\sum_{k=N+1}^{n}\left(\frac{M}{k}+\log\left(1-\frac{M}{k}\right)\right)\right)$$
$$\geqq \delta_N \exp\left(\sum_{k=N+1}^{\infty}\left(\frac{M}{k}+\log\left(1-\frac{M}{k}\right)\right)\right).$$

Since the series $\sum_{k=N+1}^{\infty}(M/k + \log(1 - M/k))$ converges, it follows that

$$\liminf_{n\to\infty} n^M\delta_n > 0$$

and (3.7) follows.

Proof of Theorem 2. We have, as in the proof of Theorem 1,

$$|S_n| = \left|\sum_{k=1}^{n} p_k\frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k R^k\right|$$
$$\leqq \sum_{k=1}^{[n/2]}|p_k|\left|\frac{a_{n-k}}{a_n} - R^k\right| + \sum_{k=[n/2]+1}^{n}|p_k|\frac{a_{n-k}}{a_n} + \sum_{k=[n/2]+1}^{\infty}|p_k|R^k \tag{3.9}$$
$$= I_n + J_n + K_n.$$

By Lemma 3 we can find N and M such that the inequality (3.3) is true for all $n \geqq N + l$ and $l = 1, 2, \cdots$. Let $n \geqq 2(N + M)$. We have then that $N + [n/2] \leqq n$ and it follows that the inequality (3.3) is true for $n \geqq 2(N + M)$ and $k = 1, 2, \cdots, [n/2]$. We have then,

$$I_n = \sum_{k=1}^{[n/2]}|p_k|\left|\frac{a_{n-k}}{a_n} - R^k\right|$$
$$\leqq \sum_{k=1}^{[n/2]} k|p_k|R^{k-1}\delta_{n-k}\prod_{j=n-k+1}^{n}\left(1+\frac{M}{j}\right)$$
$$\leqq \sum_{k=1}^{[n/2]} k|p_k|R^{k-1}\delta_{n-k}\left(1+\frac{M}{n-k+1}\right)^k$$
$$\leqq \left(1+\frac{2M}{n}\right)^{n/2}\sum_{k=1}^{[n/2]} k|p_k|R^{k-1}\delta_{n-k}.$$

On the other hand, since $n > 2M$, we have by (3.8), for $k = 1, 2, \cdots, [n/2]$,

$$\delta_n \geqq \prod_{j=n-k+1}^{n-1} \left(1 - \frac{M}{j}\right)\delta_{n-k} \geqq \left(1 - \frac{M}{n-k+1}\right)^k \delta_{n-k}$$
$$\geqq \left(1 - \frac{2M}{n}\right)^{n/2} \delta_{n-k}.$$

Hence,

$$I_n \leqq \delta_n \left(\frac{1 + 2M/n}{1 - 2M/n}\right)^{n/2} \sum_{k=1}^{[n/2]} k|p_k|R^{k-1},$$

and it follows that

$$\limsup_{n\to\infty} I_n/\delta_n \leqq e^{2M} \sum_{k=1}^{\infty} k|p_k|R^{k-1}. \tag{3.10}$$

Next, we shall estimate J_n. From (1.7) it follows that

$$\frac{a_{n-1}}{a_n} \leqq R + \delta_{n-1} \quad \text{for } n = 1, 2, \cdots$$

and so

$$\frac{a_{n-k}}{a_n} \leqq R^k \prod_{j=1}^{n} \left(1 + \frac{\delta_j}{R}\right).$$

Hence,

$$J_n = \sum_{k=[n/2]+1}^{n} |p_k| \frac{a_{n-k}}{a_n}$$
$$\leqq \prod_{j=1}^{n} \left(1 + \frac{\delta_j}{R}\right) \sum_{k=[n/2]+1}^{n} |p_k|R^k.$$

Next, since

$$K_n = \sum_{k=[n/2]+1}^{\infty} |p_k|R^k,$$

it follows that

$$J_n + K_n \leqq 2 \prod_{j=1}^{n} \left(1 + \frac{\delta_j}{R}\right) \sum_{k=[n/2]+1}^{\infty} |p_k|R^k$$

and so

$$\frac{J_n + K_n}{\delta_n} \leqq \frac{2\prod_{k=1}^{n}(1 + \delta_k/R)}{n^M}\left(\frac{1}{n^M\delta_n}\right) n^{2M} \sum_{k=[n/2]+1}^{\infty} |p_k|R^k.$$

Since $k \geqq n/2$, it follows that

$$\frac{J_n + K_n}{\delta_n} \leqq 2^{2M+1}\frac{\prod_{k=1}^{n}(1 + \delta_k/R)}{n^M}\left(\frac{1}{n^M\delta_n}\right) \sum_{k=[n/2]+1}^{\infty} k^{2M}|p_k|R^k,$$

and in view of Lemma 4 and (1.6) we find that

$$\lim_{n\to\infty} \frac{J_n + K_n}{\delta_n} = 0. \tag{3.11}$$

Thus, from (3.9), (3.10) and (3.11) it follows that

$$\limsup_{n\to\infty} \frac{1}{\delta_n}\left| \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} - \sum_{k=1}^{\infty} p_k R^k \right| \leqq e^{2M} \sum_{k=1}^{\infty} k|p_k|R^{k-1} < \infty,$$

and the theorem is proved.

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A UNIQUENESS THEOREM FOR CONVOLUTION EQUATIONS IN $L^p(R^n)$ SPACES*

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Abstract. The uniqueness problem for solutions in $L^p(R^n)$, $1 \leqq p < \infty$, to the convolution equation $T * u = f$, $f \in D(R^n)$, is discussed under the assumption that T is a finite distribution such that the null set $N(T) = \{s \in R^n : \tilde{T}(s) = 0\}$ of its Fourier transform $\tilde{T}$ is contained in the union of a finite number of hyperplanes.

1. Introduction. The convolution equation

$$T * u = f \tag{1.1}$$

is considered on R^n for a finite distribution $T \in \mathscr{E}'$ satisfying the following conditions. Denote by $\tilde{T}$ the Fourier transform of T, by $\tilde{S}$ an irreducible factor of $\tilde{T}$, by $\nabla\tilde{S}(s)$ the gradient of the function $\tilde{S}(s)$ at s, and by $N(\tilde{S})$ the real null set of $\tilde{S}$.

$$N(\tilde{S}) = \{s \in R^n : \tilde{S}(s) = 0\}.$$

Whenever $N(\tilde{S}) \neq \emptyset$ *for an irreducible factor* $\tilde{S}$ *of* $\tilde{T}$,

$$\nabla\tilde{S}(s) \neq 0 \quad on\ N(\tilde{S}). \tag{A1}$$

(A2) *The set* $N(\tilde{S})$ *consists of a finite number of hyperplanes.*
Such a class of finite distributions is denoted by $\mathfrak{C}_0$.

For the related homogeneous convolution equation

$$T * u = 0, \tag{1.1}$$

we have the following theorem.

THEOREM 1. *For* $T \in \mathfrak{C}_0$, $u = 0$ *is the only solution satisfying either of the conditions*

$$u \in L^p(R^n), \qquad 1 \leqq p < \infty; \tag{1.2}$$

$$u(x) = o(|x|^{-a}) \quad (a > 0) \quad as\ |x| \to \infty. \tag{1.3}$$

The assertion holds also for a system. If T is an $m \times m$ matrix with finite distributions as entries, then $(1.1)_h$ has only the trivial solution under condition (1.2) or (1.3) provided the determinant det T (in the sense of convolution) of T is nonsingular and belongs to the class $\mathfrak{C}_0$. In either single equation or system, the solution must be $C_0^\infty(R^n)$ if it satisfies the uniqueness condition provided $f \in C_0^\infty(R^n)$. Those assertions are proved after the proof of Theorem 1 in the next section.

The problem has been studied in K. Chen [2], [3], and [4]. The essential difference from the paper is that the condition (A2) is replaced by:

(A3) *The minimal number of nonzero principal curvatures of* $N(\tilde{S})$ *for all irreducible factors* $\tilde{S}$ *of* $\tilde{T}$ *is* k.

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Denote the class of finite distributions by $\mathfrak{C}_k$. Then for $k > 0$, the equation (1.1) has a unique continuous solution for $f \in \mathscr{D}'(R^n)$ if u satisfies the condition (1.3) with $a \geqq n - 1 - k/2$ under the method called symmetrization. In the Appendix, we prove that if $k = 0$ in (A3), then (A2) is satisfied and $T \in \mathfrak{C}_0$. Therefore the remaining case $k = 0$ is studied under a different method based on the results of R. P. Boas and G. Pólya, respectively.

The problem has also been studied by W. Littman [6], [7] under the following conditions for partial defferential equations with constant coefficients: Condition (A1) is given for $\tilde{T}$ but not for $\tilde{T}$'s factors and the condition on the number of nonzero principal curvatures is imposed globally constant not varying on location.

2. Liouville type problem—Proof of Theorem 1. If a solution u of $(1.1)_h$ satisfies the condition (1.3), the solution u also satisfies the condition (1.2) for some $p \geqq 1$. Therefore we assume from now on that u fulfills the condition (1.2).

Let us recall some required results in the proof.

THEOREM 2 (R. P. Boas [1]). *If $f(z), z \in C^1$, is an entire function of minimal exponential type and $f \in L^p(R^1)$ for some positive number p, then $f = 0$.*

THEOREM 3 (M. Plancherel and G. Pólya [8]). *For any positive number p and any entire function $F(z)$ in C^n,*

$$F(z) \equiv F(z_1, \cdots, z_n) = F(x_1 + iy_1, \cdots, x_n + iy_n) \tag{2.1}$$

of exponential type $0 \leqq c < +\infty$, if $F \in L^p(R^n)$, then

$$\int |F(z)|^p\,dx \leqq \left\{ \int |F(x)|^p\,dx \right\} \exp\{pc(|y_1| + \cdots + |y_n|), \qquad x \in R^n. \tag{2.2}$$

For simplification, we show first the following.

LEMMA 4. *Let the null space $N(\tilde{T})$ of $\tilde{T}$ consist of a hyperplane. If $u \in L^p(R^n)$, $1 \leqq p < \infty$, is a solution of $(1.1)_h$, then $u = 0$.*

Proof. By $u \in L^p(R^n)$, u and its Fourier transform $\tilde{u}$ are tempered distributions. The Fourier transformed form of $(1.1)_h$ is

$$\tilde{T}\tilde{u} = 0 \quad \text{on } R^n. \tag{2.3}$$

Then $\operatorname{supp} \tilde{u} \subset N(\tilde{T})$. Since $N(\tilde{T})$ is a hyperplane, there is a unit vector ω and a positive constant c such that $N(\tilde{T})$ can be represented by

$$s \cdot \omega = c. \tag{2.4}$$

First consider $c = 0$ and $\omega = (1, 0, \cdots, 0)$. Suppose $u \neq 0$, that is $\operatorname{supp} \tilde{u}$ is not empty. Then there is a rectangle

$$R_b = \{s \in R^n : |s_j| \leqq b_j, j = 1, \cdots, n\}$$

with $b_1 = 0$ such that

$$\operatorname{supp} \tilde{u} \cap R_b \neq \varnothing.$$

Let $\tilde{\chi}$ be a $C_0^\infty(R^n)$ function such that $0 \leqq \tilde{\chi} \leqq 1$, $\tilde{\chi} = 1$ on R_b and $\operatorname{supp} \tilde{\chi} \subset R_{b+\varepsilon}$ for some $\varepsilon > 0$. Then (2.3) yields

$$\tilde{T} \cdot (\tilde{u}\tilde{\chi}) = 0 \quad \text{on } R^n.$$

With χ as the inverse Fourier transform of $\tilde{\chi}$, χ is a tempered distribution and

$$S * v = 0, \quad v = u * \chi.$$

It is known (A. Friedman [5, p. 187]) that the convolution of $u * \chi$ in the distribution sense is the one in the function sense. W. H. Young's inequality implies $v \in L^p(R^n)$ with the same p as above. Since the support of $\tilde{v}$ is compact, $v(x)$ can be extended into an entire function $v(x + iy) = v(z)$ of finite exponential type $\leqq b + \varepsilon$; that is,

$$|v(x + iy)| \leqq C(1 + |x|)^{\alpha} \exp\{(b + \varepsilon)|y|\}, \qquad C > 0, \quad \alpha \geqq 0, \tag{2.5}$$

with $(b + \varepsilon)|y| = (b_1 + \varepsilon_1)|y_1| + \cdots + (b_n + \varepsilon_n)|y_n|$. Then by Theorem 3,

$$\int |v(x + iy)|^p \, dx \leqq \left\{ \int |v(x)|^p \, dx \right\} \exp\{p(b + \varepsilon)|y|\}. \tag{2.6}$$

For fixed $z' \equiv (z_2, \cdots, z_n) \in C^{n-1}$, let

$$f_{z'}(x_1 + iy_1) = v(x_1 + iy_1, z').$$

Treating $f_{z'}(x + iy_1)$ as a function of $x_1 + iy_1$ with z' as $n - 1$ parameters, we see that with $b_1 = 0$ in (2.5), $f_{z'}(x_1 + iy_1)$ is an entire function of minimal exponential type for each $z' \in C^{n-1}$. Therefore, (2.6) and Fubini's theorem imply $f_{z'} \in L^p(R^1)$ for almost all $z' \in C^{n-1}$. Theorem 2 yields $f_{z'} = 0$ for almost all $z' \in C^{n-1}$. That is, $v(x_1 + iy_1, z') \equiv 0$ for almost all $z' \in C^{n-1}$. But since v is an entire function, we have $v = 0$. Hence, $\tilde{u}$ has no support in the finite part of the plane $s \cdot \omega = 0$. Thus, $u = 0$.

Next suppose $c \neq 0$ and $\omega = (1, 0, \cdots, 0)$. Let $s_0 = c\omega = (c, 0, \cdots, 0)$, $\tilde{u}_1(s) = \tilde{u}(s - s_0)$, and u_1 be the inverse Fourier transform of $\tilde{u}_1$. Then $u_1(x) = e^{ix \cdot s_0} u(x) = e^{icx_1} u(x)$. We have $u_1 \in L^p(R^n)$ and supp $\tilde{u}_1$ is contained in the hyperplane $s \cdot \omega = 0$. From the proved result of the previous case, $u_1 = 0$ and then $u = 0$.

Finally suppose $c \neq 0$ and $\omega \neq \omega_0 = (1, 0, \cdots, 0)$. Let $\Phi: R^n \to R^n$ be the rotation which transforms ω to ω_0. Let

$$\tilde{u}_2(s) = \tilde{u}(\Phi(s))$$

and $u_2(x)$ be the inverse Fourier transform of $\tilde{u}_2$. By the invariance of Lebesgue measure under rotation in R^n, we have

$$u_2(x) = u(\Phi^{-1}(x))$$

which belongs to $L^p(R^n)$ and satisfies the properties of u_1. Hence, $u_2 = 0$ and then $u = 0$. This completes the proof.

Since by a partition of unity for a finite portion on a finite number of parallel hyperplanes we can use the same arguments as those in the above proof to each part in the planes, we have the same conclusion as that mentioned in the lemma. Therefore, for Theorem 1, it suffices to show the following.

LEMMA 5. *Let $T = S_0 * S_1 * S_2 \in C_0$ with $N(\tilde{S}_0) = \varnothing$, and let $N(\tilde{S}_i)$ be contained in the hyperplane*

$$P_i : s \cdot \omega^i = c^i, \qquad i = 1, 2,$$

where ω^i are two distinct unit vectors in R^n, and c^i are two positive numbers. If $u \in L^p(R^n)$, $1 \leqq p < \infty$, is a solution of the convolution equation $(1.1)_{\mathrm{h}}$, then $u = 0$.

Proof. Since $\omega^1 \neq \omega^2$, the two planes intersect at a hyperline L. The Fourier transformed form

$$\tilde{T} \cdot \tilde{u} = 0$$

of $(1.1)_h$ implies that

$$\operatorname{supp} \tilde{u} \subset N(\tilde{T}) = P_1 \cup P_2.$$

Suppose $u \neq 0$. Then $\operatorname{supp} u \neq \varnothing$. There is a finite nonnegative vector b such that

$$\operatorname{supp} \tilde{u} \cap R_b \neq \varnothing.$$

Let $\tilde{\chi} \in C_0^\infty(R^n)$ be a function such that $0 \leqq \tilde{\chi} \leqq 1$, $\tilde{\chi} = 1$ on $\operatorname{supp} \tilde{u} \cap R_b$, and $\operatorname{supp} \tilde{\chi} \subset R_{b+\varepsilon}$, $\varepsilon > 0$. Then

$$\tilde{T} \cdot \tilde{v} = 0 \quad \text{with } \tilde{v} = \tilde{\chi}\tilde{u}.$$

Furthermore, let $\tilde{\chi}_j \in C_0^\infty(R^n)$ such that $0 \leqq \tilde{\chi}_j \leqq 1, j = 0, 1, 2; \tilde{\chi}_0 = 1$ on $L \cap R_{b+\varepsilon}$, $\operatorname{supp} \tilde{\chi}_0 \subset U_\varepsilon$ which is an ε-neighborhood of $L \cap R_{b+\varepsilon}; \tilde{\chi}_j = 1$ on $(P_j \cap R_{b+\varepsilon}) \setminus U_\varepsilon$ with supports in $R_{b+2\varepsilon}, j = 1, 2$; and $\tilde{\chi}_0 + \tilde{\chi}_1 + \tilde{\chi}_2 = 1$ on $(P_1 \cup P_2) \cap \operatorname{supp} \tilde{\chi}$. Then

$$\tilde{v} = \tilde{v}_0 + \tilde{v}_1 + \tilde{v}_2 \qquad \text{with } \tilde{v}_j = \tilde{v}\tilde{\chi}_j.$$

Let v_j be the inverse Fourier transform of $\tilde{v}_j$, χ of $\tilde{\chi}$ and χ_j of $\tilde{\chi}_j$. Then, again,

$$v_j = u * [\chi * \chi_j] \in L^p(R^n), \qquad j = 0, 1, 2,$$

because χ and χ_j are rapidly decreasing functions and then $\chi * \chi_j \in L^1(R^n)$. Since $\operatorname{supp} \tilde{v}_j$ is contained in the hyperplane P_j, Lemma 4 yields $v_j = 0, j = 1, 2$, for any $\varepsilon > 0$ and any finite nonnegative vector b. Hence we have $u = v_0 \in L^p(R^n)$ with $\operatorname{supp} \tilde{v}_0 \subset L \subset P_1$. Again Lemma 4 yields $u = v_0 = 0$. This completes the proof of Lemma 5 and also of Theorem 1.

For inhomogeneous equations (1.1) with smooth and finite data, we want to get the solution with the property. We need to restrict the consideration of finite distributions further; for example, see Theorem 4.5 in K. Chen [3].

DEFINITION. A finite distribution T is called *invertible* and its Fourier transform $\tilde{T}$ is called *slowly decreasing* if there exists a positive number α such that for each point ξ in R^n one can find a point η in R^n satisfying

$$|\xi - \eta| \leqq \alpha \log(1 + |\xi|),$$
$$|\tilde{T}(\eta)| \geqq (\alpha + |\eta|)^{-\alpha}.$$

Combining the results of Theorem 4.3 in K. Chen [3] and Theorem 1, with the same argument of Theorem 4.4 in [3], we obtain the following.

COROLLARY 6. *If u is a continuous solution of the convolution equation* (1.1) *with $T \in \mathfrak{C}_0$ invertible and $f \in C_0^\infty(R^n)$ and if u satisfies either of the uniqueness conditions* (1.2) *or* (1.3), *then u is a $C_0^\infty(R^n)$-function.*

THEOREM 7. *Let T be an $m \times m$ matrix with E' entries such that its convolution determinant* det T *is nonsingular, invertible and belongs to $\mathfrak{C}_0$. If the vector-valued function u is a continuous solution of* (1.1) *with $f \in C_0^\infty(R^n)$ and fulfills the uniqueness condition* (1.2) *or* (1.3), *then u is a $C_0^\infty(R^n)$-function.*

Proof. Denote by ${}^{co}T$ the $m \times m$ matrix with ij cofactor of T as ji entries. Then

$$ {}^{co}T * T = T * {}^{co}T = (\det T) * I, $$

where I is the $m \times m$ identity matrix. Let $g = {}^{co}T * f$. Then (1.1) yields

$$ (\det T) * u_j = g_j, \qquad g_j \in C_0^\infty(R^n), $$

$j = 1, \cdots, m$. Corollary 6 implies the assertion and the proof is complete.

Appendix. As mentioned in the Introduction, the class $\mathfrak{C}_k$ of finite distributions satisfying assumptions (A1) and (A3) is studied in K. Chen [2], [3], and [4] by the method of symmetrization with $k > 0$. Now we want to characterize the geometry of $N(T)$ with $T \in \mathfrak{C}_0$. The following assertion may have been proved. Since we are unable to find a reference, we give the proof here.

LEMMA A.1. *Let $f(s)$, $s \in R^n$, be a $C^2(R^n)$-function with $N(f) \neq \varnothing$ and satisfying assumptions* (A1) *and* (A3) *with $k = 0$. Then $N(f)$ consists of parallel hyperplanes, that is, $N(f)$ satisfies* (A2).

Proof. Let s_0 be any point of $N(f)$. Since $\nabla f(s_0) \neq 0$, there is a j, $1 \leqq j \leqq n$, say $j = n$ such that

$$ \frac{\partial f(s_0)}{\partial s_n} \neq 0. $$

Hence, by the implicit function theorem, there is a connected neighborhood U_0 of s_0 on which

$$ s = (s', g(s')) \quad \text{with } s' = (s_1, \cdots, s_{n-1}) \in R^{n-1} $$

forms a C^2-diffeomorphism with $g \in C^2$ locally. Without loss of generality, we choose U_0 such that $N(f) \cap U_0$ is connected. Let

$$ s^j = \frac{\partial s}{\partial s_j} = (0, \cdots, 0, 1, 0, \cdots, 0, g_j(s')) \quad \text{with } g_j(s') = \frac{\partial g(s')}{\partial s_j}, $$

with 1 as the jth entry. Therefore,

$$ s^{ij} = (0, \cdots, 0, g_{ij}(s')) \quad \text{with } g_{ij}(s') = \frac{\partial^2 g(s')}{\partial s_i \partial s_j}. $$

Let $\nu(s) = (\nu_1(s), \cdots, \nu_n(s))$ be the normal of $N(f)$ at the point $s \in N(f)$. Then

$$ \nu_n(s) = f_n(s)/|\nabla f(s)| \neq 0 \quad \text{on } N(f) \cap U_0. $$

Let $(s^{ij}, \nu(s))$ be the inner product of s^{ij} and $\nu(s)$. Then by definition, the $n - 1$ principal curvatures of $N(f)$ at s are the $n - 1$ eigenvalues of the $(n - 1) \times (n - 1)$ matrix $(s^{ij}, \nu(s))$; that is, they are the $n - 1$ values of λ in the polynomial equation of degree $n - 1$:

$$ \det \{(s^{ij}, \nu(s)) - \lambda \delta_{ij}\} = 0. $$

In fact, the equation is

$$ \det \{g_{ij}(s')\nu_n(s) - \lambda \delta_{ij}\} = 0. \tag{A.1} $$

Since (A3) with $k = 0$ means that all the principal curvatures are zero, the coefficients of λ^h in (A.1) vanish for $h = 0, \cdots, n-2$ with $s \in N(f) \cap U_0$. In particular, deriving from the coefficients of λ^{n-2} and λ^{n-3} by $v_n(s) \neq 0$, we have on $N(f) \cap U_0$,

$$\sum g_{jj}(s') = 0, \qquad j = 1, \cdots, n-1, \tag{A.2}$$

$$\sum g_{ij}^2(s') = \sum g_{ii}(s') g_{jj}(s'), \qquad 1 \leqq i < j \leqq n-1. \tag{A.3}$$

Then from the square of (A.2) and by (A.3), we have on $N(f) \cap U_0$,

$$\sum g_{kk}^2(s') + 2 \sum g_{ij}^2(s') = 0,$$

$$1 \leqq k \leqq n-1, \quad 1 \leqq i < j < n,$$

and hence for $1 \leqq i, j \leqq n-1$,

$$g_{ij}(s') = 0.$$

Thus $g(s')$ is a linear combination of $s_1, \cdots, s_{n-1}$ for all $s \in N(f) \cap U_0$. That is, $N(f) \cap U_0$ is contained in a hyperplane for any $s \in N(f)$ by $N(f) \cap U_0$ connected.

Let V_0 be the union of all possible such neighborhoods U_0 of some fixed $s_0 \in N(f)$ with $N(f) \cap U_0$ connected. Then V_0 is an open set. Since $g(s')$ is a C^2-function on V_0, each point s in the boundary ∂V_0 of V_0 is still of the form

$$s = (s', g(s')).$$

Therefore, $\partial V_0 \subset V_0$ and V_0 is closed. Hence, $N(f) \cap V_0$ is a whole hyperplane. By assumption (A1), there are no singular points in $N(f)$. Thus, there is no intersection on hyperplanes which consist of $N(f)$. This completes the proof.

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FUNCTIONS WHOSE FOURIER TRANSFORMS DECAY AT INFINITY: QUALITATIVE CRITERIA FOR AN ADDITIONAL CASE*

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Abstract. A previous paper of Bleistein, Handelsman and Lew describes the asymptotic behavior of

$$F(\omega) = \lim_{u \to +\infty} \int_0^u \exp(i\omega t) f(t)\, dt$$

for certain functions f on $[0, +\infty)$. It estimates the growth or decay of F near $+\infty$ when f has a suitable asymptotic expansion, then establishes the decay in particular of F near $+\infty$ when f has certain qualitative properties. This note fills a gap in the preceding work; it gives a qualitative estimate for F when this transform does not decay near $+\infty$.

1. Introduction. Let f be a complex-valued function on $[0, +\infty)$ which is assumed locally integrable but need not be absolutely integrable, and let $F(\omega)$ be its Fourier transform:

$$F(\omega) = \lim_{u \to \infty} \int_0^u \exp(i\omega t) f(t)\, dt \tag{1.1}$$

whenever and wherever this limit exists. In a recent paper, Bleistein, Handelsman and Lew [1] study the behavior of this transform near $+\infty$ for a large class of functions f. They estimate the growth or decay of F near $+\infty$ whenever f has a suitable asymptotic expansion, then establish in general the decay of F near $+\infty$ whenever f has certain qualitative properties. These results extend by superposition to functions on $(-\infty, +\infty)$, hence offer a generalization of the Riemann–Lebesgue lemma. This note fills a gap in the preceding work and provides a result of qualitative type which covers functions with nondecaying transforms.

In particular, let

$$\begin{aligned} f(t) &= 0 \text{ on } [0, a) \quad \text{with } a \geqq 0 \\ &= \exp(i\gamma t^{\nu}) t^{-r} (\log t)^n \quad \text{on } [a, +\infty), \end{aligned} \tag{1.2}$$

where n is a nonnegative integer, r is a suitable complex number, γ and ν are arbitrary real numbers. If $a \neq 0, 1$ and $\operatorname{Re}(r) > 0$ then [1, Lemma 3]

$$F(\omega) = O(\omega^{-1}) \quad \text{as } \omega \to +\infty \tag{1.3}$$

whenever (γ, ν) is not in $(-\infty, 0) \times (1, +\infty)$, and

$$\begin{aligned} F(\omega) &= O(\omega^{-1} + \omega^k (\log \omega)^n) \quad \text{as } \omega \to +\infty, \\ k &= (1 - \operatorname{Re}(r) - \tfrac{1}{2}\nu)/(\nu - 1), \end{aligned} \tag{1.4}$$

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whenever (γ, ν) is in $(-\infty, 0) \times (1, +\infty)$.

However, the estimate (1.4) for the transform F even includes functions f with transforms which grow near $+\infty$, that is, functions (1.2) with

$$\gamma < 0, \quad 1 < \nu < 2, \quad r + \tfrac{1}{2}\nu \leqq 1; \tag{1.5}$$

whereas the qualitative criteria of the preceding paper cover only functions f with transforms which decay near $+\infty$, such as functions (1.2) with parameters which violate (1.5). Hence this note provides an appropriate qualitative result in the previously untreated case where F does not decay as $\omega \to +\infty$.

We shall need a condition at one point in our main theorem which restricts to some extent the variability of a given function. Hence, before stating that result, we relate this condition to some stronger and more familiar assumptions. We recall first [2, p. 269] that f on $[0, +\infty)$ is called a *slowly varying* function if, for all $u > 0$,

$$\lim_{t \to \infty} f(tu)/f(t) = 1. \tag{1.6}$$

In addition, let a be a positive number, and on $[a, +\infty)$ let p be a positive function. Then p may have the property that

$$\limsup_{t \to +\infty} p(tu)/p(t) \quad \text{is bounded on } 1 \leqq u \leqq b \text{ for some } b > 1, \tag{1.7}$$

or the property that

$$p(t + p(t))/p(t) \quad \text{is bounded on } t_0 \leqq t < +\infty \text{ for some } t_0 \geqq a. \tag{1.8}$$

If $p(t)$ is nondecreasing in t then $\limsup_{t \to +\infty} p(tu)/p(t)$ is nondecreasing in u, so that (1.7) is equivalent to

$$\limsup_{t \to +\infty} p(bt)/p(t) < +\infty \quad \text{for some } b > 1. \tag{1.9}$$

THEOREM 1. *Of the following statements,* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *where p is a positive function on $[a, +\infty)$, and a is a positive number as before*:

(i) *$t^{-r}p(t)$ satisfies* (1.6) *for some $r < 1$;*

(ii) *p satisfies* (1.7) *and $p(t) = o(t)$ near $+\infty$;*

(iii) *p satisfies* (1.8) *and $p(t) = o(t)$ near $+\infty$.*

Proof. If p satisfies (i) then

$$\lim_{t \to +\infty} p(tu)/p(u) = u^r \quad \text{on } 1 \leqq u < +\infty \tag{1.10}$$

by hypothesis, and $p(t) = o(t)$ as required. If p satisfies (ii) then

$$p(t) \leqq (b - 1)t \quad \text{on } t_0 \leqq t < +\infty \tag{1.11}$$

for some $t_0 \geqq a$, so that $t + p(t) = tu$ in (1.8) for some u in $[1, b]$.

2. Estimate. If we mention $f(+\infty)$ for a function f on $[a, +\infty)$ then we mean

$$f(+\infty) = \lim_{t \to +\infty} f(t) \quad \text{(finite or infinite)} \tag{2.1}$$

and we imply that this limit exists. Our principal result, with this convention, can be stated as follows.

THEOREM 2. *Let f be a complex-valued locally integrable function on* $[0, +\infty)$, *and let* $f(t) = g(t)\exp[-iq(t)]$ *on* $[a, +\infty)$, *where* $a \geqq 0$, *g has bounded variation on* $[a, +\infty)$, *and* $g(+\infty) = 0$, *while q is C^2, q' is positive on* $[a, +\infty)$, *and* $q'(+\infty) = +\infty$. *Moreover let* $q''(t) = p(t)^{-2}$, *where p satisfies* (1.8), $p(t) = o(t)$ *near* $+\infty$, *p is positive and nondecreasing on* $[a, +\infty)$. *Then for large enough ω there exists a unique $t(\omega)$ such that*

$$q'(t(\omega)) = \omega; \tag{2.2}$$

and for any δ in $(0, 1)$,

$$\begin{aligned} F(\omega) &= o(1) + p(t(\omega))O[g(\delta t(\omega)) + p(t(\omega))/t(\omega)] \\ &= o[1 + p(t(\omega))] \quad \textit{as } \omega \to +\infty. \end{aligned} \tag{2.3}$$

Proof. By a previous result [1, Thm. 2], $F(\omega)$ is well-defined for large enough ω, and by the Riemann–Lebesgue lemma (e.g., [3, p. 11]),

$$\int_0^a \exp(i\omega t)f(t)\,dt = o(1) \quad \text{as } \omega \to +\infty. \tag{2.4}$$

Hence without loss of generality $f(t) = 0$ on $[0, a)$. By a previous lemma [1, Lemma 2], g is a linear combination of four functions, each nonnegative and nonincreasing with limit 0 at $+\infty$. Hence without loss of generality g itself may have this form.

Clearly q'' is positive so that q' is increasing, and $q'(+\infty) = +\infty$ so that $t(\omega)$ is well-defined. Indeed $t(\omega)$ is increasing and $t(+\infty) = +\infty$, so that

$$\delta t(\omega) \leqq t(\omega) - p(t(\omega)) \tag{2.5}$$

for large enough ω since $p(t) = o(t)$ by hypothesis. If $h(t) = \omega t - q(t)$ then $h'(t) = \omega - q'(t)$, so that $1/h'$ is increasing on $(a, t(\omega))$ and $-1/h'$ is decreasing on $(t(\omega), +\infty)$. Moreover,

$$\begin{aligned} \left| \int_{t(\omega)-p(t(\omega))}^{t(\omega)+p(t(\omega))} g(t)\exp[ih(t)]\,dt \right| &\leqq 2p(t(\omega))g(t(\omega) - p(t(\omega))) \\ &\leqq 2p(t(\omega))g(\delta t(\omega)). \end{aligned} \tag{2.6}$$

If η denotes a number in $(0, 1)$ which need not be the same at each appearance, then by the mean value theorem and a lemma of Titchmarsh [1, Lemma 4], [3, p. 22],

$$\begin{aligned} \left| \int_a^{\delta t(\omega)} g(t)\exp[ih(t)]\,dt \right| &= O[g(a)/h'(t(\omega))] \\ &= O[q'(t(\omega)) - q'(\delta t(\omega))]^{-1} \\ &= O[1/t(\omega)q''(\eta t(\omega))] \\ &= O[p(t(\omega))^2/t(\omega)] \quad \text{as } \omega \to +\infty \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
\left|\int_{\delta t(\omega)}^{t(\omega)-p(t(\omega))} g(t)\exp[ih(t)]\,dt\right| &= O[g(\delta t(\omega))/h'(t(\omega)-p(t(\omega)))] \\
&= g(\delta t(\omega))O[q'(t(\omega)) - q'(t(\omega)-p(t(\omega)))] \\
&= O[g(\delta t(\omega))/p(t(\omega))q''(t(\omega)-\eta p(t(\omega)))] \\
&= O[g(\delta t(\omega))p(t(\omega))] \quad \text{as } \omega \to +\infty .
\end{aligned}
\tag{2.8}
$$

Finally, by the same considerations together with property (1.8),

$$
\begin{aligned}
\left|\int_{t(\omega)+p(t(\omega))}^{\infty} g(t)\exp[ih(t)]\,dt\right| &= O[g(t(\omega)+p(t(\omega)))/h'(t(\omega)+p(t(\omega)))] \\
&= g(t(\omega))O[q'(t(\omega)+p(t(\omega))) - q'(t(\omega))]^{-1} \\
&= O[g(t(\omega))/p(t(\omega))q''(t(\omega)+p(t(\omega)))] \\
&= O[g(t(\omega))p(t(\omega)+p(t(\omega)))^2/p(t(\omega))] \\
&= O[g(t(\omega))p(t(\omega))] \quad \text{as } \omega \to +\infty .
\end{aligned}
\tag{2.9}
$$

We obtain the desired result by combining (2.6)–(2.9).

For the function (1.2) under the conditions (1.5), we find

$$
\begin{aligned}
g(t) &= t^{-r}(\log t)^n, \qquad & t(\omega) &= |\omega/\gamma\nu|^{1/(\nu-1)}, \\
q(t) &= -\gamma t^{\nu}, \qquad & p(t) &= (\gamma\nu(1-\nu))^{-1/2}t^{1-\nu/2}.
\end{aligned}
\tag{2.10}
$$

Then the functions of (2.10) satisfy the assumptions of Theorem 2, and

$$
\begin{aligned}
F(\omega) &= o(1) + \omega^{(2-\nu)/2(\nu-1)}O[\omega^{-r/(\nu-1)}(\log \omega)^n + \omega^{-\nu/2(\nu-1)}] \\
&= O[\omega^{(2-2r-\nu)/2(\nu-1)}(\log \omega)^n] \quad \text{as } \omega \to \infty .
\end{aligned}
\tag{2.11}
$$

Thus the result of Theorem 2 coincides with the direct estimate (1.4).

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OSCILLATIONS OF nth ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT*

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Abstract. This paper is concerned with giving conditions to guarantee the existence of an oscillatory solution to the equation $y^{(n)}(t) + (-1)^{n+1}p(t)y(g(t)) = 0$, where $g(t)$ is continuous, $t \geqq g(t)$, $g(t) \to \infty$ as $t \to \infty$, and $p(t)$ is a continuous nonnegative function. The conditions given here improve the previous results obtained for this equation when $n > 2$.

1. Introduction. In the study of oscillatory solutions of differential equations with retarded arguments, several authors [2], [4], [6], and [7] have recently considered equations whose corresponding ordinary equations have all nonoscillatory solutions.

For example, consider

$$y''(t) - y(t) = 0$$

which is nonoscillatory. On the other hand,

$$y''(t) - y(t - \pi) = 0$$

has an oscillatory solution $y(t) = \cos t$.

In § 3, we will give conditions that assure that there exist oscillatory solutions to the equation

$$y^{(n)}(t) + (-1)^{n+1}p(t)y(g(t)) = 0, \qquad n > 2, \tag{1.1}$$

where $p(t)$ is a positive continuous function, $g(t)$ continuous, $g(t) \to \infty$ as $t \to \infty$, and $g(t) < t$ on $[a, \infty)$.

Section 2 will be devoted to giving the general asymptotic and oscillatory behavior of solutions to (1.1). In this section the requirement that the continuous function $g(t)$ is strictly delay can be relaxed to allow $g(t)$ to be of the form $t + \sin t$, for example.

However, it is still an open question whether there are conditions that assure the existence of oscillatory solutions to (1.1) with $n = 2m$ and $g(t)$ is not delay.

Remark. We mention here that by "solution" we mean a solution on a half-ray $[T, \infty)$ to avoid any unintentional claims of extendability.

Remark. A solution $y(t)$ on $[T, \infty)$ is oscillatory if there exists a zero of $y(t)$ on every half-ray $[s, \infty)$, $T \leqq s < \infty$.

2. Properties of solutions. Recently Kamenskii [4] and Ladas, Lakshmikantham and Papodakis [7] have classified solutions of equation (1.1). The two classifications are equivalent but the notation varies considerably. In order to minimize difficulties, we will adopt the notation used in [7].

DEFINITION 2.1. Let S denote the set of all solutions of (1.1). For each integer l, $2 \leqq 2l \leqq n$, define

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(2.1a) $$S_l^{+\infty} = \{y(t) \in S : \lim_{t\to\infty} y^{(i)}(t) = +\infty \text{ for } i = 0, 1, 2, \cdots, 2l - 1\},$$

(2.1b) $$S_l^{-\infty} = \{y(t) \in S : -y(t) \in S_l^{+\infty}\},$$

(2.1c) $$S^0 = \{y(t) \in S : y(t) \neq 0 \text{ and } \lim_{t\to\infty} y^{(i)}(t) = 0 \text{ for } i = 1, 2, \cdots, n-1, \text{ monotonically as } t \to \infty\},$$

(2.1d) $$\tilde{S} = \{y(t) \in S : y(t) \text{ oscillates}\}.$$

An elementary fact which will be used throughout the remainder of the paper will now be stated and numbered for later use.

If $y(t)$ is a positive solution of (1.1) for $t > t_1$, then there exists an integer k, $0 \leqq k \leqq m$, such that

(2.2) $$y^{(i)}(t) > 0, \qquad i = 0, 1, \cdots, 2k-1,$$
and
$$(-1)^{(i)} y^{(i)}(t) > 0, \qquad i = 2k, \cdots, n,$$

where $n = 2m$ or $n = 2m + 1$, and t is sufficiently large, say $t \geqq t_2 > t_1$. If $k = 0$, then we use only the condition

$$(-1)^i y^{(i)}(t) > 0, \qquad i = 1, \cdots, n.$$

Before stating the main results of this section, it should again be pointed out that $g(t)$ need not be delay. In particular, if $[g(t)]^n p(t) < n!$ then there exists a unique solution to the initial value problems associated with (1.1) even when $g(t)$ is not delay. This can be proved using a technique of Ryder [10].

We now prove a group of lemmas which show that if $y(t)$ is a nonoscillatory solution of (1.1) and certain integral conditions on $g(t)$ and $p(t)$ hold, then $y(t) \in S_m^+ \cup S_m^- \cup S^0$ if $n = 2m$ and $y(t) \in S^0$ if n is odd.

LEMMA 2.1. *Assume $y(t)$ is a positive solution of* (1.1) *for $n = 2m$ satisfying* (2.2) *with $k \geqq 1$ and*

(2.3) $$\int^\infty [g(t)]^{2m-1-\varepsilon} p(t)\, dt = \infty \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1.$$

If $g'(t) > 0$ and $t > g(t)$, then $k = m$ in (2.2).

Proof. Assume $1 \leqq k < m$. Multiplying (1.1) by $[g(t)]^{2(m-k)-\varepsilon}/y^{(2k-1)}(g(t))$ gives

$$\frac{[g(t)]^{2(m-k)-\varepsilon} y^{(2m)}(t)}{y^{(2k-1)}(g(t))} = p(t)[g(t)]^{2(m-k)-\varepsilon} \frac{y(g(t))}{y^{(2k-1)}(g(t))}$$
$$\geqq Np(t)[g(t)]^{2m-1-\varepsilon} \qquad (N > 0)$$

by Lemma 4 in [3] where $l = 2k$ plays the role of n in the lemma.

Next, integrating from $t_3 > t_2$ to $t > t_3$, where t_2 is given in (2.2) and t_3 is such that $t \geqq t_3$ implies $g(t) \geqq t_2$, we find that

$$\int_{t_3}^{t} \frac{[g(s)]^{2(m-k)-\varepsilon} y^{(2m)}(s)\, ds}{y^{(2k-1)}(g(s))} \geqq N \int_{t_3}^{t} p(s)[g(s)]^{2m-1-\varepsilon}\, ds. \tag{2.4}$$

Integrating the left-hand side of (2.4) by parts, we have

$$\int_{t_3}^{t} \frac{[g(s)]^{2(m-k)-\varepsilon} y^{(2m)}(s)\, ds}{y^{(2k-1)}(g(s))}$$
$$\leqq K - \int_{t_3}^{t} \frac{[2(m-k)-\varepsilon][g(s)]^{2(m-k)-\varepsilon-1} y^{(2m-1)}(s) g'(s)\, ds}{y^{(2k-1)}(g(s))}.$$

Since $y^{(2m-1)}(s)$ is negative and increasing,

$$-y^{(2m-1)}(g(s)) \geqq -y^{(2m-1)}(s).$$

Hence,

$$\int_{t_3}^{t} \frac{[g(s)]^{2(m-k)-\varepsilon} y^{(2m)}(s)\, ds}{y^{(2k-1)}(g(s))}$$
$$\leqq K - \int_{t_3}^{t} \frac{[2(m-k)-\varepsilon][g(s)]^{2(m-k)-\varepsilon-1} y^{(2m-1)}(g(s)) g'(s)\, ds}{y^{(2k-1)}(g(s))}.$$

After $2(m-k)$ such steps, we have

$$\int_{t_3}^{t} \frac{[g(s)]^{2(m-k)-\varepsilon} y^{(2m)}(s)\, ds}{y^{(2k-1)}(g(s))}$$
$$\leqq \tilde{K} + \int_{t_3}^{t} \frac{[2(m-k)-\varepsilon]\cdots[1-\varepsilon][g(s)]^{-\varepsilon} y^{(2k)}(g(s)) g'(s)\, ds}{y^{(2k-1)}(g(s))}.$$

Now applying Lemma 4 in [3] to the right-hand side, we have

$$\int_{t_3}^{t} [g(s)]^{2(m-k)-\varepsilon} \frac{y^{(2m)}(s)}{y^{(2k-1)}(g(s))}\, ds$$
$$\leqq \tilde{K} + K_1 \int_{t_3}^{t} [g(s)]^{-1-\varepsilon} g'(s)\, ds \leqq \tilde{K} - \frac{K_1[g(t)]^{-\varepsilon}}{\varepsilon} + B.$$

Thus

$$N \int_{t_3}^{t} p(s)[g(s)]^{2m-1-\varepsilon}\, ds \leqq \tilde{K} - \frac{K_1[g(t)]^{-\varepsilon}}{\varepsilon} + B$$

which contradicts (2.3).

In order to prove a similar lemma when $g(t) > t - \alpha$, $\alpha > 0$, but $g'(t)$ is not necessarily positive, we need the following lemma.

LEMMA 2.2. *If $y(t)$ is a positive solution of* (1.1) *for $n = 2m$ and $g(t) > t - \alpha$, α a constant, with $1 \leqq k < m$, then*

$$\lim_{t\to\infty} \frac{y(g(t))}{y(t)} = 1.$$

The proof of the above lemma can be found in [1], so it will not be given here.

LEMMA 2.3. *Assume $y(t)$ is a positive solution of* (1.1) *for $n = 2m$ satisfying* (2.2) *with $k \geqq 1$ and*

$$\int^{\infty} t^{2m-1-\varepsilon} p(t)\,dt = \infty \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1.$$

Then if $g(t) > t - \alpha$, $k = m$ in (2.2).

Proof. Assume $1 \leqq k < m$. Multiplying (1.1) by $t^{2(m-k)-\varepsilon}/(y^{(2k-1)}(t))$, we have

$$\frac{t^{2(m-k)-\varepsilon} y^{(2m)}(t)}{y^{(2k-1)}(t)} = p(t) t^{2(m-k)-\varepsilon} \frac{y(g(t))}{y^{(2k-1)}(t)}$$
$$\geqq M p(t) t^{2m-1-\varepsilon}$$

by using Lemma 2.2 and Lemma 4 in [3]. We now proceed to integrate by parts in the same way as in Lemma 2.1 and reach a similar contradiction.

For convenience, we will classify $g(t)$ as follows:

(i) $g(t)$ is in class G_1 if $t > g(t)$, $g(t) \to \infty$ as $t \to \infty$ and $g'(t) > 0$;

(ii) $g(t)$ is in class G_2 if $t > g(t) > t - \alpha$, $\alpha > 0$.

LEMMA 2.4. *Assume $y(t)$ is a positive solution of* (1.1) *for $n = 2m$ satisfying* (2.2) *and*

$$\int^{\infty} [g(t)]^{2m-1-\varepsilon} p(t)\,dt = \infty \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1.$$

Then $y \in S_m^{+\infty}$ whenever $g(t)$ is in G_1 or G_2.

Proof. From Lemmas 2.1 and 2.2, $\lim_{t\to\infty} y^{(i)}(t) = \infty$, $i = 0, \cdots, 2m-2$, hence we need only show that

$$\lim_{t\to\infty} y^{(2m-1)}(t) = \infty.$$

Since $y^{(2m)}(t) > 0$, then

$$\lim_{t\to\infty} y^{(2m-1)}(t) \geqq M > 0.$$

We now consider two cases.

Case (i). If $t \geqq g(t)$ and $g'(t) \geqq 0$, then by L'Hospital's rule,

$$\lim_{t\to\infty} \frac{y(g(t))}{[g(t)]^{2m-1}} \geqq M.$$

Therefore, for t large we have

$$y^{(2m)}(t) \geqq p(t) \frac{M}{2} [g(t)]^{2m-1}.$$

Hence, integrating from T to $t > T$,

$$y^{(2m-1)}(t) - y^{(2m-1)}(T) \geqq \frac{M}{2} \int_T^t p(s)[g(s)]^{2m-1}\,ds,$$

and therefore,

$$y^{(2m-1)}(t) \to \infty \quad \text{as } t \to \infty.$$

Case (ii). If $g(t) \geqq t - \alpha$, then

$$y(g(t)) > y(t - \alpha) \geqq \frac{M}{2} t^{2m-1}$$

for t large. Thus we arrive at a similar conclusion.

Remark. For $g(t)$ in G_2 we can lift the restriction that $t > g(t)$ and we have the same results if

$$\int^{\infty} t^{2m-1-\varepsilon} p(t)\, dt = \infty \quad \text{for some } \varepsilon, \quad 0 < \varepsilon < 1.$$

LEMMA 2.5. *Assume $y(t)$ is a positive solution of* (1.1) *such that for $t \geqq t_0$,*

$$(-1)^{(i)} y^{(i)}(t) > 0, \qquad i = 0, \cdots, n,$$

and

$$\int^{\infty} [g(s)]^{n-1} p(s)\, ds = \infty.$$

Then $y(t) \in S^0$ for $g(t)$ in G_1 or G_2.

Proof. It follows directly as in [7] that

$$\lim_{t \to \infty} y^{(i)}(t) = 0, \qquad i = 1, \cdots, n-1.$$

Thus, we must show that $\lim_{t\to\infty} y(t) = 0$. In order to do this we integrate (1.1) from t to $T > t$:

$$y^{(n-1)}(T) - y^{(n-1)}(t) = (-1)^n \int_t^T p(s) y(g(s))\, ds,$$

$$(-1)^{n+1} y^{(n-1)}(t) = \int_t^{\infty} p(s) y(g(s))\, ds.$$

Hence, for $n = 2m$ we have after $2m - 1$ integrations,

$$(-1) y'(t) \geqq \int_t^{\infty} \frac{(u - t)^{n-2}}{(n-2)!} p(u) y(g(u))\, du,$$

and therefore,

$$-y(T) + y(t) \geqq \int_t^T \frac{(u - t)^{n-1}}{(n-1)!} p(u) y(g(u))\, du.$$

If $y(t) \to L > 0$ as $t \to \infty$ we have

$$-L + y(t) > \frac{L}{(n-1)!} \int_t^{\infty} (u - t)^{n-1} p(u)\, du = \infty,$$

which is a contradiction.

A similar argument shows that $y(t) \to 0$, as $t \to \infty$ when n is odd.

Thus, combining the above results we have the following theorem.

THEOREM 2.1. *If* $g(t)$ *is in class* G_1 *or* G_2, *and*

$$\int^{\infty} [g(s)]^{2m-1-\varepsilon} p(s)\, ds = \infty \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1,$$

then for $n = 2m$ *the solutions of equation* (1.1) *admit the decomposition*

$$S = S_m^{+\infty} \cup S_m^{-\infty} \cup S^0 \cup \tilde{S}.$$

Since the previous lemmas held for $g(t) \geqq t - \alpha$ and $g(t)$ not necessarily a delay, we have the following decomposition corollary.

COROLLARY 2.1. *If* $g(t) \geqq t - \alpha$ *and*

$$\int^{\infty} g(s)^{2m-1-\varepsilon} p(s)\, ds = \infty, \qquad 0 < \varepsilon < 1,$$

then for $n = 2m$ *the solutions of equation* (1.1) *admit the decomposition*

$$S = S_m^{+\infty} \cup S_m^{-\infty} \cup S^0 \cup \tilde{S}.$$

The above results show that $S^{+\infty} = \bigcup S_l^{+\infty}$ used in [7] is actually $S_m^{+\infty}$ and likewise for $S^{-\infty}$.

Equation (1.1) becomes

$$y^{(n)} + p(t)y(g(t)) = 0$$

when n is odd. When $g(t) = t$, the oscillatory properties of solutions were studied by Mikusinski [8]. During the preparation of the manuscript it was brought to the author's attention that Kusano and Onose [11] studied the oscillatory properties of (1.1), n odd, when $g'(t) \geqq 0$. In Kusano and Onose's paper the authors also corrected the slight mistake found in Mikusinski's original paper.

Using the results of Lemma 2.2, we are able to extend the results of Kusano and Onose to the case where $g(t)$ is not necessarily monotone or delay.

THEOREM 2.2. *If* $g(t)$ *is in class* G_1 *or* G_2 *and*

$$\int^{\infty} [g(t)]^{n-1-\varepsilon} p(t)\, dt = \infty \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1,$$

then solutions of (1.1) *with n odd admit the decomposition*

$$S = S^0 \cup \tilde{S}.$$

Proof. Let $y(t) = S - \tilde{S}$. Without loss of generality, we assume $y(t) > 0$ and $y(g(t)) > 0$ for $t > t_0$ and there exists k such that condition (2.2) holds with $0 \leqq k \leqq m$. The theorem will follow if we can show that $k = 0$.

Since the case $g(t) \in G_1$ is proved by Kusano and Onose, we take $g(t) \in G_2$.

Assume $k \geqq 1$ and multiply (1.1) by

$$\frac{(t)^{2(m-k)+1-\varepsilon}}{y^{(2k-1)}(g(t))}, \qquad \text{where } n = 2m + 1,$$

$g(t) \geqq t - \alpha$. Now using an argument similar to the one used in Lemmas 2.1 and 2.2, we obtain

$$\tilde{K}_2 + \tilde{K}_1 \int_{t_1}^{t} (s)^{-1-\varepsilon}\, ds \leqq - \int_{t_1}^{t} s^{2m-\varepsilon} p(s)\, ds.$$

Taking the limit as $t \to \infty$ leads to a contradiction since the right side tends to $-\infty$ as $t \to \infty$.

COROLLARY 2.2. *If* $g(t) > t - \alpha$ *and*

$$\int^{\infty} (t)^{n-1-\varepsilon} p(t)\, dt = \infty \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1,$$

then solutions of (1.1) *with n odd admit the decomposition*

$$S = S^0 \cup \tilde{S}.$$

In order to show that the result is sharp in Theorem 2.2, we see that

$$y^{(n)}(t) + \frac{1}{(t^n/2)[\ln t - \ln 2 - 1]} y\left(\frac{t}{2}\right) = 0$$

has a solution $y(t) = t[\ln t - 1]$ for n odd, which is not in $S^0 \cup \tilde{S}$. On the other hand for t large,

$$\int^{t} \frac{s^{n-1}\, ds}{s^n[\ln s - \ln 2 - 1]} = \int^{t} \frac{1}{s(\ln s - c)} ds$$
$$= \ln[\ln(t) - c] - c_1$$

which approaches ∞ as $t \to \infty$.

Remark. Kusano and Onose [11] also studied the equation $y^{(2m)}(t) + p(t)y(g(t))$ for $g(t)$ monotone and differentiable. Using the techniques in the proof of Theorem 2.2, we obtain the following generalization of their results.

THEOREM 2.3. *If* $g(t)$ *is in class* G_1 *or* G_2 *and*

$$\int^{\infty} [g(t)]^{2m-1-\varepsilon} p(t)\, dt = 0 \quad \textit{for some } \varepsilon, \quad 0 < \varepsilon < 1,$$

then solutions of $y^{(2m)}(t) + p(t)y(g(t)) = 0$ *with* $n = 2m$ *are in* $\tilde{S}$.

In order to complete the decomposition for delay equations, we now state two theorems given in [2] and [7] respectively.

THEOREM 2.4 [2]. *Assume* $p(t) > 0$ *and continuous,* $g(t)$ *is nondecreasing and continuous with* $t > g(t)$ *and* $g(t) \to \infty$ *as* $t \to \infty$. *If there exists a sequence* $t_n \to \infty$ *as* $n \to \infty$ *with*

$$\int_{g(t_n)}^{t_n} \frac{(r - g(t_n))^{n-1}}{(n-1)!} p(r)\, dr \geqq 1, \tag{2.5}$$

then the bounded solutions of (1.1) *are oscillatory.*

THEOREM 2.5 [7]. *Assume that the conditions of Theorem* 2.4 *hold with* $g'(t) \geqq 0$ *and*

$$\limsup_{t \to \infty} \int_{g(t)}^{t} [g(t) - g(s)]^{n-1} p(s)\, ds > (n-1)!. \tag{2.6}$$

Then the bounded solutions of (1.1) *are oscillatory.*

Combining the results of Theorems 2.1 and 2.2 with those of Theorems 2.4 and 2.5 we have the following corollaries.

COROLLARY 2.3. *If the conditions in Theorem* 2.1 *hold along with* (2.5) *or* (2.6), *then*

$$S = S_m^{+\infty} \cup S_m^{-\infty} \cup \tilde{S}$$

when $n = 2m$.

COROLLARY 2.4. *If the conditions of Theorem* 2.2 *hold along with* (2.5) *or* (2.6), *then every solution is oscillatory when* $n = 2m + 1$.

3. Existence of oscillatory solutions. In this section, we will follow the exposition and notation given in [7] very closely.

As pointed out in [7], it is known that the retarded equation (1.1), together with the initial conditions

$$\begin{aligned} y(t) &= \phi(t), \qquad 0 \leqq t \leqq t_0, \\ y^{(i)}(t_0) &= y_i, \qquad i = 1, 2, \cdots, 2m-2, \\ y^{(2m-1)}(t_0) &= A, \end{aligned} \tag{3.1}$$

where $\phi(t)$ is continuous on $[0, t_0]$, and y_i and A are real for $i = 1, 2, \cdots, 2m-2$, has a unique solution on $[t_0, \infty]$. From now on $\phi(t)$ and y_i will remain fixed and A will be allowed to vary. The unique solution will be denoted by $y(t, A)$. Kamenskii [4] introduced a classification for the unique solutions when $n = 2$ and it was adopted and modified slightly in [7]. We will use the same notation as in [7].

$$\begin{aligned} K^{+\infty} &= \{A \in R : y(t, A) \in S^{+\infty}\}. \\ K^{-\infty} &= \{A \in R : y(t, A) \in S^{-\infty}\}. \\ K^{0} &= \{A \in R : y(t, A) \in S^{0}\}. \\ \tilde{K} &= \{A \in R : y(t, A) \in \tilde{S}\}. \end{aligned}$$

If $S^{+\infty}$ ($S^{-\infty}$) is replaced by $S_m^{+\infty}$ ($S_m^{-\infty}$), then $K^{+\infty}$ ($K^{-\infty}$) will be replaced by $K_m^{+\infty}$ ($K_m^{-\infty}$).

Under the hypotheses of Theorem 2.1, we have that

$$R = K_m^{+\infty} \cup K_m^{-\infty} \cup K^0 \cup \tilde{K},$$

and if condition (2.5) or (2.6) holds, then that

$$R = K_m^{+\infty} \cup K_m^{-\infty} \cup \tilde{K}.$$

If $K_m^{-\infty}$ and $K_m^{+\infty}$ were open intervals $(-\infty, C)$ and (B, ∞), respectively, where $B > C$, then there would exist an oscillatory solution. As pointed out in [7], the proof given in [4] goes over to $n = 2m$, with only minor modifications.

We thus have the following theorem.

THEOREM 3.1. *Under the hypotheses of Theorem* 2.3 *and condition* (2.5) *or* (2.6), *the retarded differential equation* (1.1) *with* $n = 2m$ *has at least one oscillatory solution satisfying the initial conditions* (3.1).

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A LINEAR HYPERBOLIC PROBLEM ALL OF WHOSE SOLUTIONS ARE CONSTANT AFTER FINITE TIME*

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Abstract. A class of mixed initial boundary problems for a system of two linear hyperbolic differential equations in normal hyperbolic form is considered in the region $0 \leqq t < \infty$, $0 \leqq x \leqq 1$. The boundary conditions on $x = 0$ and $x = 1$ are linear and separated but of special form. It is proved that every continuously differentiable solution, no matter what the initial condition on $t = 0$, becomes identically constant after a finite time which is independent of the initial condition. The result also holds for solutions in a wider sense. An application of the theorem to electric transmission lines terminated by electric circuits is given.

1. Statement of theorem. Consider the differential equations

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + 2\frac{\partial u_1}{\partial x} = 0, \qquad 0 \leqq t < \infty, \quad 0 \leqq x \leqq 1, \\ &\frac{\partial u_2}{\partial t} - 2\frac{\partial u_2}{\partial x} = 0, \qquad 0 \leqq t < \infty, \quad 0 \leqq x \leqq 1. \end{aligned} \tag{1}$$

We shall consider the class of mixed problems with initial conditions

$$u_1(x,0) = f_1(x), \quad u_2(x,0) = f_2(x), \qquad 0 \leqq x \leqq 1, \tag{2}$$

and boundary conditions

$$\begin{aligned} u_1(0,t) + b(t)u_2(0,t) &= C_1 + C_2 b(t), \qquad t > 0, \\ a(t)u_1(1,t) + \frac{\partial u_2(1,t)}{\partial t} &= C_1 a(t), \qquad t > 0. \end{aligned} \tag{3}$$

The functions f_1, f_2, a, and b are assumed to be smooth, and C_1 and C_2 are constants. Moreover, we suppose that

$$a(t - \tfrac{1}{2})b(t - 1) = -\alpha(t), \qquad t \geqq 1, \tag{4}$$

where

$$\alpha(t) = \begin{cases} 0 & \text{for } t \in [2n - 1, 2n] \\ 2\sin^2 \pi t & \text{for } t \in [2n, 2n + 1] \end{cases} \quad (n = \text{integer}). \tag{5}$$

THEOREM 1. *Every continuously differentiable*[1] *solution of an initial boundary problem* (1)–(5) *becomes identically constant after a finite time. That is, there exists a positive number T such that every solution satisfies*

$$u_1(x,t) = C_1, \quad u_2(x,t) = C_2, \qquad T \leqq t, \quad 0 \leqq x \leqq 1. \tag{6}$$

The number T does not depend on f_1 or f_2.

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[1] The result also holds for a solution in the wider sense defined in § 3.

In other words, backward continuation of a solution of the initial boundary problem (1), (3) from the constant state $u_1 = C_1$, $u_2 = C_2$, is not unique; many initial conditions lead to the same state. For $C_1 = C_2 = 0$, we have a linear homogeneous problem with the property that all solutions become identically zero after a finite time (this time being independent of the initial conditions).

In § 4, we will interpret this problem as describing an electric transmission line terminated at its ends by certain electrical circuits. The functions $a(t)$, $b(t)$ describe the physical parameters of these circuits. From this point of view, the theorem asserts that these circuits can be chosen so that the voltage and current on the transmission line reach prescribed constant levels in finite time, no matter what their initial values were.

The theorem will be proved in § 3 by reducing the mixed problem to a pair of differential-difference equations by the technique in [1], and then using a result of Winston and Yorke [2]. Actually, there are a large number of examples of unexpected behavior for delay-differential and functional differential equations, many of which are listed by Hale in [3]. The relation between these equations and hyperbolic mixed problems, shown in [1], [4], makes it possible to construct a similar set of examples for hyperbolic systems.

2. The example of Winston and Yorke. Consider the differential-difference equation

$$y'(t) = -\alpha(t)y(t-1), \tag{7}$$

where α is defined by (5). Let $y(t)$ be any continuous solution existing on an interval $(t_0, 2N-1)$ for an integer N, $2N - 1 \geqq t_0 + 1$. Then y has a continuation to $[2N-1, 2N]$ given by (since $y'(t) = 0$)

$$y(t) = y(2N-1), \qquad 2N - 1 \leqq t \leqq 2N.$$

Then on $2N < t < 2N + 1$ we have

$$\begin{aligned} y'(t) &= -y(2N-1)\alpha(t), \\ y(t) &= y(2N) - y(2N-1)\int_{2N}^{t} \alpha(s)\,ds \\ &= y(2N-1)\left(1 - t + 2N + \frac{\sin 2\pi t}{2\pi}\right). \end{aligned}$$

Therefore $y(2N+1) = 0$. Since $y'(t) = 0$ for $2N + 1 \leqq t \leqq 2N + 2$, it now follows that $y(t) = 0$ for all $t \geqq 2N + 1$.

3. Reduction of the mixed problem. We now indicate how to reduce the mixed problem in § 1 to the example of Winston and Yorke. We first observe that the change of variable

$$\tilde{u}_1(x,t) = u_1(x,t) - C_1, \qquad \tilde{u}_2(x,t) = u_2(x,t) - C_2,$$

reduces equations (1) and (3) to equations of the same form with C_1 and C_2 replaced by zero, and with the same coefficients in the equations. Therefore, it will suffice to consider (3) with $C_1 = 0$, $C_2 = 0$, and to show that every solution becomes identically zero after a finite time.

Let us consider more carefully what we will mean by a solution. We will not require differentiability of u_1 and u_2 but rather we interpret (1) in the wider sense of Friedrichs. That is, we require that the integrated form of (1) be satisfied. In the present case, this means that u_1 must be constant along characteristics of positive slope, $2t - x = \text{const.}$, and that u_2 must be constant along characteristics of negative slope, $2t + x = \text{const.}$ Let C_k^+ and C_k^- denote the characteristics $2t - x = k - 1$ and $2t + x = k$ $(k = 1, 2, \cdots)$, respectively. We see that the initial functions f_1 and f_2 determine $u_1(x, t)$ below C_1^+ and $u_2(x, t)$ below C_1^-. We assume that f_1 and f_2 and hence $u_1(x, t)$ and $u_2(x, t)$ are continuous. In particular, $u_1(1, t)$ and $u_2(0, t)$ are continuous on $0 \leqq t \leqq 1/2$. The first equation in (3) then determines

$$u_1(0, t) = -b(t)u_2(0, t), \qquad 0 < t \leqq \tfrac{1}{2}.$$

Since u_1 is constant on characteristics of positive slope, this determines $u_1(x, t)$ in the strip between C_1^+ and C_2^+. The function $u_1(x, t)$ will be continuous in the strip, but will be discontinuous across C_1^+ unless the corner condition

$$f_1(0) + b(0)f_2(0) = 0$$

is satisfied. Similarly, from the second equation in (3) we find $\partial u_2(1, t)/\partial t$ for $0 < t \leqq 1/2$. Requiring $u_2(1, 0) = f_2(1)$, we then obtain $u_2(1, t)$ for $0 \leqq t \leqq 1/2$. From this, $u_2(x, t)$ is found in the strip between C_1^- and C_2^-. The derivative $\partial u_2(x, t)/\partial t$ will be continuous in the strip but discontinuous along C_1^- unless the corner condition

$$a(0)f_1(1) + 2f'_2(1) = 0$$

is satisfied.

We can continue step by step, using (3) and the fact that u_1 and u_2 are constant on characteristics of positive and negative slope, respectively. In this way, we can establish the existence of unique functions u_1 and u_2 which satisfy (2) and the following conditions:

(i) $u_1(x, t)$ is constant on characteristics of positive slope and $u_2(x, t)$ is constant on those of negative slope;

(ii) $u_1(x, t)$ is continuous except perhaps across the characteristics C_k^+, $k = 1, 2, \cdots$; $u_2(x, t)$ is everywhere continuous; $\partial u_2(x, t)/\partial t$ exists and is continuous except perhaps across the characteristics C_k^-, $k = 1, 2, \cdots$;

(iii) equations (3) are satisfied for all $t > 0$ except at discontinuity points in $\{t : t = \frac{1}{2}, 1, \cdots\}$.

A pair of functions u_1, u_2 which satisfy these conditions will be called a solution of (1), (2), (3). Note that any continuously differentiable solution is also a solution in this wider sense. Because of the special form of (3), a solution becomes smoother than required by (ii). Indeed, continuity of $u_2(1, t)$ for $t > 0$ implies continuity of $u_2(0, t)$ for $t > \frac{1}{2}$ by (i), and hence, by (3), continuity of $u_1(0, t)$ for $t > \frac{1}{2}$. This, in turn, implies continuity of $u_1(1, t)$ and $\partial u_2(1, t)/\partial t$ for $t > 1$.

For any solution, it follows from (i) that

$$u_1(1, t + \tfrac{1}{2}) = u_1(0, t), \qquad u_2(0, t + \tfrac{1}{2}) = u_2(1, t), \qquad t \geqq 0. \tag{8}$$

Let

$$y_1(t) = u_1(1, t), \qquad y_2(t) = u_2(0, t), \qquad t \geqq 0. \tag{9}$$

Then

(10) $$u_1(0,t) = y_1(t+\tfrac{1}{2}), \qquad u_2(1,t) = y_2(t+\tfrac{1}{2}), \qquad t \geqq 0.$$

Introducing (9) and (10) into the boundary conditions (3) (with $C_1 = C_2 = 0$), we obtain

(11) $$\begin{aligned} &y_1(t+\tfrac{1}{2}) + b(t)y_2(t) = 0, \qquad t > 0,\\ &a(t)y_1(t) + y_2'(t+\tfrac{1}{2}) = 0, \qquad t > 0. \end{aligned}$$

From these equations we obtain

$$y_2'(t) = a(t-\tfrac{1}{2})b(t-1)y_2(t-1), \qquad t > 1.$$

Because of equation (4), it follows that $y_2(t)$ is a solution of the Winston–Yorke equation,

(12) $$y_2'(t) = -\alpha(t)y_2(t-1), \qquad t > 1.$$

Since $y_2(t) = u_2(0,t)$ is continuous for $t > \frac{1}{2}$, it follows that there is a number T, independent of the initial conditions, such that $y_2(t) = 0$ for $t \geqq T - \frac{1}{2}$. Then the first equation in (11) yields $y_1(t) = 0$ for $t \geqq T$. From (9) it follows that $u_1(1,t)$ and $u_2(0,t)$ are zero for $t \geqq T$. Finally, since $u_1(x,t)$ and $u_2(x,t)$ are constant along characteristics, they are zero for $t \geqq T$. More precisely, $u_1(x,t) = 0$ above the characteristic of positive slope through the point $(1, T)$ and $u_2(x,t) = 0$ above the characteristic of negative slope through $(0, T-\frac{1}{2})$.

4. Physical interpretation. Mixed problems with boundary conditions of the form in (3) may arise from problems of wave transmission. For example, the transmission line equations

(13) $$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}, \qquad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t},$$

can be transformed into (1) by the change of variable

(14) $$v = u_1 - u_2, \qquad i = u_1 + u_2,$$

when $L = C = \frac{1}{2}$. The boundary conditions in (3) correspond to the conditions

(15) $$[1 - b(t)]v(0,t) + [1 + b(t)]i(0,t) = 2[C_1 + C_2 b(t)],$$

(16) $$\frac{\partial}{\partial t}[i(1,t) - v(1,t)] = 2C_1 a(t) - a(t)[i(1,t) + v(1,t)].$$

It will now be shown that these boundary conditions can be realized by appropriate networks at the two ends of the transmission line. For the network in Fig. 1, Kirchhoff's laws yield

(17) $$\begin{aligned} i_2 &= \frac{d}{dt}(Cv_1) + \frac{1}{R_3}v_1,\\ i_2 &= \frac{1}{R_2}(v - v_1),\\ i_2 &= i - \frac{1}{R_1}v. \end{aligned}$$

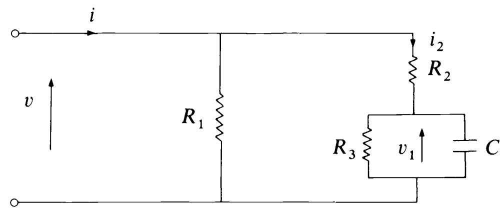

FIG. 1

Choose

$$
(18) \qquad \begin{aligned} R_1(t) &= \frac{1}{1 - 2a(t)}, \qquad & R_2(t) &= \frac{1}{2a(t)}, \\ R_3(t) &= -\frac{1}{2a(t)[1 - a(t)]}, \qquad & C(t) &= 2a(t). \end{aligned}
$$

Then from the latter two equations in (17) we obtain

$$
(19) \qquad v_1 = \frac{v - i}{2a(t)}.
$$

Substituting this expression for v_1 and using the first and third equations in (17), we obtain

$$
\frac{d}{dt}(v - i) - [1 - a(t)](v - i) = i - [1 - 2a(t)]v
$$

or

$$
\frac{di}{dt} - \frac{dv}{dt} = -a(t)(i + v).
$$

Since this corresponds to (16) for $C_1 = 0$, the circuit in Fig. 1 may be placed at the end of the transmission line at $x = 1$. The same argument shows that the network in Fig. 2 is a realization of (16) when $C_1 \neq 0$. The added element is an ideal current source.

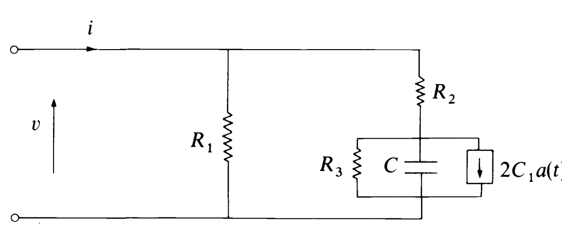

FIG. 2

For the network in Fig. 3 we have

$$
(20) \qquad -v = -E_0 + R_0 i.
$$

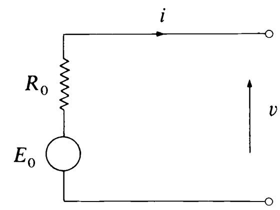

FIG. 3

Therefore, (15) can be realized by Fig. 3 when

$$R_0 = \frac{1 + b(t)}{1 - b(t)}, \qquad E_0 = \frac{2(C_1 + C_2 b(t))}{1 - b(t)}. \tag{21}$$

From the physical point of view, we prefer $C(t) \geqq 0$, and in view of (18) and (21) we want to have $a(t)$ nonzero, $b(t) \neq 1$. One possible choice is $a(t) = a$, where a is a constant greater than 2. Then from (4), $b(t)$ is proportional to $\alpha(t + 1)$ and we have $-1 < b(t) \leqq 0$. Therefore $R_0 > 0$, $R_1 < 0$, $R_2 > 0$, $R_3 > 0$, $C > 0$. The negative resistance can be built using active circuit elements. For the transmission line terminated as in Fig. 2 and Fig. 3, it follows from Theorem 1 that

$$v(x, t) = C_1 - C_2, \qquad i(x, t) = C_1 + C_2 \tag{22}$$

for $0 \leqq x \leqq 1$, $T \leqq t$. Thus, every initial state of the line is controlled to the constant state in (22) in finite time.

5. Extensions. Arguments of the above sort can be applied to mixed problems (1), (2), with boundary conditions of the form

$$\begin{aligned} &\frac{\partial u_1(0, t)}{\partial t} + b(t)u_2(0, t) = C_2 b(t), \qquad t \geqq 0, \\ &a(t)\frac{\partial u_1(1, t)}{\partial t} + \frac{\partial u_2(1, t)}{\partial t} = 0, \qquad t \geqq 0, \end{aligned} \tag{23}$$

under the condition in (4) on a and b. Indeed, Theorem 1 remains valid for this case, in the following sense.

THEOREM 2. *Let a and b satisfy* (4). *There exists a positive number T such that every continuously differentiable solution of* (1), (2), (23) *satisfies* (6) *for some constant C_1.*

Proof. The substitution $\tilde{u}_2(x, t) = u_2(x, t) - C_2$ shows that it suffices to prove the theorem in the case $C_2 = 0$. By use of equations (8), (9), (10), we can reduce the boundary conditions to

$$\begin{aligned} &y_1'(t + \tfrac{1}{2}) + b(t)y_2(t) = 0, \qquad t > 0, \\ &a(t)y_1'(t) + y_2'(t + \tfrac{1}{2}) = 0, \qquad t > 0. \end{aligned} \tag{24}$$

From these equations we obtain

$$y_2'(t) = a(t - \tfrac{1}{2})b(t - 1)y_2(t - 1).$$

As before, we deduce that $y_2(t) = 0$ for $t \geqq T - \frac{1}{2}$. The first equation in (24) now yields $y_1'(t) = 0$ for $t \geqq T$. Hence $y_1(t) = C_1$ for some constant C_1 for $t \geqq T$. We deduce that $u_1(x, t) = C_1$ and $u_2(x, t) = 0$ for $t \geqq T$.

It should be possible to obtain similar examples with simpler boundary conditions if we permit the characteristics (that is, the coefficients in (1)) to be time-varying. Theorem 2 is valid for solutions in a wider sense, not just continuously differentiable solutions.

Another possibility is the construction of an analogue for partial differential equations of the examples of pointwise degeneracy for differential delay equations due to Popov and Zverkin (see [5] and [6]).

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DUAL ORTHOGONAL SERIES*

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Abstract. The notion of a dual series problem, as exemplified by dual trigonometric series, dual Bessel series, etc., is generalized to a dual orthogonal series problem in an abstract Hilbert space. Basic existence and uniqueness theorems are proved for the abstract problem. The essence of the analytic procedure is to translate the original problem into a problem in ℓ^2 in which setting it is shown, for a certain class of problems, that Fredholm's alternative is applicable. We apply these results to dual orthogonal series associated with potential problems with mixed boundary conditions. The general theory covers, in an L^2 sense, most dual orthogonal series associated with mixed boundary conditions of the second and third kind. For such dual orthogonal series there are few restrictions, e.g., the kernel of the series can be any complete orthonormal sequence of Sturm–Liouville type, the series can be a single or multiple series, and the potential problem from which the dual orthogonal series is derived can occur in a bounded or an unbounded domain. Approximations are achieved through the principle of reduction and are rigorously shown to converge. Earlier numerical experiments have shown this procedure to be practical. A number of examples are given primarily associated with the type of boundary condition occurring in heat transfer theory.

1. Introduction. During the past two decades dual orthogonal series have been studied intensively. The development of formal solutions to particular problems has forged far ahead of rigorous theory and general methods because of the pressing demand for technologically useful results in diverse fields such as crack theory, heat transfer, design of microwave guides, etc.[1] This has raised many interesting mathematical questions to which we here respond, in part, with a rigorous theory for a general class of dual orthogonal series.

Our aim in these studies is construction of a theory covering dual orthogonal series used in applications—and these are dual Sturm–Liouville series, e.g., dual trigonometric series, dual Bessel series, etc. However, out of necessity we have chosen an abstract approach: progress has been facilitated by stripping away the inessentials which accumulate quickly when using special functions. The abstract approach has in this case the added advantage of providing a general criterion to which special cases can be fitted, thereby obviating the need for a new analysis for each new dual equation as has generally been necessary heretofore. The merit of this is enhanced by the frequent appearance of new dual orthogonal series resulting from new physical applications, e.g., [3], [4], [5], [6], [7], [8], [9].

Therefore, basic to our approach is the formulation in § 2 of the dual orthogonal series problem in an abstract Hilbert space and the solution given to this problem in § 3. These results are applied to dual Sturm–Liouville series in § 4, and some examples are presented in § 5.

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[1] A summary of the field through 1964 can be found in [1], the standard reference for dual problems. Many physical problems leading to dual orthogonal series and dual integral equations are described in [1], [2]. The reduction of potential problems to problems in dual orthogonal series is cursorily sketched in § 5 of this paper.

Briefly, from §§ 4 and 5, one sees that our results provide an L^2 existence, uniqueness, and approximation theory for dual Sturm–Liouville equations associated with potential problems in which the mixed boundary conditions are of the second and third kind (boundary conditions are defined in the Appendix). The theory does not apply if one of the mixed conditions is of the first kind. On the other hand, the boundary conditions associated with the Sturm–Liouville kernel generating the dual orthogonal series are immaterial—indeed, they can be any combination of conditions of the first three kinds or conditions appropriate to a singular Sturm–Liouville problem. We also obtain results for multiple dual orthogonal equations, such as equation (5.6a–b), which, to our knowledge, has not been done before.

Approximations to solutions are obtained by using the principle of reduction so that obtaining an Nth order approximation is equivalent to solving an Nth order system of linear algebraic equations, viz., (3.10) or (3.13). Numerical experiments for dual trigonometric series based on these equations have been carried out and show the procedure to be practical. The results have been reported earlier [7], [8]. In fact, we were stimulated to write this paper by a desire for a rigorous theory for these calculations. In § 6 areas for future research are discussed.

Included is a brief appendix with definitions of boundary conditions used in mixed boundary value problems. Nothing is new or untraditional, but we have arranged matters in a form suitable for our purposes.

2. Problem formulation. We denote by $\mathbf{R}$ a real, separable, abstract Hilbert space[2] and by ℓ^2 the Hilbert space of all real, infinite column vectors $\imath = (\imath_1, \imath_2, \cdots)$ such that $\sum \imath_i^2 < \infty$. The inner product and norm in both $\mathbf{R}$ and ℓ^2 are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$ respectively. If T is an operator, then $D(T)$ signifies its domain. Infinite matrix operators on ℓ^2 will be denoted by upper-case script letters, e.g., $\mathscr{T} = (\mathscr{T}_{kn} : k, n = 1, 2, \cdots)$, with the transformation $\mathscr{s} = \mathscr{T}\imath$ being formally defined by $\mathscr{s}_k = \sum_n \mathscr{T}_{kn}\imath_n$. One defines $D(\mathscr{T})$ as the set of $\imath \in \ell^2$ such that $\mathscr{T}\imath \in \ell^2$.

Let $\{\phi_n : n = 1, 2, \cdots\}$ be a complete orthonormal sequence in $\mathbf{R}$ and $\{a_n\}$ and $\{b_n\}$ sequences of nonnegative constants. The subspaces $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbf{R}$ are orthogonal complements [10, p. 12]. P and Q denote, respectively, the projection operators from $\mathbf{R}$ onto $\mathbf{P}$ and $\mathbf{Q}$.

The dual orthogonal series problem is this: Given $f \in \mathbf{R}$, find $\jmath \in \ell^2$ such that

$$\lim_{N \to \infty} \left\| \sum_{n=1}^{N} \jmath_n(a_n P\phi_n + b_n Q\phi_n) - f \right\| = 0. \tag{2.1}$$

When for brevity we write

$$\sum_{n=1}^{\infty} \jmath_n(a_n P\phi_n + b_n Q\phi_n) = f, \tag{2.2}$$

we mean always the relation (2.1). The sequence $\{\phi_n\}$ is called the *kernel* of the dual orthogonal series, and the sequences $\{a_n\}$ and $\{b_n\}$ *modifiers*.

[2] We follow the usual definitions for Hilbert space. Undefined terms can be found in standard textbooks on functional analysis, e.g., [10], [11].

Before proceeding we need to recall some preliminary results related to matrix operators. $\{\phi_n\}$ generates an isomorphism between $\mathbf{R}$ and ℓ^2 defined by the mapping $r \leftrightarrow \imath$, where $\imath_n = (r, \phi_n)$ [10, § 19]. $\{\phi_n\}$ generates an isomorphism between the algebra of bounded operators T on R and bounded matrix operators $\mathscr{T}$ on ℓ^2 defined by the mapping $T \leftrightarrow \mathscr{T}$ where $\mathscr{T}_{kn} = (\phi_k, T\phi_n)$ [10, § 26]. If $r \leftrightarrow \imath$ and $T \leftrightarrow \mathscr{T}$ then $Tr \leftrightarrow \mathscr{T}\imath$.

3. Existence and approximation in R. Our principal goal in this section is to prove the following theorem.

THEOREM 1. *If $\{a_n\}$ and $\{b_n\}$ are sequences of positive constants one of which is bounded above zero, and if there is a positive constant α such that $a_n/b_n \to \alpha$ as $n \to \infty$, then the dual orthogonal series equation* (2.2) *has a unique solution $\jmath \in \ell^2$. In fact, if $\mathscr{A}$ and $\mathscr{B}$ are diagonal matrices defined by $\mathscr{A}_{nn} = a_n$ and $\mathscr{B}_{nn} = b_n$, then*

$$\jmath \in D(\mathscr{A}) \cap D(\mathscr{B}). \tag{3.1}$$

Proof. Without loss of generality we assume $\{a_n\}$ is bounded above zero. Therefore, $\mathscr{A}^{-1}$ is a bounded operator with domain ℓ^2. We make the change of variable

$$\jmath = \mathscr{A}^{-1}\kappa \tag{3.2}$$

which transforms (2.2) into

$$\sum_{n=1}^{\infty} \kappa_n\left(P\phi_n + \frac{b_n}{a_n}Q\phi_n\right) = f. \tag{3.3}$$

Set $c_n = \alpha b_n/a_n$ and $g = Pf + \alpha Qf$. Then (3.3) is equivalent to the dual orthogonal equation

$$\sum_{n=1}^{\infty} \kappa_n(P\phi_n + c_nQ\phi_n) = g \tag{3.4}$$

in the sense that κ is a solution to (3.3) if and only if it is a solution to (3.4).

Let $\mathscr{P}$ and $\mathscr{Q}$ be defined by $P \leftrightarrow \mathscr{P}$ and $Q \leftrightarrow \mathscr{Q}$ under the mapping given in § 2. Then $\mathscr{P}$ and $\mathscr{Q}$ are projection operators on ℓ^2 such that $\mathscr{P} + \mathscr{Q} = \mathscr{I}$, where $\mathscr{I}$ is the identity matrix [12, p. 16]. We take the inner product of both sides of (3.4) with ϕ_n leading formally to the equation

$$(\mathscr{P} + \mathscr{Q}\mathscr{C})\kappa = \jmath, \tag{3.5}$$

where $\mathscr{C}$ is the diagonal matrix given by $\mathscr{C}_{nn} = c_n$, and $g_n = (g, \phi_n)$. Since $\mathscr{C}$ is a bounded operator, (3.5) can be (rigorously) rewritten as

$$(\mathscr{I} + \mathscr{Q}(\mathscr{C} - \mathscr{I}))\kappa = g. \tag{3.6}$$

Since $\mathscr{C}_{nn} - 1 \to 0$ as $n \to \infty$, $\mathscr{C} - \mathscr{I}$ is completely continuous [10, p. 63]. Therefore, $\mathscr{Q}(\mathscr{C} - \mathscr{I})$ is completely continuous. Consequently, by Fredholm's alternative [11, pp. 163 and 201] either the equation

$$(\mathscr{I} + \mathscr{Q}(\mathscr{C} - \mathscr{I}))m = 0 \tag{3.7}$$

has a solution $\mathscr{m} \neq 0$, or (3.6) has a unique solution for each $\mathscr{g} \in \ell^2$. Let us show that $\mathscr{m} = 0$ is the only solution to (3.7). If $\mathscr{m}$ satisfies (3.7), then

$$(\mathscr{P} + \mathscr{Q}\mathscr{C})\mathscr{m} = 0 \tag{3.8}$$

which implies $\mathscr{P}\mathscr{m} = 0$ and $\mathscr{Q}\mathscr{C}\mathscr{m} = 0$. Therefore,

$$(\mathscr{m}, \mathscr{C}\mathscr{m}) = (\mathscr{Q}\mathscr{m}, \mathscr{C}\mathscr{m}) = (\mathscr{m}, \mathscr{Q}\mathscr{C}\mathscr{m}) = 0.$$

If $\mathscr{m} \neq 0$, then $(\mathscr{m}, \mathscr{C}\mathscr{m}) > 0$. Hence, $\mathscr{m} = 0$ is the only solution to (3.7), and by Fredholm's alternative, (3.6) has a unique solution $\mathscr{k}$. Let us show that $\mathscr{k}$ satisfies (3.4). We introduce

$$h_n = \sum_{i=1}^{n} \mathscr{k}_i(P\phi_i + c_i Q\phi_i).$$

A little algebraic manipulation shows that

$$\|h_n - h_k\|^2 \leqq \sum_{i=k+1}^{n} \mathscr{k}_i^2(1 + c_i^2).$$

Since $\{c_n\}$ is bounded, $\{h_n\}$ is a strongly convergent sequence. Equation (3.5) implies that $h_n \to g$ weakly. Hence, $h_n \to g$ strongly, and $\mathscr{k}$ is the unique solution to (3.4) from which it follows that $\mathscr{j}$ is the unique solution to (2.2). Clearly, $\mathscr{j} \in D(\mathscr{A})$ which shows also that $\mathscr{j} \in D(\mathscr{B})$. This completes the proof.

The following extension of Theorem 1 is especially useful when the kernel $\{\phi_n\}$ has been generated by boundary conditions of the second or periodic kind. In such cases we often have a_1 or b_1 equal to zero, corresponding to an eigenvalue associated with ϕ_1 being zero.

THEOREM 2. *Let $\{a_n\}$ be a sequence of constants bounded above zero. Let $\{b_n\}$ be a sequence of nonnegative constants and N a positive integer such that $b_n > 0$ for $n = N + 1, N + 2, \cdots$, and for some positive constant $\alpha, a_n/b_n \to \alpha$ as $n \to \infty$. Let the subspace spanned by the N-dimensional vectors $(\mathscr{P}_{k1}, \mathscr{P}_{k2}, \cdots, \mathscr{P}_{kN})$, $k = 1, 2, \cdots$, have dimension N. Then the dual orthogonal equation* (2.2) *has a unique solution $\mathscr{j}$ satisfying* (3.1). *The theorem remains valid if the roles of a_n and b_n and $\mathscr{P}$ and $\mathscr{Q}$ are interchanged.*

Proof. Under the above hypothesis we see that the preceding proof breaks down at the point following (3.8), since it no longer follows that $\mathscr{m} \neq 0$ implies $(\mathscr{m}, \mathscr{C}\mathscr{m}) > 0$. Let us assume then that we have carried the preceding proof from its beginning to (3.8). Then $(\mathscr{m}, \mathscr{C}\mathscr{m}) = 0$ implies $\mathscr{m}_n = 0$ for $n = N + 1, N + 2, \cdots$. Since (3.8) implies $\mathscr{P}\mathscr{m} = 0$, we must have

$$\mathscr{P}_{k1}\mathscr{m}_1 + \mathscr{P}_{k2}\mathscr{m}_2 + \cdots + \mathscr{P}_{kN}\mathscr{m}_N = 0, \qquad k = 1, 2, \cdots. \tag{3.9}$$

However, the fact that the span of $(\mathscr{P}_{k1}, \mathscr{P}_{k2}, \cdots, \mathscr{P}_{kN})$ is N-dimensional implies that $\mathscr{m}_i = 0, i = 1, 2, \cdots, N$, is the only solution to the system of equations (3.9). Therefore, $\mathscr{m} = 0$ is the only solution to (3.8) so that by Fredholm's alternative, (3.6) has a unique solution, and the proof can be completed as in Theorem 1.

To compute approximations we take $N \times N$ sections of (3.6). The validity of doing this is guaranteed by the following theorem which asserts that approximants so derived converge strongly to $\mathscr{j}$.

THEOREM 3. *Assume the hypothesis of Theorem* 1 *or* 2. *If* $\{a_n\}$ *is bounded above zero, then for all sufficiently large integers N the reduced form of* (3.6), *viz.*,

$$\mathscr{k}_k(N) + \sum_{n=1}^{N} \mathscr{Q}_{kn}\left(\frac{\alpha b_n}{a_n} - 1\right)\mathscr{k}_n(N) = \mathscr{g}_k, \qquad k = 1, 2, \cdots, N, \tag{3.10}$$

possesses a unique solution, $\mathscr{k}_1(N), \mathscr{k}_2(N), \cdots, \mathscr{k}_N(N)$, *such that*

$$\lim_{N\to\infty} \sum_{n=1}^{N} (\mathscr{k}_n(N) - \mathscr{k}_n)^2 = 0, \tag{3.11}$$

where $\mathscr{k}$ *is the solution to* (3.6). *If* $\mathscr{j}_i(N) = a_i^{-1}\mathscr{k}_i(N), i = 1, 2, \cdots, N$, *then*

$$\lim_{N\to\infty} \sum_{n=1}^{N} (\mathscr{j}_n(N) - \mathscr{j}_n)^2 = 0, \tag{3.12}$$

where $\mathscr{j}$ *is the solution to* (2.2). *Let* $h = \alpha^{-1}Pf + Qf$ *and* $\mathscr{h}_n = (h, \phi_n)$. *If* $\{b_n\}$ *is bounded above zero, for sufficiently large N the reduced equation*

$$\mathscr{k}_k(N) + \sum_{n=1}^{N} \mathscr{P}_{kn}\left(\frac{1}{\alpha}\frac{a_n}{b_n} - 1\right)\mathscr{k}_n(N) = \mathscr{h}_k, \qquad k = 1, 2, \cdots, N, \tag{3.13}$$

has a unique solution such that if $\mathscr{j}_n(N) = b_n^{-1}\mathscr{k}_n(N), n = 1, 2, \cdots, N$, *then* $\mathscr{j}_n(N)$ *satisfies* (3.12), *where* $\mathscr{k}$ *is the unique solution to*

$$\left(\mathscr{I} + \mathscr{P}\left(\frac{1}{\alpha}\mathscr{A}\mathscr{B}^{-1} - \mathscr{I}\right)\right)\mathscr{k} = \mathscr{h}$$

and $\mathscr{j} = \mathscr{B}^{-1}\mathscr{k}$.

Proof. If $\{a_n\}$ is bounded above zero, then, as we saw in the proof of Theorem 1, $\mathscr{C} - \mathscr{I}$ is completely continuous. The validity of (3.11) then follows at once from the principle of reduction [13, p. 1433]. Since

$$\sum_{n=1}^{N} (\mathscr{j}_n(N) - \mathscr{j}_n)^2 = \sum_{n=1}^{N} a_n^{-2}(\mathscr{k}_n(N) - \mathscr{k}_n)^2,$$

the relation (3.11) implies (3.12). The argument is similar if $\{b_n\}$ is bounded above zero.

4. Dual Sturm–Liouville problems. In this section we relate the previous results to dual Sturm–Liouville problems. The transition from Theorems 1 and 2 to Theorem 4 is elementary. As the reader will see, all we need do is describe correctly in this concrete setting the appropriate realizations of objects in **R**.

We employ the sets σ, σ_1, and σ_2 given in the Appendix. Let $r(s)$ be a nonnegative, measurable function on σ with the set $\{s : s \in \sigma; r(s) = 0\}$ having zero measure (reference is to Lebesgue measure in $(n - 1)$-dimensional Euclidean space). We denote by $L_r^2(\sigma)$ the real Hilbert space of functions $g(s)$ Lebesgue square integrable on σ with respect to the weight function r, and by $\{\Lambda_n(s)\}$ an orthonormal sequence in $L_r^2(\sigma)$.

In the problems below, $L_r^2(\sigma)$ is the realization of **R** and $\{\Lambda_n\}$ the realization of $\{\phi_n\}$. The projection operator P is defined by

$$Pg = \begin{cases} g(s), & s \in \sigma_1, \\ 0, & s \in \sigma_2. \end{cases}$$

This determines Q, **P**, and **Q** in an obvious fashion, viz.,

$$Qg = \begin{cases} 0, & s \in \sigma_1, \\ g(s), & s \in \sigma_2, \end{cases}$$

while **P** consists of all $g \in L_r^2(\sigma)$ such that $g(s) = 0$ almost everywhere on σ_2, and **Q** consists of all $g \in L_r^2(\sigma)$ such that $g(s) = 0$ almost everywhere on σ_1. With these definitions the straightforward use of Theorems 1 and 2 yields the following theorem.

THEOREM 4. *Let $\{\Lambda_n\}$ be a complete orthonormal sequence in $L_r^2(\sigma)$. Let $\{\kappa_{1n}\}$, $\{\kappa_{2n}\}$, $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of nonnegative constants such that $\{\mu_n\}$ and $\{\nu_n\}$ are bounded and*

$$\lim_{n\to\infty} \kappa_{1n} = \kappa_1, \quad \lim_{n\to\infty} \kappa_{2n} = \kappa_2, \quad \lim_{n\to\infty} \lambda_n = \infty,$$

where κ_1 and κ_2 are positive constants. Let (at least) one of the three following assertions be true; (i)

$$\kappa_{1n}\lambda_n + \mu_n > 0 \quad \textit{and} \quad \kappa_{2n}\lambda_n + \nu_n > 0, \qquad n = 1, 2, \cdots; \tag{4.1}$$

or (ii) *for some positive integer N,*

$$\begin{aligned} &\kappa_{1n}\lambda_n + \mu_n > 0, \qquad n = N+1, N+2, \cdots, \\ &\kappa_{2n}\lambda_n + \nu_n > 0, \qquad n = 1, 2, \cdots, \end{aligned} \tag{4.2}$$

and the vectors $(\mathcal{Q}_{k1}, \mathcal{Q}_{k2}, \cdots, \mathcal{Q}_{kN})$, $k = 1, 2, \cdots$, where

$$\mathcal{Q}_{kn} = \int_{\sigma_2} \Lambda_k(s)\Lambda_n(s)r(s)\,ds,$$

span an N-dimensional space; *or* (iii)

$$\begin{aligned} &\kappa_{1n}\lambda_n + \mu_n > 0, \qquad n = 1, 2, \cdots, \\ &\kappa_{2n}\lambda_n + \nu_n > 0, \qquad n = N+1, N+2, \cdots, \end{aligned} \tag{4.3}$$

and the vectors $(\mathcal{P}_{k1}, \mathcal{P}_{k2}, \cdots, \mathcal{P}_{kN})$, $k = 1, 2, \cdots$, where

$$\mathcal{P}_{kn} = \int_{\sigma_1} \Lambda_k(s)\Lambda_n(s)r(s)\,ds,$$

span an N-dimensional space.

Then the dual orthogonal series problem given by

$$\sum_{n=1}^{\infty} \mathscr{J}_n(\mu_n + \kappa_{1n}\lambda_n)\Lambda_n(s) = f(s), \qquad s \in \sigma_1, \tag{4.4a}$$

$$\sum_{n=1}^{\infty} \mathscr{J}_n(\nu_n + \kappa_{2n}\lambda_n)\Lambda_n(s) = f(s), \qquad s \in \sigma_2, \tag{4.4b}$$

has a unique solution $j \in \ell^2$ *for each* $f \in L_r^2(\sigma)$. *In fact,* $j_n = O(k_n/\lambda_n)$ *for some* $k \in \ell^2$.

In most applications $\{\Lambda_n\}$ will be a normalized Sturm–Liouville sequence, while λ_n will be an increasing function (often the square root) of an eigenvalue associated with Λ_n. The other sequences of constants are related to the geometry of Σ and the boundary conditions on $\tau, \nu, \cdots$.

5. Applications. Five examples illustrate use of the general theory. The first example, a typical steady temperature problem in an unbounded domain, is studied in some detail to show precisely the connection between theory and applications. The second example illustrates the transition from a potential problem in an unbounded domain to one in a bounded domain. The third example illustrates a problem in which the modifier component $a_1 = 0$. The fourth example illustrates a problem in which σ is unbounded. The fifth example is a 3-dimensional potential problem giving rise to a double, dual trigonometric series.

Example 1.

$$\sum_{n=1}^{\infty} j_n(h_{11} + k_{11}\lambda_n)S_n(x) = f(x), \qquad 0 < x < c, \tag{5.1a}$$

$$\sum_{n=1}^{\infty} j_n(h_{12} + k_{12}\lambda_n)S_n(x) = f(x), \qquad c < x < 1, \tag{5.1b}$$

where $f \in L^2(0, 1)$, and (h_{11}, k_{11}), (h_{12}, k_{12}), (h_2, k_2), and (h_4, k_4) are constants satisfying (A.2), $k_{11} > 0, k_{12} > 0, h_2 + h_4 > 0, \lambda_n$ is the nth positive root of

$$(k_2 k_4 \lambda^2 - h_2 h_4)\tan\lambda = \lambda(h_2 k_4 + h_4 k_2),$$

and

$$\alpha_n = \arctan(k_4\lambda_n/h_4),$$

$$S_n(x) = \sin(\lambda_n x + \alpha_n)\left\{\frac{1}{2}\left(1 - \frac{\sin(2\lambda_n + 2\alpha_n) - \sin 2\alpha_n}{2\lambda_n}\right)\right\}^{-1/2}.$$

We note that this example is associated with the following mixed boundary value problem [8]. The steady temperature u is sought satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 1 \quad \text{and} \quad y > 0, \tag{5.2}$$

$$h_{11}u - k_{11}\frac{\partial u}{\partial y} = f(x), \qquad 0 < x < c \quad \text{and} \quad y = 0, \tag{5.3a}$$

$$h_{12}u - k_{12}\frac{\partial u}{\partial y} = f(x), \qquad c < x < 1 \quad \text{and} \quad y = 0, \tag{5.3b}$$

$$\left[h_2 u + k_2\frac{\partial u}{\partial x}\right]_{x=1} = 0 \quad \text{and} \quad \left[h_4 u - k_4\frac{\partial u}{\partial x}\right]_{x=0} = 0, \qquad y > 0, \tag{5.4}$$

$$\operatorname{grad} u = O(1), \quad \text{as } y \to \infty. \tag{5.5}$$

If we seek a solution by separating variables, we find in the usual fashion,

$$u(x, y) = \sum_{n=1}^{\infty} \mathscr{j}_n S_n(x)\, e^{-\lambda_n y}.$$

This solution satisfies (at least formally) (5.2), (5.4) and (5.5) independently of the choice of $\mathscr{j}$. We are led to the dual orthogonal equation (5.1a–b) by choosing $\mathscr{j}$ to satisfy the mixed conditions (5.3a–b).

In this example we have:

$$\sigma = (0, 1), \quad \sigma_1 = (0, c), \quad \sigma_2 = (c, 1), \quad L_r^2(\sigma) = L^2(0, 1),$$

$$\Lambda_n = S_n, \quad \mu_n = h_{11}, \quad \nu_n = h_{12}, \quad \kappa_{1n} = k_{11} \quad \text{and} \quad \kappa_{2n} = k_{12}.$$

From standard Sturm–Liouville theory, vid., e.g., [25, Chap. 3], [26, § 3.13], it follows that $\lambda_n > 0$, $n = O(\lambda_n)$, and $\{S_n(x)\}$ is a complete orthonormal sequence in $L^2(0, 1)$. The reader can now verify that the hypothesis of Theorem 4, with (4.1) applying, is fulfilled. Hence, *the dual orthogonal equation* (5.1a–b) *has a unique solution* $\mathscr{j} \in \ell^2$ *such that* $\mathscr{j}_n = O(\mathscr{k}_n/n)$ *for some* $\mathscr{k} \in \ell^2$.

In computing approximations to $\mathscr{j}$ we need explicitly the terms a_n, b_n, α, and $\mathscr{Q}_{kn}$ appearing in (3.10). They are given by: $a_n = h_{11} + k_{11}\lambda_n$, $b_n = h_{12} + k_{12}\lambda_n$, $\alpha = k_{11}/k_{12}$, and $\mathscr{Q}_{kn} = \int_c^1 S_k S_n\, dx$. This last expression can be integrated in closed form so that the numerical implementation of (3.10) is straightforward. See [8] for an example in which numerical details have been carried out.

By a proper choice of the boundary conditions, (5.1a–b) includes the classic kernels $\{\sqrt{2}\sin n\pi x\}$, $\{\sqrt{2}\cos(n - 1/2)\pi x\}$, and $\{\sqrt{2}\sin(n - 1/2)\pi x\}$. For example, if $k_2 = 0$ and $k_4 = 0$, then (5.1a–b) becomes

$$\sum_{n=1}^{\infty} \mathscr{j}_n(h_{11} + k_{11}\pi n)(\sqrt{2}\sin n\pi x) = f(x), \qquad 0 < x < c,$$

$$\sum_{n=1}^{\infty} \mathscr{j}_n(h_{12} + k_{12}\pi n)(\sqrt{2}\sin n\pi x) = f(x), \qquad c < x < 1.$$

Example 2. We now seek a solution to the corresponding problem on a rectangle, i.e., we replace (5.2) and (5.5) by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 1 \quad \text{and} \quad 0 < y < \beta,$$

$$h_3 u + k_3 \frac{\partial u}{\partial y} = g(x), \qquad 0 < x < 1 \quad \text{and} \quad y = \beta,$$

where $g \in L^2(0, 1)$, and h_3 and k_3 are constants satisfying (A.2).

Most analytic methods previously used have been restricted to unbounded regions, and extending the analysis to bounded regions has been formidable, cf. [6]. Here we shall see, except for the algebra being more tedious, the analysis is essentially unchanged from Example 1. We develop a separated variable solution as in Example 1 and obtain

$$u(x, y) = \sum_{n=1}^{\infty} \left\{ \mathscr{j}_n \left[-\sinh \lambda_n y + \left(\frac{h_3 \tanh \lambda_n \beta + k_3 \lambda_n}{h_3 + k_3 \lambda_n \tanh \lambda_n \beta} \right) \cosh \lambda_n y \right] + G_n \cosh \lambda_n y \right\} S_n(x),$$

where

$$G_n = \frac{\operatorname{sech} \lambda_n\beta \int_0^1 g(x)S_n(x)\,dx}{h_3 + k_3\lambda_n \tanh \lambda_n\beta}.$$

Using (5.3a–b) we arrive at the dual orthogonal equation

$$\sum_{n=1}^{\infty} j_n\left\{h_{11}\left(\frac{h_3 \tanh \lambda_n\beta + k_3\lambda_n}{h_3 + k_3\lambda_n \tanh \lambda_n\beta}\right) + k_{11}\lambda_n\right\}S_n(x)$$

$$= f(x) - \sum_{n=1}^{\infty} G_nS_n(x), \qquad 0 < x < c,$$

$$\sum_{n=1}^{\infty} j_n\left\{h_{12}\left(\frac{h_3 \tanh \lambda_n\beta + k_3\lambda_n}{h_3 + k_3\lambda_n \tanh \lambda_n\beta}\right) + k_{12}\lambda_n\right\}S_n(x)$$

$$= f(x) - \sum_{n=1}^{\infty} G_nS_n(x), \qquad c < x < 1.$$

Exactly as in Problem 1, one can verify that the hypothesis of Theorem 4, with (4.1) applying, is satisfied. Therefore, *the above dual trigonometric equation has a unique solution* $j \in \ell^2$ *such that* $j_n = O(k_n/n)$ *for some* $k \in \ell^2$.

Example 3. Here we envisage the dual Legendre equation

$$\sum_{n=2}^{\infty} j_n(n-1)\left\{\sqrt{\left(\frac{2n-1}{2}\right)}P_{n-1}(\cos\theta)\right\} = f(\theta), \qquad 0 < \theta < \pi/2,$$

$$\sum_{n=1}^{\infty} j_n(n-1+h)\left\{\sqrt{\left(\frac{2n-1}{2}\right)}P_{n-1}(\cos\theta)\right\} = 0, \qquad \pi/2 < \theta < \pi,$$

where $f \cdot \sin\theta \in L^2(0, \pi/2)$, h is positive, and P_n is a Legendre polynomial. This corresponds to the following steady temperature problem. Heat is forced into the upper hemisphere of a thermally homogeneous sphere of radius 1 and leaves through the lower hemisphere by Newtonian cooling. The governing equations are

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0, \qquad 0 < \rho < 1 \quad\text{and}\quad 0 < \theta < \pi,$$

$$\frac{\partial u}{\partial\rho} = f(\theta), \qquad 0 < \theta < \pi/2,$$

$$\frac{\partial u}{\partial\rho} + hu = 0, \qquad \pi/2 < \theta < \pi.$$

Here $\kappa_{1n} = 1$, $\kappa_{2n} = 1$, $\lambda_n = n - 1$, $\mu_n = 0$, and $\nu_n = h$. We note that

$$\kappa_{11}\lambda_1 + \mu_1 = 0 \quad\text{and}\quad \kappa_{1n}\lambda_n + \mu_n > 0, \qquad n = 2, 3, \cdots.$$

Therefore we must apply Theorem 4 using (4.2). Now, $\mathcal{Q}_{11} = 1/2$, so that the span of $\{\mathcal{Q}_{k1}\}$, $k = 1, 2, \cdots$, has dimension 1. The remaining hypotheses in Theorem 4 can be verified exactly as in Example 1, so that *the above dual Legendre equation has a unique solution* $j \in \ell^2$ *such that* $j_n = O(k_n/n)$ *for some* $k \in \ell^2$.

Example 4. We choose an example in which σ is unbounded, viz., σ is the interval $0 < x < \infty$. The kernel is the Laguerre functions $\{e^{-x/2}L_n(x): n = 0, 1, \cdots\}$, where L_n is the Laguerre polynomial of order n as defined by Szegö [24, p. 96]. These functions form a complete orthonormal sequence in $L^2(0, \infty)$ [24, p. 104]. Certain dual Laguerre equations have been formally analyzed [14], [15], [16] but are of a somewhat different type than that given below:

$$\sum_{n=1}^{\infty} j_n(h_1 + k_1\sqrt{n-1})(e^{-x/2}L_{n-1}(x)) = e^{-x/2}f(x), \qquad 0 < x < c,$$

$$\sum_{n=1}^{\infty} j_n(h_2 + k_2\sqrt{n-1})(e^{-x/2}L_{n-1}(x)) = e^{-x/2}f(x), \qquad c < x < \infty,$$

where (h_1, k_1) and (h_2, k_2) satisfy (A.2) and $h_1 + h_2$ is positive and $e^{-x/2}f \in L^2(0, \infty)$. Pursuing arguments used in the first three examples, one can readily verify that *the above dual Laguerre equation has a unique solution $j \in \ell^2$ such that $j_n = O(k_n/\sqrt{n})$ for some $k \in \ell^2$.*

Example 5. This example displays a multiple dual orthogonal series. The steady temperature is sought in a semi-infinite rectangular parallelepiped with zero temperature on the side walls and mixed boundary conditions of second and third type along the base. The governing equations are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < x, y < 1 \quad \text{and} \quad z > 0,$$

$$u = 0 \qquad \begin{cases} 0 < x < 1, \text{ and } y = 0 \text{ or } y = 1, \text{ and } z > 0, \\ x = 0 \text{ or } x = 1, \text{ and } 0 < y < 1, \text{ and } z > 0, \end{cases}$$

$$h_1 u - k_1\frac{\partial u}{\partial z} = f(x, y), \qquad 0 < x < c, \quad 0 < y < d, \quad z = 0,$$

$$h_2 u - k_2\frac{\partial u}{\partial z} = f(x, y), \qquad c < x < 1, \quad 0 < y < 1, \quad \text{or } 0 < x < c,$$
$$d < y < 1, \text{ and } z = 0,$$

where (h_1, k_1) and (h_2, k_2) satisfy (A.2) and k_1 and k_2 are positive. We are led to the following double, dual trigonometric series by seeking a solution through separating variables:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn}[h_1 + k_1\pi(m^2 + n^2)^{1/2}](2\sin m\pi x \sin n\pi y)$$
$$= f(x, y), \qquad 0 < x < c \text{ and } 0 < y < d, \tag{5.6a}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn}[h_2 + k_2\pi(m^2 + n^2)^{1/2}](2\sin m\pi x \sin n\pi y)$$
$$= f(x, y), \qquad c < x < 1 \quad \text{and} \quad 0 < y < 1, \text{ or}$$
$$0 < x < c \quad \text{and} \quad d < y < 1. \tag{5.6b}$$

Let $p(m, n)$ be the rectangular mapping from pairs of positive integers onto the positive integers. Specifically,

$$p(m, n) = \begin{cases} m + (n - 1)^2, & m \leqq n, \\ m^2 - n + 1, & m > n. \end{cases}$$

We define λ_p and $\Lambda_p(x, y)$ by

$$\lambda_{p(m,n)} = \pi(m^2 + n^2)^{1/2} \quad \text{and} \quad \Lambda_{p(m,n)}(x, y) = 2 \sin m\pi x \sin n\pi y.$$

Thus $\lambda_p \to \infty$ as $p \to \infty$, and $\{\Lambda_p(x, y)\}$ is a complete orthonormal sequence in the Hilbert space of functions Lebesgue square integrable on $0 \leqq x, y \leqq 1$. We can now formally rewrite (5.6a–b) as

$$\text{(5.7a)} \quad \sum_{p=1}^{\infty} \mathscr{j}_p(h_1 + k_1\lambda_p)\Lambda_p(x, y) = f(x, y), \qquad 0 < x < c \quad \text{and} \quad 0 < y < d,$$

$$\text{(5.7b)} \quad \sum_{p=1}^{\infty} \mathscr{j}_p(h_2 + k_2\lambda_p)\Lambda_p(x, y) = f(x, y), \qquad \begin{array}{l} c < x < 1 \quad \text{and} \quad 0 < y < 1, \text{ or} \\ 0 < x < c \quad \text{and} \quad d < y < 1. \end{array}$$

By applying Theorem 4 we see that (5.7a–b) has a unique solution $\mathscr{j}$ such that $\sum_p \mathscr{j}_p^2\lambda_p^2 < \infty$. Hence, *the double dual trigonometric equation* (5.6a–b) *has a unique solution* $\{c_{mn}\}$ *such that*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn}^2(m^2 + n^2) < \infty$$

for each f Lebesgue square integrable on $0 \leqq x, y \leqq 1$.

6. Discussion. In terms of the mixed boundary value problems of mathematical physics the reader will have noted that our theory covers with reasonably complete generality dual orthogonal series associated with mixed boundary conditions of the second and third kind, but does not apply if one of the mixed conditions is of the first kind. Contrariwise, with a few exceptions, the only dual orthogonal series which have been analyzed earlier [1] have been those associated with potential problems in unbounded domains and in which the mixed conditions are of the first and second kind.

What is the reason for this difference? Roughly speaking, we believe it to be as follows. Solutions to potential problems in which the mixed conditions are of the second and third kind are more regular in their behavior at the boundary than those in which there is a mixed condition of the first kind [17], [18], [19]. This characteristic is reflected when one translates the original potential problem, as we have done, into ℓ^2 in the following way: When the mixed boundary conditions are of the second and third kind, the key matrix, $\mathscr{C} - \mathscr{I}$, in (3.6) is completely continuous, whereas if one of the mixed conditions is of the first kind, one cannot construct $\mathscr{C}$ as we have done (see discussion following (6.2a–b) below). On the other hand, when (i) the kernel $\{\Lambda_n\}$ is associated with boundary conditions of the first and second kind, (ii) the mixed conditions are also of the first and second kind, and (iii) the domain is unbounded, then the formulas tend to be of a neat and precise type, e.g., (6.1a–b), and with such tidy formulas the employment of special functions has a greater chance of producing useful results.

In making a Hilbert space approach to problems in which one of the mixed conditions is of the first kind, there may be difficulties of a more intrinsic nature. To see this, we consider an important example due to Tranter [20] (the original analysis being subsequently simplified [22], [23], cf. [21]):

$$\text{(6.1a)} \qquad \sum_{n=1}^{\infty} \frac{k_n}{n - 1/2}\left[\sqrt{\left(\frac{2}{\pi}\right)} \cos(n - 1/2)x\right] = 1, \qquad 0 < x < \pi/2,$$

$$\text{(6.1b)} \qquad \sum_{n=1}^{\infty} k_n\left[\sqrt{\left(\frac{2}{\pi}\right)} \cos(n - 1/2)x\right] = 0, \qquad \pi/2 < x < \pi.$$

Tranter has shown formally that a solution to this problem is given by

$$k_n = \frac{4\pi}{\sqrt{2}} \frac{P_{n-1}(0)}{\Gamma^2(1/4)}, \qquad n = 1, 2, \cdots.$$

Using Laplace's asymptotic estimate for $P_n(0)$ [24, p. 188] we see that

$$k_n = \begin{cases} \dfrac{4\sqrt{\pi}(-1)^{(n-1)/2}}{\Gamma^2(1/4)(n-1)^{1/2}} + O(n^{-1}), & n = 1, 3, \cdots, \\ 0, & n = 2, 4, \cdots. \end{cases}$$

Since $k \notin \ell^2$, the Hilbert space approach, used here, of translating the original problem into an ℓ^2 problem may prove troublesome. A similar phenomenon is shown in a dual Legendre series studied by Collins [1, p. 174; 27], cf. [21].

Shepherd, in an early and prescient study [28], avoided this difficulty by integrating that part of a dual trigonometric equation associated with the boundary condition of the second kind and was able to obtain an explicit and rigorous solution. Specifically, he studied the dual orthogonal equation

$$\text{(6.2a)} \qquad k_1 \frac{1}{\sqrt{\pi}} + \sum_{n=2}^{\infty} k_n\left(\sqrt{\frac{2}{\pi}} \cos(n - 1)x\right) = \cos mx, \qquad 0 < x < \pi/2,$$

$$\text{(6.2b)} \qquad \sum_{n=2}^{\infty} k_n\left(\sqrt{\frac{2}{\pi}} \sin(n - 1)x\right) = -\sin mx, \qquad \pi/2 < x < \pi,$$

where (6.2a) has resulted from an integration.

Finally, we note that the ratio b_n/a_n associated with (6.1a–b) tends to infinity as $n \to \infty$. This is the *technical reason* for our being unable to apply Theorems 1 and 2 to dual orthogonal series associated with potential problems with a mixed condition of the first kind.

Thus, future research is naturally directed along two lines: first, to establish rigorously the conditions under which various "closed form" solutions to specific equations are valid in order to understand the inherent limitations of a theory encompassing dual orthogonal series associated with potential problems in which one of the mixed conditions is of the first kind, and, second, to construct a theory for such series.

Appendix. Let σ be an open, connected set on the surface of an n-dimensional region Σ in which u satisfies a second order partial differential equation, and let u satisfy on σ the relation

$$h(s)u + k(s)\frac{\partial u}{\partial n} = f(s), \qquad s \in \sigma, \tag{A.1}$$

where n denotes outward normal. The condition (A.1) is called *constant* if $h(s)$ and $k(s)$ are constant, and *homogeneous* if $f \equiv 0$. In the constant case we usually impose the restrictions

$$h \geqq 0, \quad k \geqq 0, \quad \text{and} \quad h + k > 0. \tag{A.2}$$

It is called a boundary condition of the *first kind* if $k \equiv 0$, of the *second kind* if $h \equiv 0$, and of the *third kind* if $h(s) > 0$ and $k(s) > 0$, for almost all $s \in \sigma$. It is called a *constant mixed condition* (often shortened to *mixed condition*) if the following holds: (i) σ is the union of two disjoint, open, connected sets (say σ_1 and σ_2) and the points in $\bar{\sigma}_1 \cap \bar{\sigma}_2$ which are interior points of $\bar{\sigma}_1 \cup \bar{\sigma}_2$ (here $\bar{\sigma}$ is the closure of σ, etc.); (ii) on σ_1 and σ_2, u satisfies the constant boundary conditions

$$h_i u + k_i \frac{\partial u}{\partial n} = f(s), \qquad s \in \sigma_i \quad (i = 1, 2), \tag{A.3}$$

and

$$k_1 h_2 \neq k_2 h_1. \tag{A.4}$$

Condition (A.4) is a nontriviality condition. If it is false, the mixed boundary condition reduces to a constant boundary condition. Other names for mixed boundary conditions are *broken* or *discontinuous* boundary conditions. Sometimes in the literature one sees a boundary condition of the third kind referred to as a mixed boundary condition. That is a malapropism.

In solving potential problems with mixed boundary conditions by the method of separation of variables the following situation obtains. An orthogonal set of coordinates is used. A solution for Laplace's equation is sought in Σ whose surface is the union of the closure of disjoint, open, connected sets, say $\sigma, \tau, \upsilon, \cdots$, such that on each of these sets one of the orthogonal coordinates is constant. u satisfies a constant, *homogeneous*, boundary condition on each of the subsurfaces $\tau, \upsilon, \cdots$ (though constant on each subsurface, the condition can be different on different subsurfaces), while on σ, u satisfies the mixed condition (A.3). A separated variable solution in the form of an infinite series is sought which satisfies the differential equation in Σ and the boundary conditions on $\tau, \upsilon, \cdots$. Each term in the series is multiplied by an unknown coefficient. The coefficients are chosen to satisfy (A.3), and this gives rise to a dual orthogonal series problem. When we speak of the dual orthogonal series being *associated with mixed conditions*, we refer only to boundary conditions on σ and not on $\tau, \upsilon, \cdots$. Each term in the series solution contains a function, say $\Lambda_n(s)$, usually called the eigenfunction, such that $\{\Lambda_n\}$ is a complete orthonormal set in the Hilbert space of all functions Lebesgue square integrable on σ with respect to an appropriate weight function.

When we speak of *the boundary conditions associated with* $\{\Lambda_n\}$, we refer only to those boundary conditions on τ, υ, $\cdots$ that enter into the Sturm–Liouville problem determining $\{\Lambda_n\}$.

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THE SZEGÖ RECURSION RELATION AND INVERSES OF POSITIVE DEFINITE TOEPLITZ MATRICES*

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Abstract. Those matrices which are inverses of positive definite Toeplitz matrices are characterized. Further, those sequences of polynomials which are Szegö polynomials (orthonormal with respect to a suitable measure on the unit circle) are characterized and a recursion relation is given which yields the corresponding Toeplitz matrix.

1. Introduction. The problem of characterizing classes of orthogonal polynomials by the recursion relations which they satisfy was first considered by J. Favard [4] in 1935. The same problem for the polynomials orthogonal with respect to a suitable weight function on the unit circle introduced by G. Szegö was solved by F. V. Atkinson [3].

In this paper we find that we are able to utilize the results of Szegö, Atkinson, and Aronszajn [2] to completely characterize those matrices which are inverses of positive definite Toeplitz matrices and to identify the Toeplitz matrix associated with a given sequence of Szegö polynomials and to give a useful recursion relation which enables us to calculate Szegö polynomials with ease.

2. Development. Let $\sigma(\theta)$ be a nondecreasing function of bounded variation (nonconstant) on the interval $[0, 2\pi)$. We may define a class of polynomials $\{p_n(z)\}$ satisfying

$$\deg p_n(z) = n, \tag{1}$$

$$\frac{1}{2\pi}\int_0^{2\pi} p_k(z)\bar{p}_j(z)\,d\sigma(\theta) = \delta_{kj} \quad (\text{where } z = e^{i\theta}), \tag{2}$$

$$p_{nn} > 0, \quad \text{where } p_{nn} \text{ is the leading coefficient of } p_n(z). \tag{3}$$

We suppose that σ has an infinite set of points of increase if the sequence $\{p_n(z)\}$ is to be infinite. These polynomials were first studied by G. Szegö and excellent expositions of their theory may be found in [5], [6] and [9].

The polynomials satisfying (1), (2) and (3) satisfy the recursion relation

$$p_{nn}zp_n(z) = p_{n+1,n+1}p_{n+1}(z) - p_{n+1,0}p_{n+1}^*(z), \tag{4}$$

or equivalently,

$$p_{nn}p_{n+1}(z) = p_{n+1,n+1}zp_n(z) - p_{n+1,0}p_n^*(z) \tag{4'}$$

and the condition

$$p_{n+1,n+1}^2 = \sum_{k=0}^{n+1} |p_{k,0}|^2, \tag{5}$$

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where

$$p_k(z) = \sum_{j=0}^{k} p_{kj} z^j$$

and

$$p_k^*(z) = z^k \bar{p}(1/z) \quad [9, \text{p. } 293].$$

In 1935, J. Favard [4] first considered the problem of determining orthogonality properties which follow from a given recursion relation satisfied by a sequence of real polynomials. More recently, Atkinson [3, Chap. 7] showed that the relations (4′) and (5) satisfied by a sequence of polynomials $\{p_n(z)\}$ satisfying (1) and (3) implies the existence of a nondecreasing function $\sigma(\theta)$ of bounded variation on $[0, 2\pi)$ for which equation (2) becomes valid.

Let us begin with a sequence of polynomials $\{p_n(z)\}$ satisfying (1), (3), (4) and (5) and let $\sigma(\theta)$ be the function for which (2) is valid. We define the moments of σ to be the sequence $\{\varphi_k\}$ given by

$$\varphi_k = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\, d\sigma(\theta), \qquad -\infty < k < \infty.$$

We may define an inner product on the space of polynomials of degree n or less by

$$\langle p(z), q(z)\rangle_\sigma = \frac{1}{2\pi}\int_0^{2\pi} p(z)\bar{q}(z)\, d\sigma(\theta),$$

where p, q are polynomials of degree less than or equal to n. Identifying a polynomial, $p(z)$, with its sequence of coefficients $(p_0, p_1, \cdots, p_n)$ where $p(z) = \sum_{k=0}^{n} p_k z^k$, it is easily verified that the above inner product is equivalent to $\langle p(z), q(z)\rangle_\sigma = \langle \tilde{p}, \Phi\tilde{q}\rangle$, where $\tilde{p}, \tilde{q}$ are the sequences of coefficients of p, q respectively, the inner product is the usual one, and Φ is the Toeplitz matrix given by $\Phi = (\varphi_{k-j})$. It is easy to verify directly that Φ is positive definite since it is simply the Gram matrix of the linearly independent set $1, e^{i\theta}, e^{2i\theta}, \cdots, e^{in\theta}$ on $[0, 2\pi)$ with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$ defined earlier.

DEFINITION. We shall refer to the inner product given by $\langle \tilde{p}, \Phi\tilde{q}\rangle$ as the Φ *inner product* between p and q and we shall not distinguish in an inner product between a polynomial and its sequence of coefficients.

We are now in a position to characterize those matrices which are inverses of positive definite Toeplitz matrices.

3. Characterization of the inverse matrices.

PROPOSITION 1. *An $(n + 1) \times (n + 1)$ matrix S is the inverse of a positive definite Toeplitz matrix if and only if $S = PP^*$, where P is upper-triangular and the polynomials formed from the columns of p satisfy the relations* (3), (4), (5).

Proof. First suppose that S is the inverse of the positive definite Toeplitz matrix, Φ. Then if we orthonormalize the sequence $1, z, z^2, \cdots, z^n$ with respect to the Φ inner product we obtain polynomials $p_k(z)$, $0 \leqq k \leqq n$, satisfying (1), (3), (4) and (5). Let us define a function

$$S_n(w, z) = \sum_{k=0}^{n} \bar{p}_k(w) p_k(z) = \sum_{k=0}^{n}\sum_{j=0}^{n} \bar{s}_{kj}^n \bar{w}^k z^j,$$

where the last term defines the constants s_k^{nj}. We form the conjugate of the matrix of coefficients:

$$T = (s^n_{kl}) = \begin{bmatrix} s^n_{00} & s^n_{01} & \cdots & s^n_{0n} \\ s^n_{10} & & & \vdots \\ \vdots & & & \vdots \\ s^n_{n0} & \cdots\cdots & & s^n_{nn} \end{bmatrix}.$$

The matrix, T, may be shown to be the inverse of the matrix Φ [7, Proposition] and so we conclude $T = S$. Further, if we write $p_k(z) = \sum_{j=0}^n p_{kj}z^j$ then it is easy to verify from the definition of S that $s^n_{kj} = \sum_{t=0}^n p_{tk}\bar{p}_{tj}$, where we define $p_{jk} = 0$ if $j < k$. Writing this in matrix form, we have

$$S = \begin{bmatrix} p_{00} & p_{10} & \cdot & \cdot & \cdot & p_{n0} \\ 0 & p_{11} & \cdot & \cdot & \cdot & p_{n1} \\ 0 & 0 & p_{22} & \cdot & \cdot & p_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & p_{nn} \end{bmatrix} \begin{bmatrix} \bar{p}_{00} & 0 & \cdot & \cdot & 0 \\ \bar{p}_{10} & \bar{p}_{11} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \bar{p}_{n0} & \cdot & \cdot & \cdot & \bar{p}_{nn} \end{bmatrix}$$

The polynomials formed from the columns of P are the orthogonal polynomials $\{p_k(z)\}$ and so satisfy (1), (3), (4), (5).

Conversely, suppose that S is an $(n + 1) \times (n + 1)$ matrix of the form PP^*, where the polynomials $p_k(z)$ (formed from the kth column of P) satisfy (3), (4), (5) for $0 \leqq k \leqq n$. Then there is a bounded step function $\sigma(\theta)$ defined on $[0, 2\pi)$ for which (2) is valid for $0 \leqq k \leqq n$ [3, pp. 175–176]. Defining

$$\varphi_k = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\, d\sigma(\theta),$$

we see that the matrix $\Phi = (\varphi_{k-j})$, $0 \leqq k, j \leqq n$, is a positive definite Toeplitz matrix and the polynomials $p_0, p_1(z), \cdots, p_n(z)$ are orthogonal with respect to the Φ inner product. As before, it follows that matrix $S = PP^*$ is the inverse of the positive definite Toeplitz matrix, Φ, which was to be shown.

COROLLARY 1.1. *An $n \times n$ matrix S is positive definite if and only if $S = PP^*$, where P is upper triangular and $p_{kk} > 0$ for $0 \leqq k \leqq n - 1$.*

Proof. If S is positive definite, we need only take for the columns of P the coefficients of the orthonormal set $\{p_k(z)\}$ of polynomials of degree k corresponding to the inner product defined by the inverse, Φ, of S. The proof that $S = PP^*$ is identical to that given in [7, Proposition]. The converse is clear.

4. Characterization of the Szegö polynomials. We now consider the problem of generating the moments of the distribution function and thereby the positive definite Toeplitz matrix which defines the inner product with respect to which a sequence of polynomials $\{p_k(z)\}$ satisfying (1), (3), (4), (5) is orthonormal.

PROPOSITION 2. *Let the sequence of polynomials $\{p_k(z)\}$ satisfy conditions* (1), (3), (4), (5). *The m ments $\{\varphi_k\}$ of the corresponding distribution function may be calculated recursively by*:

$$\varphi_0 = 1/p_{00}^2,$$

$$\varphi_{n+1} = -(1/p_{n+1,n+1}) \sum_{k=0}^{n} \varphi_k p_{n+1,k},$$

and

$$\varphi_{-k} = \bar{\varphi}_k.$$

The polynomials $p_0, p_1(z), \cdots, p_n(z)$ *are orthonormal with respect to the inner product defined by*

$$\varphi = (\varphi_{k-j}), \qquad 0 \leqq k, j \leqq n.$$

Proof. Let $S_n = PP^*$ be the $(n+1) \times (n+1)$ matrix generated as in the proof of Proposition 1, and let $\Phi_n = (\varphi_{k-j})$, $0 \leqq k, j \leqq n$. Since $\Phi_n = S_n^{-1}$, and the last column of S_n satisfies

$$s_{k,n+1}^{n+1} = p_{n+1,k} p_{n+1,n+1},$$

it follows that the dot product between the zeroth row vector of φ_n and the last column of S_n is zero, i.e.,

$$\sum_{k=0}^{n-1} \varphi_k s_{k,n+1}^{n+1} = 0 \quad \text{so } \varphi_{n+1} s_{n+1,n+1}^{n+1} = -\sum_{k=0}^{n} \varphi_k s_{k,n+1}^{n+1}$$

or

$$\varphi_{n+1} = -(1/p_{n+1,n+1}) \sum_{k=0}^{n} \varphi_k p_{n+1,k}.$$

Further,

$$1 = \langle p_0, \Phi p_0 \rangle = \varphi_0 p_{00}^2, \quad \text{so } \varphi_0 = 1/p_{00}^2.$$

The last assertion is clear from our previous discussion.

The following recursion relation characterizes and allows us to easily calculate Szegö polynomials.

PROPOSITION 3.

(A) *Let* $p_{00} > 0$ *and define* $p_0(z) = p_{00}$.

(B) *If* $p_n(z)$ *has been calculated, let the scalar* λ_n *be chosen so that* $|\lambda_n| < 1/p_{nn}$. *We then generate* $p_{n+1}(z)$ *by the recursion relation*

(C) $$p_{n+1}(z) = (1 - |\lambda_n|^2 p_{nn}^2)^{-1/2}(z p_n(z) - \lambda_n p_{nn} p^*(z)),$$

where $p^*(z) = z^n \bar{p}_n(1/z)$.

The polynomials $p_n(z)$ *generated in this way satisfy* (1), (3), (4′) (*and so* (4)), *and* (5).

Conversely, if the sequence of polynomials $\{p_n(z)\}$ *satisfies the relations* (1), (3), (4), (5), *then it satisfies* (A), (B), (C), *where* $\lambda_n = \sum_{k=0}^{n} \varphi_{k+1} p_{nk}$ *and the* φ_k *are generated as in Proposition* 2 (*or correspond to a given positive definite Toeplitz matrix or nondecreasing bounded distribution function on* $[0, 2\pi)$).

Proof. Let the sequence of polynomials $\{p_n(z)\}$ be generated by (A), (B), (C). Then it is clear that $\deg p_n(z) = n$ and $p_{nn} > 0$ since $p_{00} > 0$ and $(1 - |\lambda_n|^2 p_{nn}^2)^{1/2} > 0$ (proceed by induction). Thus, we need only verify conditions (4′) and (5).

Equating the coefficients of z^{n-1} in (C) we obtain

(6) $$p_{n+1,n+1} = p_{nn}(1 - |\lambda_n|^2 p_{nn}^2)^{-1/2}$$

and so

$$p_{n+1,n+1}^2 - |\lambda_n|^2 p_{nn}^2 p_{n+1,n+1}^2 = p_{nn}^2$$

from which we conclude

$$1-|\lambda_n|^2 p_{nn}^2 = p_{nn}^2/p_{n+1,n+1}^2.$$

Equating constant terms in (C) gives $p_{n+1,0} = -\lambda_n p_{nn}^2(1-|\lambda_n|^2 p_{nn}^2)^{-1/2}$ which yields

(7) $$-p_{nn}\lambda_n(1-|\lambda_n|^2 p_{nn}^2)^{-1/2} = p_{n+1,0}/p_{nn}.$$

Using these relations, (C) becomes

(8) $$p_{n+1}(z) = (p_{n+1,n+1}/p_{nn})zp_n(z) - (p_{n+1,0}/p_{nn})p_n^*(z)$$

which is (4′).

By equating coefficients of z^0 and z^{n+1} in (C), we obtain

$$p_{n+1,0} = (1-|\lambda_n|^2 p_{nn}^2)^{-1/2}(-\lambda_n p_{nn}^2),$$

$$p_{n+1,n+1} = (1-|\lambda_n|^2 p_{nn}^2)^{-1/2}(-\lambda_n p_{nn}^2),$$

which yields the equality

$$p_{nn}^2 + |p_{n+1,0}|^2 = p_{n+1,n+1}^2.$$

This implies, in particular, that

$$p_{00}^2 + |p_{10}|^2 = p_{11}^2$$

and (5) follows by induction.

Conversely, let the polynomials $\{p_n(z)\}$ satisfy (1), (3), (4), (5). Then we may write

(9) $$zp_n(z) = \sum_{j=0}^{n+1} \gamma_j p_j(z).$$

Taking the Φ inner product on both sides with $p_k(z)$ (Φ is given by Proposition 2), we obtain

$$\langle zp_n(z), \Phi p_k(z)\rangle = \gamma_k.$$

But also the left-hand side is

$$\langle \Phi zp_n(z), p_k(z)\rangle = \lambda_n \bar{p}_k(0)$$

provided $0 \leqq k \leqq n$, where

$$\lambda_n = \sum_{k=0}^{n} \varphi_{k+1} p_{nk}.$$

If $k = n+1$, $\langle zp_n(z), \Phi p_{n+1}(z)\rangle = p_{nn}/p_{n+1,n+1}$. We may now write (9) in the form

$$zp_n(z) = \lambda_n \sum_{k=0}^{n} \bar{p}_k(0)p_k(z) + (p_{nn}/p_{n+1,n+1})p_{n+1}(z).$$

We observe, now, that

$$\sum_{k=0}^{n} \bar{p}_k(0)p_k(z) = p_{nn}p_n^*(z) \quad [9, \text{p. } 290, 11.3.5]$$

and so we have obtained

$$zp_n(z) = p_{nn}\lambda_n p_n^* + (p_{nn}/p_{n+1,n+1})p_{n+1}(z) \tag{10}$$

with λ_n as above.

Let us define

$$q_{n+1}(z) = zp_n(z) - p_{nn}\lambda_n p_n^*(z),$$

where $q_{n+1}(z) = \text{const.}\, p_{n+1}(z)$ and clearly

$$p_{n+1}(z) = q_{n+1}(z)/\|q_{n+1}(z)\|_\Phi . \tag{11}$$

Calculating the Φ-norm of q_{n+1}:

$$\|q_{n+1}\|_\Phi^2 = \langle q_{n+1}, \Phi q_{n+1}\rangle = \langle q_{n+1}, \Phi(p_{nn}z^{n+1} + r_n(z))\rangle = \langle \Phi q_{n+1}, p_{nn}z^{n+1}\rangle$$

(since p_{n+1} is orthogonal to all polynomials of degree less than $n + 1$ and r_n is a polynomial of degree n or less). We may easily verify that

$$\Phi q_{n+1} = \Phi(zp_n(z) - p_{nn}\lambda_n p_n^*(z)) = \begin{bmatrix} * \\ \vdots \\ (1/p_{nn}) - |\lambda_n|^2 p_{nn} \end{bmatrix}.$$

Combining this result with (11) gives the desired equality.

This result combined with our previous work clearly yields a simple algorithm for inverting positive definite Toeplitz matrices [7].

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EQUICONVERGENCE THEOREMS FOR SERIES WHOSE TERMS SATISFY A DIFFERENCE EQUATION*

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Abstract. We discuss the error of expansions of functions in series of function $\{p_n(z)\}$ which are defined by a generating function. If $p_n(z)$ satisfies a linear difference equation of a certain kind, then the error of the expansion may be simply related to the error of a much simpler Taylor series. Some of our formulas are of practical value in summing expansions in $p_n(z)$ with given coefficients, and we give several applications to expansions in Pollaczek polynomials $P_n^\lambda(x; a, b)$.

1. Introduction. In this paper we are concerned with the convergence properties of expansions in functions $\{p_n(z)\}$ which are defined by a generating function

$$K(z, w) = \sum_{n=0}^{\infty} p_n(z)w^n.$$

It is found that if $p_n(z)$ satisfies a linear recursion relationship of a rather general kind, the error of an èxpansion in these functions can be simply related to the error of a much simpler expansion, that is, a Taylor series with related coefficients. In the course of our analysis, formulas are given which are of practical value in "summing" the $p_n(z)$ series when the sum of the related Taylor series is known, and we present applications of our results to expansions in the so-called Pollaczek polynomials $P_n^\lambda(z; a, b)$. Also, certain neat statements can be made about the convergence of the $p_n(z)$ series when the related function is an entire function of exponential order.

2. Formulas. Let

$$K(z, w) = \sum_{n=0}^{\infty} p_n(z)w^n \tag{1}$$

for (z, w) belonging to some region of $C \times C$. In particular, we assume $z \in Z \subset C$ and for each fixed $z \in Z$, (1) has a nonzero radius of convergence. Also let

$$f(z) = \sum_{n=0}^{\infty} \mathscr{L}_n(f)p_n(z) \tag{2}$$

converge absolutely. We define

$$\Phi(z) = \sum_{n=0}^{\infty} \sigma_n \mathscr{L}_n(f)z^n, \qquad \sigma_n > 0. \tag{3}$$

(In general $\{\sigma_n\}$ is a sequence which will insure that (3) is analytic in some neighborhood of the origin.) Then

$$\sigma_k \mathscr{L}_k(f) = \frac{1}{2\pi i}\int_C \frac{\phi(t)}{t^{k+1}}\,dt, \tag{4}$$

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where ϕ is analytic on and within C, a simple closed contour encircling the origin.[1]

Let

$$K^*(z,t) = \frac{1}{t}\sum_{n=0}^{\infty}\frac{q_n(z)}{t^n}, \qquad q_n(z) = \frac{p_n(z)}{\sigma_n}. \tag{5}$$

If we assume that

$$\overline{\lim_{n\to\infty}}\ |q_n(z)|^{1/n} = \zeta(z) < \infty, \tag{6}$$

then (5) will converge if $|t| > \zeta(z)$, z fixed.

We then can write

$$\begin{aligned} E_n[f(z)] &= \frac{1}{2\pi i}\int_C \phi(t)K_n^*(z,t)\,dt, \\ E_n[f(z)] &= f(z) - \sum_{k=0}^{n-1}\mathscr{L}_k(f)p_k(z), \\ K_n^*(z,t) &= \frac{1}{t}\sum_{k=n}^{\infty}\frac{q_k(z)}{t^k} \end{aligned} \tag{7}$$

provided that on C, ϕ is analytic and $|t| > \zeta(z)$.

For $n = 0$ this becomes

$$f(z) = \frac{1}{2\pi i}\int_C \phi(t)K^*(z,t)\,dt. \tag{8}$$

Most practical interest attaches to the generating function (1) when $p_n(z)$ satisfies a recursion relationship

$$\sum_{v=0}^{r} A_v(n)p_{n+v}(z) = 0, \qquad n = 0, 1, 2, \cdots, \tag{9}$$

where not all the A_v are zero. Often the $A_v(n)$ are rational functions of n, for example, if $p_n(z)$ is one of the classical orthogonal polynomials. When this happens is made clear by the following theorem.

THEOREM 1. *$p_n(z)$ satisfies (9) with $A_v(n)$ a rational function of n if and only if K satisfies the differential equation*

$$\sum_{v=0}^{s}\frac{\partial^v}{\partial w^v}K(z,w)B_v(w) = 0, \tag{10}$$

where the B_v are polynomials in w not all of which are zero.

The proof of this theorem is just a matter of series manipulations. Since it depends on properties of generating functions so well described by other writers (see [2, vol. 3; 3]) we omit it. For instance, to show (10) implies (9), one would write

$$\frac{\partial^r}{\partial w^r}K = \sum_{n=0}^{\infty} p_{n+r}(z)(n+1)_r w^n,$$

substitute in (10), and rearrange. Note that A_v, B_v in general will depend on z.

[1] For a discussion of the theory of (1), (2), (3), see [1].

Now, with the exception of certain logarithmic cases, the consideration of which introduces only mechanical, not conceptual, difficulties into our analysis, the work of Birkhoff and Trjitzinsky [4], [5] shows that if p_n satisfies (9), then we can express it as a linear combination

$$p_n(z) = \sum_{h=1}^{r'} V_h(n), \qquad r' \leqq r, \tag{11}$$

where the $V_h(n)$ are functions of z which have the following asymptotic representations:

$$\begin{aligned} V_h(n) &\sim e^{Q_h(n)} s_h(n), \qquad n \to \infty, \\ Q_h(n) &= \mu_{0,h} n \ln n + \mu_{1,h} n + \mu_{2,h} n^{(\rho-1)/\rho} + \cdots + \mu_{\rho,h} n^{1/\rho}, \\ s_h(n) &= n^{\theta_h}[\alpha_{0,h} + \alpha_{1,h} n^{-1/\rho} + \cdots], \end{aligned} \tag{12}$$

where ρ is an integer $\geqq 1$, $\alpha_{0,h} \neq 0$ and $\mu_{0,h}$ is an integral multiple of $1/\rho$. (12) is a natural generalization of a Poincaré type asymptotic expansion. For the properties of such expansions, see the cited references.

We now assume that σ_n can also be chosen so that

$$q_n(z) = \sum_{h=1}^{r'} W_h(n), \tag{13}$$

where a representation like (12) holds for each $W_h(n)$ but $\mu_{0,h} = 0$, $1 \leqq h \leqq r'$.

In [6], it is shown that $K_n^*(z, t)$ can be written as a linear combination of functions $U_h(n, t)$,

$$K_n^*(z, t) = \sum_{h=1}^{r'} U_h(n, t), \tag{14}$$

where each U_h has the asymptotic expansion

$$\begin{aligned} U_h(n, t) &\sim \frac{e^{Q_h(n)}}{t^n} s_h^*(n, t), \qquad n \to \infty, \\ s_h^*(n, t) &= n^{\theta_h}[\beta_{0,h} + \beta_{1,h} n^{-1/\rho} + \cdots]. \end{aligned} \tag{15}$$

The leading constants in the latter series may be found by the method of undetermined coefficients explained in the above reference. We have, for example,

$$\begin{aligned} \beta_{0,h} &= \frac{\alpha_{0,h}}{t - w_h}, \\ \beta_{1,h} &= \frac{\alpha_{1,h}}{t - w_h} + \frac{w_h \mu_{2,h}(\rho - 1)\alpha_{o,h}}{\rho(t - w_h)^2} \qquad (\rho > 1) \\ &= \frac{\alpha_{1,h}}{t - w_h} + \frac{w_h \theta_h \alpha_{0,h}}{(t - w_h)^2} \qquad (\rho = 1), \end{aligned} \tag{16}$$

where $w_h = e^{\mu_{1,h}} \neq 0$ and where $\theta_h, \alpha_{j,h}, Q_h, \mu_{j,h}$ are the parameters corresponding to the decomposition (12).

Now from the development in [5] the $\beta_{j,h}$ are seen to be continuous functions (in fact, analytic) if $t \neq w_h$. An easy argument using Theorem 7.13 of Rudin [7] shows the asymptotic representation (15) holds uniformly on all compact t sets not containing w_h. Note also that w_h is strictly within C, since by convergence of (5) and (15),

$$\overline{\lim_{n\to\infty}} \left| \frac{q_n(z)}{t^n} \right|^{1/n} = \frac{\max_h |w_h|}{|t|} < 1, \qquad t \in C. \tag{17}$$

Thus the representation (15) holds uniformly on C.

Substituting (14) in (7) gives

$$E_n[f(z)] = \sum_{h=1}^{r'} Y_h(n), \tag{18}$$

$$Y_h(n) \sim e^{Q_h(n)} n^{\theta_h} [\gamma_{0,h} + \gamma_{1,h} n^{-1/\rho} + \cdots], \qquad n \to \infty. \tag{19}$$

Since the $\gamma_{j,h}$ above depend on n, (19) must be interpreted as

$$\begin{aligned} &e^{-Q_h(n)} n^{-\theta_h} Y_h(n) - \sum_{j=0}^{r} \gamma_{j,h} n^{-j/\rho} = O(\gamma_{r+1,h} n^{-(r+1)/\rho}), \\ &n \to \infty, \qquad r = 0, 1, 2, \cdots. \end{aligned} \tag{20}$$

We have

$$\gamma_{0,h} = \frac{\alpha_{0,h}}{2\pi i} \int_C \frac{\phi(t)\, dt}{t^n (t - w_h)}, \tag{21}$$

$$\begin{aligned} \gamma_{1,h} &= \frac{1}{2\pi i} \int_C \frac{\phi(t)}{t^n} \left(\frac{\alpha_{1,h}}{t - w_h} + \frac{\lambda}{(t - w_h)^2} \right) dt, \\ \lambda &= w_h \mu_{2,h} \alpha_{0,h} (\rho - 1)/\rho \qquad (\rho > 1) \\ &= w_h \theta_h \alpha_{0,h} \quad (\rho = 1). \end{aligned} \tag{22}$$

Since ϕ is analytic on and within C we have

$$\begin{aligned} \frac{R_n[\phi, \alpha]}{\alpha^n} &= \frac{1}{2\pi i} \int_C \frac{\phi(t)\, dt}{t^n (t - \alpha)}, \\ R_n[\phi, \alpha] &= \phi(\alpha) - \sum_{k=0}^{n-1} \sigma_k \mathscr{L}_k(f) \alpha^k. \end{aligned} \tag{23}$$

Also,

$$\frac{R_{n-1}[\phi', \alpha]}{\alpha^n} - \frac{n R_n[\phi, \alpha]}{\alpha^{n+1}} = \frac{1}{2\pi i} \int_C \frac{\phi(t)\, dt}{t^n (t - \alpha)^2}. \tag{24}$$

We find that

$$\gamma_{0,h} = \frac{\alpha_{0,h}}{w_h^n} R_n[\phi, w_h], \tag{25}$$

$$\gamma_{1,h} = \frac{(\alpha_{1,h} - n\lambda)}{w_h^n} R_n[\phi, w_h] + \frac{\lambda}{w_h^n} R_{n-1}[\phi', w_h]. \tag{26}$$

THEOREM 2. *Let* $\phi(t)K_n^*(z, t)$, *where* ϕ, K_n^* *are given by* (3) *and* (5), *be analytic in some annulus* $A = \{t \mid r_1 < |t| < r_2\}$. *Let* $q_n(z)$ *have the decomposition* (13), *where*

$$W_h(n) \sim e^{Q_h(n)} s_h(n), \qquad n \to \infty, \tag{27}$$

$\mu_{0,h} = 0, s_h, Q_h$ *as in* (12). *Then* $E_n[f(z)]$ *has the asymptotic representation* (18), (19) *with leading coefficients* $\gamma_{0,h}$, $\gamma_{1,h}$ *related to the error of the Taylor series for* $\phi(t)$ *by* (25), (26).

Note r_1 is the singularity of largest modulus of K_n^*, r_2 the singularity of smallest modulus of ϕ. If $r_1 < r_2$ then a path C can be determined so the analysis above holds. Also, the fact that (1) has a nonzero radius of convergence is not essential to the analysis, only that the series for K_n^* converge. Thus the p_n may be generated by a "formal" power series.

Let C be a circle of radius R. Then

$$\left| \frac{R_n[\phi, \alpha]}{\alpha^n} \right| \leqq M_\phi(R)/R^{n-1}(R - |\alpha|), \tag{28}$$

where $M_\phi(R)$ is the maximum of $|\phi(t)|$ on $t \in C$. It follows that there exist constants A_h such that

$$|Y_h(n)| < \frac{e^{\operatorname{Re} Q_h(n)} n^{\operatorname{Re} \theta_h}}{R^{n-1}(R - |w_h|)} |\alpha_{0,h}| M_\phi(R) \left| 1 + \frac{A_h}{n^{1/\rho}} \right|, \qquad n > n_0, \tag{29}$$

and this can be a useful upper bound in (18). It is interesting to see what happens when ϕ is entire. Let

$$M_\phi(R) = O(e^{R^{\sigma + \varepsilon}}), \qquad R \to \infty \qquad \text{for all } \varepsilon > 0. \tag{30}$$

Then the function

$$g_n(R) = e^{R^{\sigma+\varepsilon}}/R^n$$

occurs in (27). This function attains a minimum at

$$R_0 = \left(\frac{n}{\sigma + \varepsilon} \right)^{1/(\sigma+\varepsilon)} \tag{31}$$

and Stirling's formula shows that

$$g_n(R_0) = \frac{C n^{-1/2}}{\Gamma(n/(\sigma + \varepsilon))} [1 + O(n^{-1})]. \tag{32}$$

We have shown the following result.

THEOREM 3. *Let* $\phi(t)$ *be an entire function of order* σ *and let the hypotheses of Theorem* 2 *hold. Then for every* $\varepsilon > 0$,

$$E_n[f(z)] = \Gamma\left(\frac{n}{\sigma + \varepsilon} \right)^{-1} \sum_{h=1}^{r} O_h[n^{\operatorname{Re} \theta_h - 1/2} e^{\operatorname{Re} Q_h(n)}], \qquad n \to \infty. \tag{33}$$

The order terms in the sum depend on ε.

3. Applications. As an example of the application of some of the previous formulas, we will consider the generating function for the Pollaczek polynomials,

$P_n^\lambda(z; a, b)$, [1, vol. 2, p. 218]. We will take $\sigma_n \equiv 1$.

$$(34)\qquad K(z, w) = \left(1 - \frac{w}{w_1}\right)^{-\lambda + iA}\left(1 - \frac{w}{w_2}\right)^{-\lambda - iA} = \sum_{n=0}^{\infty} P_n^\lambda(z; a, b) w^n,$$

where

$$(35)\qquad \begin{aligned} w_1 &= z + i\sqrt{1 - z^2}, \\ w_2 &= \frac{1}{w_1} = z - i\sqrt{1 - z^2}, \\ A &= -(az + b)/\sqrt{1 - z^2}, \qquad a, b, \lambda \text{ real}. \end{aligned}$$

Here we mean

$$(36)\qquad \begin{aligned} &\sqrt{1 - z^2} = |1 - z^2|^{1/2}\, e^{(i/2)(\phi_1 + \phi_2 - \pi)}, \\ &\phi_1 = \arg(z - 1), \quad \phi_2 = \arg(z + 1), \qquad -\pi < \phi_1 \leqq \pi, \quad 0 \leqq \phi_2 < 2\pi. \end{aligned}$$

Darboux's method may be applied to (34) to obtain an asymptotic representation for $p_n(z)$.

We get $r' = 2$,

$$(37)\qquad \begin{aligned} &p_n(z) = P_n^\lambda(z; a, b) = V_1(n) + V_2(n), \\ &V_1(n) \sim \frac{(1 - w_1/w_2)^{-\lambda - iA}}{\Gamma(\lambda - iA)} w_2^n n^{\lambda - 1 - iA}\left[1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right], \\ &V_2(n) \sim \frac{(1 - w_2/w_1)^{-\lambda + iA}}{\Gamma(\lambda + iA)} w_1^n n^{\lambda - 1 + iA}\left[1 + \frac{d_1}{n} + \frac{d_2}{n^2} + \cdots\right], \qquad n \to \infty. \end{aligned}$$

This reveals the remarkable fact that the powers of n in the algebraic portion of the asymptotic expansion of the Pollaczek polynomials *depend on z*, viz., $\theta_1 = \lambda - 1 - iA$, $\theta_2 = \lambda - 1 + iA$. For none of the classical orthogonal polynomials is this the case. (The Gegenbauer polynomials result when $a = b = 0$ so for these polynomials, $\theta_1 = \theta_2 = \lambda - 1$.)

For these Pollaczek polynomials we further have $\rho = 1$, $\mu_{0,1} = \mu_{0,2} = 0$,

$$e^{\mu_{1,h}} = w_2, \quad e^{\mu_{2,h}} = w_1$$

and $\alpha_{0,h}$ may be read off (37).

K as a function of w has branch cuts at w_1 and w_2. We assume a branch cut is established between these two points to make K single-valued. Now consider

$$(38)\qquad \phi(z) = \left(1 - \frac{z}{\gamma}\right)^\mu, \qquad |\gamma| > \max(|w_1|, |w_2|).$$

We have

$$(39)\qquad K^*(z, t) = \frac{1}{t}\left(1 - \frac{w_2}{t}\right)^{-\lambda + iA}\left(1 - \frac{w_1}{t}\right)^{-\lambda - iA},$$

$$\alpha_1 = -\lambda + iA, \quad \alpha_2 = -\lambda - iA,$$

so

$$f(z) = \frac{1}{2\pi i} \int_C \frac{(1 - t/\gamma)^\mu}{t} \left(1 - \frac{w_2}{t}\right)^{-\lambda + iA} \left(1 - \frac{w_1}{t}\right)^{-\lambda - iA} dt \tag{40}$$

and C is a circle on which

$$\max(|w_1|, |w_2|) < |t| < |\gamma|,$$

a branch cut having been made from γ to $\infty e^{i \arg \gamma}$.

Making the change of variable $w_1 u + (1 - u)w_2 = t$ in the integral and expanding $t^{2\lambda - 1}, (1 - t/\gamma)^\mu$ gives

$$\begin{aligned} f(z) = \frac{1}{2\pi i}(w_1^2 - 1)^{1 - 2\lambda}\left(1 - \frac{w_2}{\gamma}\right)^\mu \sum_{m,n=0}^{\infty} \frac{(-\mu)_m(-1)^m(1 - 2\lambda)_n}{m!n!w_2^n(\gamma - w_2)^m} \\ \cdot (w_2 - w_1)^{m+n} \int_{C'} u^{\alpha_1 + m + n}(u - 1)^{\alpha_2}\, du, \end{aligned} \tag{41}$$

where C' is a simple closed curve around [0, 1] in the clockwise direction. But this is a known integral for the beta function [1, vol. 1, p. 15]. We thus have obtained the expansion

$$\begin{aligned} &(w_1^2 - 1)^{1 - 2\lambda}\left(1 - \frac{w_2}{\gamma}\right)^\mu \frac{\Gamma(1 - \lambda + iA)\Gamma(1 - \lambda - iA)}{\Gamma(2 - 2\lambda)\pi} \sin[\pi(1 - \lambda - iA)] \\ &\cdot F_1\left(1 - \lambda + iA, -\mu, 1 - 2\lambda, 2 - 2\lambda; \frac{1 - w_1^2}{1 - \gamma w_1}, 1 - w_1^2\right) \\ &= \sum_{n=0}^{\infty} P_n^\lambda(z; a, b)\frac{(-\mu)_n \gamma^{-n}}{n!}. \end{aligned} \tag{42}$$

The function F_1 is Appell's hypergeometric function [2, vol. 1, p. 224 (6)].

If instead we start with the function

$$\phi(z) = e^{\xi z}, \tag{43}$$

a similar analysis yields the expansion

$$f(z) = \Phi_2(\lambda - iA, \lambda + iA, 1, w_2\xi, w_1\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} P_n^\lambda(z; a, b). \tag{44}$$

Now for t fixed,

$$R_n[\phi, t] = R_n[e^{\xi t}] = \frac{(\xi t)^n}{n!}[1 + O(n^{-1})], \tag{45}$$

so

$$\gamma_{0,h} = \frac{\alpha_{0,h}\xi^n}{n!} \tag{46}$$

and so

$$E_n[f(z)] = Y_1(n) + Y_2(n), \tag{47}$$

$$\begin{aligned} Y_1(n) &\sim \frac{(\xi w_2)^n}{n!} n^{\lambda - 1 - iA} \alpha_{0,1} \left[1 + \frac{a_1}{n} + \cdots\right], \qquad n \to \infty, \\ Y_2(n) &\sim \frac{(\xi w_1)^n}{n!} n^{\lambda - 1 + iA} \alpha_{0,2} \left[1 + \frac{b_1}{n} + \cdots\right], \qquad n \to \infty. \end{aligned} \tag{48}$$

Roughly speaking, this means that $E_n[f(z)]$ behaves like $\xi^n P_n^\lambda(z; a, b)/n!$ as $n \to \infty$. Since in this case $\mathscr{L}_n(f)$ can be estimated asymptotically by Stirling's formula, the theory in [3] or a straightforward majorization argument based on the estimates (37) could be used to obtain an asymptotic formula for $E_n[f(z)]$. However, $\mathscr{L}_n(f)$ will not in general be so tractable.

Similar formulas pertaining to the expansion (42) can also be readily deduced.

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ASYMPTOTIC ANALYSIS OF NONLINEAR DIFFUSION AND RELATED MULTIDIMENSIONAL INTEGRALS*

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Abstract. In many important physical systems involving both diffusion and nonlinearity it often occurs that initially diffusion is the dominant mechanism. The question then arises as to whether or not linearization provides a uniformly valid first approximation for large times. A detailed examination of several model equations, both deterministic and stochastic, reveals that often the nonlinearity has a cumulative effect on a long time scale which must eventually be included in the first approximation. In particular, it is proved that linearization is not uniformly valid for Burgers' model of turbulence. A major part of the analysis involves constructing asymptotic expansions for an interesting class of multidimensional integrals.

1. Introduction and general discussion. The linear phenomenon of diffusive decay has its mathematical origins in the theory of heat conduction. Indeed, the mathematical description of the diffusion process is usually presented in terms of solutions of the classical heat equation. However, many important physical systems involve the combined effects of nonlinearity and diffusion which greatly increases the mathematical complexity. As a result our understanding of such problems is rather meager. In [1] the reader will find an excellent account of the rich variety of phenomena that can occur due to the interplay between diffusion and simple nonlinearities.

Confronted with the difficult task of constructing solutions for problems involving nonlinearity and diffusion, many authors have resorted to a study of various limiting situations. Perhaps the most commonly made assumption is that the diffusion is the controlling mechanism, or, equivalently, that the nonlinearity is "weak." This assumption is usually based on either the "smallness" of the initial data or certain plausibility arguments which essentially contend that dissipation will eventually force the nonlinear terms to be negligible. Regardless of the validity of such an assumption, its advantages are apparent. A perturbation approach becomes feasible in which the leading term in the perturbation expansion represents a linear diffusion process. Higher order terms represent corrections due to the nonlinearity in the system. We shall refer to this approximation of the solution as a regular perturbation expansion (RPE).

Of course, if the RPE is uniformly valid (see (3.12) for a necessary condition), then in a sense the problem is trivial mathematically. Moreover, one is unlikely to gain much of an understanding of the fully nonlinear case. However, it turns out that for many important physical systems the RPE is not uniformly valid. This is often very difficult to discern and as a result many authors have erroneously asserted the uniform validity of the RPE in special cases. The purpose of the present work is to examine in detail several simple model equations, both random and deterministic, in the hope of elucidating the conditions under which one might expect

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nonuniformities to arise in the general case. Several of the models are physically important and have been treated incorrectly in recent works.

In an earlier paper [2], a singular perturbation technique was devised for constructing formal uniformly valid asymptotic solutions in situations where the RPE is nonuniform. The method itself was motivated by a study of several simple exact solutions. In each case considered the RPE was expressed in terms of Fourier integrals. Examination of the behavior of the RPE was based on the assumed equivalence of ordering in Fourier space. However, a rigorous establishment of the possible regions of validity of the RPE necessitates an asymptotic evaluation of the Fourier integrals themselves.

In § 2 we construct asymptotic expansions for several integrals which occur frequently in nonlinear diffusion problems. A typical integral is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\frac{e^{-x^2t}-e^{-y^2t}}{x^2-y^2}\,dx\,dy, \tag{1.1}$$

where f is a real-valued function and t is a large, positive parameter. The task of constructing expansions, for $t \to +\infty$, for (1.1) and related integrals is facilitated by the use of Cauchy principal values.

In § 3 we consider the Cauchy problem for certain deterministic differential equations. A detailed study of the exact solution of a model equation illustrates the effect of higher dimensions in reducing the importance of weak nonlinear terms in diffusion problems. In addition, we conclude from this study that estimating the significance of neglected nonlinear terms by using only the solution to the linear problem does not in general provide a sufficient test of the uniform validity of the RPE. Accordingly, results such as those discussed in [3], which deals with the problem of small disturbances in a dissipative gas, must be interpreted with some care.

The second part of § 3 involves a study of a class of problems involving quadratic nonlinearities. It is shown that no apparent nonuniformities arise in the associated RPE except in certain special cases. One such case corresponds to a physical process involving mass-diffusion and absorption due to nonlinear chemical reaction. We establish that certain assertions made in [4] regarding the uniform validity of the RPE for second order reactions are incorrect. The technique discussed in [2] provides a means of obtaining a uniformly valid solution for this case.

Although the main emphasis of this paper is on a study of nonlinear diffusion problems, § 4 is devoted to a brief examination of the long time effect of small variable coefficient terms on the diffusion operator. We consider the Cauchy problem for the deterministic case and following the treatment in § 3 establish conditions under which one might expect the RPE to be nonuniform.

In §§ 5 and 6 we consider the Cauchy problem for two stochastic model equations. Section 5 is devoted to a detailed study of the so-called final period of decay problem for Burgers' model of turbulence. In [5] it is argued that the RPE (or equivalently linearization) is uniformly valid during this time regime. We present a rigorous proof that this is not the case. In § 6 we examine a limiting case of the passive scalar problem which has served as a convenient model for testing closure schemes designed for the problem of homogeneous turbulence. We establish that a RPE based on the smallness of the velocity field is not uniformly valid. Moreover, the results of the analysis bring out the doubtful validity of cumulant discard approximations.

Aside from their importance as regards the problem of homogeneous turbulence the two equations studied in §§ 5 and 6 are representative of those arising in many physical situations. The reader interested in other examples of statistical initial value problems involving weak nonlinearities may wish to refer to [6].

The Appendix contains a brief discussion of several integrals which appear in the main body of the paper.

2. Asymptotic analysis of the basic integrals. The integrals which we shall consider are of the form

$$\int_{R^N} g(x)h(x,t)\,dx, \tag{2.1}$$

where x denotes an N-dimensional vector in the real N-dimensional Euclidean space R^N, t is a real parameter and g and h are (with a few exceptions) real-valued functions. The function h will in general involve exponential behavior reflecting the fact that the integrals arise in a study of diffusion processes. Our main concern will be to determine asymptotic approximations of these integrals for large values of the parameter t.

The approach that we shall follow relies on the use of straightforward generalizations of Watson's lemma and on the properties of Cauchy principal value integrals. For our purposes it proves convenient to use the following form of Watson's lemma.

WATSON'S LEMMA. *Let $g(x)$ be a real-valued function which is $L^1[0, \infty)$, i.e.,*

$$\int_0^\infty |g(x)|\,dx < \infty, \tag{2.2}$$

and which has the expansion

$$g(x) = \sum_{n=1}^\infty a_n x^{n/k-1}, \tag{2.3}$$

for $0 \leqq x < \delta$, $\delta > 0$ and k a positive integer. Then the function $I(t) = \int_0^\infty e^{-xt}g(x)\,dx$, $t \geqq 0$, has the asymptotic expansion

$$I(t) \sim \sum_{n=1}^\infty a_n \Gamma\left(\frac{n}{k}\right) t^{-n/k} \quad \textit{for } t \to \infty. \tag{2.4}$$

For a proof of this classical result the reader may wish to refer to [7].

As Hardy has shown in detailed investigations [8], one can operate with Cauchy principal values largely as with ordinary integrals. We shall make repeated use of the following formula for parametric differentiation of principal values:

$$\frac{d}{dx} P\int_{-\infty}^\infty \frac{g(x,y)}{y-x}\,dy = P\int_{-\infty}^\infty \frac{1}{y-x}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) g(x,y)\,dy, \tag{2.5}$$

which holds, for example, if g is $C^1(R^2)$ and

$$\int_{-\infty}^\infty \left|\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) g(x,y)\right| dy < \infty.$$

A second property involves interchanging the order of integration for iterated principal value integrals. Here we must use the Poincaré–Bertrand formula [9]; namely,

$$P_x\frac{1}{x}P_y\frac{1}{y+x} + P_y\frac{1}{y}P_x\frac{1}{x+y} = \pi^2\delta(x)\delta(y) + P_x\frac{1}{x}P_y\frac{1}{y}, \tag{2.6}$$

where the above notation is a symbolism for

$$\begin{aligned} &P\int_{-\infty}^{\infty}\frac{dx}{x}P\int_{-\infty}^{\infty}dy\frac{g(x,y)}{y+x} + P\int_{-\infty}^{\infty}\frac{dy}{y}P\int_{-\infty}^{\infty}dx\frac{g(x,y)}{y+x} \\ &\qquad = \pi^2 g(0,0) + P\int_{-\infty}^{\infty}\frac{dx}{x}P\int_{-\infty}^{\infty}\frac{dy}{y}g(x,y). \end{aligned} \tag{2.7}$$

A sufficient condition for the validity of (2.7) is that g is $L^1(R^2)$ and uniformly Lipschitz continuous on R^2.

We are now in a position to construct asymptotic expansions for several important integrals. We shall begin by briefly considering two elementary one-dimensional integrals. The first one is associated with the Fourier transform representation of the solution to the unperturbed diffusion equation; namely,

$$I_1(t) = \int_{-\infty}^{\infty} g(x)\exp(-x^2t - ix\lambda t^{1/2})\,dx, \tag{2.8}$$

where λ is a function of the positive parameter t. Assuming that g is analytic at $x = 0$ and $L^1(R^1)$ it readily follows from a direct modification of the procedure leading to Watson's lemma that (for $t \to \infty$ with $\lambda \leqq O(1)$)

$$I_1 \sim \frac{e^{-\lambda^2/4}}{t^{1/2}}\sum_{m=0}^{\infty}\frac{g^{(m)}(0)}{m!t^{m/2}}\int_{-\infty}^{\infty}\left(\xi - i\frac{\lambda}{2}\right)^m e^{-\xi^2}\,d\xi, \tag{2.9}$$

where

$$g^{(m)}(0) = \left.\frac{d^m g(x)}{dx^m}\right]_{x=0}.$$

Equation (2.8) can be generalized to an N-dimensional integral, in which case a result similar to (2.9) holds in terms of radial derivatives of $g(x)$ evaluated at the origin with the leading term (assuming $g(0) \neq 0$) being $O(1/t^{N/2})$.

The second integral is

$$I_2(t) = \int_0^{\infty} g(x)\Delta(-x^2)\,dx, \tag{2.10}$$

where

$$\Delta(\xi) \equiv (e^{\xi t} - 1)/\xi. \tag{2.11}$$

For simplicity we shall assume that g is analytic at $x = 0$ and $L^1[0, \infty)$. It proves convenient to rewrite (2.10) as

$$
\begin{aligned}
I_2(t) &= g(0)\int_0^\infty \Delta(-x^2)\,dx + g'(0)\int_0^\infty x\,e^{-x^2}\Delta(-x^2)\,dx \\
&\quad + \int_0^\infty \frac{g(x) - g(0) - g'(0)x\,e^{-x^2}}{x^2}\,dx \\
&\quad - \int_0^\infty \frac{g(x) - g(0) - g'(0)x\,e^{-x^2}}{x^2}\,e^{-x^2 t}\,dx.
\end{aligned}
\tag{2.12}
$$

The factor e^{-x^2} has been included to insure the existence of the third (t-independent) integral on the right-hand side of (2.12). It follows from (2.4) that

$$
\begin{aligned}
I_2(t) &\sim \sqrt{\pi} g(0)t^{1/2} + \frac{g'(0)}{2}\log(1+t) + \int_0^\infty \frac{g(x) - g(0) - g'(0)x\,e^{-x^2}}{x^2}\,dx \\
&\quad - \sum_{m=2}^{\infty}\left\{\frac{g^{(m)}(0)((m-3)/2)!}{2m!\,t^{(m-1)/2}} + \frac{g'(0)(-1)^m}{2(m-1)t^{m-1}}\right\}.
\end{aligned}
\tag{2.13}
$$

Upon expanding $\log(1+t)$ as

$$
\log(1+t) \sim \log t + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{mt^m},
$$

certain terms in (2.13) cancel leaving

$$
\begin{aligned}
I_2(t) &\sim \sqrt{\pi} g(0)t^{1/2} + \frac{g'(0)}{2}\log t + \int_0^\infty \frac{g(x) - g(0) - g'(0)x\,e^{-x^2}}{x^2}\,dx \\
&\quad - \sum_{m=2}^{\infty}\frac{g^{(m)}(0)(\frac{1}{2}m - \frac{3}{2})!}{2m!\,t^{(m-1)/2}}.
\end{aligned}
\tag{2.14}
$$

As we shall find, the occurrence of the $\log t$ term in (2.14) has significance for diffusion problems in R^2. The $O(1)$ term in (2.14) is also of some interest. In particular we note that it requires global information about g. Moreover, the choice of the function e^{-x^2} is essential (in the sense that any equivalent function f must satisfy

$$
\int_0^\infty (f(x) - e^{-x^2})/x\,dx = 0).
$$

We shall now examine three double integrals. The first one arises in a study of the passive scalar problem (see § 6). It is given by

$$
I_3(t) = \int_D g(x)\exp[(\lambda^2 - 1)x^2 t]\,d\lambda\,dx, \tag{2.15}
$$

where D denotes the domain $[0, 1] \times R^1$ in the (λ, x)-plane. For simplicity we shall assume that g is $C^\infty(R^1)$ and that g and its derivatives to all orders are $L^1(R^1)$. In this case the double integral is equivalent to the iterated integrals. An additional restriction on g will be made at an appropriate point in the analysis.

It seems clear that the origin and the line $\lambda = 1$ play an important role in the asymptotic evaluation of this integral because of the vanishing of the argument of the exponential factor. Of course it is easy to find the limiting behavior of I_4, but we are interested in obtaining the entire expansion. Although by no means obvious, by making use of the identity

$$-2\sqrt{\pi t}x\, e^{-x^2 t}\int_0^1 e^{x^2\lambda^2 t}\, d\lambda = P\int_{-\infty}^{\infty}\frac{e^{-y^2 t}}{y-x}\, dy, \tag{2.16}$$

we can transform (2.15) to a more tractable form. The proof of (2.16) is given in the Appendix, equation (A.3). Utilizing (2.16) the integral in (2.15) becomes

$$I_3(t) = \frac{1}{2\sqrt{\pi t}} P_x \frac{1}{x} P_y \frac{1}{x-y} g(x)\, e^{-y^2 t}. \tag{2.17}$$

In order to take advantage of the exponential factor we must interchange the order of integration. This can be effected by using (2.7) which leads to

$$I_3(t) = \frac{1}{2\sqrt{\pi t}} P_y \frac{e^{-y^2 t}}{y} P_x \frac{g(x)}{x-y} + \frac{\pi^2 g(0)}{2\sqrt{\pi t}} - \frac{1}{2\sqrt{\pi t}} P_y \frac{e^{-y^2 t}}{y} P_x \frac{g(x)}{x}. \tag{2.18}$$

The value of the last integral in (2.18) is zero because the integrand is an odd function of y. If g is such that the function

$$P\int_{-\infty}^{\infty}\frac{g(x)}{x-y}\, dx$$

is analytic at $y = 0$, then the first integral on the right-hand side can be expanded as in the case of the one-dimensional integrals. This leads us to the resulting asymptotic expansion

$$I_3(t) \sim \frac{\pi^{3/2} g(0)}{2t^{1/2}} + \frac{1}{\sqrt{\pi t}}\sum_{m=0}^{\infty}\frac{(m-\frac{1}{2})!}{2(2m+1)!t^m} P\int_{-\infty}^{\infty}\frac{1}{x}\frac{d^{2m+1}g(x)}{dx^{2m+1}}\, dx. \tag{2.19}$$

The form of the expansion in (2.19) is rather surprising. While the value of g at the origin determines the dominant contribution, the form of the higher order terms depends upon global information about g. This situation should be contrasted with (2.9). It is clear that the presence of the critical curve $\lambda = 1$ is responsible for this phenomenon.

Although the use of principal values proved convenient in the determination of (2.19) this expansion, as well as (2.14), can also be obtained by a method involving the use of Mellin transforms [10]. (This fact was kindly pointed out to the author by J. S. Lew.) However, for the following integrals the use of principal values is both natural and crucial. The only other method by which the author has managed to obtain expansions for these integrals involves lengthy calculations with generalized Fourier transforms.

A study of diffusion equations involving variable coefficients in § 4 depends on the long time behavior of the following double integral:

$$\begin{aligned} I_4(t) &= \int_{R^2} g(x, y)\Delta(x^2 - y^2)\, e^{-x^2 t}\, dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\frac{e^{-y^2 t} - e^{-x^2 t}}{x^2 - y^2}\, dx\, dy. \end{aligned} \tag{2.20}$$

Rather than give a detailed description of conditions on g we shall simply assume for the moment that g is $L^1(R^2)$ and uniformly Lipschitz continuous on R^2. It would appear from the form of the integrand in (2.20) that the x- and y-axes are critical curves; for along these curves the argument of one or the other of the exponentials vanishes. In fact, we shall show that the asymptotic behavior of I_4 is affected only by the nature of g near the x- and y-axes. It should be noted that along the axes the t-dependent factor in (2.20) resembles that in the integrand of (2.10).

Expressing $(x^2 - y^2)^{-1}$ as $(2x)^{-1}[(x - y)^{-1} + (x + y)^{-1}]$, we can rewrite (2.20) as

$$\begin{aligned} I_4(t) &= P_x\frac{1}{2x}P_y\left(\frac{1}{x - y} + \frac{1}{x + y}\right)g(x, y)\, e^{-y^2 t} \\ &\quad - P_x\frac{e^{-x^2 t}}{2x}P_y\left(\frac{1}{x - y} + \frac{1}{x + y}\right)g(x, y), \end{aligned} \tag{2.21}$$

since each of the integrals on the right exists separately. As we did for I_3 we use (2.7) to interchange the order of integration in the first integral so that

$$\begin{aligned} I_4(t) &= \pi^2 g(0, 0) + P_y\frac{e^{-y^2 t}}{2y}P_x\frac{1}{x - y}[g(x, y) + g(-x, y)] \\ &\quad + P_x\frac{e^{-x^2 t}}{2x}P_y\frac{1}{y - x}[g(x, y) + g(x, -y)]. \end{aligned} \tag{2.22}$$

A term of the form $P_x(1/x)P_y(1/y)g(x, y)\, e^{-y^2 t}$ does not appear because of cancellations due to the terms $(x - y)^{-1}$ and $(x + y)^{-1}$. If g is such that the function $P_x(x - y)^{-1}[g(x, y) + g(-x, y)]$ is analytic at $y = 0$ and the function $P_y(y - x)^{-1} \cdot [g(x, y) + g(x, -y)]$ is analytic at $x = 0$, then the asymptotic expansion of I_4 is given by

$$\begin{aligned} I_4(t) &\sim \pi^2 g(0, 0) + \frac{1}{t^{1/2}}\sum_{m=0}^{\infty}\frac{(m - \frac{1}{2})!}{(2m + 1)!2t^m}\left\{P_x\frac{1}{x}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)^{2m+1}\right. \\ &\quad \cdot [g(x, y) + g(-x, y)]_{y=0} + P_y\frac{1}{y}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)^{2m+1} \\ &\quad \left. \cdot [g(x, y) + g(x, -y)]_{x=0}\right\}. \end{aligned} \tag{2.23}$$

The last double integral which we shall consider arises in a study of diffusion problems involving quadratic nonlinearities in § 3.

$$I_5(t) = \int_{R^2} g(x, y)\Delta(x^2 - y^2) \exp(-cx^2t + iAxt^{1/2})\, dx\, dy, \tag{2.24}$$

where c and A are real constants with $c > 1$. As with I_4 we can express I_5 as

$$\begin{aligned} I_5(t) = &-P_x \frac{\exp(-cx^2t + iAxt^{1/2})}{2x} P_y \frac{1}{x - y}[g(x, y) + g(x, -y)] \\ &+ P_x \frac{1}{2x} P_y \frac{1}{x - y}[g(x, y) + g(x, -y)] \\ &\cdot \exp[-y^2t + (1 - c)x^2t + iAxt^{1/2}]. \end{aligned} \tag{2.25}$$

The first integral can be handled as above; however, the second one requires more study. We shall assume that g is analytic at the origin and $L^1(R^2)$. Because of the exponential decay as $x^2 + y^2 \to \infty$ only the neighborhood of the origin makes a relevant contribution to the integral. Hence, disregarding an error term which is exponentially small for large t, the second integral in (2.25) becomes

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\partial^{m+n} g(0, 0)}{\partial x^m \partial y^n} \frac{[1 + (-1)^n]}{m!n!} P_x \frac{1}{x} P_y \frac{1}{x - y} x^m y^n \\ &\cdot \exp[-y^2t + (1 - c)x^2t + iAxt^{1/2}]. \end{aligned} \tag{2.26}$$

On setting $x' = x\sqrt{t}$ and $y' = y\sqrt{t}$, (2.26) can be expressed (upon dropping primes) as

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\partial^{m+2n} g(0, 0)}{\partial x^m\, \partial y^{2n}} \frac{1}{m!(2n)!t^{(m+2n)/2}} J_{m,2n}(c - 1, 1, A, 0), \tag{2.27}$$

where we have introduced

$$J_{m,n}(a, b, A, B) \equiv P_x \frac{1}{2x} P_y \frac{1}{x - y} x^m y^n \exp[-ax^2 - by^2 + iAx + iBy], \tag{2.28}$$

which is well-defined for a and b positive, A and B real. Since $J_{m,n}$ can be expressed as

$$J_{m,n}(a, b, A, B) = \frac{1}{i^{m+n}} \frac{\partial^{m+n} J_{0,0}(a, b, A, B)}{\partial A^m\, \partial B^n}, \tag{2.29}$$

we need only reduce $J_{0,0}$ to an ordinary integral. This can be accomplished by applying (A.4) which yields

$$\begin{aligned} J_{0,0}(a, b, A, B) = P \int_{-\infty}^{\infty} dx \int_0^1 d\xi \Big\{ &\sqrt{\pi b} \\ &\cdot \exp(-ax^2 + iAx + iBx - bx^2 + bx^2\xi^2) \\ &- i\sqrt{\frac{\pi}{b}} B \frac{\exp(-ax^2 + iAx + iBx - iBx\xi - \xi^2B^2/(4b))}{x} \Big\}. \end{aligned} \tag{2.30}$$

The first part of this integral can be simplified by interchanging the order of integration, integrating with respect to x and making the change of variables $\xi = \sqrt{(a+b)/b}\sin\theta$. The second part reduces to an ordinary double integral upon applying (A.4). This leaves us with

$$\begin{aligned} J_{0,0}(a,b,A,B) &= \pi\int_0^{\tan^{-1}\sqrt{b/a}} \exp\left[-\frac{(A+B)^2}{4(a+b)}\sec^2\theta\right]d\theta \\ &\quad + \frac{\pi B}{2\sqrt{ab}}\int_0^1\int_0^1 (A+B-B\xi) \\ &\quad \cdot \exp\left[-\frac{\xi^2B^2}{4b} - \frac{\eta^2}{4a}(A+B-B\xi)^2\right]d\eta\, d\xi. \end{aligned} \tag{2.31}$$

Provided a and b are positive, we can differentiate $J_{0,0}$ an arbitrary number of times with respect to a, b, A and B. There is no difficulty in taking A or B to be zero.

Finally, if the function $P_y(x-y)^{-1}[g(x,y)+g(x,-y)]$ is analytic at $x=0$, then the asymptotic expansion of (2.24) is given by

$$\begin{aligned} I_5(t) &\sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial^{m+2n}g(0,0)}{\partial x^m\,\partial y^{2n}}\,\frac{J_{m,2n}(c-1,1,A,0)}{m!(2n)!t^{(m+2n)/2}} \\ &\quad + \sum_{m=1}^{\infty}\frac{e^{-A^2/4c}}{2m!(ct)^{m/2}}P_y\frac{1}{y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^m[g(x,y)+g(x,-y)]_{x=0} \\ &\quad \cdot \int_{-\infty}^{\infty}\left(\xi+\frac{iA}{2\sqrt{c}}\right)^{m-1}e^{-\xi^2}\,d\xi, \end{aligned} \tag{2.32}$$

where $J_{m,2n}$ is defined by (2.28), (2.29) and (2.31).

We conclude this section by considering one triple integral which arises in the study of equations with variable coefficients:

$$I_6(t) = \int_{R^3} g(x,y,z)E(x^2-y^2;x^2-z^2)\,e^{-x^2t}\,dx\,dy\,dz, \tag{2.33}$$

where

$$E(\eta;\xi) \equiv \frac{\Delta(\eta)-\Delta(\xi)}{\eta-\xi}. \tag{2.34}$$

For the moment the only restriction that we place on g is that it be $L^1(R^3)$ and uniformly Lipschitz continuous on R^3. Setting $(z^2-y^2)^{-1} = (2y)^{-1}[(y-z)^{-1} + (y+z)^{-1}]$, splitting (2.33) into two integrals and applying formula (2.7) we obtain

$$\begin{aligned} I_6(t) &= \pi^2\int_{-\infty}^{\infty} g(x,0,0)\Delta(-x^2)\,dx + \mathscr{P}_{yz}P_xP_y\Delta(x^2-y^2)\,e^{-x^2t} \\ &\quad \cdot P_z\frac{1}{z-y} \\ &\quad \cdot\left[\frac{g(x,y,z)+g(x,y,-z)-g(x,y,z-y)-g(x,y,y-z)}{2y}\right], \end{aligned} \tag{2.35}$$

where $\mathscr{P}_{yz}$ denotes a cyclic summation on y and z. The factors involving $g(x, y, z - y)$ and $g(x, y, y - z)$ have been added to make the integrand well-behaved as $y \to 0$; because of the oddness they contribute nothing to the integral. Now we shall assume that g is such that the function $\psi(x, y)$, given by

$$\begin{aligned}\psi(x, y) \equiv {} & P_z \frac{1}{z - y} \\ & \cdot \frac{g(x, y, z) + g(x, y, -z) - g(x, y, z - y) - g(x, y, y - z)}{2y},\end{aligned} \tag{2.36}$$

satisfies the conditions which make (2.23) valid (with ψ in place of g in (2.23)). Further, we shall assume that $g(x, 0, 0)$ is $L^1(R^1)$ and analytic at $x = 0$. Thus, we can use the results of (2.14) and (2.23) to deduce that

$$\begin{aligned}I_6(t) \sim {} & 2\pi^{5/2} g(0, 0, 0) t^{1/2} + \pi^2 P \int_{-\infty}^{\infty} \frac{g(x, 0, 0) - g(0, 0, 0)}{x^2}\, dx \\ & + \pi^2 \mathscr{P}_{yz} P_y \frac{1}{y} \frac{\partial g(0, y, 0)}{\partial y} \\ & - \sum_{m=0}^{\infty} \frac{(m - \frac{1}{2})!}{(2m + 1)!\, t^{m+1/2}} \\ & \quad \cdot \left\{ \pi^2 (2m + 1) \frac{\partial^{2m+2} g(0, 0, 0)}{\partial x^{2m+2}} - \frac{1}{2} \mathscr{P}_{yz} \left[P_y \frac{1}{y} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^{2m+1} \right.\right. \\ & \quad \cdot [\psi(x, y) + \psi(x, -y)]_{x=0} + P_x \frac{1}{x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^{2m+2} \\ & \quad \left.\left. \cdot [\psi(x, y) + \psi(-x, y)]_{y=0} \right] \right\},\end{aligned}$$

where ψ is defined in (2.36). Thus, in order to obtain the coefficients for a particular g we are faced with calculating 2-fold iterated Cauchy principal value integrals. The contributions to (2.37) are associated with the critical nature of the axes and the x, y- and x, z-planes.

The integrals which we have examined play an important role in the sequel. They were chosen to illustrate the techniques necessary for an asymptotic evaluation as well as the unusual form which the resulting asymptotic expansions can take. The reader will note that often we placed conditions on the function g which were much too restrictive; however, our main interest was in obtaining the form of the asymptotic expansions rather than in determining the weakest conditions under which they are valid.

3. The Cauchy problem for nonlinear diffusion. In this section we consider the Cauchy problem for a class of partial differential equations involving "weak nonlinearities." In particular, we investigate in some detail the influence of quadratic nonlinearity on the diffusion operator. To be specific, consider the equation

$$Lu \equiv \frac{\partial u}{\partial t} - \nabla^2 u = \varepsilon f(u), \quad x = (x_1, \cdots, x_N) \in R^N, \quad t > 0, \tag{3.1}$$

where f is a nonlinear term involving the real-valued function u and its spatial derivatives,

$$\nabla^2 \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2},$$

and $\varepsilon > 0$ is a small parameter. Further, u is required to satisfy the initial condition

$$u(x, 0; \varepsilon) = h(x), \qquad x \in R^N. \tag{3.2}$$

The parameter ε is taken to be a measure of the nonlinearity (at least for suitably small values of t). As we shall see, this translates into certain restrictions on the moments of h. Associated with (3.1)–(3.2) is the so-called regular perturbation expansion (RPE) given by

$$\sum_{j=0}^{\infty} u_j(x, t)\varepsilon^j, \tag{3.3}$$

where the first two terms satisfy

$$Lu_0 = 0, \qquad u_0(x, 0) = h(x), \tag{3.4}$$

$$Lu_1 = f(u_0), \qquad u_1(x, 0) = 0. \tag{3.5}$$

The higher order terms in (3.3) are determined by iterating in (3.1) upon the lower order terms subject to the initial conditions

$$u_j(x, 0) = 0, \quad x \in R^N, \quad j \geqq 1. \tag{3.6}$$

From the perspective of one faced with having to construct approximate solutions for a particular case of (3.1)–(3.2) the RPE has a natural appeal. The nonlinear problem is replaced by a sequence of tractable linear problems. Moreover, a study of simple examples provides some assurance that under fairly general conditions this linearization process has an initial region of validity $0 \leqq t \leqq T(\varepsilon)$. To one willing to accept the RPE as an approximate solution the determination of $T(\varepsilon)$ becomes of crucial importance; for times larger than $T(\varepsilon)$ the essential part of the nonlinear effects must be incorporated in the basic approximation. Unfortunately it appears to be quite difficult to establish a rule which permits a direct determination of $T(\varepsilon)$ for a particular case of (3.1)–(3.2). However, the behavior of the RPE itself can be of some help; for if nonuniformities arise in the RPE on a time scale $\tilde{T}(\varepsilon)$ then clearly we must have $T \leqq O(\tilde{T})$. Again appeal to simple examples for which exact solutions are available suggests that for many problems T and $\tilde{T}$ are the same.

In this section we shall briefly consider special cases of (3.1)–(3.2) in order to illustrate the effect on T of the form of the nonlinearity and the dimension of the x-space. There are several equivalent ways of representing the terms in the RPE. In each case one is confronted with the difficult task of estimating the behavior of these terms for various ranges of x and t. Fortunately this task becomes much easier when the initial data is such that we are permitted to work with ordinary Fourier transforms; e.g., when h is $L^1(R^N)$. For the most part we shall restrict our attention to initial data that satisfy this condition. The Fourier transforms of u and h are defined by

$$A_l = A(k_l, t; \varepsilon) = \frac{1}{(2\pi)^N} \int_{R^N} u(x, t; \varepsilon)\, e^{ik_l \cdot x}\, dx, \tag{3.7}$$

$$H_l = H(k_l) = \frac{1}{(2\pi)^N} \int_{R^N} h(x)\, e^{ik_l \cdot x}\, dx, \tag{3.8}$$

with the corresponding relations

$$u(x, t; \varepsilon) = \int_{R^N} A_l\, e^{-ik_l \cdot x}\, dk_l, \qquad H(x) = \int_{R^N} H_l\, e^{-ik_l \cdot x}\, dx. \tag{3.9}$$

Associated with the RPE for $u(x, t; \varepsilon)$ in physical space is a RPE for A_l in Fourier space (the expansion obtained by transforming (3.3) term by term). Although it proves convenient to perform much of the analysis in Fourier space, it is important to note that a Fourier space expansion which is uniform for $k_l \in R^N$ and $t \in [0, \infty)$ does not necessarily lead (upon term-by-term integration) to a physical space expansion which is uniform for $x \in R^N$ and $t \in [0, \infty)$. This is illustrated by the uniform Fourier space expansion ($\varepsilon \to 0$)

$$\sum_{p=1}^{\infty} \varepsilon^p\, e^{-p^2k^2/4t}, \qquad k \in R^1, \quad t > 0, \tag{3.10}$$

which is the transform of

$$(\pi t)^{1/2} \sum_{p=1}^{\infty} \frac{\varepsilon^p}{p}\, e^{-x^2t/p^2}, \qquad x \in R^1, \quad t > 0. \tag{3.11}$$

Comparing the ratios of successive terms in the latter expansion we find that it is not uniform in any region in $R^1 \times [0, \infty)$ where $xt^{1/2} > O(1)$.

Before considering the first example we simply note that if the RPE is to represent a uniformly valid asymptotic expansion of a bounded solution of (3.1) and (3.2) then it is necessary, but not sufficient, that

$$u(x, t; \varepsilon) - \sum_{i=0}^{\infty} u_i(x, t)\varepsilon^i = O(\varepsilon^{m+1}), \qquad m = 0, 1, \cdots. \tag{3.12}$$

In this condition, the "O" symbol is for $\varepsilon \to 0$ and it implies estimates that are independent of x for $x \in R^N$ and of t for $t \geqq 0$.

The following example is especially useful for illustrating the effect of higher dimensions and the "degree of the nonlinearity" when f has the form u^m. Consider the equation

$$Lu = -\varepsilon u(x, t; \varepsilon)[u(0, t; \varepsilon)]^p, \qquad x \in R^N, \quad t > 0, \tag{3.13}$$

with the initial condition

$$u(x, 0; \varepsilon) = h(x), \qquad x \in R^N, \qquad h \in L^1(R^N), \tag{3.14}$$

and where p is any real number. We shall deal only with nonnegative solutions so that there is no ambiguity in the meaning of the right-hand side of (3.13). As the reader may verify, the exact solution of this problem can be readily expressed in terms of Fourier integrals as

$$u(x,t;\varepsilon) = \int_{R^N} \frac{H_l \exp(-|k_l|^2 t - ik_l \cdot x)}{\{1 + \varepsilon p \int_0^t [\int_{R^N} H_m e^{-|k_m|^2 t}\, dk_m]^p\, dt\}^{1/p}}\, dk_l, \tag{3.15}$$

where H_l is defined in (3.8). When $\varepsilon = 0$, (3.15) reduces to the solution of the unperturbed diffusion equation. The modification introduced when $\varepsilon > 0$ is entirely determined by the multiplicative factor

$$\left\{1 + \varepsilon p \int_0^t \left[\int_{R^N} H_m e^{-|k_m|^2 t}\, dk_m\right]^p dt\right\}^{-1/p}. \tag{3.16}$$

It is easily shown that the RPE results from expanding (3.16) in a binomial series. We shall use the fact that H_m is a continuous function converging to zero as $|k_m| \to \infty$ (since $h \in L^1(R^N)$) to assess the long time behavior of (3.16). Rather than attempt a detailed study of (3.16) we shall restrict our attention to those values of p and N which correspond to problems of physical interest. Finally, we note that for $\varepsilon < 0$ this example serves as a good model for studying the problem of the blowing up of solutions (cf. [11]).

(i) *One dimension with* $p = 1$. In this case (3.16) reduces to

$$1 + \varepsilon \int_{-\infty}^{\infty} H_m \Delta(-k_m^2)\, dk_m, \tag{3.17}$$

where $\Delta(y)$ is defined in (2.11). With the assumed conditions on H_m it follows that for $t \to \infty$,

$$\begin{aligned} &1 + \varepsilon \int_{-\infty}^{\infty} H_m \Delta(-k_m^2)\, dk_m \\ &\quad = 1 + \varepsilon\left[2\sqrt{\pi t} H_0 + P\int_{-\infty}^{\infty} \frac{H_m - H_0}{k_m^2}\, dk_m + o(1)\right], \end{aligned} \tag{3.18}$$

where

$$H_0 = H(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\, dx. \tag{3.19}$$

With stronger conditions on h we could apply the results of (2.14). In any event it is clear from (3.18) that the RPE represents a uniformly valid asymptotic expansion if and only if the initial data is such that $H_0 = 0$. If $H_0 \neq 0$, the RPE becomes disordered when $H_0 \varepsilon t^{1/2} = O(1)$; or, assuming ε to represent a scaling of H_0 such that $H_0 = O(1)$, the RPE becomes disordered when $t = O(1/\varepsilon^2)$. As we shall see, such a result is characteristic of quadratic nonlinearities in one dimension.

(ii) *Two dimensions with* $p = 1$. For ease of presentation we shall assume that $h(x_1, x_2) = h(x_1^2 + x_2^2)$. The results are qualitatively the same for general h. For this case we have from (3.16)

$$\begin{aligned} 1 + 2\pi\varepsilon \int_0^{\infty} H(y)\Delta(-y^2) y\, dy = 1 + 2\pi\varepsilon\Bigg[&\sqrt{\pi} H(0)\log t \\ &+ \int_0^{\infty} \frac{H(y) - H(0)e^{-y^2}}{y}\, dy + o(1)\Bigg], \quad t \to \infty, \end{aligned} \tag{3.20}$$

where

$$H(0) = \frac{1}{2\pi}\int_0^\infty h(r)r\,dr. \tag{3.21}$$

The reader should consult (2.14) for the general expansion. Thus, the RPE becomes disordered when

$$H(0)\varepsilon \log t = O(1).$$

The effect of the nonlinearity is weaker for two dimensions than for one dimension. This is to be anticipated on physical grounds. Nevertheless it is clear that when $H(0) \neq 0$, one must eventually account for the nonlinear term.

An important conclusion to be drawn from this result concerns the reliability of estimating the significance of neglected nonlinear terms by using only the solution to the linear problem. For example, it can readily be shown that by setting $\varepsilon = 0$ in (3.15) we have the estimates (for $t \to \infty$)

$$\frac{\partial u(x,t;0)}{\partial t} = O\left(\frac{1}{t^2}\right), \qquad u(x,t;0)u(0,t;0) = O\left(\frac{1}{t^2}\right), \tag{3.22}$$

for $x_1^2 + x_2^2 \leqq O(t)$. From this one could conclude that linearization is self-consistent (for $0 < \varepsilon \ll 1$). However, the exact solution shows that the nonlinear term has a cumulative effect represented by the $\log t$ term and, hence, that linearization does not lead to a uniformly valid first approximation. Results such as those stated in [3] must be qualified with this example in mind.

(iii) *Three and higher dimensions with* $p = 1$. It is not difficult to show that the integral in (3.16) is $O(1)$ for $t \to \infty$, $N \geqq 3$. Thus, we conclude that for three and higher dimensions the RPE represents a uniformly valid asymptotic solution to (3.13).

(iv) *One dimension with p a positive integer greater than or equal to* 2. In this case (3.16) becomes

$$\left\{1 + \varepsilon p \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_{m_1}H_{m_2}\cdots H_{m_p}\Delta(-k_{m_1}^2 \cdots -k_{m_p}^2) \cdot dk_{m_1}\cdots dk_{m_p}\right\}^{-1/p}. \tag{3.23}$$

The asymptotic behavior of the integral in (3.23) can be obtained by transforming to p-dimensional spherical coordinates. It is easily verified that (3.23) becomes $\{1 + O(\varepsilon \log t)\}^{-1/2}$ when $p = 2$ and $\{1 + O(\varepsilon t^{1-p/2})\}^{-1/p}$ for $p \geqq 3$. The conclusion we draw from this is that cubic nonlinearities cause the RPE to become disordered when $t = O(e^{1/\varepsilon})$; whereas, no nonuniformities arise for higher degree nonlinearities.

The preceding example is very useful for illustrating the importance of weak nonlinear terms in diffusion problems. Other examples for which exact solutions can be deduced are studied in [2]. We turn now to an investigation of an important class of problems involving quadratic nonlinearities. For simplicity, we shall restrict our attention to a one-dimensional version of (3.1); namely,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \varepsilon N u^2, \qquad x \in R^1, \quad t > 0, \tag{3.24}$$

with $u(x, 0; \varepsilon) = h(x) \in L^1(R^1)$. The equivalent Fourier space representation of (3.24) is given by

$$\frac{\partial A_l}{\partial t} + \sigma_l A_l = \varepsilon \int K_{mn} A_m A_n \, \delta_{l,mn} \, dk_{mn}, \tag{3.25}$$

with $A_l = H_l$ at $t = 0$. The function $K_{mn} = K(k_m, k_n)$ is related to the operator N (and can be chosen to be symmetric in its arguments), $\delta(k)$ is the Dirac delta function and $\sigma_l = k_l^2$. Further, the following abbreviated notations will be used:

$$\begin{aligned} \delta_{l_1 l_2 \cdots l_r, m_1 m_2 \cdots m_s} &= \delta\left(\sum_{\alpha=1}^{r} k_{l_\alpha} - \sum_{\beta=1}^{s} k_{m_\beta}\right), \\ dk_{m_1 m_2 \cdots m_r} &= \prod_{p=1}^{r} dk_{m_p}, \\ \sigma_{l_1 l_2 \cdots l_r, m_1 m_2 \cdots m_s} &= \sum_{\alpha=1}^{r} k_{l_\alpha}^2 - \sum_{\beta=1}^{s} k_{m_\beta}^2, \end{aligned} \tag{3.26}$$

and the limits of all integrations are taken to be from $-\infty$ to $+\infty$.

Construction of the RPE for A_l is straightforward; upon setting

$$A_l = a_l e^{-\sigma_l t}, \qquad a_l \sim \sum_{r=0}^{\infty} \varepsilon^r a_{rl}, \tag{3.27}$$

we obtain the first few terms as

$$a_{0l} = H_l, \tag{3.28}$$

$$a_{1l} = \int K_{mn} H_m H_n \Delta(\sigma_{l,mn}) \, \delta_{l,mn} \, dk_{mn}, \tag{3.29}$$

$$a_{2l} = 2 \int K_{mn} K_{pq} H_n H_p H_q E(\sigma_{l,npq}; \sigma_{l,mn}) \, \delta_{l,mn} \, \delta_{m,pq} \, dk_{mnpq}, \tag{3.30}$$

where $\Delta(x)$ and $E(x; y)$ are defined in (2.11) and (2.34). As pointed out earlier, in order to ascertain the behavior of the corresponding RPE for u we must transform the RPE for A_l back to physical space. A close examination of the general form of the higher order terms in (3.27) indicates that we need only consider the first two terms of the expansion to check for obvious nonuniformities. Of course the fact that nonuniformities do not occur does not prove that the RPE is a uniform asymptotic expansion of the solution. In order to make our task easier we shall assume that the initial data, represented by $h(x)$, is of exponential type at infinity, as well as $L^1(R^1)$, so that we can make use of (2.9) and (2.32) to obtain (for $t \to \infty$)

$$u_0(x, t) \sim \frac{e^{-\lambda^2/4}}{t^{1/2}} \sum_{m=0}^{\infty} \frac{H^{(m)}(0)}{m! t^{m/2}} \int \left(\xi + i\frac{\lambda}{2}\right)^m e^{-\xi^2} \, d\xi, \tag{3.31}$$

$$u_1(x,t) \sim \sum_{m,n=0}^{\infty} \frac{\partial^{m+2n} g(0,0)}{\partial y^m \partial z^{2n}} \frac{J_{m,2n}(1,1,\sqrt{2}\lambda,0)}{m!(2n)!t^{(m+2n)/2}}$$

$$(3.32) \qquad + \sum_{m=1}^{\infty} \frac{e^{-\lambda^2/4}}{2m!(2t)^{m/2}} P_z \frac{1}{z}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^m [g(y,z) + g(y,-z)]_{y=0}$$

$$\cdot \int [\xi + i(\lambda/2)]^{m-1} e^{-\xi^2}\, d\xi,$$

where

$$(3.33) \qquad \lambda = \frac{x}{t^{1/2}} \leqq O(1), \qquad g(y,z) = K\left(\frac{y+z}{\sqrt{2}}, \frac{y-z}{\sqrt{2}}\right) H\left(\frac{y+z}{\sqrt{2}}\right) H\left(\frac{y-z}{\sqrt{2}}\right),$$

and $J_{m,2n}$ is defined by (2.28)–(2.31). The choice of scaling of x with respect to t given in (3.33) is the natural one for linear diffusion.

In order to compare u_0 with u_1 first assume that $H_0 \neq 0$, in which case $u_0 = O(1/t^{1/2})$. From (3.32) it follows that $u_1 = O(1)$ if $K(0,0) \neq 0$ and $u_1 \leqq O(1/t^{1/2})$ if $K(0,0) = 0$. On the other hand, if $H_0 = 0$, $u_0 \leqq O(1/t)$ and the behavior of u_1 depends upon the form of K and H. We shall examine two special cases in detail.

(i)

$$(3.34) \qquad Nu^2 = -u^2.$$

This case is important in chemical reaction processes. In a recent work [4], Bentwich studied (3.34) for initial data satisfying

$$(3.35) \qquad h(x) = c + \int_{-\infty}^{\infty} H_l e^{-ik_l x}\, dk_l,$$

$$(3.36) \qquad h(x) = \int_{-\infty}^{\infty} H_l e^{-ik_l x}\, dk_l,$$

where in (3.35) c denotes a positive constant and $H_0 = 0$, and in (3.36) $H_0 \neq 0$. For (3.35) he finds that the RPE becomes disordered when $t = O(1/\varepsilon)$. He then employs a singular perturbation procedure, first used in [2], to formally obtain a uniform asymptotic expansion for the solution. However, for (3.36) he erroneously concludes that the RPE is uniformly valid. Indeed, the present results show that when $H_0 \neq 0\,(K \equiv -1)$ the leading terms in the expressions for u_0, u_1 and u_2 yield

$$\left(\frac{\pi}{t}\right)^{1/2} H_0 e^{-\lambda^2/4} + \varepsilon\pi H_0^2 \int_0^{\pi/4} \exp[-(\lambda^2/4)\sec^2\theta]\, d\theta + \varepsilon^2 t^{1/2} 8\pi^{3/2} H_0^3$$

$$(3.37) \qquad \cdot \int_0^1 \int_0^{\pi/6} \frac{\exp[-(\lambda^2/4)(1-\Phi\sec^2\theta)]}{\sqrt{1-\Phi\sec^2\theta}}\, d\theta\, d\Phi.$$

It can be shown that the leading term in u_m, $m \geqq 0$, is $O(t^{(m+1)/2})$. Thus, the RPE becomes disordered when $t = O(1/\varepsilon^2)$. A technique for constructing an expansion for $u(x,t;\varepsilon)$ valid for this range of t is discussed in [2].

If $H_0 = 0$, then $u_0 \leqq O(1/t)$. We have the possibility of $u_1 = O(1/t^{1/2})$ if

$$P_z \frac{1}{z}\frac{\partial}{\partial z}\left[H\left(\frac{z}{\sqrt{2}}\right)H\left(-\frac{z}{\sqrt{2}}\right)\right] \neq 0, \tag{3.38}$$

again leading to a nonuniformity.

(ii)

$$Nu^2 = uu_x. \tag{3.39}$$

Many nonlinear dissipative systems reduce to Burgers' equation for appropriate limiting situations. For this case, $K(y, z) = (y + z)/2$. Thus, if $H_0 \neq 0$, it appears that the RPE for Burgers' equation, assuming $h \in L^1(R^1)$, is uniformly valid. Indeed, the asymptotic solution which corresponds to propagation of a pulse [12] is

$$u(x, t) \sim \frac{e^{-\lambda^2/4}}{\varepsilon t^{1/2}[\pi^{1/2}/(e^{\varepsilon\pi H_0} - 1) + \int_{-\lambda/2}^{\infty} e^{-y^2}\,dy]}, \qquad t \to \infty. \tag{3.40}$$

For ε small (3.40) may be expanded in powers of ε and the solution is readily shown to agree (at least for the first few terms) with the long time behavior of the RPE.

If $H_0 = 0$, the largest that u_1 can be is $O(1/t)$. Depending upon the behavior of H_l at the origin the RPE may or may not be uniform. Each case must be investigated individually.

We conclude this section with a few general remarks. The effect of higher dimensions for equations of the form (3.1) is much the same as for the simple example in (3.13). In the special case when $f(u) = -u^2$, $N = 3$, $H(0) \neq 0$, the RPE becomes disordered when $\varepsilon H_0 \log t = O(1)$. In higher dimensions, no apparent nonuniformities arise. Finally, if $f(u) = -u^m$, m a positive integer greater than or equal to 3, and $N = 1$, the RPE is uniformly valid except when $m = 3$, in which case the disorder arises when $\varepsilon H_0^2 \log t = O(1)$.

4. Diffusion equations with variable coefficients. The long time effect of small variable coefficient terms on the diffusion operator is of physical as well as mathematical interest. Here we briefly consider a special class of such problems represented by the equation

$$Lu = \frac{\partial u}{\partial t} - \nabla^2 u = \varepsilon M u f, \qquad x \in R^N, \quad t > 0, \tag{4.1}$$

where Muf denotes a general variable coefficient term, linear in u and its spatial derivatives and linear in a known time-independent function f and its spatial derivatives. For example, $Muf = \nabla \cdot (f \nabla u)$, corresponding to diffusion in a medium with variable diffusivity given by $1 - \varepsilon f(x)$. As in the case of the nonlinear problems, $\varepsilon > 0$ is a small parameter. The smallness of the parameter does not imply that f is small pointwise, but rather that certain global properties of f are not too large. We are interested in investigating the ordinary initial value problem, so we require that u satisfy

$$u(x, 0; \varepsilon) = h(x), \qquad x \in R^N. \tag{4.2}$$

The essential features of the analysis are brought out by a study of the one-dimensional problem. If we assume that f and h are both $L^1(R^1)$, the first three terms of the RPE are given by

$$\begin{aligned}(4.3)\qquad &\int H_l \exp(-\sigma_l t - ik_l\lambda t^{1/2})\,dk_l + \varepsilon \int K_{mn}H_mF_n\Delta(\sigma_{l,m}) \\ &\qquad\cdot \exp(-\sigma_l t - ik_l\lambda t^{1/2})\,\delta_{l,mn}\,dk_{lmn} \\ &+ \varepsilon^2 \int K_{mn}K_{pq}H_pF_nF_qE(\sigma_{l,p};\sigma_{l,m}) \exp(-\sigma_l t - ik_l\lambda t^{1/2}) \\ &\qquad\cdot \delta_{l,mn}\,\delta_{m,pq}\,dk_{lmpq},\end{aligned}$$

where we have adopted the notation of § 3. The function $F_l = F(k_l)$ denotes the transform of f and $K_{mn} = K(k_m, k_n)$ is related to the operator M. Asymptotic expansions describing the long time behavior of the second and third integrals in (4.3) are obtained by a slight modification of (2.23) and (2.37) respectively. We shall not write out these expressions. If we assume that $H_0 \neq 0$, the leading terms are given by

$$\begin{aligned}(4.4)\qquad &\left(\frac{\pi}{t}\right)^{1/2} H_0\, e^{-\lambda^2/4} + 2\varepsilon\pi K(0,0)H_0F_0 \int_0^{\pi/2} e^{-\lambda^2\sec^2\theta/4}\,d\theta \\ &+ \varepsilon^2 t^{1/2}\pi^2K^2(0,0)H_0F_0^2 \int_{-\infty}^{\infty} \frac{1-e^{-y^2}}{y^2}\, e^{-i\lambda y}\,dy.\end{aligned}$$

An apparent nonuniformity exists if both $K(0,0)$ and $F(0)$ are different from zero, in which case the RPE becomes disordered when

$$\varepsilon t^{1/2}F(0) = O(1).$$

In particular, it would appear that a perturbation of the diffusivity of the form $-\varepsilon f(x)$, corresponding to $Muf = (\partial/\partial x)[f(\partial u/\partial x)]$ and $K_{mn} = -k_m(k_m + k_n)$, does not influence the solution (of the $\varepsilon = 0$ problem) to leading order (in ε). For the only case in which it appears that nonuniformities arise, namely, $Muf = uf$, a singular perturbation solution of the form

$$(4.5)\qquad A_l = H_0\, e^{-\sigma_l t} + F_0 \int_0^{\varepsilon^2 t} G(s) \exp[\sigma_l^2(s - \varepsilon^2 t)]\,ds,$$

where

$$(4.6)\qquad G(s) = H_0\left[\sqrt{\frac{\pi}{s}} + 2\pi^{3/2}F_0\, e^{\pi^2F_0^2 s} \int_{-\pi F_0 s^{1/2}}^{\infty} e^{-\xi^2}\,d\xi\right],$$

has been suggested in [2] for $t \geqq O(1/\varepsilon^2)$.

If, on the other hand, $H_0 = 0$ in (4.3), the situation becomes more complicated and nonuniformities may appear depending upon the behavior of certain integrals involving H, F and K. We shall not go into details but simply note that the asymptotic evaluation of the relevant integrals can be carried out in any particular case by using the results in § 2.

Consideration of (4.1) in higher dimensions leads to conclusions paralleling those in § 3. For example, if $Muf = uf$, it can be shown that the RPE becomes disordered when $\varepsilon^2 \log tF(0)H(0) = O(1)$ in two dimensions and that it is uniformly valid in three and higher dimensions.

5. Final period of decay for Burgers' model of turbulence. The Burgers' model of turbulence is a one-dimensional version of three-dimensional, real turbulence. Its basic dynamical equation is the Burgers' equation, which for convenience we shall write in the form

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 2\varepsilon u \frac{\partial u}{\partial x}, \qquad x \in R^1, \quad t > 0. \tag{5.1}$$

The parameter ε is related to the Reynolds number. In (5.1) it is understood that $u(x, t; \varepsilon)$ is a stationary random function of position, which has known initial mean value properties.

Associated with the problem of homogeneous turbulence is the so-called final period of decay (see [13]), referring to that time regime for which dissipation has forced the nonlinear terms to be weak. It is generally assumed that the neglect of the nonlinear terms during the final period of decay provides a uniformly valid first approximation as $t \to \infty$. Whether or not this assumption is justified remains an open question. Other authors have made the same assumption in studies of the Burgers' model of turbulence (e.g., see [5]). However, as we shall show, in general, linearization is invalid for the Burgers' model of turbulence.

The parameter ε can be taken as a measure of the nonlinearity, where we assume that $0 < \varepsilon \ll 1$. Alternately one may suppose that the initial conditions are such that this is the case. The procedure which we shall follow involves constructing the RPE for the cumulants and then assessing the asymptotic behavior of the various terms. The system of equations governing the time evolution of the cumulants forms an infinite set of coupled integro-differential equations. The reader less familiar with this subject may wish to refer to [13], [14]. The equations for the hierarchy are (using the notation of § 3)

$$\frac{\partial Q_l^{(2)}}{\partial t} + 2\sigma_l Q_l^{(2)} = \varepsilon \mathscr{P}_{ll'}\left[ik_l \int Q_{ml'}^{(3)}\, dk_m \right], \tag{5.2}$$

$$\frac{\partial Q_{ll'}^{(3)}}{\partial t} + \sigma_{ll'l''} Q_{ll'}^{(3)} = \varepsilon \mathscr{P}_{ll'l''}\left[ik_l \int Q_{ml'l''}^{(4)}\, dk_m + 2ik_l Q_{l'}^{(2)} Q_{l''}^{(2)} \right], \tag{5.3}$$

$$\begin{aligned} \frac{\partial Q_{ll'l''}^{(4)}}{\partial t} + \sigma_{ll'l''l'''} Q_{ll'l''}^{(4)} = \varepsilon \mathscr{P}_{ll'l''l'''}\bigg[& ik_l \int Q_{ml'l''l'''}^{(5)} \\ & + 2ik_l \mathscr{P}_{l'l''l'''}\{Q_{l'l''}^{(3)} Q_{l'''}^{(2)}\} \bigg], \end{aligned} \tag{5.4}$$

$$\frac{\partial Q^{(5)}_{ll'l''l'''}}{\partial t} + \sigma_{ll'l''l'''l''''} Q^{(5)}_{ll'l''l'''} = \varepsilon \mathscr{P}_{ll'l''l'''l''''}\Bigg[ik_l \int Q^{(6)}_{ml'l''l'''l''''}\, dk_m + 2ik_l \mathscr{P}_{l'l''l'''l''''}\{Q^{(4)}_{l'l''l'''}Q^{(2)}_{l''''} + Q^{(3)}_{l'l''}Q^{(3)}_{l'''l''''}\}\Bigg], \tag{5.5}$$

$\vdots$

$$\frac{\partial Q^{(n)}_{ll'\cdots l^{(n-2)}}}{\partial t} + \sigma_{ll'\cdots l^{(n-1)}} Q^{(n)}_{ll'\cdots l^{(n-2)}} = \varepsilon \mathscr{P}_{ll'\cdots l^{(n-1)}}\Bigg[ik_l \int Q^{(n+1)}_{ml'\cdots l^{(n-1)}}\, dk_m + \text{terms involving products of the form } Q^{(n-r)}Q^{(r+1)},\quad r = 1, 2, \cdots, n-2\Bigg]. \tag{5.6}$$

Here the $Q^{(n)}_{ll'\cdots l^{(n-2)}} = Q^{(n)}(k_l, k_{l'}, \cdots, k_{l^{(n-2)}}, t; \varepsilon)$ are the Fourier space cumulants which are related to the physical space cumulants $R^{(n)}(r, r', \cdots, r^{(n-2)}, t; \varepsilon)$ by $(n-1)$-dimensional transforms, that is,

$$R^{(n)}(r, r', \cdots, r^{(n-2)}, t; \varepsilon) = \int Q^{(n)}(k_l, k_{l'}, \cdots, k_{l^{(n-2)}}, t; \varepsilon) \cdot \exp\left[-ik_l r - \cdots - ik_{l^{(n-2)}} r^{(n-2)}\right] dk_{ll'\cdots l^{(n-2)}}. \tag{5.7}$$

For example, $R^{(2)}(r, t; \varepsilon)$ is the two-point correlation function and $Q^{(2)}(k_l, t; \varepsilon)$ is the energy spectrum. The $\mathscr{P}_{ll'\cdots l^{(n)}}$ notation is used to imply cyclic summation over $l, l', \cdots, l^{(n)}$ and the variable $k_{l^{(n-1)}}$ in the equation for $Q^{(n)}_{l\cdots l^{(n-2)}}$ is defined by $k_l + \cdots + k_{l^{(n-1)}} = 0$.

The RPE for the Fourier space cumulants can be expressed as

$$Q^{(n)}_{ll'\cdots l^{(n-2)}} \sim \sum_{j=0}^{\infty} \varepsilon^j Q^{(n)}_{jll'\cdots l^{(n-2)}}, \qquad n \geqq 2, \tag{5.8}$$

where the j subscript denotes the perturbation ordering. The initial values of the $Q^{(n)}$ will be denoted by $H^{(n)}$, these being assumed independent of ε. It is a perfectly straightforward matter to determine the terms in (5.8). For our purposes we need only the following expressions:

$$Q^{(n)}_{0ll'\cdots l^{(n-2)}} = H^{(n)}_{ll'\cdots l^{(n-2)}}\, e^{-\sigma_{ll'\cdots l^{(n-1)}} t}, \qquad k_l + \cdots + k_{l^{(n-1)}} = 0, \quad n \geqq 2, \tag{5.9}$$

$$Q^{(2)}_{1l} = \mathscr{P}_{ll'}\left[ik_l \int H^{(3)}_{ml'} \Delta(\sigma_{l,m(l-m)})\, dk_m\right] e^{-2\sigma_l t}, \tag{5.10}$$

$$Q^{(3)}_{1ll'} = \mathscr{P}_{ll'l''}\Bigg[ik_l \int H^{(4)}_{ml'l''} \Delta(\sigma_{l,m(l-m)})\, dk_m + 2ik_l H^{(2)}_{l'} H^{(2)}_{l''} \Delta(\sigma_{l,l'l''})\Bigg] e^{-\sigma_{ll'l''} t}, \tag{5.11}$$

$$
\begin{aligned}
Q^{(4)}_{1ll'l''} &= \mathscr{P}_{ll'l''l'''}\Big[ik_l \int H^{(5)}_{ml'l''l'''}\Delta(\sigma_{l,m(l-m)})\,dk_m \\
&\qquad + 2ik_l\mathscr{P}_{l'l''l'''}\{H^{(3)}_{l'l''}H^{(3)}_{l'''l''''}\Delta(\sigma_{l,l'''(l+l''')})\}\Big]e^{-\sigma_{ll'l''l'''}t},
\end{aligned} \tag{5.12}
$$

$\vdots$

$$
\begin{aligned}
Q^{(n)}_{1ll'\cdots l^{(n-2)}} &= \mathscr{P}_{ll'\cdots l^{(n-1)}}\Big[ik_l \int H^{(n+1)}_{ml'\cdots l^{(n-1)}}\Delta(\sigma_{l,m(l-m)})\,dk_m \\
&\qquad + \text{terms involving products of the form } H^{(n-r)}H^{(r+1)} \\
&\qquad \cdot\Delta(\sigma_{l\cdots})\Big]e^{-\sigma_{ll'\cdots l^{(n-1)}}t},
\end{aligned} \tag{5.13}
$$

$$
\begin{aligned}
Q^{(2)}_{2l} &= \mathscr{P}_{ll'}\Big\{k_l \int [k_lH^{(4)}_{nm(l-m)}E(\sigma_{ll',mn(l-m)(l+n)};\sigma_{l,m(l-m)}) \\
&\qquad - 2k_mH^{(4)}_{nl'(l-m)}E(\sigma_{l,m(l-m)(m-n)};\sigma_{l,m(l-m)})]\,dk_{mn} \\
&\qquad + 2k_l\int [k_lH^{(2)}_mH^{(2)}_{(l-m)}E(2\sigma_{l,m(l-m)};\sigma_{l,m(l-m)}) \\
&\qquad - 4k_mH^{(2)}_lH^{(2)}_{(l-m)}E(-2\sigma_{(l-m)};\sigma_{l,m(l-m)})]\,dk_m\Big\}e^{-2\sigma_lt},
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
Q^{(3)}_{2ll'} &= -\mathscr{P}_{ll'l''}\Big\{k_l\int [\mathscr{P}_{l'l''}k_{l'}H^{(5)}_{nml''(l-m)}E(\sigma_{ll',mn(l-m)(l'-n)};\sigma_{l,m(l-m)}) \\
&\qquad + 2k_lk_mH^{(5)}_{nl'l''(l-m)}E(\sigma_{l,n(l-m)(n-m)};\sigma_{l,m(l-m)})]\,dk_{mn} \\
&\qquad + 2k_lH^{(3)}_{l'l''}\int H^{(2)}_{(l-m)}E(-2\sigma_{(l-m)};\sigma_{l,m(l-m)})\,dk_m \\
&\qquad + k_l\mathscr{P}_{l'l''}\int [2k_mH^{(3)}_{(l-m)l'}H^{(2)}_{l''}E(\sigma_{l,l''(l-m)(m+l'')};\sigma_{l,m(l-m)}) \\
&\qquad + k_{l'}H^{(3)}_{m(l-m)}H^{(2)}_{l''}E(\sigma_{l',l''m(l-m)};\sigma_{l,m(l-m)}) \\
&\qquad + 2k_{l'}H^{(3)}_{ml''}H^{(2)}_{(l-m)}E(\sigma_{ll',m(l-m)(l-m)(m+l'')}; \\
&\qquad \cdot\sigma_{l,m(l-m)}) \\
&\qquad + 2k_{l''}H^{(2)}_{l'}(H^{(3)}_{m(-l'')} - H^{(3)}_{(-m)l''}) \\
&\qquad \cdot E(\sigma_{l,l'm(l''-m)};\sigma_{l,l'l''})]\,dk_m\Big\}e^{-\sigma_{ll'l''}t}.
\end{aligned} \tag{5.15}
$$

Corresponding expressions can be written down for the other terms in (5.8). It should be noted that subscripts of the form $(l - m)$ in (5.10)–(5.15) refer to the variable $k_l - k_m$.

As indicated above our aim is to prove that, in general, the RPE for the physical space cumulants are not uniformly valid for $t \to \infty$. Before establishing this result we shall remark briefly on the behavior of the RPE for the Fourier space cumulants.

It is relatively easy to verify that the expansion for $Q^{(n)}$, $n \geqq 2$, becomes disordered as $t \to \infty$, in every domain in the $(n-1)$-dimensional Fourier space. In particular, provided that each $H^{(n)}$ is nonzero at the origin, the expansion for $Q^{(n)}$, $n \geqq 3$, becomes disordered in a neighborhood of the origin of radius $O(\varepsilon)$ when $t = O(1/\varepsilon^2)$. A close examination of the $Q^{(n)}$ indicates that the latter region is the most crucial insofar as the effect that it has on the RPE for the corresponding physical space cumulants. In fact, our assertion in [2] that the RPE for the physical space cumulants are nonuniform was based on examination of this region. Here, using the analysis of § 2, we establish the validity of this assertion rigorously.

The RPE for the physical space cumulants have the form

$$R^{(n)}(r, r', \cdots, r^{(n-2)}, t; \varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j R_j^{(n)}(r, r', \cdots, r^{(n-2)}, t), \qquad n \geqq 2. \tag{5.16}$$

In order to ascertain the asymptotic behavior of the various terms in (5.16) we need to specify the smoothness of the initial conditions. Although it would be sufficient simply to require that the initial data is L^1, at certain points in the analysis we shall make further implicit, but obvious, assumptions in order to bring out more of the structure. Assuming that each $H^{(n)}$ is nonzero at the origin we can establish the asymptotic behavior of the zeroth order term in (5.16) by means of (5.7)–(5.9), the dominated convergence theorem and a little algebra:

$$\begin{aligned} R_0^{(n)}(r, r', \cdots, r^{(n-2)}, t) \sim{}& \frac{1}{\sqrt{n}}\left(\frac{\pi}{t}\right)^{(n-1)/2} H^{(n)}(0, 0, \cdots, 0) \\ &\cdot \exp\left\{\left[(1-n)\sum_{j=0}^{n-2}(r^{(j)})^2 + 2\sum_{0 \leqq j < k \leqq n-2} r^{(j)} r^{(k)}\right]\Big/ 4nt\right\}, \\ & \qquad n \geqq 2, \end{aligned} \tag{5.17}$$

for

$$\frac{r}{t^{1/2}}, \frac{r'}{t^{1/2}}, \cdots, \frac{r^{(n-2)}}{t^{1/2}} \leqq O(1).$$

If $H^{(n)}$ is zero at the origin for some n, then (5.17) will be replaced by $R_0^{(n)} = o(1/t^{(n-1)/2})$ for that value of n. In particular, if $H^{(2)}(k_l)$ is analytic at the origin and $H^{(2)}(0) = 0$, then it follows from (2.8) that

$$\begin{aligned} R_0^{(2)}(r, t) \sim \frac{d^2 H^{(2)}(0)}{dk^2} \frac{e^{-r^2/8t}}{4\sqrt{2}t^{3/2}} \int_{-\infty}^{\infty} \left(\xi - \frac{ir}{2\sqrt{2}t^{1/2}}\right)^2 e^{-\xi^2}\, d\xi, \\ r/t^{1/2} \leqq O(1), \end{aligned} \tag{5.18}$$

since $H^{(2)}$ is an even function.

Next we shall examine the long time behavior of the first nonlinear correction term $R_1^{(n)}$, $n \geqq 2$. From (5.10) and (2.32) it follows that

$$R_1^{(2)} = O(H_{00}^{(3)}/t^{1/2}), \qquad t \to \infty. \tag{5.19}$$

The coefficient k_l in (5.10) is responsible for the $1/t^{1/2}$ decay. Thus $R_0^{(2)}$ and $R_1^{(2)}$ have the same decay rate and to this order the RPE for $R^{(2)}$ appears to be uniform.

Consider next the expression for $R_1^{(3)}$. From (5.11) we have that

$$R_1^{(3)}(r, r', t) = \int \mathscr{P}_{ll'l''}\left\{ik_l \int H^{(4)}_{ml'l''}\Delta(\sigma_{l,m(l-m)})\,dk_m + 2ik_l H_{l'}^{(2)}H_{l''}^{(2)}\Delta(\sigma_{l,l'l''})\right\} \cdot \exp\left[-\sigma_{ll'l''}t - ik_l\lambda t^{1/2} - ik_{l'}\lambda' t^{1/2}\right] dk_{ll'}, \tag{5.20}$$

where

$$\lambda = \frac{r}{t^{1/2}}, \quad \lambda' = \frac{r'}{t^{1/2}} \quad \text{and} \quad k_l + k_{l'} + k_{l''} = 0. \tag{5.21}$$

It is easily verified that the term involving $H^{(4)}$ leads to a contribution of at most $O(H^{(4)}_{000}/t)$. The dominant contribution comes from the last term. Although we could exhibit the complete expansion by modifying the analysis leading to (2.32) we shall content ourselves with just the leading term, which is

$$\begin{aligned} R_1^{(3)}(r, r', t) = \frac{\pi[H^{(2)}(0)]^2}{t^{1/2}} \int_0^1 \frac{1}{\sqrt{3-s}\sqrt{1+s}} &\left\{ -\lambda \exp\left[-\frac{\lambda^2}{8(3-s)} - \frac{\lambda'^2}{8(1+s)}\right] \right. \\ &- \lambda' \exp\left[-\frac{\lambda'^2}{8(3-s)} - \frac{\lambda^2}{8(1+s)}\right] \\ &\left. + \frac{(\lambda+\lambda')}{2}\exp\left[-\frac{(\lambda-\lambda')^2}{8(1+s)} - \frac{(\lambda+\lambda')^2}{8(3-s)}\right]\right\} ds \\ &+ O\left(\frac{1}{t}\right), \qquad \lambda, \lambda' \leqq O(1), \quad t \to \infty. \end{aligned} \tag{5.22}$$

From (5.17) we have that $R_0^{(3)} = O(1/t)$. Comparing this result with (5.22) we see that the RPE for $R^{(3)}$ becomes disordered when $t = O(1/\varepsilon^2)$ provided that $r^2 + r'^2 = O(t)$. No apparent disorder arises when $r^2 + r'^2 = O(1)$. Although this result is sufficient to establish mathematically that, in general, linearization is invalid for the Burgers' model of turbulence, physically it may not be important if the correlation lengths associated with the initial values for the physical space cumulants are $O(1)$. It turns out that the other first order terms $R_1^{(n)}$, $n \geqq 4$, exhibit the same behavior as that discussed for $R_1^{(3)}$.

In order to establish that the RPE for the $R^{(n)}$ become disordered near the origin we must next consider the long time behavior of the second order terms $R_2^{(n)}$. First let us examine the expression for $R_2^{(2)}$ given by (5.7) and (5.14):

$$\begin{aligned} R_2^{(2)}(r, t) = \int \exp(-2\sigma_l t - ik_l\lambda t^{1/2}) &\left\{ \mathscr{P}_{ll'}k_l \int [k_l H^{(4)}_{mn(l-m)} E(\sigma_{ll',mn(l-m)(l+n)};\ \sigma_{l,m(l-m)}) \right. \\ &- 2k_m H^{(4)}_{nl'(l-m)} E(\sigma_{l,m(l-m)(m-n)};\ \sigma_{l,m(l-m)})]\,dk_{mn} \\ &+ 2k_l \int [k_l H_m^{(2)} H_{l-m}^{(2)} E(2\sigma_{l,m(l-m)};\ \sigma_{l,m(l-m)}) \\ &\left. - 4k_m H_l^{(2)} H_{l-m}^{(2)} E(-2\sigma_{(l-m)};\ \sigma_{l,m(l-m)})]\,dk_m \right\} dk_l. \end{aligned} \tag{5.23}$$

It can be shown that the terms involving $H^{(4)}$ in (5.25) lead to a contribution of the form $O(H^{(4)}_{000}/t^{1/2})$. The terms involving $H^{(2)}$ are potentially more important, so we shall discuss them in more detail. The two integrals which we must examine are of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 g(x,y)\left[\frac{\exp\{[x^2-y^2-(x-y)^2]t\}-1}{x^2-y^2-(x-y)^2}\right]^2 \cdot \exp[-2x^2t - i\lambda x t^{1/2}]\,dx\,dy \tag{5.24}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(x-y)g(x,y)\left[\frac{\dfrac{1-\exp(-2y^2t)}{2y^2} - \dfrac{\exp\{[x^2-y^2-(x-y)^2]t\}-1}{x^2-y^2-(x-y)^2}}{x^2+y^2-(x-y)^2}\right] \cdot \exp(-2x^2t - i\lambda x t^{1/2})\,dx\,dy. \tag{5.25}$$

In the Appendix, (A.8) and (A.13), we outline a procedure for evaluating these integrals in terms of the results of § 2. Sparing the reader the details we find that the asymptotic behavior of the terms involving $H^{(2)}$ in (5.23) is given by

$$\begin{aligned} &2\pi[H_0^{(2)}]^2\left\{-e^{-\lambda^2/8} + 2\int_0^{\pi/6} e^{-(\lambda^2/8)\sec^2\theta}\,d\theta \right. \\ &\quad + 2\frac{d^2}{d\lambda^2}\left[3\int_0^{\pi/6}(1-4\sin^2\theta)e^{-(\lambda^2/8)\sec^2\theta}\,d\theta \right. \\ &\quad \left.\left. - 2\int_0^{\pi/4}(1-2\sin^2\theta)\,e^{-(\lambda^2/8)\sec^2\theta}\,d\theta\right]\right\} \\ &\quad + \frac{H^{(2)}(0)[d^2H^{(2)}(0)/dk^2]}{t} \\ &\quad \cdot\left\{\frac{\pi}{4}e^{-\lambda^2/8} + \frac{7}{2}J_{2,0}(3,1,-\sqrt{2}\lambda,0) - J_{2,0}(1,1,-\lambda,0)\right. \\ &\quad + \frac{1}{2}J_{0,2}(3,1,\sqrt{2}\lambda,0) + 4\frac{d^4}{d\lambda^4} \\ &\quad \cdot\left[\int_0^{\pi/4}(1-2\sin^2\theta)\,e^{-(\lambda^2/8)\sec^2\theta}\,d\theta\right. \\ &\quad \left.\left. - \int_0^{\pi/6}(1-4\sin^2\theta)\,e^{-(\lambda^2/8)\sec^2\theta}\,d\theta\right]\right\} + O\left(\frac{1}{t^{3/2}}\right), \\ &\qquad\qquad \lambda = \frac{r}{t^{1/2}} \leqq O(1). \end{aligned} \tag{5.26}$$

In particular,

$$R_2^{(2)}(0,t) \sim \pi\left(\frac{9\sqrt{3}-12-\pi}{3}\right)[H_0^{(2)}]^2. \tag{5.27}$$

Suppose that $H_0 \neq 0$; then $R_0^{(2)} = O(H_0^{(2)}/t^{1/2})$ and $R_2^{(2)} = O([H_0^{(2)}]^2)$, for $r \leqq O(t^{1/2})$. Hence, the RPE for $R^{(2)}$ becomes disordered when

$$t = O\left(\frac{1}{\varepsilon^4[H_0^{(2)}]^2}\right). \tag{5.28}$$

On the other hand, suppose that $H_0^{(2)} = 0$ but that $d^2H_0^{(2)}/dk^2 \neq 0$. Then from (5.18) and (5.26) we have that $R_0^{(2)} = O(1/t^{3/2})$ and $R_2^{(2)} = O(1/t^{3/2})$, so that it would appear in this case that the RPE for $R^{(2)}$ is well-behaved to this order. In the same manner, it can be shown that the terms involving $H^{(3)}H^{(2)}$ in (5.15) lead to an $O(1/t^{1/2})$ contribution for $R_2^3(r, r', t)$ for $r^2 + r'^2 \leqq O(t)$. Similar nonuniformities arise for the other cumulants.

Having established that linearization is invalid for the Burgers' model of turbulence when $H_0^{(2)} \neq 0$, we shall briefly discuss the implications of the perturbation analysis. It appears that the RPE are useful only for $0 \leqq t < O(1/\varepsilon^2)$. However, as is common in perturbation problems, the form of the nonuniformity suggests how one should go about rescaling the dependent and independent variables so as to obtain expansions that are valid for $t \geqq O(1/\varepsilon^2)$. As one might suspect from the behavior of the expansion for $R^{(2)}$, a second rescaling becomes necessary when $t = O(1/\varepsilon^4)$, which in turn necessitates the introduction of a third set of expansions. The interested reader may wish to refer to [2] for details. We simply note here that the final set of moment equations (valid for $t \geqq O(1/\varepsilon^4)$) are (5.2)–(5.6) with ε set equal to one. In other words, we are faced with solving the fully nonlinear set of equations. However, there is one simplification. The initial conditions depend only on $H_0^{(2)}$, which suggests that the physical space cumulants have a universal structure (apart from multiplication by powers of $H_0^{(2)}$).

We conclude this section with a few brief remarks on the final period of decay for Burgers' model of turbulence in higher dimensions. The counterpart of (5.1) is

$$\frac{\partial u_i}{\partial t} - \nabla^2 u_i = 2\varepsilon(u \cdot \nabla)u_i, \quad x \in R^N, \quad t > 0, \quad i = 1, \cdots, N, \tag{5.29}$$

for the N-dimensional vector u. The equations for the hierarchy of cumulants resemble (5.2)–(5.6) with far more subscripts. We shall restrict our remarks to the case where the Fourier space cumulants are smooth functions near the origin. Then terms like the last one in (5.11) cause the RPE for the third order cumulants to become disordered when $t = O(1/\varepsilon^2)$ provided the spatial separations, measured by the independent variables (see (5.21)), are sufficiently large. This result holds independently of the number of spatial dimensions. Similar results hold for the higher order cumulants. However, the counterpart of (5.23) does not lead to nonuniformities in higher dimensions. Mathematically, this is because the effect of higher dimensions is to bring in additional factors of k_l and k_m (in (5.23)) which weakens the contribution near the origin. Physically, this is because the diffusion mechanism is stronger in higher dimensions, which reduces the cumulative effect of the nonlinearity.

Whether or not the RPE are uniformly valid when the cumulants are not smooth near the origin is a more difficult question to answer. Of course, in this case it is more meaningful to consider the final period of decay for homogeneous turbulence. The author is presently studying this problem.

6. Diffusion of a passive scalar in a random velocity field. The closure difficulties inherent in a statistical study of turbulence are notorious. Various models such as the Burgers' model of turbulence have received extensive study for the purpose of testing closure schemes. One such model which has attracted much attention is the diffusion of a passive scalar in a random velocity field [15]. For a passive scalar in an incompressible fluid with uniform properties, the concentration $\psi(x, t)$ satisfies the equation

$$\partial\psi/\partial t + u \cdot \nabla\psi = \kappa\nabla^2\psi, \quad x \in R^3, \quad t > 0, \tag{6.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity field and κ is the molecular diffusivity. (Since $\nabla \cdot u = 0$, the convection term in (6.1) could just as well be written as $\nabla \cdot (\psi u)$.)

Given the velocity field u, (6.1) is a linear partial differential equation with random coefficients. However, upon multiplying (6.1) by various powers of ψ and averaging we are confronted with a closure problem quite similar to that of turbulent dynamics. This occurs because (6.1) is nonlinear in stochastic quantities, although it is linear in the concentration variable.

In order to circumvent the closure problem many authors have resorted to cumulant discard approximations tantamount to assuming that u and ψ are to a certain degree statistically independent. Our purpose in this section is to examine the validity of such approximations by considering the limiting situation in which the convection term in (6.1) is small in comparison with the diffusion term. The advantage of this limiting situation is that formally the hierarchy of equations for the correlation tensors are uncoupled to leading order. However, as we shall show, the associated RPE for the correlation tensors are not uniformly valid.

In order to make our presentation simpler to follow we shall consider a one-dimensional version of (6.1). To be specific, we shall examine the one-dimensional equation

$$\frac{\partial\psi}{\partial t} - \frac{\partial^2\psi}{\partial x^2} = -\varepsilon\frac{\partial}{\partial x}(u\psi), \quad x \in R^1, \quad t > 0, \tag{6.2}$$

where $0 < \varepsilon \ll 1$. We shall assume that u and ψ are stationary random functions of position both having zero means. It will also be convenient to assume that u is independent of time in each realization, although our conclusions can be shown to hold in the case where u itself satisfies a diffusion equation to leading order. We shall *not* assume that u is a Gaussian process *nor* that u and ψ are statistically independent.

As ψ and u are stationary random functions of x, the physical space moments involving products of ψ alone or of u alone depend only on the relative geometry. It is consistent to require the mixed moments involving products of ψ and u to satisfy the same condition. The physical space cumulants are defined by

$$\begin{aligned} R^{m,n}(r, r', \cdots, r^{(m+n-2)}, t) = {} & \langle\psi(x,t)\psi(x+r,t)\cdots\psi(x+r^{(m-2)},t) \\ & \cdot u(x+r^{(m-1)})u(x+r^{(m)})\cdots u(x+r^{(m+n-2)})\rangle \\ & - \sum_{\alpha\beta\gamma\cdots} \langle\psi(x,t)\cdots u(x+r^{(\alpha)})\rangle \\ & \cdot \langle\psi(x+r^{(\beta)},t)\cdots u(x+r^{(\gamma)})\rangle \cdots, \end{aligned} \tag{6.3}$$

where the superscripts m and n denote the number of ψ's and u's, respectively, occurring in the first moment on the right of (6.3). The summation contains all the necessary combinations of products of moments, involving $\psi(x,t), \psi(x+r,t), \cdots, u(x+r^{(m+n-2)})$, to ensure that the cumulant $R^{m,n}$ tends to zero as the relative separations tend to infinity. For example,

$$
\begin{aligned}
&R^{2,0}(r,t) = \langle \psi(x,t)\psi(x+r,t)\rangle, \qquad R^{0,2}(r) = \langle u(x)u(x+r)\rangle,\\
&R^{1,2}(r,r',t) = \langle \psi(x,t)u(x+r)u(x+r')\rangle,\\
&R^{2,2}(r,r',r'',t) = \langle \psi(x,t)\psi(x+r,t)u(x+r')u(x+r'')\rangle\\
&\qquad - \langle \psi(x,t)\psi(x+r,t)\rangle\langle u(x+r')u(x+r'')\rangle\\
&\qquad - \langle \psi(x,t)u(x+r')\rangle\langle \psi(x+r,t)u(x+r'')\rangle\\
&\qquad - \langle \psi(x,t)u(x+r'')\rangle\langle \psi(x+r,t)u(x+r')\rangle.
\end{aligned}
\tag{6.4}
$$

In order that we may express the RPE in terms of Fourier integrals we shall make the physically reasonable stipulation that initially each cumulant be integrable. The related Fourier space cumulants are defined by

$$
\begin{aligned}
R^{m,n}(r,r',\cdots,r^{(m+n-2)},t) = \int Q^{m,n}_{ll'\cdots l(m+n-2)} \exp(-k_{l'}r - ik_{l''}r' - \cdots\\
- k_{l(m+n-1)}r^{(m+n-2)})\delta_{ll'\cdots l(m+n-1)}\, dk_{ll'\cdots l(m+n-1)},
\end{aligned}
\tag{6.5}
$$

where we have employed the notation of § 2. The initial values of the $Q^{m,n}$ will be denoted by

$$
Q^{m,n}_{ll'\cdots l(m+n-2)} = H^{m,n}_{ll'\cdots l(m+n-2)}, \qquad t = 0. \tag{6.6}
$$

The hierarchy of equations for the time evolution of the cumulants is found in the usual manner by first multiplying (6.2) by various powers of ψ and u and then taking ensemble averages. Because the equation for the general cumulant $R^{m,n}$ can only involve other cumulants of the form $R^{p,s}, 0 \leqq p \leqq m$, it is appropriate to group the equations for the cumulants according to their first superscript. For our purposes it will suffice to list only the following equations (for the Fourier space cumulants):

$$
\frac{\partial Q^{1,1}_l}{\partial t} + \sigma_l Q^{1,1}_l = -\varepsilon i k_l \int Q^{1,2}_{ml'}\, dk_m, \qquad k_l + k_{l'} = 0, \tag{6.7}
$$

$$
\frac{\partial Q^{1,2}_{ll'}}{\partial t} + \sigma_l Q^{1,2}_{ll'} = -\varepsilon i k_l \left\{ \int Q^{1,3}_{ml'l''}\, dk_m + \mathscr{P}_{l'l''} Q^{1,1}_{-l'} Q^{0,2}_{l''} \right\}, \tag{6.8}
$$

$$
\begin{aligned}
\frac{\partial Q^{1,3}_{ll'l''}}{\partial t} + \sigma_l Q^{1,3}_{ll'l''} = -\varepsilon i k_l \Big\{ \int Q^{1,4}_{ml'l''l'''}\, dk_m\\
+ \mathscr{P}_{l'l''l'''}[Q^{1,2}_{(-l'-l'')l'} Q^{0,2}_{l''} + Q^{1,1}_{-l'} Q^{0,3}_{l''l'''}] \Big\},
\end{aligned}
\tag{6.9}
$$

$\vdots$

$$
\frac{\partial Q^{2,0}_l}{\partial t} + 2\sigma_l Q^{2,0}_l = -\varepsilon i \mathscr{P}_{ll'} k_l \int Q^{2,1}_{ml'}\, dk_m, \qquad k_l + k_{l'} = 0, \tag{6.10}
$$

$$\frac{\partial Q^{2,1}_{ll'}}{\partial t} + \sigma_{ll'} Q^{2,1}_{ll'} = -\varepsilon i \mathscr{P}_{ll'} k_l \left\{ \int Q^{2,2}_{ml'l''}\, dk_m + Q^{2,0}_{l'} Q^{0,2}_{l''} + Q^{1,1}_{-l''} Q^{1,1}_{l'} \right\}, \qquad k_l + k_{l'} + k_{l''} = 0, \tag{6.11}$$

$$\begin{aligned} \frac{\partial Q^{2,2}_{ll'l''}}{\partial t} + \sigma_{ll'} Q_{ll'l''} = -\varepsilon i \mathscr{P}_{ll'} k_l \Big\{ \int Q^{2,3}_{ml'l''l'''}\, dk_m + Q^{2,0}_{l'} Q^{0,3}_{l''l'''} \\ + Q^{1,2}_{(l+l'')l''} Q^{1,1}_{l'} + \mathscr{P}_{l''l'''}[Q^{2,1}_{l'(-l'-l'')} Q^{0,2}_{l'''} + Q^{1,2}_{l'l''} Q^{1,1}_{-l'''}] \Big\}, \end{aligned} \tag{6.12}$$

$$\vdots$$

$$\frac{\partial Q^{3,0}_{ll'}}{\partial t} + \sigma_{ll'l''} Q^{3,0}_{ll'} = -\varepsilon i \mathscr{P}_{ll'l''} k_l \left\{ \int Q^{3,1}_{ml'l''}\, dk_m + \mathscr{P}_{l'l''} Q^{2,0}_{l'} Q^{1,1}_{l''} \right\}. \tag{6.13}$$

The way in which we have grouped the above equations brings out the fact that the hierarchy of equations, while badly coupled, is actually a linear system. The RPE for the cumulants has the form

$$Q^{m,n}_{ll'\cdots l(m+n-2)} \sim \sum_{j=0}^{\infty} \varepsilon^j Q^{m,n}_{jll'\cdots l(m+n-2)}, \tag{6.14}$$

where the j subscript denotes the perturbation ordering. Recognizing the special way in which the hierarchy of equations is coupled, we shall first examine the equations of evolution for the $Q^{1,n}$, $n \geqq 1$. Substituting (6.14) into (6.7)–(6.9) we find that

$$Q^{1,n}_{0ll'\cdots l(n-1)} = H^{1,n}_{ll'\cdots l(n-1)}\, e^{-\sigma_l t}, \qquad \sigma_l = k_l^2, \tag{6.15}$$

$$Q^{1,1}_{1l} = -ik_l \int H^{1,2}_{ml'} \Delta(\sigma_{l,m})\, e^{-\sigma_l t}\, dk_m, \tag{6.16}$$

$$Q^{1,2}_{1ll'} = -ik_l \left\{ \int H^{1,3}_{ml'l''} \Delta(\sigma_{l,m})\, dk_m + \mathscr{P}_{l'l''} H^{1,1}_{-l'} H^{0,2}_{l''} \Delta(\sigma_{l,l'}) \right\} e^{-\sigma_l t}, \tag{6.17}$$

$$\begin{aligned} Q^{1,3}_{1ll'l''} = -ik_l \Big\{ \int H^{1,4}_{ml'l''l'''} \Delta(\sigma_{l,m})\, dk_m + \mathscr{P}_{l'l''l'''} \\ \cdot [H^{1,2}_{(-l'-l'')l'} H^{0,2}_{l'''} \Delta(\sigma_{l,(l'+l'')}) + H^{1,1}_{-l'} H^{0,3}_{l''l'''} \Delta(\sigma_{l,l'})] \Big\} e^{-\sigma_l t}, \end{aligned} \tag{6.18}$$

$$\begin{aligned} Q^{1,1}_{2l} = -k_l \Big[\int k_m H^{1,3}_{p-l(l-m)} E(\sigma_{l,p}; \sigma_{l,m})\, dk_{mp} \\ + H^{1,1}_l \int k_m H^{0,2}_{(l-m)} E(0; \sigma_{l,m})\, dk_m \\ + H^{0,2}_l \int k_m H^{1,1}_{(m-l)} E(\sigma_{l,(l-m)}; \sigma_{l,m})\, dk_m \Big] e^{-\sigma_l t}, \end{aligned} \tag{6.19}$$

$$
\begin{aligned}
Q_{2ll'}^{1,2} = -k_l\Bigg[\bigg[\int k_m H^{1,4}_{pl'l''(l-m)} E(\sigma_{l,p}\,;\sigma_{l,m})\,dk_{mp} + Q^{1,1}_{-l''} Q^{1,1}_{l'}\bigg\}, \\
&+ H^{1,2}_{ll'} \int k_m H^{0,2}_{(l-m)} E(0\,;\sigma_{l,m})\,dk_m \\
&+ \mathscr{P}_{l'l''} H^{0,2}_{l'} \int k_m H^{1,2}_{(l'+m)l''} E(\sigma_{l,(l'+m)}\,;\ \sigma_{i,m})\,dk_m \\
&- \mathscr{P}_{l'l''} H^{0,2}_{l''} k_{l'} \int H^{1,2}_{ml'} E(\sigma_{l,l'}\,;\sigma_{l,m})\,dk_m \\
&+ H^{0,3}_{l'l''} \int k_m H^{1,1}_{(m-l)} E(\sigma_{l,(l-m)}\,;\sigma_{l,m})\,dk_m \\
&+ \mathscr{P}_{l'l''} H^{1,1}_{-l'} \int k_m H^{0,3}_{l''(l-m)} E(\sigma_{l,l'}\,;\sigma_{l,m})\,dk_m\Bigg] e^{-\sigma_l t}.
\end{aligned}
\tag{6.20}
$$

In order to assess the long time behavior of the RPE for the physical space cumulants we must transform the above expressions back to physical space by means of (6.5). Examination of (6.15) indicates that the effect of diffusion on $Q^{1,n}_{0ll'\cdots l(n-1)}$ is restricted to the variable k_l. Consequently it is easy to show that $R_0^{1,n} = O(t^{-1/2})$, $n \geqq 1$, provided

$$
\int H^{1,n}_{0l'\cdots l(n-1)} \exp(-k_{l'}r - \cdots - k_{l(n)}r^{(n-1)})\delta_{l'\cdots l(n)}\,dk_{l'\cdots l(n)} \neq 0. \tag{6.21}
$$

Next we consider the first order terms in (6.14). From (6.16) and (2.23) it follows that $R_1^{1,1}$ is at most $O(t^{-1/2})$. Thus, to this point, no nonuniformity is evident in the RPE for $R^{1,1}$. The situation turns out to be the same for the other cumulants $R^{1,n}$, $n \geqq 1$. It can be verified that the contribution from the term involving $H^{1,n+1}$ in the expression for $Q_1^{1,n}$ is at most $O(t^{-1/2})$. Further, all terms involving products of lower order cumulants, such as the last term in (6.17), lead to contributions of at most $O(t^{-1/2})$. The reader should note that a careless application of the ideas developed in [2] regarding the equivalence of ordering in Fourier space on the basis of $k_l t^{1/2} = O(1)$ might cause one to conclude erroneously that terms such as the last one in (6.17) lead to an $O(1)$ contribution.

To establish that the RPE for the $R^{1,n}$, $n \geqq 1$, are nonuniform we must consider the long time behavior of the second order terms. For the most part we shall restrict our discussion to $R_2^{1,1}$ as the other expressions $R_2^{1,n}$, $n \geqq 1$, lead to similar conclusions. Applying the results of (2.37) we find that the term involving $H^{1,3}$ in (6.19) accounts for an $O(t^{-1/2})$ contribution. It is the remaining terms in (6.19) which are responsible for the potential nonuniformity. In the Appendix, (A.16) and (A.19), we outline a procedure for evaluating these terms. As part of the analysis we are required to determine the long time behavior of the integral in (2.15). The end result is that the second term on the right-hand side of (6.19) leads to an $O(t^{-1/2})$ contribution at most; whereas, the last term makes an $O(1)$ contribution to $R_2^{1,1}$. Consequently, it appears that the RPE for $R^{1,1}$ becomes disordered when $t = O(1/\varepsilon^4)$.

As indicated at the beginning of this section our primary reason for studying the perturbation problem is to investigate the validity of cumulant discard approximations. The fact that the RPE for the cumulants are nonuniform, while nontrivial to prove, is in itself not surprising. However, the manner in which the nonuniformity arises is instructive. For example, all of the terms in (6.20) with the exception of the one involving $H^{1,4}$ and the one involving $H_{ll'}^{1,2}$ yield $O(1)$ contributions to $R_2^{1,2}$. This suggests that these terms must be retained casting serious doubt on a closure scheme which neglects any of them. Consideration of the other second order terms $R_2^{1,n}$ results in the same conclusion. Thus, it does not appear that closure schemes based on cumulant discard approximations are mathematically justifiable even for the perturbation problem.

One might hope to gain more insight into the passive scalar problem by further pursuing the small parameter limit. However, the author has been unable to devise a simple singular perturbation procedure for removing the nonuniformity in the RPE for the $R^{1,n}$. The matching technique employed in [2] does not seem to be appropriate for this problem because of additional complexity associated with the space variables. Essentially there are two length scales present: one is related to the diffusion process and the parameter ε (namely, $x = O(1/\varepsilon^2)$) and the other to the velocity field. Both scales are important for large times ($t \geqq O(1/\varepsilon^4)$).

We conclude this section with a few brief remarks concerning the nature of the RPE for the other cumulants $R^{m,n}$, $m \geqq 2$, $n \geqq 0$. It turns out that the behavior of these expansions, although more complicated, is qualitatively the same as that for the cumulants $R^{1,n}$. There is, however, one important difference. Certain of the expansions become disordered on the time scale $t = O(1/\varepsilon^2)$. For example, consider

$$Q_{1ll'}^{2,1} = -i\mathscr{P}_{ll'}k_l\left\{\int H_{ml'l''}^{2,2}\Delta(\sigma_{l,m})\,dk_m + H_{l'}^{2,0}H_{l''}^{0,2}\Delta(\sigma_{l,l'}) + H_{-l''}^{1,1}H_{l'}^{1,1}\Delta(\sigma_{l,l''})\right\}e^{-\sigma_{ll'}t}, \qquad k_l + k_{l'} + k_{l''} = 0. \tag{6.22}$$

It is easy to show that $R_0^{2,1} = O(1/t)$ and from (6.22) that $R_1^{2,1} = O(1/t^{1/2})$. The latter behavior is due to the forcing terms involving $H^{2,0}$ and $H^{1,1}$ in (6.22). A more thorough study shows that as in the case for the $R^{1,n}$ the time scale $t = O(1/\varepsilon^4)$ is the important one for accounting for the coupling of the hierarchy of equations.

Appendix. We consider here several integrals referred to in the main body of the paper. The first one is

$$K_1(a, A, x) = P\int_{-\infty}^{\infty} \frac{\exp(-ay^2 + iAy)}{y - x}\,dy, \qquad a > 0, \quad A \text{ and } x \text{ real}. \tag{A.1}$$

Our aim is to reduce K_1 to an ordinary integral. This can be accomplished by first examining the special case when $A = 0$. Differentiating both sides of (A.1) with respect to x and applying (2.5) we obtain

$$\frac{\partial K_1(a,0,x)}{\partial x} = -2\sqrt{\frac{\pi}{a}} - 2axK_1(a,0,x). \tag{A.2}$$

This equation can readily be integrated yielding

$$K_1(a,0,x) = -2\sqrt{\pi a}\, e^{-ax^2} \int_0^1 x\, e^{ax^2\eta^2}\, d\eta, \tag{A.3}$$

where we have made use of the condition $K_1(a,0,0) = 0$. The desired result follows upon differentiating both sides of (A.1) with respect to A and solving the resulting differential equation for K_1 subject to (A.3). This leads to

$$\begin{aligned} K_1(a,A,x) = e^{iAx}\Bigg[& 2\sqrt{\pi a}x\, e^{-ax^2} \int_0^1 e^{ax^2\eta^2}\, d\eta \\ & - i\sqrt{\frac{\pi}{a}}A \int_0^1 \exp\left(\frac{-\eta^2 A^2}{4a} - iAx\eta\right) d\eta \Bigg]. \end{aligned} \tag{A.4}$$

It also proves convenient to express the following principal value integral as an ordinary iterated integral:

$$K_2(a,c,A) = aP_x\frac{1}{x}P_y\frac{1}{y-x}\Delta(ax^2 - ay^2)\exp(-cx^2t + iAx), \tag{A.5}$$

where $c \geqq a \geqq 0$, A real. This integral can be handled in the same manner as K_1. Differentiate both sides of (A.5) with respect to a and use $K_2(0,c,A) = 0$. After some simplification we obtain

$$K_2(a,c,A) = -2\pi a \int_0^1 d\xi \int_0^{\sin^{-1}\sqrt{\xi a/c}} \exp\left(-\frac{A^2}{4c}\sec^2\theta\right) d\theta, \tag{A.6}$$

with the special case

$$K_2(a,c,0) = -\pi a\left[\left(2 - \frac{c}{a}\right)\sin^{-1}\left(\frac{a}{c}\right)^{1/2} + \sqrt{\frac{c}{a} - 1}\right]. \tag{A.7}$$

The following two integrals arose in the study of Burgers' model of turbulence in § 5. We shall manipulate them into a form convenient for obtaining asymptotic expansions.

$$\begin{aligned} K_3(\lambda,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & x^2 g(x,y)[\Delta(x^2 - y^2 - (x-y)^2)]^2 \\ & \cdot \exp(-2x^2t - i\lambda x t^{1/2})\, dx\, dy, \end{aligned} \tag{A.8}$$

where $t > 0$, λ real. We need only assume that g is $L^1(R^2)$. First make the change of variables $x = \sqrt{2}x'$, $y = (x' + y')/\sqrt{2}$. Upon dropping primes, (A.8) becomes

$$\begin{aligned} K_3(\lambda,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & 4x^2 g\left(\sqrt{2}x, \frac{x+y}{\sqrt{2}}\right)\left[\frac{\Delta(2x^2 - 2y^2) - \Delta(x^2 - y^2)}{x^2 - y^2}\right] \\ & \cdot \exp(-4x^2t - i\sqrt{2}\lambda x t^{1/2})\, dx\, dy. \end{aligned} \tag{A.9}$$

Upon setting $(x^2 - y^2)^{-1} = (2x)^{-1}[(x-y)^{-1} + (x+y)^{-1}]$ we can rewrite (A.9) as

$$K_3(\lambda, t) = P_x 4x \exp(-4x^2 t - i\lambda\sqrt{2}xt^{1/2}) P_y h(x, y)[\Delta(x^2 - y^2) - \Delta(2x^2 - 2y^2)]$$

$$+ P_x 4x \exp(-4x^2 t - i\lambda\sqrt{2}xt^{1/2})[g(\sqrt{2}x, \sqrt{2}x) - g(\sqrt{2}x, 0)] \tag{A.10}$$

$$\cdot P_y \frac{1}{y - x} [\Delta(x^2 - y^2) - \Delta(2x^2 - 2y^2)],$$

where

$$h(x, y) = \frac{g(\sqrt{2}x, (x + y)/\sqrt{2}) + g(\sqrt{2}x, (x - y)/\sqrt{2}) - g(\sqrt{2}x, \sqrt{2}x) - g(\sqrt{2}x, 0)}{y - x}. \tag{A.11}$$

The function h will be smooth if g is smooth. It is now an easy matter to establish the asymptotic behavior of K_3 for $t \to \infty$. The integral involving h can be handled using (2.23). If g is analytic at the origin, we can expand g in the last integral and integrate term by term. Consequently the last integral in (A.10) has the asymptotic expansion

$$4 \sum_{m=0}^{\infty} \frac{d^m}{dx^m}[g(\sqrt{2}x, \sqrt{2}x) - g(\sqrt{2}x, 0)]_{x=0} \frac{1}{m!t^{m/2}} \frac{1}{(-\sqrt{2}i)^{m+2}} \tag{A.12}$$

$$\cdot \frac{d^{m+2}}{d\lambda^{m+2}}[K_2(1, 4, -\lambda\sqrt{2}) - K_2(2, 4, -\lambda\sqrt{2})],$$

where K_2 is given in (A.6).

Next we examine

$$K_4(\lambda, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(x - y)g(x, y)\frac{\Delta(-2y^2) - \Delta(x^2 - y^2 - (x - y)^2)}{x^2 + y^2 - (x - y)^2} \tag{A.13}$$

$$\cdot \exp(-2x^2 t - i\lambda x t^{1/2})\, dx\, dy.$$

With a little algebra we can rewrite (A.13) as

$$K_4(\lambda, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{h(x, y) - h(x, 0)}{2y^2}[1 - e^{-2y^2 t}]\exp(-2x^2 t - i\lambda x t^{1/2})\,dx\,dy$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y)\Delta(x^2 - y^2 - (x - y)^2)\exp(-2x^2 t - i\lambda x t^{1/2})\,dx\,dy$$

$$+ 2\sqrt{2\pi}\int_{-\infty}^{\infty} h(x, 0)\exp(-2x^2 t - i\lambda x t^{1/2})\,dx \tag{A.14}$$

$$+ \tfrac{1}{2}P_x g(x, 0)\exp(-2x^2 t - i\lambda x t^{1/2}) P_y \frac{1}{y}(y - x)$$

$$\cdot [\Delta(-2y^2) - \Delta(x^2 - y^2 - (x - y)^2)],$$

where

$$\text{(A.15)}\qquad \begin{aligned} h(x,y) &= \frac{g(x,y)-g(x,0)}{2y}(y-x), \qquad y \neq 0, \\ h(x,0) &= -\frac{x}{2}\frac{\partial g(x,y)}{\partial y}\bigg]_{y=0}. \end{aligned}$$

The asymptotic behavior of K_4 for $t \to \infty$ can be determined from (A.14) by a straightforward application of previous results.

The last two integrals which we shall examine are associated with the passive scalar problem.

$$\text{(A.16)}\qquad \begin{aligned} K_5(\lambda,t,c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &xyg(x,y)\frac{\Delta(x^2-y^2)-t}{x^2-y^2} \\ &\cdot \exp(-cx^2t - i\lambda xt^{1/2})\,dx\,dy, \end{aligned}$$

where $c \geqq 1$. A convenient form for obtaining an asymptotic evaluation for $t \to \infty$ is given by

$$\text{(A.17)}\qquad \begin{aligned} K_5(\lambda,t,c) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &y\left[\frac{g(x,y)-g(x,-y)-g(x,x)+g(x,-x)}{x-y}\right] \\ &\cdot \Delta(x^2-y^2)\exp(-cx^2t - i\lambda xt^{1/2})\,dx\,dy \\ &+ \tfrac{1}{2}P_x \exp(-cx^2t - i\lambda xt^{1/2})[g(x,x)-g(x,-x)] \\ &\quad\cdot P_y\frac{y}{y-x}\Delta(x^2-y^2) \\ &+ \frac{t}{2}P_x \exp(-cx^2t - i\lambda xt^{1/2})P_y\frac{y}{y-x}[g(x,y)-g(x,-y)]. \end{aligned}$$

The right-hand side of (A.17) consists of three terms. The first can be handled by (2.23) or (2.32) depending upon whether $c = 1$ or $c > 1$, respectively. In either case, the contribution is at most $O(t^{-1/2})$. If g is sufficiently smooth and $g(0, y)$ is an even function of y, then the third term in (A.17) is also at most $O(t^{-1/2})$. Such is the case with the passive scalar problem. Finally, the second term can be expressed as

$$\text{(A.18)}\qquad \begin{aligned} \sqrt{\pi t}\int_{-\infty}^{\infty} dx \int_0^1 &d\eta[g(x,-x)-g(x,x)][1 + x^2t(\eta^2-1)] \\ &\cdot \exp(x^2t(\eta^2-c) - i\lambda xt^{1/2}). \end{aligned}$$

If $c > 1$, then (A.18) can be evaluated very simply. However, if $c = 1$, we are confronted with a more difficult problem requiring (2.19) for its resolution. Again, in either case, (A.18) is at most $O(t^{-1/2})$.

The last integral is

$$\text{(A.19)}\qquad \begin{aligned} K_6(\lambda,t,c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &xyg(x,y)\left[\frac{\Delta(x^2-(x-y)^2)-\Delta(x^2-y^2)}{y^2-(x-y)^2}\right] \\ &\cdot \exp(-cx^2t - i\lambda xt^{1/2})\,dx\,dy, \end{aligned}$$

where $c \geqq 1$. We can rewrite (A.19) as

$$K_6(\lambda, t, c) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[g(x, x+y) - g(x, y) + x\frac{g(x, x+y) - g(x, x/2)}{2y+x}\right.$$

$$\text{(A.20)} \qquad \left. - x\frac{g(x, y) - g(x, x/2)}{2y - x}\right]\Delta(x^2 - y^2)\exp(-cx^2t - i\lambda x t^{1/2})\,dx\,dy$$

$$+ \frac{2\sqrt{\pi t}}{3}\int_{-\infty}^{\infty} dx \int_0^1 d\eta\, g(x, x/2)\exp(-cx^2t - i\lambda x t^{1/2})$$

$$\cdot[\exp[x^2t(3+\eta^2)/4] - \exp(x^2\eta^2 t)].$$

The first term on the right-hand side of (A.20) is at most $O(t^{-1/2})$. The last term can be evaluated in a straightforward manner if $c > 1$; however, if $c = 1$ we must use (2.19). In either case we find that the last term is $O(1)$ provided $g(0, 0) \neq 0$.

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ON SUMMATION OF SERIES OF HYPERBOLIC FUNCTIONS*

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Abstract. This paper presents a method of summation of four series of hyperbolic functions. The method is based partly on partial fraction decompositions of hyperbolic functions and partly on values of the Weierstrass elliptic function at half-periods. The series are summed in closed form in terms of two special coefficients when the parameter involved in the series takes on the special values 1, $\sqrt{3}$ or $1/\sqrt{3}$.

1. Introduction. The purpose of this paper is to present a method of summation of the following four series of hyperbolic functions:

$$\text{(1)}\qquad \begin{aligned} \mathrm{I}_{2s}(c) &= \sum_{n=1}^{\infty} \frac{1}{\sinh^{2s} n\pi c}, & \mathrm{II}_{2s}(c) &= \sum_{n=1}^{\infty} \frac{1}{\cosh^{2s} n\pi c}, \\ \mathrm{III}_{2s}(c) &= \sum_{n=1}^{\infty} \frac{1}{\sinh^{2s} (2n-1)\pi c/2}, & \mathrm{IV}_{2s}(c) &= \sum_{n=1}^{\infty} \frac{1}{\cosh^{2s} (2n-1)\pi c/2}, \end{aligned}$$

where $s \geqq 1$ and $c = 1, \sqrt{3}$ or $1/\sqrt{3}$. The investigation is motivated by an attempt to separate the series from the following summations:

$$\text{(2)}\qquad \begin{aligned} &\sum_{n=1}^{\infty} \left(\frac{1}{\sinh^2 n\pi} + \frac{15}{2\sinh^4 n\pi} + \frac{15}{2\sinh^6 n\pi} \right) = \frac{1}{126}, \\ &\sum_{n=1}^{\infty} \left\{ \frac{1}{\cosh^2 (2n-1)\pi/2} - \frac{15}{2\cosh^4 (2n-1)\pi/2} + \frac{15}{2\cosh^6 (2n-1)\pi/2} \right\} = 0, \\ &\sum_{n=1}^{\infty} \left\{ \frac{1}{\cosh^2 (2n-1)\pi c/2} - \frac{3}{2\cosh^4 (2n-1)\pi c/2} \right\} \\ &\quad - \sum_{n=1}^{\infty} \left(\frac{1}{\sinh^2 n\pi c} + \frac{3}{2\sinh^4 n\pi c} \right) = \frac{1}{60}. \end{aligned}$$

These summations were previously obtained by the author in evaluation of the Weierstrass elliptic function at half-periods [1]–[3]; the parameter c was equal to $\sqrt{3}$ or $1/\sqrt{3}$.

The summations of $\mathrm{I}_2(1)$ and $\mathrm{IV}_2(1)$ in particular were proposed by Shafer and solved independently by Livingston and Rayleigh [4]. The former's method is based on Cauchy's residue theorem while the latter's is based on the lemniscate function. Later, Kiyek and Schmidt [5] extended the summations to the preceding four series when $s \geqq 1$ and $c = 1$. Their method is based on partial fraction decomposition of the Weierstrass zeta function. In this paper, a method of summation will be presented, which is based partly on partial fraction decompositions of hyperbolic functions and partly on values of the Weierstrass elliptic function at half-periods. This method results in a substantial reduction of computation as compared with Kiyek and Schmidt's method. The resulting summations are

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expressed in closed form in terms of a special coefficient σ_4 when $c = 1$ and in terms of a second special coefficient σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$. These two coefficients will be defined in the following section.

2. The coefficients σ_4 and σ_6. Define a coefficient σ_{2s}^*, for $s \geqq 2$, by the double series

$$\sigma_{2s}^* = \sum_{n,m=-\infty}^{\infty}{}' \frac{1}{(2m\omega + 2n\omega')^{2s}}, \tag{3}$$

where the prime on the summation sign denotes the omission of simultaneous zeros of m and n from the double summation. In this paper, we are concerned only with the case $2\omega = 1$. When there is a need to emphasize the period $2\omega'$, the coefficient is written as $\sigma_{2s}^*(2\omega')$. The coefficients σ_4 and σ_6 are now defined as

$$\sigma_4 = \sigma_4^*(i), \quad \sigma_6 = \sigma_6^*(e^{\pi i/3}). \tag{4}$$

By symmetry, they are both real. Their values are given in the Appendix.

3. Summation of the series when $s = 1$. From the following partial fraction decompositions of hyperbolic functions [6, p. 36],

$$\begin{aligned} \pi \coth \pi x &= \frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{1}{m^2 + x^2}, \\ \pi \tanh \pi x &= 8x \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2 + 4x^2}, \end{aligned} \tag{5}$$

we find by differentiation,

$$\begin{aligned} \frac{1}{\sinh^2 \pi x} &= \frac{1}{\pi^2 x^2} - \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{m^2 - x^2}{(m^2 + x^2)^2}, \\ \frac{1}{\cosh^2 \pi x} &= \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - 4x^2}{\{(2m-1)^2 + 4x^2\}^2}. \end{aligned} \tag{6}$$

Put in turn $x = nc$ and $(2n-1)c/2$ in each expression and sum up from $n = 1$ to ∞; this gives

$$\begin{aligned} \mathrm{I}_2(c) &= \frac{1}{\pi^2 c^2} S_2 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^2 - n^2c^2}{(m^2 + n^2c^2)^2}, \\ \mathrm{II}_2(c) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - 4n^2c^2}{\{(2m-1)^2 + 4n^2c^2\}^2}, \\ \mathrm{III}_2(c) &= \frac{4}{\pi^2 c^2} U_2 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4m^2 - (2n-1)^2c^2}{\{4m^2 + (2n-1)^2c^2\}^2}, \\ \mathrm{IV}_2(c) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - (2n-1)^2c^2}{\{(2m-1)^2 + (2n-1)^2c^2\}^2}, \end{aligned} \tag{7}$$

where S_2 and U_2 are the case $s = 1$ of the following series,

$$S_{2s} = \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \frac{(2\pi)^{2s}}{2(2s)!}|B_{2s}|, \qquad s \geqq 1,$$

(8)

$$U_{2s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2s}} = \left(1 - \frac{1}{2^{2s}}\right) S_{2s}, \qquad s \geqq 1,$$

in which B_{2s} are the Bernoulli numbers. When $s = 1$, $B_2 = 1/6$ so that $S_2 = \pi^2/6$ and $U_2 = \pi^2/8$. Note that the order of summation of the preceding four series is not interchangeable.

To proceed further, denote

$$\sigma_2^*(ci) = 2\left(1 - \frac{1}{c^2}\right) S_2 + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^2 - n^2c^2}{(m^2 + n^2c^2)^2}. \tag{9}$$

It is noted that the definition of σ_{2s}^* in (3) does not extend to the case $s = 1$. Consider a Weierstrass elliptic function of double periods $2\omega = 1$ and $2\omega' = ci$. We find from the double series definition of the function [7, p. 355] that at the half-periods, $1/2$, $ci/2$ and $(1 + ci)/2$, respectively,

$$e_1(ci) = 8U_2 - \sigma_2^*(ci) + 16 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - 4n^2c^2}{\{(2m-1)^2 + 4n^2c^2\}^2},$$

(10)

$$e_2(ci) = -\frac{8}{c^2} U_2 - \sigma_2^*(ci) + 16 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4m^2 - (2n-1)^2c^2}{\{4m^2 + (2n-1)^2c^2\}^2},$$

$$e_3(ci) = -\sigma_2^*(ci) + 16 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - (2n-1)^2c^2}{\{(2m-1)^2 + (2n-1)^2c^2\}^2}.$$

Consequently, the desired series are

$$\mathrm{I}_2(c) = \frac{1}{6} - \frac{1}{2\pi^2}\sigma_2^*(ci),$$

(11)

$$\mathrm{II}_2(c) = -\frac{1}{2} + \frac{1}{2\pi^2}\{e_1(ci) + \sigma_2^*(ci)\},$$

$$\mathrm{III}_2(c) = -\frac{1}{2\pi^2}\{e_2(ci) + \sigma_2^*(ci)\},$$

$$\mathrm{IV}_2(c) = \frac{1}{2\pi^2}\{e_3(ci) + \sigma_2^*(ci)\},$$

which hold for any c in general, save purely imaginary values. The preceding three functions at half-periods are the three roots of the cubic equation

$$W^3 - 15\sigma_4^*(ci)W - 35\sigma_6^*(ci) = 0. \tag{12}$$

They are real and distinct if c is real; furthermore,

$$e_1(ci) > e_3(ci) > e_2(ci). \tag{13}$$

When $c = 1$, the two coefficients are

$$\sigma_4^*(i) = \sigma_4, \quad \sigma_6^*(i) = 0. \tag{14}$$

When $c = \sqrt{3}$, we consider $2\omega' = 2e^{\pi i/3} = 1 + i\sqrt{3}$. It can be shown [7, p. 379] that

$$\sigma_4^*(i\sqrt{3}) = \tfrac{1}{16}(35\sigma_6)^{2/3}, \qquad \sigma_6^*(i\sqrt{3}) = \tfrac{11}{32}\sigma_6. \tag{15}$$

Again, when $c = 1/\sqrt{3}$, we consider $2\omega' = 2e^{\pi i/6}/\sqrt{3} = 1 + i/\sqrt{3}$. It can similarly be shown that

$$\sigma_4^*(i/\sqrt{3}) = \tfrac{9}{16}(35\sigma_6)^{2/3}, \qquad \sigma_6^*(i/\sqrt{3}) = -\tfrac{297}{32}\sigma_6. \tag{16}$$

Hence, the three roots of the cubic equation are found as shown in Table 1, where

TABLE 1

c	1	$\sqrt{3}$	$1/\sqrt{3}$
$e_1(ci)$	u	$(2\sqrt{3} + 1)v/4$	$3(2\sqrt{3} - 1)v/4$
$e_2(ci)$	$-u$	$-(2\sqrt{3} - 1)v/4$	$-3(2\sqrt{3} + 1)v/4$
$e_3(ci)$	0	$-v/2$	$3v/2$

$$u = (15\sigma_4)^{1/2}, \quad v = (35\sigma_6)^{1/3}. \tag{17}$$

To evaluate σ_2^*, consider a Weierstrass zeta function of double periods $2\omega = 1$ and $2\omega' = ci$. From the double series definition of the function [7, p. 354], we find at the half-period 1/2,

$$\zeta(\tfrac{1}{2}|ci) = \tfrac{1}{2}\sigma_2^*(ci). \tag{18}$$

Again from the definition, we find for any period $2\omega'$ in general,

$$\zeta(\tfrac{1}{2}|2\omega') = 2\zeta(\tfrac{1}{2}|4\omega') + \tfrac{1}{2}e_2(4\omega'). \tag{19}$$

Consequently, when $2\omega' = e^{\pi i/3}$,

$$\begin{aligned}\zeta(\tfrac{1}{2}|e^{\pi i/3}) &= 2\zeta(\tfrac{1}{2}|1 + i\sqrt{3}) + \tfrac{1}{2}e_2(1 + i\sqrt{3})\\ &= 2\zeta(\tfrac{1}{2}|i\sqrt{3}) + \tfrac{1}{2}e_3(i\sqrt{3}),\end{aligned} \tag{20}$$

and when $2\omega' = e^{\pi i/6}/\sqrt{3}$,

$$\begin{aligned}\zeta(\tfrac{1}{2}|e^{\pi i/6}/\sqrt{3}) &= 2\zeta(\tfrac{1}{2}|1 + i/\sqrt{3}) + \tfrac{1}{2}e_2(1 + i/\sqrt{3})\\ &= 2\zeta(\tfrac{1}{2}|i/\sqrt{3}) + \tfrac{1}{2}e_3(i/\sqrt{3}).\end{aligned} \tag{21}$$

Hence, with the aid of the values

$$\zeta(\tfrac{1}{2}|i) = \frac{\pi}{2}, \quad \zeta(\tfrac{1}{2}|e^{\pi i/3}) = \frac{\pi}{\sqrt{3}}, \quad \zeta(\tfrac{1}{2}|e^{\pi i/6}/\sqrt{3}) = \pi\sqrt{3}, \tag{22}$$

we find

(23)

c	1	$\sqrt{3}$	$1/\sqrt{3}$
$\sigma_2^*(ci)$	π	$\pi/\sqrt{3} + v/4$	$\pi\sqrt{3} - 3v/4$

With the values in Table 1 and (23), the series in (11) are readily evaluated.

The results are given in Table 2. For convenience, we write

$$a = \sqrt{3}. \tag{24}$$

TABLE 2

Values of the series in (1) *for* $s = 1$

c	1	$\sqrt{3}$	$1/\sqrt{3}$
$\mathrm{I}_2(c)$	$\frac{1}{6} - \frac{1}{2\pi}$	$\frac{1}{6} - \frac{a}{6\pi} - \frac{v}{8\pi^2}$	$\frac{1}{6} - \frac{a}{2\pi} + \frac{3v}{8\pi^2}$
$\mathrm{II}_2(c)$	$-\frac{1}{2} + \frac{1}{2\pi} + \frac{u}{2\pi^2}$	$-\frac{1}{2} + \frac{a}{6\pi} + \frac{a+1}{4\pi^2}v$	$-\frac{1}{2} + \frac{a}{2\pi} + \frac{3(a-1)}{4\pi^2}v$
$\mathrm{III}_2(c)$	$-\frac{1}{2\pi} + \frac{u}{2\pi^2}$	$-\frac{a}{6\pi} + \frac{a-1}{4\pi^2}v$	$-\frac{a}{2\pi} + \frac{3(a+1)}{4\pi^2}v$
$\mathrm{IV}_2(c)$	$\frac{1}{2\pi}$	$\frac{a}{6\pi} - \frac{v}{8\pi^2}$	$\frac{a}{2\pi} + \frac{3v}{8\pi^2}$

4. Summation of the series when $s \geqq 2$. Write (6) in the form

$$\begin{aligned} \frac{\pi^2}{\sinh^2 \pi x} &= \frac{1}{x^2} - \sum_{m=1}^{\infty} \left\{ \frac{1}{(m+ix)^2} + \frac{1}{(m-ix)^2} \right\}, \\ \frac{\pi^2}{\cosh^2 \pi x} &= 4 \sum_{m=1}^{\infty} \left\{ \frac{1}{(2m-1+2ix)^2} + \frac{1}{(2m-1-2ix)^2} \right\}. \end{aligned} \tag{25}$$

Differentiate both sides $(2s-2)$ times, put in turn $x = nc$ and $(2n-1)c/2$ in the resulting expressions and then sum up from 1 to ∞. Using the notation

$$\begin{aligned} \frac{1}{(2s-1)!} \frac{d^{2s-2}}{dx^{2s-2}} \frac{\pi^2}{\sinh^2 \pi x} &= \pi^{2s} \sum_{k=1}^{s} \frac{A_{2s,2k}}{\sinh^{2k} \pi x}, \\ \frac{1}{(2s-1)!} \frac{d^{2s-2}}{dx^{2s-2}} \frac{\pi^2}{\cosh^2 \pi x} &= \pi^{2s} \sum_{k=1}^{s} (-1)^{k+1} \frac{A_{2s,2k}}{\cosh^{2k} \pi x}, \end{aligned} \tag{26}$$

one obtains for $s \geqq 2$,

$$\sum_{k=1}^{s} A_{2s,2k} \mathrm{I}_{2k}(c) = \frac{S_{2s}}{(\pi c)^{2s}} + \frac{1}{\pi^{2s}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{(-1)^s}{(m+nci)^{2s}} + \frac{(-1)^s}{(m-nci)^{2s}} \right\},$$

$$\sum_{k=1}^{s} (-1)^k A_{2s,2k} \mathrm{II}_{2k}(c) = \frac{2^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{(-1)^s}{(2m-1+2nci)^{2s}} + \frac{(-1)^s}{(2m-1-2nci)^{2s}} \right\},$$

$$
\begin{aligned}
\sum_{k=1}^{s} A_{2s,2k}\mathrm{III}_{2k}(c) = \frac{2^{2s}}{(\pi c)^{2s}}U_{2s} + \frac{2^{2s}}{\pi^{2s}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[&\frac{(-1)^s}{\{2m+(2n-1)ci\}^{2s}} \\
&+ \frac{(-1)^s}{\{2m-(2n-1)ci\}^{2s}}\Bigg],
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\sum_{k=1}^{s} (-1)^k A_{2s,2k}\mathrm{IV}_{2k}(c) = \frac{2^{2s}}{\pi^{2s}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[&\frac{(-1)^s}{\{2m-1+(2n-1)ci\}^{2s}} \\
&+ \frac{(-1)^s}{\{2m-1-(2n-1)ci\}^{2s}}\Bigg].
\end{aligned}
$$

Note that the order of summation of the double series is now interchangeable.

Define an elliptic function of double periods $2\omega = 1$ and $2\omega' = ci$ as a derivative of the Weierstrass elliptic function of the same double periods in the form

$$
\begin{aligned}
W_{2s}(z|ci) &= \frac{1}{(2s-1)!}\frac{d^{2s-2}}{dz^{2s-2}}W_2(z|ci) \\
&= \sum_{n,m=-\infty}^{\infty}\frac{1}{(z-m-nci)^{2s}}, \qquad s \geqq 2.
\end{aligned} \tag{28}
$$

We find at half-periods,

$$
\begin{aligned}
W_{2s}(\tfrac{1}{2}|ci) = 2^{2s+1}U_{2s} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg\{&\frac{2^{2s+1}}{(2m-1+2nci)^{2s}} \\
&+ \frac{2^{2s+1}}{(2m-1-2nci)^{2s}}\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
W_{2s}(\tfrac{1}{2}ci|ci) = (-1)^s\frac{2^{2s+1}}{c^{2s}}U_{2s} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[&\frac{2^{2s+1}}{\{2m+(2n-1)ci\}^{2s}} \\
&+ \frac{2^{2s+1}}{\{2m-(2n-1)ci\}^{2s}}\Bigg],
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
W_{2s}(\tfrac{1}{2}+\tfrac{1}{2}ci|ci) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[&\frac{2^{2s+1}}{\{2m-1+(2n-1)ci\}^{2s}} \\
&+ \frac{2^{2s+1}}{\{2m-1-(2n-1)ci\}^{2s}}\Bigg].
\end{aligned}
$$

Further, from (3), we find that when $2\omega = 1$ and $2\omega' = ci$,

$$
\sigma_{2s}^{*}(ci) = 2\left\{1 + \frac{(-1)^s}{c^{2s}}\right\}S_{2s} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{2}{(m+nci)^{2s}} + \frac{2}{(m-nci)^{2s}}\right\}. \tag{30}
$$

It follows therefore, since $A_{2s,2s} = 1$, that

$$
\begin{aligned}
\mathrm{I}_{2s}(c) &= \frac{(-1)^s}{2\pi^{2s}}\{\sigma^*_{2s}(ci) - 2S_{2s}\} - \sum_{k=1}^{s-1} A_{2s,2k}\mathrm{I}_{2k}(c), \\
\mathrm{II}_{2s}(c) &= \frac{1}{2\pi^{2s}} W_{2s}(\tfrac{1}{2}|ci) - \frac{2^{2s}}{\pi^{2s}} U_{2s} - \sum_{k=1}^{s-1} (-1)^{s+k} A_{2s,2k}\mathrm{II}_{2k}(c), \\
\mathrm{III}_{2s}(c) &= \frac{(-1)^s}{2\pi^{2s}} W_{2s}(\tfrac{1}{2}ci|ci) - \sum_{k=1}^{s-1} A_{2s,2k}\mathrm{III}_{2k}(c), \\
\mathrm{IV}_{2s}(c) &= \frac{1}{2\pi^{2s}} W_{2s}(\tfrac{1}{2} + \tfrac{1}{2}ci|ci) - \sum_{k=1}^{s-1} (-1)^{s+k} A_{2s,2k}\mathrm{IV}_{2k}(c),
\end{aligned} \tag{31}
$$

so that the desired series can be evaluated recurrently from the series for $s = 1$ and the values of σ^*_{2s} and W_{2s} at half-periods.

Note that if σ^*_{2s} in (3) is defined instead by (30), its validity can be extended to $s = 1$ so as to include σ^*_2 in (9). Note also that the functions W_{2s} at half-periods and σ^*_{2s} are connected by

$$W_{2s}(\tfrac{1}{2}|ci) + W_{2s}(\tfrac{1}{2}ci|ci) + W_{2s}(\tfrac{1}{2} + \tfrac{1}{2}ci|ci) = (2^{2s} - 1)\sigma^*_{2s}(ci). \tag{32}$$

σ^*_{2s} satisfies the following recurrence relation:

$$\tfrac{1}{3}(s-3)(2s+1)E_{2s} = E_4E_{2s-4} + E_6E_{2s-6} + \cdots + E_{2s-4}E_4, \qquad s \geqq 4, \tag{33}$$

where

$$E_{2s} = (2s-1)\sigma^*_{2s}(ci), \qquad s \geqq 2, \tag{34}$$

so that σ^*_{2s} can be expressed in terms of the initial coefficients σ^*_4 and σ^*_6. Hence, it can be expressed in terms of σ_4 when $c = 1$ and in terms of σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$.

Furthermore, the function W_4 at half-periods is given by

$$W_4 = W_2^2 - 5\sigma^*_4(ci), \tag{35}$$

where W_2 is the Weierstrass elliptic function at half-periods. The subsequent functions W_{2s} at half-periods satisfy the following recurrence relation:

$$\tfrac{1}{3}(s-1)(2s-3)F_{2s} = F_2F_{2s-2} + F_4F_{2s-4} + \cdots + F_{2s-2}F_2, \qquad s \geqq 3, \tag{36}$$

where

$$F_{2s} = (2s-1)W_{2s}, \qquad s \geqq 1, \tag{37}$$

so that the functions W_{2s} at half-periods can be expressed in terms of W_2 at half-periods and the value of σ^*_4. Hence, they can be expressed in terms of σ_4 when $c = 1$ and in terms of σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$.

Finally, to evaluate $A_{2s,2k}$, we differentiate both sides of either equation in (26) twice and equate the coefficients. The following recurrence relation is obtained, for $s \geqq 1$ and $1 \leqq k \leqq s$:

$$A_{2s+2,2k} = \frac{1}{2s(2s+1)}\{(2k-1)(2k-2)A_{2s,2k-2} + 4k^2A_{2s,2k}\}. \tag{38}$$

TABLE 3

Values of the series in (1) *for* $s = 2$ *and* $s = 3$

c		$2s = 4$	$2s = 6$
$\mathrm{I}_{2s}(c)$	1	$-\frac{11}{90} + \frac{1}{3\pi} + \frac{u^2}{30\pi^4}$	$\frac{191}{1890} - \frac{4}{15\pi} - \frac{u^2}{30\pi^4}$
	$\sqrt{3}$	$-\frac{11}{90} + \frac{a}{9\pi} + \frac{v}{12\pi^2} + \frac{v^2}{32\pi^4}$	$\frac{191}{1890} - \frac{4a}{45\pi} - \frac{v}{15\pi^2} - \frac{v^2}{32\pi^4} - \frac{11v^3}{2240\pi^6}$
	$1/\sqrt{3}$	$-\frac{11}{90} + \frac{a}{3\pi} - \frac{v}{4\pi^2} + \frac{9v^2}{32\pi^4}$	$\frac{191}{1890} - \frac{4a}{15\pi} + \frac{v}{5\pi^2} - \frac{9v^2}{32\pi^4} + \frac{297v^3}{2240\pi^6}$
$\mathrm{II}_{2s}(c)$	1	$-\frac{1}{2} + \frac{1}{3\pi} + \frac{u}{3\pi^2} + \frac{u^2}{3\pi^4}$	$-\frac{1}{2} + \frac{4}{15\pi} + \frac{4u}{15\pi^2} + \frac{u^2}{3\pi^4} + \frac{u^3}{5\pi^6}$
	$\sqrt{3}$	$-\frac{1}{2} + \frac{a}{9\pi} + \frac{a+1}{6\pi^2}v + \frac{2+a}{8\pi^4}v^2$	$-\frac{1}{2} + \frac{4a}{45\pi} + \frac{2(a+1)v}{15\pi^2} + \frac{2+a}{8\pi^4}v^2 + \frac{3(5a+8)}{160\pi^6}v^3$
	$1/\sqrt{3}$	$-\frac{1}{2} + \frac{a}{3\pi} + \frac{a-1}{2\pi^2}v + \frac{9(2-a)}{8\pi^4}v^2$	$-\frac{1}{2} + \frac{4a}{15\pi} + \frac{2(a-1)v}{5\pi^2} + \frac{9(2-a)}{8\pi^4}v^2 + \frac{81(5a-8)}{160\pi^6}v^3$
$\mathrm{III}_{2s}(c)$	1	$\frac{1}{3\pi} - \frac{u}{3\pi^2} + \frac{u^2}{3\pi^4}$	$-\frac{4}{15\pi} + \frac{4u}{15\pi^2} - \frac{u^2}{3\pi^4} + \frac{u^3}{5\pi^6}$
	$\sqrt{3}$	$\frac{a}{9\pi} - \frac{a-1}{6\pi^2}v + \frac{2-a}{8\pi^4}v^2$	$-\frac{4a}{45\pi} + \frac{2(a-1)v}{15\pi^2} - \frac{2-a}{8\pi^4}v^2 + \frac{3(5a-8)}{160\pi^6}v^3$
	$1/\sqrt{3}$	$\frac{a}{3\pi} - \frac{a+1}{2\pi^2}v + \frac{9(2+a)}{8\pi^4}v^2$	$-\frac{4a}{15\pi} + \frac{2(a+1)v}{5\pi^2} - \frac{9(2+a)}{8\pi^4}v^2 + \frac{81(5a+8)}{160\pi^6}v^3$
$\mathrm{IV}_{2s}(c)$	1	$\frac{1}{3\pi} - \frac{u^2}{6\pi^4}$	$\frac{4}{15\pi} - \frac{u^2}{6\pi^4}$
	$\sqrt{3}$	$\frac{a}{9\pi} - \frac{v}{12\pi^2} - \frac{v^2}{32\pi^4}$	$\frac{4a}{45\pi} - \frac{v}{15\pi^2} - \frac{v^2}{32\pi^4} + \frac{3v^3}{320\pi^6}$
	$1/\sqrt{3}$	$\frac{a}{3\pi} + \frac{v}{4\pi^2} - \frac{9v^2}{32\pi^4}$	$\frac{4a}{15\pi} + \frac{v}{5\pi^2} - \frac{9v^2}{32\pi^4} - \frac{81v^3}{320\pi^6}$

In particular, for $s \geqq 1$,

$$A_{2s,2s} = 1. \tag{39}$$

The values of the series for $s = 2$ and $s = 3$ thus found are given in Table 3 in terms of u and v defined in (17), and a defined in (24). For convenience of reference, the first few values of σ_{2s}^* and W_{2s} at half-periods, as well as those of $A_{2s,2k}$, are shown in Tables 4 and 5, respectively.

We note that in the case $c = 1$, the sums of the series for $s = 1$ and $s = 2$ are given by Kiyek and Schmidt in terms of the gamma function. When the gamma function is converted into σ_4 by the relation given in the Appendix, they are in agreement with the present results. In addition, the results also confirm the relations in (2).

TABLE 4

Values of $\sigma_{2s}^*(ci)$ and W_{2s} at half-periods when $2\omega = 1$ and $2\omega' = ci$

(1) For $c = 1$, s	$\sigma_{2s}^*(i)/u^s$	$W_{2s}(\frac{1}{2}\|i)/u^s$	$W_{2s}(\frac{1}{2}i\|i)/u^s$	$W_{2s}(\frac{1}{2}+\frac{1}{2}i\|i)/u^s$
2	1/15	2/3	2/3	−1/3
3	0	2/5	−2/5	0
4	1/525	8/35	8/35	1/35
5	0	2/15	−2/15	0
6	2/53625	64/825	64/825	−2/825
(2) For $c = \sqrt{3}$, s	$\sigma_{2s}^*(ai)/v^s$	$W_{2s}(\frac{1}{2}\|ai)/v^s$	$W_{2s}(\frac{1}{2}ai\|ai)/v^s$	$W_{2s}(\frac{1}{2}+\frac{1}{2}ai\|ai)/v^s$
2	1/16	$(2+a)/4$	$(2-a)/4$	−1/16
3	11/1120	$3(5a+8)/80$	$-3(5a-8)/80$	3/160
4	3/1792	$3(16+9a)/224$	$3(16-9a)/224$	−3/1792
5	1/3584	$(149a+256)/1792$	$-(149a-256)/1792$	−1/3584
6	683/14,350,336	$3(1280+737a)/39424$	$3(1280-737a)/39424$	15/157,696
(3) For $c = 1/\sqrt{3}$, s	$\sigma_{2s}^*\left(\frac{i}{a}\right)/v^s$	$W_{2s}\left(\frac{1}{2}\middle\|\frac{i}{a}\right)/v^s$	$W_{2s}\left(\frac{i}{2a}\middle\|\frac{i}{a}\right)/v^s$	$W_{2s}\left(\frac{1}{2}+\frac{i}{2a}\middle\|\frac{i}{a}\right)/v^s$
2	9/16	$9(2-a)/4$	$9(2+a)/4$	−9/16
3	−297/1120	$81(5a-8)/80$	$-81(5a+8)/80$	−81/160
4	243/1792	$243(16-9a)/224$	$243(16+9a)/224$	−243/1792
5	−243/3584	$243(149a-256)/1792$	$-243(149a+256)/1792$	243/3584
6	497,907/14,350,336	$2187(1280-737a)/39424$	$2187(1280+737a)/39424$	10935/157,696

TABLE 5

Values of $A_{2s,2k}$

$2s$	$2k = 2$	$2k = 4$	$2k = 6$	$2k = 8$	$2k = 10$	$2k = 12$
12	4/155,925	62/4725	256/945	19/15	2	1
10	2/2835	17/189	7/9	5/3	1	
8	4/315	2/5	4/3	1		
6	2/15	1	1			
4	2/3	1				
2	1					

Appendix. Values of σ_4 and σ_6. The two coefficients σ_4 and σ_6 are related to elliptic integrals by

$$\text{(A.1)} \qquad \sigma_4 = \frac{4}{15}K^4\left(\sin\frac{\pi}{4}\right), \quad \sigma_6 = \frac{64\sqrt{3}}{315}K^6\left(\sin\frac{\pi}{12}\right),$$

where $K(k)$ is the complete elliptic integral of the first kind in Legendre's form with modulus k. These elliptic integrals are in turn related to the gamma function

[8, pp. 524–526] by

$$K\left(\sin\frac{\pi}{4}\right) = \frac{1}{4\pi^{1/2}}\Gamma^2(1/4), \qquad K\left(\sin\frac{\pi}{12}\right) = \frac{\pi^{1/2}}{3^{3/4}}\frac{\Gamma(1/6)}{2\Gamma(2/3)}. \tag{A.2}$$

Hence, we have

$$\sigma_4 = \frac{1}{960\pi^2}\Gamma^8(1/4), \qquad \sigma_6 = \frac{2\pi^3}{25515}\frac{\Gamma^6(1/6)}{\Gamma^6(2/3)}. \tag{A.3}$$

The two coefficients are also related to one of the Jacobian theta functions. It is well known that when $2\omega = 1$, the three Weierstrass elliptic functions at half-periods [9, p. 393] are

$$\begin{gathered} e_1(2\omega') = \frac{\pi^2}{3}(\theta_3^4 + \theta_4^4), \qquad e_2(2\omega') = -\frac{\pi^2}{3}(\theta_2^4 + \theta_3^4), \\ e_3(2\omega') = \frac{\pi^2}{3}(\theta_2^4 - \theta_4^4), \end{gathered} \tag{A.4}$$

where θ_2, θ_3 and θ_4 are three Jacobian theta functions defined by

$$\begin{gathered} \theta_2 = 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}, \qquad \theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \\ \theta_4 = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}. \end{gathered} \tag{A.5}$$

In these expressions, the value of q corresponding to $2\omega' = ci$ is

$$q = e^{-\pi c}. \tag{A.6}$$

The three Jacobian theta functions are connected by

$$\theta_3^4 = \theta_2^4 + \theta_4^4. \tag{A.7}$$

When $c = 1$, $q = e^{-\pi}$. The three functions in (A.4) are

$$\begin{gathered} \frac{\pi^2}{3}(\theta_3^4 + \theta_4^4) = (15\sigma_4)^{1/2}, \qquad -\frac{\pi^2}{3}(2\theta_3^4 - \theta_4^4) = -(15\sigma_4)^{1/2}, \\ \frac{\pi^2}{3}(\theta_3^4 - 2\theta_4^4) = 0. \end{gathered} \tag{A.8}$$

Consequently, we find

$$\theta_3^4 = 2\theta_4^4, \tag{A.9}$$

and

$$\sigma_4 = \frac{\pi^4}{60}\theta_3^8. \tag{A.10}$$

Again, when $c = \sqrt{3}$, $q = e^{-\pi\sqrt{3}}$. The three functions are now

$$\begin{gathered} \frac{\pi^2}{3}(\theta_3^4 + \theta_4^4) = \frac{2\sqrt{3}+1}{4}(35\sigma_6)^{1/3}, \qquad -\frac{\pi^2}{3}(2\theta_3^4 - \theta_4^4) = -\frac{2\sqrt{3}-1}{4}(35\sigma_6)^{1/3}, \\ \frac{\pi^2}{3}(\theta_3^4 - 2\theta_4^4) = -\tfrac{1}{2}(35\sigma_6)^{1/3}. \end{gathered} \tag{A.11}$$

We find

$$\theta_4^4 = \frac{2 + \sqrt{3}}{4}\theta_3^4, \tag{A.12}$$

and

$$\sigma_6 = \frac{\pi^6}{105\sqrt{3}}\theta_3^{12}. \tag{A.13}$$

It is thus seen that the two cofficients are related to the Jacobian theta function θ_3 by (A.10) and (A.13). On account of rapid convergence of the series representing this function, the relations (A.10) and (A.13) are suitable for computation of the two coefficients to high precision. Using a $224S$ value of π taken from Shanks and Wrench [10] and a $224S$ value of e from Lehmer [11], the values found in Table 6 rounded to $221S$ are obtained.

TABLE 6

$\sigma_4 =$	3.15121	20021	53897	53821	76899	42248	68855	66455	19354	51485
	24384	70540	35738	42598	37682	74612	16108	69439	55074	50822
	34067	97840	76434	94604	88644	39664	38519	91157	48984	67849
	99631	69871	53948	35729	19980	22725	78430	90624	47216	07569
	29979	87305	05866	80605						
$\sigma_6 =$	5.86303	16934	23401	59797	02134	43837	82343	75153	76204	12955
	75122	82731	11230	49523	95831	56859	89351	55366	27614	95871
	40705	48300	13181	76095	79616	29185	72528	01856	60542	90818
	19106	35839	48652	82224	01194	34321	49029	07968	05036	80277
	40339	78330	80895	$8017\bar{0}$						

The computation was carried out on an IBM 370 computer by using a multi-precision arithmetic package prepared by Dr. T. C. Ting. This package can extend the computation to $224S$. A check on the author's previous $101S$ values [12] reveals no discrepancy. It is noted that two different relations with Jacobian theta functions were used in the previous computation. The present series of θ_3 converges more rapidly since the values of q used are both smaller.

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AN APPROXIMATION OF HARMONIC FUNCTIONS IN E^3 BY POTENTIALS OF UNIT CHARGES*

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Abstract. Harmonic functions defined on bounded, open, connected and complement connected regions D in E^3 are uniformly approximated by sums of point charge potentials each of unit charge and positioned on ∂D.

MAIN THEOREM. *For every harmonic function f in D, there exist a sequence of constants C_n and a family of finite sequences $\{y_{n,1}, y_{n,2}, \cdots, y_{n,n}\}(n = n_j \to \infty)$ of points on ∂D such that*

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to f(x) \quad \textit{as } n \to \infty,$$

uniformly on every compact subset of D. Certain restrictions are placed on ∂D.

1. Introduction. Let D be a bounded, open, connected region in E^3 with connected complement. A family of finite sequences $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}(n = n_j \to \infty)$ of points on ∂D is called an *asymptotically neutral family* relative to D if there exist constants C_n such that

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to 0 \quad \text{as } n = n_j \to \infty,$$

uniformly on every compact subset of D.

In this paper we assume that the boundary ∂D contains an *asymptotically neutral family* relative to D, and we show how any harmonic function in D can be approximated uniformly by sums of point charge potentials, each of unit charge and positioned on ∂D.

MAIN THEOREM. *For every harmonic function f in D, there exist a sequence of constants C_n and a family of finite sequences $\{y_{n,1}, y_{n,2}, \cdots, y_{n,n}\}(n = n_j \to \infty)$ of points on ∂D such that*

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to f(x) \quad \textit{as } n \to \infty,$$

uniformly on every compact subset of D.

Sufficient conditions for the existence of *asymptotically neutral families* on ∂D are given by the author in [3]. Specifically, if ∂D is a Lyapunov surface with a local parametric representation having Hölder continuous third partial derivatives, then ∂D contains an *asymptotically neutral family*.

Results of this type in E^2 have been proved by J. Korevaar requiring no restrictions on the boundary [2]. Similar methods are used in this paper with nontrivial modifications to prove the main theorem in E^3.

2. Notations. We restrict our consideration to E^3, the points of which are denoted by x, y. The Euclidean distance between x and y is denoted by $\|x - y\|$. Integrals over 2-dimensional surfaces are denoted by $\int (\cdot)\, d\sigma$, $d\sigma$ being the surface element. Integrals over 3-dimensional regions are denoted by $\int (\cdot)\, dx$.

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A Lyapunov surface in E^3 is a closed, bounded 2-dimensional surface S satisfying the following conditions:

(i) At each point of the surface there exists a well-defined tangent plane, and hence a well-defined normal.

(ii) There exist constants A and λ, $0 < \lambda \leqq 1$, such that if θ is the angle between the normals at any two points x and y of S, then θ satisfies a Hölder condition $\theta \leqq A\|x - y\|^{\lambda}$.

(iii) There is a constant d such that for all points y of S, the portion of the surface inside a sphere of radius d about y intersects lines parallel to the normal at y in at most one point.

From condition (i) we can construct, at each point y of a Lyapunov surface, a rectangular coordinate system (ξ, η, ζ) with the ζ-axis along the normal to the surface at y. From condition (iii), the subregion of S contained in a Lyapunov sphere about y can be represented by a function $\Phi(\xi, \eta)$ over a region Λ in the (ξ, η)-plane.

Lyapunov regions are regions bounded by Lyapunov surfaces. For interesting properties of Lyapunov regions see Gunter [1].

Let $f(\xi, \eta)$, defined in a region $\Lambda \subset E^2$, be bounded and possess bounded, continuous derivatives up to order k,

$$\left| \frac{\partial^{\tau} f}{\partial \xi^{\tau_1} \partial \eta^{\tau_2}} \right| < A,$$

$\tau_1 + \tau_2 = \tau$, $\tau = 0, 1, 2, \cdots, k$, such that the derivatives of order k are λ-Hölder continuous with the same constant A (see [1]). The class of such functions is denoted by $H_k(A, \lambda)$.

The surface S belongs to the class $L_k(A, \lambda)$, if $\Phi(\xi, \eta) \in H_k(A, \lambda)$, where A and λ are independent of the choice of y on S. Note that Lyapunov surfaces belong to the class $L_1(A, \lambda)$.

Let μ be a function defined on S. If (ξ, η, ζ) are the coordinates of a point y of S, we may define μ on a region Λ in the (ξ, η)-plane by putting $\mu(\xi, \eta) = \mu(\Phi(\xi, \eta)) = \mu(y)$. A function μ defined on S belongs to the class $H_k(A, \lambda)$ if $\mu(\xi, \eta) \in H_k(A, \lambda)$ on Λ where A and λ are independent of the choice of y.

3. Preliminaries. It is well known that if U is harmonic in a bounded region D or in an unbounded region D with the added condition that U is harmonic at infinity ($U \to 0$ as $R \to \infty$), then U can be expressed as the sum of a single and double layer potential

$$U(x) = \frac{1}{4\pi} \int_S \left\{ \frac{1}{\|x - y\|} \frac{\partial U}{\partial N} - U \frac{\partial}{\partial N} \left(\frac{1}{\|x - y\|} \right) \right\} d\sigma(y), \tag{3.1}$$

where S is the boundary of D, and $\partial/\partial N$ denotes the derivative in the direction of the outward normal. Furthermore, if U is harmonic in a region including D and its boundary S we can solve an exterior Dirichlet problem relative to S and find a function V, harmonic in the complement of the closure of the domain D, $\bar{D}^c$, which assumes the same boundary values as U,

$$\lim_{z \to x} V(z) = U(x), \qquad x \in S, \quad z \in \bar{D}^c.$$

Applying Green's identity to the functions $V(y)$ and $1/\|x - y\|$, with $x \in S$, $y \in \bar{D}^c$, we have

$$\int_S \left\{ \frac{1}{\|x - y\|} \frac{\partial V(y)}{\partial N} - V(y) \frac{\partial}{\partial N} \left(\frac{1}{\|x - y\|} \right) \right\} d\sigma(y)$$

$$= \int_{D^c} \left\{ \frac{1}{\|x - y\|} \Delta V - V \Delta \left(\frac{1}{\|x - y\|} \right) \right\} dy = 0. \tag{3.2}$$

Since $U = V$ on S, one may replace V by U in (3.2) and substitute

$$\int_S \frac{1}{\|x - y\|} \frac{\partial V}{\partial N} d\sigma(y) \quad \text{for} \quad \int_S U(y) \frac{\partial}{\partial N} \left(\frac{1}{\|x - y\|} \right) d\sigma(y)$$

in (3.1) to arrive at the single layer representation of U,

$$U(x) = \frac{1}{4\pi} \int_S \frac{1}{\|x - y\|} \frac{\partial(U - V)}{\partial N} d\sigma(y). \tag{3.3}$$

Next we examine the expression $\partial(U - V)/\partial N$. The first part, $\partial U/\partial N$, is certainly continuous on S since U is by assumption harmonic in a region containing D and its boundary S. For S sufficiently smooth, $\partial V/\partial N$ will also be continuous on S, in fact much more holds. Explicitly, if $S \in L_k(B, \lambda)$ and $U \in H(k, A, \lambda)$ on S then $\partial V/\partial N$ is in the class $H(k - 2, cA, \lambda')$, [1, p. 101]. We arrive at the representation for U,

$$U(x) = \frac{1}{4\pi} \int_S \frac{1}{\|x - y\|} g(y) \, d\sigma(y), \tag{3.4}$$

where $g(y)$ is a continuous function on S.

The integral in (3.4) can now be represented as a limit of Riemann sums, where the convergence is uniform on compact subsets K of the domain D:

$$U(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\lambda_{nk}}{\|x - y_{nk}\|}, \qquad y_{nk} \in S. \tag{3.5}$$

Finally, for completeness, we state the following important result which plays a supporting role in § 4.

THEOREM (Harnack). *Suppose f_n is harmonic in a region D, and the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a limit function f on every compact subset of D. Then f is harmonic in D. Moreover, the partial derivatives of f_n converge uniformly to the corresponding partial derivatives of f on every compact subset of D.*

4. Density of an asymptotically neutral family. Our analysis in § 5 of functions which can be approximated by sums of unit charge potentials requires that we first investigate the density of an *asymptotically neutral family*. We begin with the following lemma.

LEMMA 1. *An asymptotically neutral family (A.N.F.) relative to D is dense in the boundary ∂D.*

Proof. Assume not. Let N be an open neighborhood on ∂D which contains no points of the A.N.F. For convenience, with no loss in generality, translate and rotate the region D so that the x_1-axis passes through the neighborhood N on the surface ∂D and the origin of E^3 is in D. Let $\widetilde{\partial D} = \partial D \setminus N$. The complement of ∂D, denoted by ∂D^c, consists of a bounded and an unbounded component. In contrast $\widetilde{\partial D}^c$ is an open set with only one component.

The A.N.F. $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}(n = n_j \to \infty)$ on ∂D satisfies

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to 0 \quad \text{as } n = n_j \to \infty, \tag{4.1}$$

uniformly on every compact subset of D. Taking partial derivatives with respect to x_1 in (4.1) and applying Harnack's theorem, we find that

$$\sum_{k=1}^{n} \frac{(x_1 - y_{1nk})}{\|x - y_{nk}\|^3} \to 0 \quad \text{as } n = n_j \to \infty, \tag{4.2}$$

uniformly on the compact subsets of D.

Let

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{(x_1 - y_{1nk})}{\|x - y_{nk}\|^3}.$$

The family $\{F_n\}$ is uniformly bounded on every compact subset of $\widetilde{\partial D^c}$. By (4.2) the sequence $\{F_n\}_{n=1}^{\infty}$ converges to zero uniformly on the compact subsets of D. Thus, $\{F_n\}_{n=1}^{\infty}$ must converge to zero uniformly on the compact subsets of the extended region $\widetilde{\partial D^c}$ (a uniformly bounded sequence of harmonic functions is a normal family and $\widetilde{\partial D^c}$ is connected and contains D). Now pick $x_1^* > 0$, where $(x_1^*, 0, 0) \in \widetilde{\partial D^c}$. For large values of x_1^*,

$$\frac{(x_1^* - y_{1nk})}{\|x^* - y_{nk}\|^3} \simeq \frac{1}{\|x^* - y_{nk}\|^2} \simeq \frac{1}{\|x^*\|^2}.$$

Thus, $F_n(x^*) \simeq 1/\|x^*\|^2$ for all n. This contradicts the convergence of F_n to zero on compact subsets of $\widetilde{\partial D^c}$. The contradiction completes the proof.

5. Main theorem proof. Let f be harmonic in D. It will suffice to give the proof for a fixed compact subset K of D. Since K is a finite distance from ∂D, we can construct a surface S in the region between ∂D and K which surrounds K and is sufficiently smooth, i.e., $S \in L_k(B, \lambda)$, $k \geqq 3$. We first examine the class of functions, denoted by $A(K)$, which contains all finite sums of potentials formed from unit charges located on ∂D or uniform limits, modulo constants, of such finite sums. Thus,

$$A(K) = \left\{ f : f = \sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n, x \in K, y_{nk} \in \partial D \right\}$$
$$\cup \{g : \text{there exists } f_n, \text{ as above, such that } f_n \to g \text{ uniformly on } K\}.$$

Clearly, constants and single unit charge potentials, $1/\|x - y\|$, belong to $A(K)$. Our first nontrivial observation is that $-1/\|x - y\|$ belongs to $A(K)$.

To prove this let $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}(n = n_j \to \infty)$ be our A.N.F. on ∂D. Lemma 1 shows that this family is dense in ∂D. We now select a sequence $\{y_{nk_n}\}$ $(n = n_j \to \infty)$ such that

$$y_{nk_n} \to y \quad \text{as } n = n_j \to \infty.$$

The A.N.F. satisfies

$$\left(\sum_{k=1}^{n}{}' \frac{1}{\|x - y_{nk}\|}\right) + \frac{1}{\|x - y_{nk_n}\|} + C_n \to 0 \quad \text{as } n = n_j \to \infty,$$

where $\sum'$ denotes the sum with the term for which $k = k_n$ deleted. Hence,

$$\sum_{k=1}^{n}{}' \frac{1}{\|x - y_{nk}\|} + C_n \to -\frac{1}{\|x - y\|} \quad \text{as } n = n_j \to \infty,$$

uniformly on every compact subset of D. This puts $-1/\|x - y\|$ in our class $A(K)$.

If τ is any tangential direction at $y \in \partial D$, we next show that for any real scalar λ,

$$\lambda \frac{\partial}{\partial \tau}\left(\frac{1}{\|x - y\|}\right)$$

belongs to $A(K)$.

Let $y' \to y$ along an arc on D associated with the tangential direction τ. Then

$$\frac{\lambda(1/\|x - y'\| - 1/\|x - y\|)}{\|y' - y\|} \to \lambda \frac{\partial}{\partial \tau}\left(\frac{1}{\|x - y\|}\right) \tag{5.1}$$

as $y' \to y$, uniformly on K. Consider

$$\frac{[\lambda/\|y' - y\|]}{\lambda/\|y' - y\|} \frac{\lambda(1/\|x - y'\| - 1/\|x - y\|)}{\|y' - y\|}, \tag{5.2}$$

where $[\cdot]$ denotes the integral part. Let

$$\left[\frac{\lambda}{\|y' - y\|}\right] = n.$$

Then, (5.2) can be written in the form

$$\frac{n}{\|x - y'\|} + \frac{n}{\|x - y\|}. \tag{5.3}$$

But $n/\|x - y'\|$ and $-n/\|x - y\|$ are in $A(K)$, hence, expression (5.2) is an element of $A(K)$. Now

$$\frac{[\lambda/\|y' - y\|]}{\lambda/\|y' - y\|} \to 1 \quad \text{as } y' \to y,$$

hence, expression (5.2) converges to $\lambda(\partial/\partial\tau)(1/\|x - y\|)$. We conclude that

$$\lambda \frac{\partial}{\partial \tau}\left(\frac{1}{\|x - y\|}\right) \in A(K).$$

After these preliminaries, we complete the proof of the theorem. Let $H(D)|K$ denote the class of all harmonic functions on D restricted to K. Let $A_0(K)$ denote the closed subspace of $H(D)|K$ spanned by 1 and the functions $(\partial/\partial\tau)(1/\|x - y\|)$, $y \in \partial D$. Clearly,

$$A_0(K) \subset A(K) \subset H(D)|K. \tag{5.4}$$

Our theorem will be proved if we can show that equality holds in (5.4).

Applying the Hahn–Banach theorem and the Riesz representation theorem to the closed subspace $A_0(K) \subset H(D)|K$, we see that in order to show $A_0(K) = H(D)|K$, it is sufficient to demonstrate that any measure $d\mu$ which is orthogonal to $A_0(K)$ over K is also orthogonal to $H(D)|K$. Let $d\mu$ be a measure on K such that

$$\int_K d\mu(x) = 0, \qquad \int_K \frac{\partial}{\partial \tau}\left(\frac{1}{\|x - y\|}\right) d\mu(x) = 0 \tag{5.5}$$

for all y on S, and all tangential directions τ at y.

We now introduce the potential

$$V(z) = \int_K \frac{1}{\|x - z\|}\, d\mu(x). \tag{5.6}$$

The function $V(z)$ is harmonic off K, including infinity, and vanishes at infinity. Also, by (5.5),

$$\frac{\partial}{\partial \tau} V(y) = \int_K \frac{\partial}{\partial \tau} \frac{1}{\|x - y\|}\, d\mu(x) = 0$$

for all y on S and all tangential directions τ at y. Hence, V is constant on S. However, a function which is harmonic at infinity with mass zero cannot have an extremum at infinity [4]. Thus $V(z) = 0$ throughout the unbounded component of the complement of K. In other words, $d\mu(x) \perp 1/\|x - z\|$ (over K) for all z in a neighborhood of S and further out. Thus by (3.5), $d\mu$ is orthogonal to all harmonic functions (over K). That is, $A_0(K) = H(D)|K$.

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $x'(t) = -a(t)f(x(t - r(t)))$*

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Abstract. In this paper sufficient conditions are given for all bounded solutions of $x'(t) = -a(t)f(x(t - r(t)))$ to tend to zero as $t \to \infty$. Also, conditions are given which insure the existence of nontrivial bounded solutions.

1. Introduction. We consider the one-dimensional delay-differential equation

$$x'(t) = -a(t)f(x(t - r(t))), \tag{1}$$

where $a, r:[0, \infty) \to R, f:R \to R, a(t) > 0, q \geqq r(t) \geqq 0$ for some $q \geqq 0, xf(x) > 0$ for $x \neq 0$ and $a(t)$, $r(t)$ and $f(x)$ are continuous.

The purpose of this paper is first of all to give conditions on $a(t)$, $r(t)$ and $f(x)$ which insure that all bounded solutions of (1) tend to zero as $t \to \infty$ and then to give conditions which guarantee that (1) has bounded solutions.

Let C denote the set of continuous functions $\phi:[-q, 0] \to R$, where q is the bound given on $r(t)$, and let $\|\phi\| = \sup_{s\in[-q,0]}|\phi(s)|$ for $\phi \in C$. Further, for $b > 0$, let $C_b = \{\phi \in C : \|\phi\| \leqq b\}$. If $x(\cdot)$ is continuous on the interval $[t - q, t]$, we denote by x_t the function in C for which $x_t(s) = x(t + s)$ for $s \in [-q, 0]$. Then equation (1) is a special case of the one-dimensional delay-differential equation

$$x'(t) = F(t, x_t(\cdot)), \tag{2}$$

where $F:[0, \infty) \times C \to R$. We say $x(\cdot) = x(\cdot, t_0, \phi)$ is a *solution of* (2) through (t_0, ϕ) on the interval $[t_0, T)$ if $x:[t_0 - q, T) \to R$ is continuous and continuously differentiable on (t_0, T) and satisfies (2) on $[t_0, T)$ with $x_{t_0} = \phi$.

A solution $x(\cdot)$ of (2) on $[t_0, T)$, $0 \leqq t_0 < T \leqq \infty$, is said to be *noncontinuable* either if $T = \infty$, or if $T < \infty$ and for every positive ε, $x(\cdot)$ cannot be extended as a solution of (2) to $[t_0, T + \varepsilon)$.

We need the following basic results for delay-differential equations.

THEOREM 1. *Let $F:[0, \infty) \times C \to R$ be continuous.*

(i) *For each $t_0 \geqq 0$ and $\phi \in C$, there exist $T > t_0$ and a solution $x(\cdot)$ of* (2) *on $[t_0, T)$ such that $x_{t_0} = \phi$.*

(ii) *If $|F|$ is bounded on $[t_0, T] \times C_b$ for each t_0, T and $b \geqq 0$ and if $x(\cdot)$ is a noncontinuable solution of* (2) *on $[t_0, T)$, $T < \infty$, then* $\limsup_{t\to T^-}|x(t)| = \infty$.

(iii) *Any solution on $[t_0, T)$ can be extended to an interval on which it is noncontinuable.*

For equation (1) we have $F(t, \phi) = -a(t)f(\phi(-r(t)))$ and the properties of Theorem 1 hold.

2. On solutions tending to zero. We are now ready to give our main result for this paper.

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THEOREM 2. *Suppose* $\int^\infty a(t)\,dt = \infty$ *and either* $b(t)r(t) \to 0$ *as* $t \to \infty$ *or* $a(t) \to 0$ *as* $t \to \infty$, *where* $b(t) = \sup_{s \in [0,t]} a(s)$. *Then all bounded solutions of* (1) *tend to zero as* $t \to \infty$.

Proof. By Theorem 1 any bounded solution of (1) can be extended to an interval of the form $[t_0, \infty)$. Suppose $x(t)$ is a bounded solution which eventually has constant sign, say $x(t) > 0$ for large t. Then $x'(t) = -a(t)f(x(t - r(t))) < 0$ and it follows that $\lim_{t\to\infty} x(t) = c$ exists. If $c > 0$, then there exists $t^* \geqq t_0$ such that $f(x(t - r(t))) \geqq d > 0$ for $t \geqq t^*$ and some $d > 0$. Thus $x'(t) \leqq -a(t)d$ for $t \geqq t^*$ which implies $x(t) \leqq x(t^*) - d\int_{t^*}^t a(s)\,ds \to -\infty$ as $t \to \infty$. This contradicts that $x(t)$ is bounded and, therefore, $c = 0$. A similar argument holds if $x(t)$ is eventually negative. Hence, if $x(t)$ is a bounded solution which does not tend to zero, it must be oscillatory. Then there exist $\varepsilon > 0$ and sequences $\{t_n\}$, $\{t_n^*\} \uparrow \infty$ such that, for each n, either $x(t_n) = 0$, $x(t_n^*) = \varepsilon$, $x'(t_n^*) \geqq 0$ and $0 < x(t) < \varepsilon$ if $t_n < t < t_n^* < t_{n+1}$ or $x(t_n) = 0$, $x(t_n^*) = -\varepsilon$, $x'(t_n^*) \leqq 0$ and $0 > x(t) > -\varepsilon$ if $t_n < t < t_n^* < t_{n+1}$. As a similar argument holds for both cases, we assume the former. Integrating $x'(t)$ from t_n to t_n^*, we obtain

$$\varepsilon = |x(t_n^*) - x(t_n)| \leqq \int_{t_n}^{t_n^*} a(s)|f(x(s - r(s)))|\,ds \leqq M \int_{t_n}^{t_n^*} a(s)\,ds, \tag{3}$$

where M is a bound on $|f(x(\cdot))|$. Suppose $b(t)r(t) \to 0$ as $t \to \infty$. Now $b(t)$ is continuous and monotone increasing and, from (3), we have

$$\varepsilon \leqq M \int_{t_n}^{t_n^*} b(s)\,ds \leqq Mb(t_n^*)(t_n^* - t_n)$$

or

$$t_n^* - t_n \geqq \varepsilon / Mb(t_n^*). \tag{4}$$

Let n be chosen sufficiently large such that $b(t_n^*)r(t_n^*) < \varepsilon/M$. Then

$$r(t_n^*) < \varepsilon/Mb(t_n^*) \leqq t_n^* - t_n,$$

which implies $t_n < t_n^* - r(t_n^*) \leqq t_n^*$. Thus, $x(t_n^* - r(t_n^*)) > 0$ and we have $x'(t_n^*) = -a(t_n^*)f(x(t_n^* - r(t_n^*))) < 0$. This contradicts $x'(t_n^*) \geqq 0$. Now, suppose $a(t) \to 0$ as $t \to \infty$. From (3) and the mean value theorem for integrals, we have

$$\varepsilon \leqq M \int_{t_n}^{t_n^*} a(s)\,ds = Ma(\xi)(t_n^* - t_n)$$

or

$$t_n^* - t_n \geqq \varepsilon/Ma(\xi),$$

where $t_n < \xi < t_n^*$. Since $a(t) \to 0$, it follows that $a(\xi) \to 0$ as $n \to \infty$. Then n can be chosen sufficiently large such that $t_n^* - t_n > q$. It follows that $t_n < t_n^* - r(t_n^*) \leqq t_n^*$ and we obtain a contradiction as before. This completes the proof.

From Theorem 2 it follows that if $a(t)$ is bounded with $\int^\infty a(t)\,dt = \infty$ and if $r(t) \to 0$, then all bounded solutions of (1) tend to zero as $t \to \infty$.

Before discussing conditions which will guarantee the existence of bounded solutions of (1), we wish to relate Theorem 2 to a recent result of Ladas [10, Thm. 4.1(b)]. In Theorem 2 $f(x(t - r(t)))$ can be replaced by $f(x(t), x(t - r(t)))$, where f satisfies the condition: if x and y have the same sign, then $f(x, y)$ has that sign. With this condition and the requirement that $\int^{\infty} t^{n-1} a(t)\,dt = \infty$, Ladas considered the equation

$$x^{(n)}(t) + a(t) f(x(t), x(g(t))) = 0, \tag{5}$$

where $g(t) \leqq t$ for $t \geqq 0$, $g(t) \to \infty$ as $t \to \infty$, $a(t) > 0$ and a, f and g are continuous. Among his results, he proved that for odd n, each bounded solution of (5) is either oscillatory or tends to zero. For $n = 1$ and $g(t) = t - r(t)$, Theorem 2 provides simple conditions for which the bounded oscillatory solutions also tend to zero. For other possible forms of the argument of f, we refer to [13].

The remainder of this paper is primarily concerned with obtaining various new results for the equation

$$x'(t) = -a(t) x^{\gamma}(t - r(t)), \tag{6}$$

where $\gamma > 0$ is the quotient of odd integers. In [14] Yorke points out that for $a(t) = \text{const.}$, $r(t) \equiv q > 0$ and $\gamma = \frac{1}{3}$, (6) has a nontrivial orbitally asymptotically stable periodic solution. However, from Theorem 2 we see that if, for instance, $a(t)$ is bounded and either $a(t) \to 0$ or $r(t) \to 0$, then there is no value of γ for which (6) can have a nontrivial periodic solution. In [14] Yorke has also proved a significant stability theorem which can be applied to (6) for $\gamma \geqq 1$. We now combine Yorke's theorem with Theorem 2 to obtain an additional stability result for (6). For stability definitions we refer to [14] and [7, p. 47].

THEOREM 3. *Suppose, in* (6), $\gamma \geqq 1$ *and* $a(t) \leqq \alpha$ *for some* $\alpha > 0$.

(i) *If* $0 \leqq \alpha q \leqq 3/2$, *then the zero solution of* (6) *is uniformly stable.*

(ii) *If* $0 < \alpha q < 3/2$ *and* $a(t) \geqq \alpha_0$ *for some* $\alpha_0 > 0$, *then the zero solution of* (6) *is uniformly asymptotically stable.*

(iii) *If* $0 \leqq \alpha q \leqq 3/2$, $\int^{\infty} a(t)\,dt = \infty$ *and either* $a(t) \to 0$ *or* $r(t) \to 0$ *as* $t \to \infty$, *then the zero solution of* (6) *is asymptotically stable.*

Proof. (i) and (ii) follow immediately as they are special cases of Yorke's result [14, Thm. 1.1]. We apply (i) and Theorem 2 to show that (iii) holds. By (i) if $0 \leqq \alpha q \leqq 3/2$, then a solution with sufficiently small initial condition must be bounded. But, by Theorem 2, such a solution tends to zero as $t \to \infty$ and this completes the proof.

Although the uniform asymptotic stability conclusion in (ii) is stronger than the conclusion in (iii), it is required in (ii) that $a(t)$ be bounded strictly away from zero. This condition is unnecessary in order to conclude asymptotic stability. Also, by considering the equation $x' = -x/(t + 1)$, where $a(t) = 1/(t + 1)$, $r(t) \equiv 0$ and $\gamma = 1$, we see that uniform asymptotic stability cannot be obtained under the conditions given in (iii).

If $\gamma = 1$, then (6) is a linear homogeneous equation and it is well known that all solutions are bounded if the zero solution is stable [9, p. 97]. Thus, for the linear case, Theorem 3 provides conditions for which all solutions tend to zero. The author has been unable to find any results in the literature along these lines which include the nonlinear case $\gamma \neq 1$, and Theorem 4 below is the only type of result we have obtained in that direction. The following simple lemma will be useful.

LEMMA 1. *If* $x(t)$ *is a noncontinuable solution of* (1) *on* $[t_0, T)$ *with* $T < \infty$, *then* $r(T) = 0$.

Proof. If $x(t)$ is noncontinuable on $[t_0, T)$ with $T < \infty$, then $x(t)$ is unbounded on this interval. Suppose $r(T) > 0$. Then there exists $t^* \in [t_0, T)$ such that $t_0 - q \leqq t - r(t) \leqq t^*$ for all $t \in [t_0, T)$. Since $x(\cdot)$ is bounded on $[t_0 - q, t^*]$, $x(t - r(t))$ is bounded for $t \in [t_0, T)$. This implies that $x'(t) = -a(t)f(x(t - r(t)))$ is bounded and this contradicts that $x(t)$ is unbounded on the finite interval $[t_0, T)$. It follows that $r(T) = 0$.

If $r(t) > 0$ for all t, then we can conclude from Lemma 1 that each solution of (1) is defined on an interval of the form $[t_0, \infty)$. The author does not know if the same result holds when we just require $r(t) \geqq 0$.

THEOREM 4. *Suppose $r(t) > 0$ and suppose there exists $N > 0$ such that*

$$b(t)r(t)\exp\left(\int_0^t a(s)\,ds\right) \leqq N \quad \text{for } t \geqq 0. \tag{7}$$

Then for $\gamma < 1$, all solutions of (6) *are bounded. If, in addition, $\int^\infty a(t)\,dt = \infty$, then all solutions tend to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (6). Since $r(t) > 0$, it follows from Lemma 1 that $x(t)$ is defined on an interval of the form $[t_0 - q, \infty)$. We will first show that if $|x(t) - x(t - r(t))|$ is bounded for $t \geqq t_0$, then $x(t)$ is bounded. Suppose otherwise. Then there exist $M > 0$ and $t^* \geqq t_0$ such that $|x(t) - x(t - r(t))| \leqq M$ for $t \geqq t_0$, $|x(t^*)| \geqq 2M$ and $x'(t^*)$ and $x(t^*)$ have the same sign, say $x'(t^*) > 0$, $x(t^*) > 0$. But since $|x(t) - x(t - r(t))| \leqq M$, it follows that $x(t^* - r(t^*)) \geqq M > 0$ or $x'(t^*) = -a(t^*)x^\gamma(t^* - r(t^*)) < 0$ which is a contradiction. Now consider

$$|x(t)| \leqq G(t) \equiv |x(t_0)| + \int_{t_0}^t a(s)|x^\gamma(s - r(s))|\,ds.$$

Then $G'(t) = a(t)|x^\gamma(t - r(t))| \geqq 0$ so $G(t)$ is monotone increasing for $t \geqq t_0$. If $G(t)$ is bounded, then $x(t)$ is bounded. Suppose $G(t)$ is unbounded. Let $t_1 \geqq t_0 + q$ be chosen such that $G(t) \geqq 1$ for $t \geqq t_1$. Then, for $t \geqq t_1$, we have $|x(t - r(t))| \leqq G(t - r(t)) \leqq G(t)$ and

$$G'(t) = a(t)|x(t - r(t))|^\gamma \leqq a(t)G^\gamma(t) \leqq a(t)G(t). \tag{8}$$

We apply Gronwall's inequality to (8) to obtain

$$|x(t)| \leqq G(t) \leqq G(t_1)\exp\left(\int_{t_1}^t a(s)\,ds\right) \quad \text{for } t \geqq t_1, \tag{9}$$

where $G(t_1) = |x(t_0)| + \int_{t_0}^{t_1} a(s)|x^\gamma(s - r(s))|\,ds$. By the mean value theorem for derivatives,

$$|x(t) - x(t - r(t))| = |x'(\xi)|r(t), \qquad t - r(t) \leqq \xi \leqq t. \tag{10}$$

Let $t_2 \geqq t_1 + q$. From (9) and (10) we have for $t \geqq t_2$,

$$\begin{aligned}|x(t) - x(t - r(t))| &= a(\xi)|x^\gamma(\xi - r(\xi))|r(t)\\ &\leqq b(\xi)G^\gamma(\xi - r(\xi))r(t) \leqq b(t)r(t)G^\gamma(t)\\ &\leqq b(t)r(t)G(t) \leqq b(t)r(t)G(t_1)\exp\left(\int_{t_1}^t a(s)\,ds\right) \leqq G(t_1)N.\end{aligned}$$

Thus, $|x(t) - x(t - r(t))|$ is bounded and it follows that $x(t)$ is bounded. Now, suppose $\int^\infty a(t)\,dt = \infty$. Then from (7) $b(t)r(t) \to 0$ and, by Theorem 2, all solutions tend to zero.

As a final remark it is perhaps appropriate to give some additional history of the problem studied in this paper. In recent years several papers concerning the one-dimensional delay-differential equation have been written in response to a research problem of Bellman [1]. One of the earliest such papers was authored by K. Cooke [3] in which he examined the linear equation

$$x'(t) = -ax(t - r(t)). \tag{11}$$

He showed that if $r(t) \geqq 0$ is continuous with $r(t) \to 0$ as $t \to \infty$ and $\int^\infty r(t)\,dt < \infty$, then each solution $x(\cdot)$ of (11) satisfies

$$\lim_{t\to\infty} x(t)\,e^{at} = c \tag{12}$$

for some constant c and conversely, for each constant c there is a solution which satisfies (12). This result has been extended by Grossman and Yorke [6] and Chow [2] to include the case where $a(\cdot)$ is a bounded function of t. Similar results have also been obtained in [4], [5] and [8]. Also, in [11] and [12], Winston has obtained results along these lines for nonlinear equations where the nonlinearity is due to state dependent delays. We have not attempted to obtain results as sharp as these investigators. At the same time, however, our restrictions on $a(t)$, $r(t)$ and $f(x)$ have, in general, not been as stringent as those imposed by others.

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BEST RATIONAL FUNCTION APPROXIMATION FOR LAPLACE TRANSFORM INVERSION*

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Abstract. Rational function approximations $\bar{g}_n(p)$ in partial form to a function $\bar{f}(p)$ of the Laplace transform operator p are determined to have inverses $g_n(t)$ which approximate the inverse $f(t)$ of $\bar{f}(p)$ in a least squares sense. Examples are presented and compared with Padé table and other rational approximations.

1. Introduction. In a previous paper the author [1] has considered the general question of the generation of rational function approximations for Laplace transform inversion. The original idea of using rational approximations for this purpose goes back to the work of Luke [2], [3]. Since then the method has been successfully applied by a number of authors. For a general account of the use of rational approximations, including those of Padé in the inversion of Laplace transforms, the reader is referred to Luke [4]. Examples of applications are also given in the work of Longman [1], [5], [6], [7], [8], [9], [10], Akin and Counts [11], [12], Counts and Akin [13], and Murphy [14]. However Luke and the other workers have not considered the important question as to how to obtain "good" rational approximations for this purpose, although questions of "suitability" and "convenience" were raised by the author [1]. A desirable characteristic (Longman [1]) for Laplace transform inversions arising in most physical problems is that our rational approximations $\bar{g}_n(p)$ to $\bar{f}(p)$ should have no poles in the right-hand half-plane $\operatorname{Re}(p) > 0$.

The purpose of the present paper is to establish a practical criterion for a "best" rational approximation, and to demonstrate its application. The criterion adopted is a least squares one in the t-space, requiring however only a knowledge of $\bar{f}(p)$ but not of its inverse $f(t)$, apart from a general assumption of square integrability. The details are presented in the next section.

2. Theory. We define the Laplace transform $\bar{f}(p)$ of a real function $f(t)$ in the usual way:

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

and we assume meanwhile that $f(t)$ is square integrable, that is to say that

$$K = \int_0^\infty [f(t)]^2\, dt \tag{2}$$

exists and is finite.

Now let us suppose that we are given $\bar{f}(p)$ or have obtained it as the operational solution of a physical problem, and wish to obtain a sequence of approximations to $f(t)$. (It often happens in applications of the Laplace transform that the Bromwich integral and other inversion formulas are not convenient for numerical computation.) We approximate $\bar{f}(p)$ by a rational function $\bar{g}_n(p)$ which

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we may as well consider to be already expanded in partial fractions

$$\bar{g}_n(p) = \sum_{r=1}^{n} A_r/(p + \alpha_r). \tag{3}$$

We have supposed here that the n poles $p = -\alpha_r$ of $\bar{g}_n(p)$ are all simple, and that the numerator of $\bar{g}_n(p)$ is of lower degree than the denominator, so that the expansion (3) is valid. We further suppose that all the α_r have positive real parts. The inverse $g_n(t)$ of $\bar{g}_n(p)$ is

$$g_n(t) = \sum_{r=1}^{n} A_r e^{-\alpha_r t}, \tag{4}$$

and we now make the demand that the A's and α's be chosen so that $g_n(t)$ is the "best" approximation to $f(t)$.

This requirement is made explicit by demanding that the integral

$$I_n = \int_0^\infty [g_n(t) - f(t)]^2 \, dt \tag{5}$$

(whose existence is established by the above conditions) shall be a minimum. By elementary calculation we find

$$\begin{aligned} I_n &= \int_0^\infty \left[\sum_{r=1}^{n} A_r e^{-\alpha_r t} - f(t) \right]^2 dt \\ &= \int_0^\infty \left[\sum_{r=1}^{n} \sum_{k=1}^{n} A_r A_k e^{-(\alpha_r + \alpha_k)t} - 2f(t) \sum_{r=1}^{n} A_r e^{-\alpha_r t} + \{f(t)\}^2 \right] dt. \end{aligned} \tag{6}$$

An essential feature of our method is that, apart from the constant K (equation (2)), I_n is determined in terms of $\bar{f}(p)$ rather than $f(t)$. We have

$$I_n = \sum_{r=1}^{n} \sum_{k=1}^{n} \frac{A_r A_k}{\alpha_r + \alpha_k} - 2 \sum_{r=1}^{n} A_r \bar{f}(\alpha_r) + K. \tag{7}$$

We now seek α_r, A_r to minimize I_n. For this we require

$$\begin{aligned} \frac{\partial I_n}{\partial A_r} &= 2 \sum_{k=1}^{n} \frac{A_k}{\alpha_r + \alpha_k} - 2\bar{f}(\alpha_r) = 0, & r = 1, 2, \cdots, n, \\ \frac{\partial I_n}{\partial \alpha_r} &= -2 \sum_{k=1}^{n} \frac{A_r A_k}{(\alpha_r + \alpha_k)^2} - 2A_r \bar{f}'(\alpha_r) = 0, & r = 1, 2, \cdots, n. \end{aligned} \tag{8}$$

Thus we need to solve

$$\begin{aligned} A_r &: \sum_{k=1}^{n} \frac{A_k}{\alpha_r + \alpha_k} = \bar{f}(\alpha_r), & r = 1, 2, \cdots, n, \\ \alpha_r &: \sum_{k=1}^{n} \frac{A_k}{(\alpha_r + \alpha_k)^2} = -\bar{f}'(\alpha_r), & r = 1, 2, \cdots, n. \end{aligned} \tag{9}$$

These are $2n$ equations for the n A's and n α's. Here we have assumed that no A_r vanishes, and for later reference each equation is preceded by the parameter with respect to which I_n has been minimized in deriving the equation.

It is of interest to note the significance of these equations in the p-plane. We find

$$
\begin{aligned}
\bar{g}_n(p) &= \bar{f}(p), & p &= \alpha_1, \alpha_2, \cdots, \alpha_n, \\
\bar{g}_n'(p) &= \bar{f}'(p), & p &= \alpha_1, \alpha_2, \cdots, \alpha_n,
\end{aligned} \tag{10}
$$

or, in words, $\bar{g}_n(p)$ and its derivative $\bar{g}_n'(p)$ are made equal to $\bar{f}(p)$ and its derivative $\bar{f}'(p)$ respectively at *minus* the poles of $\bar{g}_n(p)$.

Now, unfortunately, the solution of equations (9) for the A's and α's is not a simple matter when n is larger than 1. Furthermore the solution may not be unique and some solutions may yield a smaller minimum of I_n than others. Complex solutions of (9) are of course allowed, but only those which yield real functions $g_n(t)$. This means that complex α's must occur in conjugate pairs, and that the two corresponding A's are also conjugate to each other.

For any given solution of (9) we can calculate I_n except for the constant K from (7); this enables us to compare different solutions and choose the best, that is, the one that yields the smallest I_n. It may also happen, in some cases, that although we do not know $f(t)$ we can calculate K by Parseval's theorem [15, p. 267],

$$
\int_0^\infty e^{-at}[f(t)]^2\,dt = \frac{1}{2\pi}\int_0^\infty [\bar{f}(a/2 + iy)]^2\,dy, \tag{11}
$$

which for $a = 0$ yields

$$
K = \int_0^\infty [f(t)]^2\,dt = \frac{1}{2\pi}\int_0^\infty [\bar{f}(iy)]^2\,dy. \tag{12}
$$

In order to solve the equations (9) in practical cases an iteration method was used, and this is described in the next section.

3. Solution of the equations. Analytic solution of the equations (9) is not in general tractable, but we may hope to solve them by an iterative procedure in which we seek to minimize I_n successively with respect to its $2n$ parameters A_1, $A_2, \cdots, A_n, \alpha_1, \alpha_2, \cdots, \alpha_n$. With this in mind we solve each of the equations (9) for the parameter with respect to which it represents a minimization. This yields the system of equations

$$
\begin{aligned}
A_r &= 2\alpha_r\left[\bar{f}(\alpha_r) - \sum_{\substack{k=1\\k\neq r}}^{n} \frac{A_k}{\alpha_r + \alpha_k}\right], & r &= 1, 2, \cdots, n, \\
\alpha_r &= \frac{1}{2}\left[A_r \Big/ \left\{-\bar{f}'(\alpha_r) - \sum_{\substack{k=1\\k\neq r}}^{n} \frac{A_k}{(\alpha_r + \alpha_k)^2}\right\}\right]^{1/2}, & r &= 1, 2, \cdots, n.
\end{aligned} \tag{13}
$$

Starting from an initial guess for the A's and α's we proceed by iteration and hope for convergence to a solution. This has been done on a computer for a number of examples. At each step of the iteration we compute also I_n (with the term K if known—otherwise without) and check that I_n decreases as the iteration proceeds. In some cases a poor initial guess leads to divergence or to a complex solution $g_n(t)$, but then another initial guess often leads to a meaningful solution. Some results of applying the method are given in the next section.

4. Some examples. Let us consider first the simple example

$$\bar{f}(p) = (p + 1)^{-2} \tag{14}$$

for which we know the inverse

$$f(t) = t\, e^{-t}. \tag{15}$$

For the case $n = 1$ we have for A, α (dropping the suffix 1) the equations

$$\begin{aligned} A/\alpha &= 2(\alpha + 1)^{-2}, \\ A/\alpha^2 &= 8(\alpha + 1)^{-3}, \end{aligned} \tag{16}$$

having the solution

$$\alpha = 1/3, \quad A = 3/8,$$

so that

$$g_1(t) = (3/8)\, e^{-t/3}. \tag{17}$$

Using $K = 1/4$ we readily find here that

$$I_1 = 5/128 \doteqdot 0.0391.$$

We may compare this with the [0, 1] Padé approximant to $\bar{f}(p)$, namely

$$\bar{h}(p) = 1/(1 + 2p)$$

having the inverse

$$h(t) = \tfrac{1}{2}\, e^{-t/2}.$$

Then we find

$$I_1 = \int_0^\infty [h(t) - f(t)]^2\, dt = 1/18 \doteqdot 0.0556.$$

We next consider the example

$$\bar{f}(p) = (1/p) \log (1 + p), \tag{18}$$

which has as inverse the exponential integral

$$f(t) = E_1(t) = \int_t^\infty \frac{e^{-u}}{u}\, du; \tag{19}$$

(see [16, p. 251, no. 5]). This function is tabulated in Abramowitz and Stegun [17, pp. 228–251] and in Jahnke, Emde and Lösch [18, pp. 17–22], and was considered in the author's previous paper [1], where approximate inversion was carried out by means of diagonal elements of the Padé table of the Maclaurin expansion of (19), as well as by means of rational approximations obtained by a transformation due to Levin [19].

Equations (9) were solved in the form (13) for values of n from 1 to 4, and in Table 1 we give the values obtained for the A's and α's and for the I_n. Here we were able to use the known value for K:

$$K = \int_0^\infty [E_1(t)]^2\,dt = 2\log 2; \tag{20}$$

see [17, p. 230, no. 5.1.33].

TABLE 1
Values of the A's and α's for $\bar{f}(p) = (1/p)\log(1+p)$ for $n = 1, 2, 3, 4$. The last column gives the integral I_n for the error (equation (5)).

n	α_r	A_r	I_n
1	3.92155363	3.18724852	0.0911
2	2.36902424	1.98881870	
	34.2657866	3.40540679	0.0125
3	1.89681580	1.49002978	
	13.2520248	2.29828967	
	207.922736	3.40964161	0.0024
4	1.66883035	1.20384539	
	8.07965471	1.83690663	
	63.1163866	2.31144962	
	993.435106	3.40978572	0.00056

Using these results, values of $g_n(t)$ were calculated for a few selected values of t. They are compared in Table 2 with $E_1(t)$. The results compare favorably with earlier results obtained using Padé and Levin approximations [1].

TABLE 2
Values of $E_1(t)$ and $g_n(t)$ for $t = 0(0.5)2.0$ and $n = 1, 2, 3, 4$

t	$E_1(t)$	$g_1(t)$	$g_2(t)$	$g_3(t)$	$g_4(t)$
0	∞	3.1872	5.3942	7.1980	8.7620
0.5	0.5598	0.4486	0.6084	0.5802	0.5550
1.0	0.2194	0.0631	0.1861	0.2236	0.2275
1.5	0.1000	0.0089	0.0569	0.0866	0.0985
2.0	0.0489	0.0013	0.0174	0.0335	0.0428

5. Further developments. It is planned to apply the method to a number of inversion problems arising in physical applications of the Laplace transform. A tabulation of A's and α's for a number of functions $\bar{f}(p)$ would seem to be useful, and might find application in the inversion of operational solutions of physical problems of the type

$$\int_0^\infty \bar{f}(p, \xi)\phi(\xi)\,d\xi. \tag{21}$$

Such solutions arise, for example, in theoretical seismology [6].

As a variant on the method we can consider minimizing the integrals

$$I_n(a) = \int_0^\infty e^{-at}[g_n(t) - f(t)]^2\, dt. \tag{22}$$

This introduces a weight function e^{-at} which is useful if we desire to emphasize the accuracy of our approximations $g_n(t)$ for small values of t. This is also useful in cases where $f(t)$ is not square integrable, that is, where K (equation (2)) does not exist. We then may still have a finite value for

$$K(a) = \int_0^\infty e^{-at}[f(t)]^2\, dt, \tag{23}$$

and we may be able to find this from $\bar{f}(p)$ using Parseval's theorem (equation (11)). We can also then relax the conditions on $\bar{g}_n(p)$ to some extent, and allow poles on the imaginary axis in the p-plane.

On the other hand, this weight function does not unduly complicate the algebra. For $I_n(a)$ we now have the formula

$$I_n(a) = \sum_{r=1}^{n} \sum_{k=1}^{n} \frac{A_r A_k}{\alpha_r + \alpha_k + a} - 2 \sum_{r=1}^{n} A_r \bar{f}(\alpha_r + a) + K(a), \tag{24}$$

and the equations (9) are now to be replaced by

$$\begin{aligned} A_r &: \sum_{k=1}^{n} \frac{A_k}{\alpha_r + \alpha_k + a} = \bar{f}(\alpha_r + a), \qquad r = 1, 2, \cdots, n, \\ \alpha_r &: \sum_{k=1}^{n} \frac{A_k}{(\alpha_r + \alpha_k + a)^2} = -\bar{f}'(\alpha_r + a), \qquad r = 1, 2, \cdots, n. \end{aligned} \tag{25}$$

It is hoped that the results of these further investigations will be published in due course.

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EXPONENTIAL DECAY OF FUNCTIONALS OF SOLUTIONS OF A PSEUDOPARABOLIC EQUATION*

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Abstract. Exponential decay estimates are derived for quadratic functionals of solutions of boundary value problems associated with a third order diffusion-like equation. Spatial as well as time decay estimates are included.

1. Introduction. In this paper we derive two decay estimates for quadratic functionals of solutions of boundary value problems associated with the differential equation

$$u_{,ii} + au_{,iit} - u_{,t} = 0. \tag{1}$$

One estimate gives the exponential decay of a functional with increasing distance from part of the boundary and the other gives the exponential decay with increasing time.

Equation (1) is a member of the class of equations referred to as *pseudoparabolic* by Showalter and Ting [1] and we adopt that terminology here. Equations of this type occur in the mathematical description of a variety of physical processes, for instance, in the theory of nonsteady flow of second order fluids [2]; in the theory of seepage of homogeneous fluids through a fissured rock [3] and in the study of imprisoned resonant radiation in a gas [4], [5], [6].

For results related to the spatial decay estimate see the author's paper [7] and those of Edelstein [8], [9] where the spatial decay of solutions of second order parabolic equations is studied. Results on decay estimates of weak solutions of pseudoparabolic equations are given in [1].

2. Notation and definitions. We shall be concerned with (1) defined in the $(n + 1)$-dimensional cylindrical domain

$$D_T = B \times (0, T], \qquad 0 < T < \infty,$$

where B is a bounded domain in n-dimensional Euclidean space R_n. The lateral surface of the cylinder D_T will be denoted by S_T.

The summation convention will be employed throughout so that repeated subscripts (except t and τ) are to be summed from 1 to n. The symbol $,i$ will be used to denote partial differentiation with respect to the variable x_i.

3. Spatial decay. We consider the problem

$$u_{,ii} + au_{,iit} - u_{,t} = 0 \quad \text{in } D_{t_0}, \qquad t_0 \leqq T, \tag{2}$$

$$u(x, 0) = 0 \quad \text{in } \bar{B}, \tag{3}$$

$$\frac{\partial}{\partial n}(u + au_{,t}) + \alpha(u + au_{,t}) = 0 \quad \text{on } [\partial B - C_0] \times [0, t_0]. \tag{4}$$

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We assume that the boundary ∂B of B has a plane part C_0 and that B lies entirely on one side of C_0. An unspecified energy flux flows into the region through $C_0 \times [0, t_0]$, a portion of the lateral surface S_{t_0}. As usual, $\partial/\partial n$ denotes the normal derivative and a and α are nonnegative constants. Without loss of generality we assume also that $a \leqq 1$. (For one possible physical interpretation of this boundary value problem see [4].)

Let C_s be the intersection with B of a plane parallel to and at a distance s from C_0 and let B_s stand for the set of all points of B whose distance from C_0 is greater than s (we assume that B_s is connected). Then we shall prove the following:

$$U(s,t) \leqq U(0,t)\, e^{-s/c}, \qquad 0 < t \leqq t_0, \tag{5}$$

where

$$U(s,t) = \int_0^t \int_{B_s} [u_{,i}u_{,i} + (u_{,\tau})^2 + a^2 u_{,i\tau}u_{,i\tau}]\, dx\, d\tau$$

and u is a smooth solution of (2)–(4). *The explicit constant c will be determined in the proof.*

Proof. As the first step in the derivation of this estimate we multiply (2) by $u_{,t}$ and integrate by parts over B_s to obtain

$$\int_{B_s} (u_{,t})^2\, dx = \int_{C_s} u_{,t}\frac{\partial}{\partial n}(u + au_{,t})\, d\sigma + \int_{\partial B_s - C_s} u_{,t}\frac{\partial}{\partial n}(u + au_{,t})\, d\sigma$$
$$- \int_{B_s} u_{,it}(u + au_{,t})_{,i}\, dx.$$

Integrating this with respect to t from 0 to t_0 and using (4) yields

$$\int_0^{t_0}\int_{B_s} (u_{,t})^2\, dx\, dt + \int_0^{t_0}\int_{B_s} u_{,it}(u + au_{,t})_{,i}\, dx\, dt = \int_0^{t_0}\int_{C_s} u_{,t}\frac{\partial}{\partial n}(u + au_{,t})\, d\sigma\, dt$$
$$- \alpha \int_0^{t_0}\int_{\partial B_s - C_s} u_{,t}(u + au_{,t})\, d\sigma\, dt. \tag{6}$$

However,

$$-\int_0^{t_0}\int_{\partial B_s - C_s} u_{,t}u\, d\sigma\, dt = -\frac{1}{2}\int_0^{t_0}\int_{\partial B_s - C_s} (u^2)_{,t}\, d\sigma\, dt = -\frac{1}{2}\int_{\partial B_s - C_s} u^2(x, t_0)\, d\sigma$$

and

$$\int_0^{t_0}\int_{B_s} u_{,ti}u_{,i}\, dx\, dt = \frac{1}{2}\int_0^{t_0}\int_{B_s} (u_{,i}u_{,i})_{,t}\, dt\, dx = \frac{1}{2}\int_{B_s} \frac{\partial u(x,t_0)}{\partial x_i}\cdot\frac{\partial u(x,t_0)}{\partial x_i}\, dx, \tag{7}$$

by (3). Using these expressions in (6) we obtain the equality

$$\int_0^{t_0}\int_{B_s} (u_{,t})^2\, dx\, dt + a\int_0^{t_0}\int_{B_s} u_{,it}u_{,it}\, dx\, dt + \frac{\alpha}{2}\int_{\partial B_s - C_s} u^2(x, t_0)\, d\sigma$$
$$+ a\alpha\int_0^{t_0}\int_{\partial B_s - C_s} (u_{,t})^2\, d\sigma\, dt + \frac{1}{2}\int_{B_s} u_{,i}(x, t_0)u_{,i}(x, t_0)\, dx \tag{8}$$
$$= \int_0^{t_0}\int_{C_s} u_{,t}\frac{\partial}{\partial n}(u + au_{,t})\, d\sigma\, dt.$$

Now multiply (2) by u and again integrate by parts to obtain

$$\int_{B_s} u,_t u\,dx = \frac{1}{2}\int_{B_s} (u^2),_t\,dx = \int_{C_s} u\frac{\partial}{\partial n}(u + au,_t)\,d\sigma + \int_{\partial B_s - C_s} u\frac{\partial}{\partial n}(u + au,_t)\,d\sigma - \int_{B_s} u,_i(u + au,_t),_i\,dx.$$

Upon integrating with respect to t, rearranging and using (3) we have

$$\frac{1}{2}\int_{B_s} u^2(x, t_0)\,dx + \int_0^{t_0}\int_{B_s} [u,_i u,_i + au,_i u,_{it}]\,dx\,dt = \int_0^{t_0}\int_{\partial B_s - C_s} u\frac{\partial}{\partial n}(u + au,_t)\,d\sigma\,dt + \int_0^{t_0}\int_{C_s} u\frac{\partial}{\partial n}(u + au,_t)\,d\sigma\,dt. \tag{9}$$

But

$$\begin{aligned}\int_0^{t_0}\int_{\partial B_s - C_s} u\frac{\partial}{\partial n}(u + au,_t)\,d\sigma\,dt &= -\alpha\int_0^{t_0}\int_{\partial B_s - C_s} u(u + au,_t)\,d\sigma\,dt\\ &= -\alpha\int_0^{t_0}\int_{\partial B_s - C_s} u^2\,d\sigma\,dt - \frac{a\alpha}{2}\int_{\partial B_s - C_s} u^2(x, t_0)\,d\sigma\end{aligned}$$

using (3) and (4). Thus this equality along with (7) and (9) implies

$$\begin{aligned}&\int_0^{t_0}\int_{B_s} u,_i u,_i\,dx\,dt + \frac{1}{2}\int_{B_s} u^2(x, t_0)\,dx + \frac{a}{2}\int_{B_s} u,_i(x, t_0)u,_i(x, t_0)\,dx\\ &\quad + \alpha\int_0^{t_0}\int_{\partial B_s - C_s} u^2\,d\sigma\,dt + \frac{a\alpha}{2}\int_{\partial B_s - C_s} u^2(x, t_0)\,d\sigma\\ &= \int_0^{t_0}\int_{\partial B_s - C_s} u\frac{\partial}{\partial n}(u + au,_t)\,d\sigma\,dt.\end{aligned} \tag{10}$$

Addition of (8) and (10) gives

$$\begin{aligned}&U(s, t) + \frac{1}{2}\int_{B_s} u^2(x, t_0)\,dx + \frac{1}{2}(1 + a)\int_{B_s} u,_i(x, t_0)u,_i(x, t_0)\,dx\\ &\quad + \frac{\alpha}{2}(1 + a)\int_{\partial B_s - C_s} u^2(x, t_0)\,d\sigma + \alpha\int_0^{t_0}\int_{\partial B_s - C_s} [u^2 + a(u,_t)^2]\,d\sigma\,dt\\ &\leqq \int_0^{t}\int_{C_s} (u + u,_t)\frac{\partial}{\partial n}(u + au,_t)\,d\sigma\,dt.\end{aligned} \tag{11}$$

We now drop the positive terms on the left-hand side of (11), and use the arithmetic–geometric mean and Schwarz inequalities to obtain

$$\begin{aligned}U(s, t) &\leqq \frac{\gamma}{2}\int_0^{t_0}\int_{C_s} (u + u,_t)^2\,d\sigma\,dt + \frac{1}{\gamma}\int_0^{t_0}\int_{C_s} [u,_i u,_i + a^2 u,_{it}u,_{it}]\,d\sigma\,dt\\ &\leqq \frac{\gamma}{2}\int_0^{t_0}\int_{C_s} (1 + \beta)\left[(u,_t)^2 + \frac{1}{\beta}u^2\right]d\sigma\,dt + \frac{1}{\gamma}\int_0^{t_0}\int_{C_s} [u,_i u,_i + a^2 u,_{it}u,_{it}]\,d\sigma\,dt\end{aligned} \tag{12}$$

for arbitrary positive γ and β. But u is initially zero and thus we can write

$$\int_0^t u^2(x,\tau)\,d\tau \leqq \frac{\pi^2}{4t^2}\int_0^t (u_{,\tau})^2\,d\tau.$$

This inequality in (12) yields

$$(13)\quad U(s,t) \leqq \frac{\gamma}{2}(1+\beta)\frac{1+4t_0^2}{\beta\pi^2}\int_0^{t_0}\int_{C_s} u^2\,d\sigma\,dt + \frac{1}{\gamma}\int_0^{t_0}\int_{C_s}(u_{,i}u_{,i} + a^2u_{,it}u_{,it})\,d\sigma\,dt.$$

The optimal choice for β is $\beta = 2t_0/\pi$ and this value of β leads to the choice for γ of $\gamma = \sqrt{2}/(1 + 2t_0/\pi)$; with these values of γ and β inequality (13) becomes

$$(14)\qquad U(s,t) \leqq \frac{1+2t_0/\pi}{\sqrt{2}}\int_0^{t_0}\int_{C_s}(u^2 + u_{,i}u_{,i} + a^2u_{,it}u_{,it})\,d\sigma\,dt.$$

Now we can also express $U(s,t)$ by

$$U(s,t) = \int_s^l\int_0^t\int_{C_s}[u_{,i}u_{,i} + (u_{,\tau})^2 + a^2u_{,i\tau}u_{,i\tau}]\,dx\,d\tau\,ds',$$

where l is the maximum diameter of B normal to C_0; thus, by (14),

$$\frac{dU}{ds} + \frac{1}{c}u \leqq 0,$$

where $c = (1 + 2t_0/\pi)/\sqrt{2}$ and our estimate is an immediate consequence of this inequality.

4. Time decay. In this section we consider u to be a sufficiently smooth solution of the boundary value problem

$$(15)\qquad u_{,ii} + au_{,iit} - u_{,t} = 0 \quad \text{in } D_T,$$

$$(16)\qquad u = f(x) \quad \text{on } B,$$

$$(17)\qquad u = 0 \quad \text{on } S_T.$$

We also redefine B_τ as $B_\tau = D_\tau \cap \{t = \tau\}$. If we let

$$(18)\qquad V(\tau) = \int_{B_\tau}(u^2 + au_{,i}u_{,i})\,dx,$$

then the main result of this section is the following:

$$(19)\qquad V(t) \leqq V(0)\,e^{-2kt},$$

where $k = \lambda_1/(1 + a\lambda_1)$, λ_1 being the smallest fixed membrane eigenvalue for the region B.

The proof is simple. If we differentiate (18), integrate by parts and use (15) and (17) we find that

$$\frac{dV}{dt} = -2\int_{B_\tau} u_{,i}u_{,i}\,dx.$$

But we can write

$$\frac{dV}{dt} \leqq -2\gamma \int_{B_\tau} u_{,i}u_{,i}\,dx - 2(1-\gamma)\lambda_1 \int_{B_\tau} u^2\,dx, \qquad \gamma > 0,$$

where we have used the inequality

$$\int_{B_\tau} u^2\,dx \leqq \frac{1}{\lambda_1}\int_{B_\tau} u_{,i}u_{,i}\,dx$$

which is valid by (17). Choosing $\gamma = \lambda_1 a/(1 + a\lambda_1)$, we have

$$\frac{dV}{dt} \leqq -\frac{2\lambda_1}{1 + a\lambda_1} V$$

from which the result follows immediately. Inequality (19) is a best possible result since equality holds when $u(x,t) = \exp[-\lambda_1 t/(1 + a\lambda_1)]u_1(x)$, where $u_1(x)$ is the first fixed membrane eigenfunction.

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INEQUALITIES FOR JACOBI POLYNOMIALS AND DIRICHLET AVERAGES*

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Abstract. Upper approximations are derived for the absolute values of single and double Dirichlet averages of x^n. These averages are homogeneous polynomials of degree n in several complex variables. Special cases yield upper approximations to the absolute value of a Jacobi polynomial with any complex values of the indices and argument. A Jacobi polynomial is represented in a new way by a triple sum.

1. Introduction. In discussing the convergence of series of Jacobi polynomials, one finds a need for upper approximations to the polynomials which hold more generally than those previously available. The asymptotic formulas of Darboux [6, p. 196] show that

$$\limsup_{n\to\infty} |P_n^{(\alpha,\delta)}(\cos\theta)|^{1/n} = e^{|\mathrm{Im}\,\theta|}. \tag{1.1}$$

(For a definition of the Jacobi polynomial $P_n^{(\alpha,\delta)}$ see (5.1) below. In the complex plane whose points represent values of $\cos\theta$, the locus of points with $\mathrm{Im}\,\theta = \text{const.}$ is an ellipse with foci 1 and -1.) In view of (1.1) it seems reasonable to seek a proof of the following theorem.

THEOREM 1.1. *Let α and δ be any complex numbers. There exists a sequence of positive numbers f_n, depending on α and δ, such that, for every complex value of θ,*

$$|P_n^{(\alpha,\delta)}(\cos\theta)| \leqq f_n(\alpha,\delta)\, e^{n|\mathrm{Im}\,\theta|}, \qquad n = 0, 1, 2, \cdots, \tag{1.2a}$$

$$\lim_{n\to\infty} [f_n(\alpha,\delta)]^{1/n} = 1. \tag{1.2b}$$

It seems that the only known result of type (1.2) is a very precise one [6, Thm. 7.32.1], [5, § 10.18] which gives the maximum value for real θ of $P_n^{(\alpha,\delta)}(\cos\theta)$ if α and δ are real and greater than -1. We aim for less precision in the present paper, but we shall prove Theorem 1.1 and exhibit explicit formulas for f_n. The proof depends on a new representation (5.11) of the Jacobi polynomial by a triple sum.

We shall derive first some simple inequalities for certain homogeneous polynomials in several variables denoted by R_n and $\mathscr{R}_n$. They are the single and double Dirichlet averages of x^n. One of the inequalities for R-polynomials can be specialized to obtain a result of type (1.2) for Gegenbauer polynomials ($\alpha = \delta$). The corresponding inequality for $\mathscr{R}$-polynomials yields a less precise result of type (1.2) for general Jacobi polynomials and provides the proof of Theorem 1.1. A variant of this theorem will be used in a subsequent paper to discuss the expansion of analytic functions in series of Jacobi polynomials with complex indices [4].

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2. Dirichlet averages. Denote the set of nonnegative integers by $\mathbb{N}$, the real line by $\mathbb{R}$, the complex plane by $\mathbb{C}$, and the right half-plane by $\mathbb{C}_> = \{x \in \mathbb{C} : \operatorname{Re} x > 0\}$. Suppose $b \in \mathbb{C}^k_>$ and define $c = \sum_{i=1}^k b_i$. On the set E of all k-tuples $(u_1, \cdots, u_k)$ of nonnegative weights with $\sum_{i=1}^k u_i = 1$, we define the Dirichlet measure

$$d\mu_b(u) = [B(b)]^{-1} \prod_{i=1}^k u_i^{b_i - 1} \, du_1 \cdots du_{k-1}, \qquad k \geqq 2, \tag{2.1}$$

where B is the beta function in several variables,

$$B(b) = [\Gamma(c)]^{-1} \prod_{i=1}^k \Gamma(b_i). \tag{2.2}$$

Note that $\mu_b(E) = 1$. Let $z \in \mathbb{C}^k$ and denote a convex combination of its components by $u \cdot z = \sum_{i=1}^k u_i z_i$. The Dirichlet average [2] of x^n, $n \in \mathbb{N}$, is a homogeneous polynomial of degree n,

$$R_n(b, z) = \int (u \cdot z)^n \, d\mu_b(u). \tag{2.3}$$

The integration extends over the set E. If $k = 1$ we define $R_n(b, z) = z^n$.

Suppose further that $\beta \in C^\kappa_>$ and define $\gamma = \sum_{j=1}^\kappa \beta_j$. Let v be a κ-tuple of nonnegative weights with $\sum_{j=1}^\kappa v_j = 1$, let Z be a $k \times \kappa$ matrix of complex numbers, and define $u \cdot Z \cdot v = \sum_{i=1}^k \sum_{j=1}^\kappa u_i Z_{ij} v_j$. The double Dirichlet average [3] of x^n is

$$\mathscr{R}_n(b, Z, \beta) = \iint (u \cdot Z \cdot v)^n \, d\mu_b(u) \, d\mu_\beta(v). \tag{2.4}$$

Before extending these definitions to general complex values of b and β, we note some elementary inequalities. Let $\operatorname{Re} b = (\operatorname{Re} b_1, \cdots, \operatorname{Re} b_k)$ and define the norms

$$|z| = \max\{|z_1|, \cdots, |z_k|\}, \qquad |Z| = \max\{|Z_{11}|, \cdots, |Z_{k\kappa}|\}. \tag{2.5}$$

The total variation measure $|\mu_b|$ is given by

$$\begin{aligned} d|\mu_b|(u) &= |B(b)|^{-1} \prod_{i=1}^k u_i^{\operatorname{Re} b_i - 1} \, du_1 \cdots du_{k-1} \\ &= |B(b)|^{-1} B(\operatorname{Re} b) \, d\mu_{\operatorname{Re} b}(u). \end{aligned} \tag{2.6}$$

It follows that

$$|R_n(b, z)| \leqq \int |u \cdot z|^n \, d|\mu_b|(u) \leqq |z|^n \int d|\mu_b|(u) = |z|^n |B(b)|^{-1} B(\operatorname{Re} b).$$

A similar procedure for $\mathscr{R}_n$ completes the proof of the following inequalities.

THEOREM 2.1. *Let* $n \in \mathbb{N}$, $b \in \mathbb{C}^k_>$, $z \in \mathbb{C}^k$, $\beta \in \mathbb{C}^\kappa_>$, $Z \in \mathbb{C}^{k\kappa}$. *Define the norms* $|z|$ *and* $|Z|$ *by* (2.5). *Then*

$$|R_n(b,z)| \leqq \frac{B(\operatorname{Re} b)}{|B(b)|}|z|^n, \tag{2.7}$$

$$|\mathscr{R}_n(b,Z,\beta)| \leqq \frac{B(\operatorname{Re} b)B(\operatorname{Re}\beta)}{|B(b)B(\beta)|}|Z|^n. \tag{2.8}$$

3. General complex parameters. We now proceed to define the polynomials when the real parts of b and β are not necessarily positive. If $a \in \mathbb{C}$ let

$$(a,0) = 1, \qquad (a,n) = a(a+1)\cdots(a+n-1), \qquad n-1 \in \mathbb{N}. \tag{3.1}$$

Then, if $b \in \mathbb{C}^k_>$ and $m \in \mathbb{N}^k$,

$$\int \prod_{i=1}^{k} u_i^{m_i}\, d\mu_b(u) = \frac{B(b+m)}{B(b)} = \frac{(b_1,m_1)\cdots(b_k,m_k)}{\left(c, \sum_{i=1}^{k} m_i\right)}. \tag{3.2}$$

Multinomial expansion of $(u \cdot z)^n$ in (2.3) shows that

$$(c,n)R_n(b,z) = n!\sum \frac{(b_1,m_1)\cdots(b_k,m_k)}{m_1!\cdots m_k!} z_1^{m_1}\cdots z_k^{m_k}, \tag{3.3}$$

where the summation extends over all nonnegative integers $m_1, \cdots, m_k$ whose sum is n. Since the right side is a polynomial in the components of b, we may use (3.3) as a definition of the left side for all $b \in \mathbb{C}^k$, $k \geqq 1$.

Likewise, multinomial expansion of $(u \cdot Z \cdot v)^n$ in (2.4) gives

$$\begin{aligned}(c,n)(\gamma,n)&\mathscr{R}_n(b,Z,\beta)\\ &= n!\sum \frac{\left(b_1, \sum_{j=1}^{\kappa} m_{1j}\right)\cdots\left(b_k, \sum_{j=1}^{\kappa} m_{kj}\right)\left(\beta_1, \sum_{i=1}^{k} m_{i1}\right)\cdots\left(\beta_\kappa, \sum_{i=1}^{k} m_{i\kappa}\right)}{m_{11}!\cdots m_{k\kappa}!}\\ &\qquad \cdot Z_{11}^{m_{11}}\cdots Z_{k\kappa}^{m_{k\kappa}},\end{aligned} \tag{3.4}$$

where the summation extends over all nonnegative integers $m_{11}, \cdots, m_{k\kappa}$ whose sum is n. The right side serves to define the left side for all $b \in \mathbb{C}^k$ and all $\beta \in \mathbb{C}^\kappa$.

We now prove a form of Vandermonde's theorem and a generalization thereof.

THEOREM 3.1. *Let* $n \in \mathbb{N}$, $b \in \mathbb{C}^k$, $\beta \in \mathbb{C}^\kappa$. *Define* $c = \sum_{i=1}^{k} b_i$ *and* $\gamma = \sum_{j=1}^{\kappa}\beta_j$. *Then*

$$\sum \frac{(b_1,m_1)\cdots(b_k,m_k)}{m_1!\cdots m_k!} = \frac{(c,n)}{n!}, \tag{3.5}$$

where the summation extends over all nonnegative integers $m_1, \cdots, m_k$ whose sum is n. Also,

$$\sum \frac{\left(b_1, \sum_{j=1}^{\kappa} m_{1j}\right) \cdots \left(b_k, \sum_{j=1}^{\kappa} m_{kj}\right)\left(\beta_1, \sum_{i=1}^{k} m_{i1}\right) \cdots \left(\beta_\kappa, \sum_{i=1}^{k} m_{i\kappa}\right)}{m_{11}! \cdots m_{k\kappa}} = \frac{(c, n)(\gamma, n)}{n!}, \tag{3.6}$$

where the summation extends over all nonnegative integers $m_{11}, \cdots, m_{k\kappa}$ whose sum is n.

Proof. Since both sides of (3.5) are polynomials in the components of b, it suffices to prove the equation when $\operatorname{Re} b > 0$. In this case (2.3) shows that $R_n(b, z) = 1$ if $z_1 = \cdots = z_k = 1$. Substituting these values in (3.3) gives (3.5). The proof of (3.6) using (2.4) and (3.4) is exactly similar.

In addition to the norms $|z|$ and $|Z|$ defined by (2.5), we introduce the norms

$$\|z\| = \sum_{i=1}^{k} |z_i|, \qquad \|Z\| = \sum_{i=1}^{k} \sum_{j=1}^{\kappa} |Z_{ij}|. \tag{3.7}$$

THEOREM 3.2. *Let $n \in \mathbb{N}$, $b \in \mathbb{C}^k$, $z \in \mathbb{C}^k$, $\beta \in \mathbb{C}^\kappa$, $Z \in \mathbb{C}^{k\kappa}$. Define norms as in* (2.5) *and* (3.7), *and let $c = \sum_{i=1}^{k} b_i$ and $\gamma = \sum_{j=1}^{\kappa} \beta_j$. Then*

$$|(c, n)R_n(b, z)| \leqq (\|b\|, n)|z|^n, \tag{3.8}$$

$$|(c, n)R_n(b, z)| \leqq (|b|, n)\|z\|^n, \tag{3.9}$$

$$|(c, n)(\gamma, n)\mathscr{R}_n(b, Z, \beta)| \leqq (\|b\|, n)(\|\beta\|, n)|Z|^n, \tag{3.10}$$

$$|(c, n)(\gamma, n)\mathscr{R}_n(b, Z, \beta)| \leqq (|b|, n)(|\beta|, n)\,\|Z\|^n. \tag{3.11}$$

Proof. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we find from (3.1) that

$$|(a, n)| \leqq (|a|, n). \tag{3.12}$$

Hence, by (3.3),

$$|(c, n)R_n(b, z)| \leqq n! \sum \frac{(|b_1|, m_1) \cdots (|b_k|, m_k)}{m_1! \cdots m_k!} |z_1|^{m_1} \cdots |z_k|^{m_k}. \tag{3.13}$$

The right side does not exceed

$$n!\,|z|^n \sum \frac{(|b_1|, m_1) \cdots (|b_k|, m_k)}{m_1! \cdots m_k!} = |z|^n(\|b\|, n),$$

where we have used (3.5). This proves (3.8), and the proof of (3.10), using (3.6), is exactly similar.

To prove (3.9), note that

$$(|b_1|, m_1) \cdots (|b_k|, m_k) \leqq (|b|, m_1) \cdots (|b|, m_k) \leqq (|b|, n), \tag{3.14}$$

for the middle member is a product of n factors which do not exceed the corresponding n factors of $(|b|, n)$. Thus (3.13) implies, by use of the multinomial theorem,

$$|(c, n)R_n(b, z)| \leqq (|b|, n)n! \sum \frac{|z_1|^{m_1} \cdots |z_k|^{m_k}}{m_1! \cdots m_k!} = (|b|, n)\|z\|^n.$$

The proof of (3.11) is exactly similar; it suffices to note that

$$\left(|b_1|, \sum_{j=1}^{\kappa} m_{1j}\right) \cdots \left(|b_k|, \sum_{j=1}^{\kappa} m_{kj}\right) \leqq (|b|, n), \tag{3.15}$$

with a corresponding inequality for β.

Equation (3.3) leaves R_n undefined if $(c, n) = 0$. We shall be particularly interested in the case where all components of b have a common value, say v. Then R_n is a rational function of v which may have either a pole or a removable singularity when $(c, n) = (kv, n) = 0$. We show next that the singularity is removable if v is zero or a negative integer.

THEOREM 3.3. *Let $n, k - 1, r$ be nonnegative integers such that $n > kr$. If $b = (v, \cdots, v) \in \mathbb{C}^k$, the singularities of $R_n(b, z)$ and $\mathcal{R}_n(b, Z, \beta)$ at $v = -r$ are removable. Similar statements hold for $\mathcal{R}_n$ with respect to β and with respect to b and β jointly.*

Proof. By (3.3), if $(kv, n) \neq 0$,

$$R_n(b, z) = \frac{n!}{(kv, n)} \sum \frac{(v, m_1) \cdots (v, m_k)}{m_1! \cdots m_k!} z_1^{m_1} \cdots z_k^{m_k}. \tag{3.16}$$

Since this is a rational function of v, it suffices to show that $-r$ is not a pole. By (3.1), (kv, n) has a simple zero at $-r$. Since the summation extends over all nonnegative integers $m_1, \cdots, m_k$ whose sum is $n > kr$, at least one of the m_i in each term of the sum must exceed r. Thus the numerator of every term has a zero at $-r$, and the function tends to a finite limit as $v \to -r$. The proof for $\mathcal{R}_n$, using (3.4), is entirely similar. If $\beta = (v', \cdots, v')$, each term of $\mathcal{R}_n$ is the product of a rational function of v and a rational function of v'. Therefore it is immaterial whether v and v' tend to the respective limits $-r$ and $-r'$ separately or together. This completes the proof.

We shall henceforth assume that removable singularities have been removed by requiring continuity. Since (3.8)–(3.11) provide no estimates of $|R_n|$ and $|\mathcal{R}_n|$ at such points, and very large overestimates nearby, we now consider these cases separately.

THEOREM 3.4. *Let $n, k - 1, r$ be nonnegative integers such that $n > kr$. Let $v \in \mathbb{C}$ and define $M = \max\{|v|, |v + r + 1|\}$. Then inequalities (3.8)–(3.11) remain valid if we put $b = (v, \cdots, v)$, replace (c, n) by $k(kv, kr)(kv + kr + 1, n - kr - 1)$, replace $(\|b\|, n)$ by $k(k|v| + 1, n - 1)$ if $r = 0$, and replace $(|b|, n)$ by $|(v, r)|(M, n - r - 1)$. Moreover, (3.10) and (3.11) remain valid if corresponding replacements are made with regard to β (for b, c, k, v, r, M read β, γ, κ, v', r', M') or if replacements are made with regard to both b and β.*

Note. The replacement for (c, n) is nonzero if $|v + r| < 1/k$. Hence we obtain upper estimates of $|R_n|$ and $|\mathcal{R}_n|$ on a neighborhood of the removable singularity at $v = -r$.

Proof. Consider a single term in the sum (3.16) and suppose that m_i is one of the summation indices which exceeds r in this term. Then, if $(kv, n) \neq 0$,

$$\begin{aligned}\frac{(v, m_i)}{(kv, n)} &= \frac{v(v+1)\cdots(v+r)\cdots(v+m_i-1)}{kv(kv+1)\cdots(kv+kr)\cdots(kv+n-1)} \\ &= \frac{(v,r)(v+r+1, m_i-r-1)}{(kv,kr)k(kv+kr+1, n-kr-1)}.\end{aligned} \tag{3.17}$$

Thus,

$$\begin{aligned}&|k(kv,kr)(kv+kr+1,n-kr-1)R_n(b,z)| \\ &\leqq n!\sum\frac{(|v|,m_1)\cdots|(v,r)(v+r+1,m_i-r-1)|\cdots(|v|,m_k)}{m_1!\cdots m_k!}|z_1|^{m_1}\cdots|z_k|^{m_k},\end{aligned} \tag{3.18}$$

where i may take different values from 1 to k in different terms of the sum. Since both sides are continuous functions of v, the inequality holds without exception. If $r \geqq 1$, then $|v| + r \geqq 1$ and

$$|(v,r)(v+r+1,m_i-r-1)| \leqq (|v|,r)(|v|+r)(|v|+r+1,m_i-r-1) = (|v|,m_i).$$

Thus, by (3.5), the right side of (3.18) does not exceed

$$(k|v|,n)|z|^n = (\|b\|,n)|z|^n.$$

If $r = 0$ this method fails. We then assume $v \neq 0$ and divide both sides of (3.8) by $|v|$. The resulting inequality compares continuous functions of v and therefore is valid even if $v = 0$.

To modify (3.9) we return to (3.18) and note that

$$\begin{aligned}(|v|,m_1)&\cdots|(v,r)(v+r+1,m_i-r-1)|\cdots(|v|,m_k) \\ &\leqq |(v,r)|(M,m_1)\cdots(M,m_i-r-1)\cdots(M,m_k) \\ &\leqq |(v,r)|(M,n-r-1).\end{aligned}$$

Thus the right side of (3.18) does not exceed $|(v,r)|(M,n-r-1)\|z\|^n$.

The proofs for (3.10) and (3.11) are entirely similar.

4. Gegenbauer polynomials. Equation (3.3) leads at once to the generating relation

$$\prod_{i=1}^{k}(1-tz_i)^{-b_i} = \sum_{n=0}^{\infty} t^n\frac{(c,n)}{n!}R_n(b,z), \qquad |t|\cdot|z| < 1. \tag{4.1}$$

Putting $b = (v, v)$ and $z = (e^{i\theta}, e^{-i\theta})$ gives the left side the value $(1 - 2t\cos\theta + t^2)^{-v}$, which is the generating function of the Gegenbauer polynomials. Thus,

$$C_n^v(\cos\theta) = \frac{(2v,n)}{n!}R_n(v,v;e^{i\theta},e^{-i\theta}), \qquad n \in \mathbb{N}. \tag{4.2}$$

This is a polynomial in v as well as in $\cos\theta$. Inequalities (3.8) and (3.9) yield, for all complex v and θ,

$$|C_n^v(\cos\theta)| \leqq \frac{(2|v|, n)}{n!} e^{n|\mathrm{Im}\,\theta|}, \tag{4.3}$$

$$|C_n^v(\cos\theta)| \leqq \frac{(|v|, n)}{n!}[2\cosh(\mathrm{Im}\,\theta)]^n. \tag{4.4}$$

The first inequality is ordinarily the more useful and is always the sharper when θ is real or n is sufficiently large. In particular it gives for Legendre polynomials ($v = \frac{1}{2}$) the famous inequality $|P_n(\cos\theta)| \leqq 1$ for real θ. However, the second inequality is the sharper if $|\mathrm{Im}\,\theta|$ is large but n is not.

From (2.7) we obtain the additional inequality

$$|C_n^v(\cos\theta)| \leqq \frac{|(2v, n)| B(\mathrm{Re}\,v, \mathrm{Re}\,v)}{n!\,|B(v, v)|} e^{n|\mathrm{Im}\,\theta|}, \qquad \mathrm{Re}\,v > 0. \tag{4.5}$$

This is sharper than (4.3) if $\mathrm{Im}\,v \neq 0$ and n is large.

We now return to (4.1) and prove a result needed in the next section.

LEMMA 4.1. *Let* $m \in \mathbb{N}$, $v \in \mathbb{C}$, $x \in \mathbb{C}$. *Then*

$$R_{2m+1}(v, v; x, -x) = 0, \tag{4.6}$$

$$(v + \tfrac{1}{2}, m) R_{2m}(v, v; x, -x) = (\tfrac{1}{2}, m) x^{2m}. \tag{4.7}$$

Proof. In (4.1) put $b = (v, v)$ and $z = (x, -x)$ to obtain

$$(1 - t^2x^2)^{-v} = \sum_{n=0}^{\infty} t^n \frac{(2v, n)}{n!} R_n(v, v; x, -x).$$

The binomial series of the left side is

$$(1 - t^2x^2)^{-v} = \sum_{m=0}^{\infty} t^{2m} \frac{(v, m)}{m!} x^{2m}.$$

Comparison of coefficients of t^n in these two series leads to (4.6) and (4.7) with the help of the identity

$$(2v, 2m) = 2^{2m}(v, m)(v + \tfrac{1}{2}, m). \tag{4.8}$$

5. Jacobi polynomials. The representation [6, p. 68]

$$P_n^{(\alpha,\delta)}(x) = 2^{-n} \sum_{m=0}^{n} \binom{\alpha + n}{m} \binom{\delta + n}{n - m} (x + 1)^m (x - 1)^{n-m} \tag{5.1}$$

shows, by comparison with (3.3), that

$$P_n^{(\alpha,\delta)}(x) = \frac{(1 + \alpha + \delta + n, n)}{2^n n!} R_n(-\alpha - n, -\delta - n; x + 1, x - 1), \qquad n \in \mathbb{N}. \tag{5.2}$$

Both sides are polynomials in α, δ and x. If $\alpha = \delta$, a quadratic transformation of

R_n puts it in a form comparable to the right side of (4.2):

$$
\begin{aligned}
P_n^{(\alpha,\alpha)}(\cos\theta) &= \frac{(1+\alpha,n)}{n!} R_n(\tfrac{1}{2}+\alpha, \tfrac{1}{2}+\alpha; e^{i\theta}, e^{-i\theta}) \\
&= \frac{(1+\alpha,n)}{(1+2\alpha,n)} C_n^{1/2+\alpha}(\cos\theta).
\end{aligned} \tag{5.3}
$$

Applied to (5.2), the inequalities of § 3 do not give results of the desired type (1.2). However, (5.3) and (4.3) yield

$$
|P_n^{(\alpha,\alpha)}(\cos\theta)| \leqq \frac{|(1+\alpha,n)|(|1+2\alpha|,n)}{|(1+2\alpha,n)|n!} e^{n|\operatorname{Im}\theta|}, \quad (1+2\alpha,n) \neq 0, \tag{5.4}
$$

and this is of type (1.2). Since the left side is continuous in α and the right side approaches a finite limit as α tends to $-\frac{1}{2}$ or a negative integer, we can obtain an inequality of type (1.2) in a neighborhood of any such point. In particular, in a region containing the points $-\frac{1}{2}$ and -1, we find by cancelling factors in the numerator and denominator that

$$
|P_n^{(\alpha,\alpha)}(\cos\theta)| \leqq \frac{|(2+\alpha,n-1)|(|1+2\alpha|+1,n-1)}{2|(3+2\alpha,n-2)|n!} e^{n|\operatorname{Im}\theta|}, \qquad n \geqq 2, \quad (3+2\alpha,n-2) \neq 0. \tag{5.5}
$$

On the other hand, (5.4) is useless if $\frac{1}{2}+\alpha = -r$, where r is a positive integer with $2r < n$. At such a point (and for better estimates nearby) we use (5.3) and Theorem 3.4 to obtain

$$
|P_n^{(\alpha,\alpha)}(\cos\theta)| \leqq \frac{(|1+2\alpha|,n)|(1+\alpha,n)|}{2|(1+2\alpha,2r)(2+2\alpha+2r,n-2r-1)|n!} e^{n|\operatorname{Im}\theta|}, \tag{5.6}
$$

where $r-1 \in \mathbb{N}$, $n > 2r$, and $|\frac{1}{2}+\alpha+r| < \frac{1}{2}$.

For general values of α and δ, we shall use instead of (5.3) a new representation of the Jacobi polynomial as an $\mathscr{R}$-polynomial.

LEMMA 5.1. *Let* $n \in \mathbb{N}$, $b \in \mathbb{C}^k$, $z \in \mathbb{C}^k$, $\beta \in \mathbb{C}^\kappa$, $\zeta \in \mathbb{C}^\kappa$. *Define* $c = \sum_{i=1}^k b_i$, $\gamma = \sum_{j=1}^\kappa \beta_j$, *and assume* $(c,n)(\gamma,n) \neq 0$. *Define* $Z_{ij} = z_i + \zeta_j$ *for all* $i = 1, \cdots, k$ *and* $j = 1, \cdots, \kappa$. *Then*

$$
\mathscr{R}_n(b, Z, \beta) = \sum_{m=0}^{n} \binom{n}{m} R_{n-m}(b,z) R_m(\beta,\zeta). \tag{5.7}
$$

Proof. Since both sides are rational functions of the components of b and β, it suffices to prove (5.7) when $b \in \mathbb{C}^k_>$ and $\beta \in \mathbb{C}^\kappa_>$. In (2.4) substitute $u \cdot Z \cdot v = u \cdot z + v \cdot \zeta$, apply the binomial theorem, and integrate using (2.3).

LEMMA 5.2. *Let* $n \in \mathbb{N}$, $(\alpha,\delta) \in \mathbb{C}^2$, $(x,y) \in \mathbb{C}^2$. *Define* $Z = \begin{pmatrix} x+y & -x+y \\ x-y & -x-y \end{pmatrix}$. *Then*

$$
\mathscr{R}_{2n+1}(\tfrac{1}{2}+\alpha, \tfrac{1}{2}+\alpha; Z; \tfrac{1}{2}+\delta, \tfrac{1}{2}+\delta) = 0, \tag{5.8}
$$

$$
\begin{aligned}
&(1+\alpha,n)(1+\delta,n)\mathscr{R}_{2n}(\tfrac{1}{2}+\alpha, \tfrac{1}{2}+\alpha; Z; \tfrac{1}{2}+\delta, \tfrac{1}{2}+\delta) \\
&\quad = (\tfrac{1}{2},n)(1+\alpha+\delta+n,n) R_n(-\alpha-n, -\delta-n; x^2, y^2).
\end{aligned} \tag{5.9}
$$

Proof. Since the right sides of both equations are polynomials in α and δ, we may suppose $(1 + 2\alpha, 2n)(1 + 2\delta, 2n) \neq 0$. For every $p \in \mathbb{N}$, (5.7) implies

$$\begin{aligned}
&\mathscr{R}_p(\tfrac{1}{2} + \alpha, \tfrac{1}{2} + \alpha; Z; \tfrac{1}{2} + \delta, \tfrac{1}{2} + \delta) \\
&\quad = \sum_{r=0}^{p} \binom{p}{r} R_{p-r}(\tfrac{1}{2} + \alpha, \tfrac{1}{2} + \alpha; y, -y) R_r(\tfrac{1}{2} + \delta, \tfrac{1}{2} + \delta; x, -x).
\end{aligned}$$

If p is odd, the right side vanishes by (4.6). If $p = 2n$ we may put $r = 2m$ and use (4.7). The right side becomes

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{2n}{2m} \frac{(\tfrac{1}{2}, n-m)(\tfrac{1}{2}, m)}{(1+\alpha, n-m)(1+\delta, m)} x^{2m} y^{2n-2m} \\
&\quad = \frac{2^{-2n}(2n)!(-1)^n}{(1+\alpha, n)(1+\delta, n)} \sum_{m=0}^{n} \frac{(-\alpha - n, m)(-\delta - n, n-m)}{m!(n-m)!} x^{2m} y^{2n-2m} \\
&\quad = \frac{(\tfrac{1}{2}, n)(1 + \alpha + \delta + n, n)}{(1+\alpha, n)(1+\delta, n)} R_n(-\alpha - n, -\delta - n; x^2, y^2).
\end{aligned}$$

COROLLARY 5.3. *Let* $n \in \mathbb{N}$, $(\alpha, \delta) \in \mathbb{C}^2$, $\theta \in \mathbb{C}$. *Define*

$$Z = \begin{pmatrix} e^{i\theta/2} & -e^{-i\theta/2} \\ e^{-i\theta/2} & -e^{i\theta/2} \end{pmatrix}.$$

Then

$$P_n^{(\alpha,\delta)}(\cos\theta) = \frac{(1+\alpha, n)(1+\delta, n)}{(\tfrac{1}{2}, n)n!} \mathscr{R}_{2n}(\tfrac{1}{2} + \alpha, \tfrac{1}{2} + \alpha; Z; \tfrac{1}{2} + \delta, \tfrac{1}{2} + \delta). \tag{5.10}$$

Proof. Put $x = \cos(\theta/2)$, $y = i\sin(\theta/2)$ in (5.9) and use (5.2), which completes the proof.

If $\operatorname{Re}\alpha > -\tfrac{1}{2}$ and $\operatorname{Re}\delta > -\tfrac{1}{2}$, we may combine (5.10) and (2.4) to obtain a representation of the Jacobi polynomial by a double integral. Simple changes of the integration variables transform this into a double integral found by Braaksma and Meulenbeld [1, (2.3)]. On the other hand, for unrestricted α and δ, (5.10) and (3.4) yield the apparently new representation

$$\begin{aligned}
P_n^{(\alpha,\delta)}(\cos\theta) &= \frac{1}{2^{2n}(\tfrac{1}{2} + \alpha, n)(\tfrac{1}{2} + \delta, n)} \\
&\quad \cdot \sum (-1)^{q+s} \frac{(\tfrac{1}{2} + \alpha, p+q)(\tfrac{1}{2} + \alpha, r+s)(\tfrac{1}{2} + \delta, p+r)(\tfrac{1}{2} + \delta, q+s)}{p!q!r!s!} \\
&\quad \cdot \exp[i(p - q - r + s)\theta/2],
\end{aligned} \tag{5.11}$$

where the summation extends over all nonnegative integers p, q, r, s whose sum is $2n$.

THEOREM 5.4. *Let* $n \in \mathbb{N}$, $(\alpha, \delta) \in \mathbb{C}^2$, $\theta \in \mathbb{C}$, *and assume* $(\tfrac{1}{2} + \alpha, n)(\tfrac{1}{2} + \delta, n) \neq 0$. *Then*

$$|P_n^{(\alpha,\delta)}(\cos\theta)| \leqq f_n(\alpha, \delta)\, e^{n|\operatorname{Im}\theta|}, \tag{5.12}$$

where

$$f_n(\alpha, \delta) = \frac{(|1 + 2\alpha|, 2n)(|1 + 2\delta|, 2n)}{2^{2n}(2n)!\,|(\frac{1}{2} + \alpha, n)(\frac{1}{2} + \delta, n)|}. \tag{5.13}$$

Proof. Apply (3.10) to (5.10) and use (4.8), which completes the proof.

The function f_n defined by (5.13) satisfies (1.2b), as one can verify with the help of [5, (1.18(4))]. However, (5.4) is a sharper result if $\alpha = \delta$ and n is large. The singularity of f_n when α or δ is $-\frac{1}{2}$ is removable. For example,

$$\lim_{\delta \to -1/2} \frac{(|1 + 2\delta|, 2n)}{|(\frac{1}{2} + \delta, n)|} = \frac{(2n)!}{n!},$$

and hence,

$$|P_n^{(\alpha, -1/2)}(\cos \theta)| \leqq \frac{(|1 + 2\alpha|, 2n)}{2^{2n} n!\,|(\frac{1}{2} + \alpha, n)|} e^{n|\mathrm{Im}\,\theta|}, \qquad (\tfrac{1}{2} + \alpha, n) \neq 0. \tag{5.14}$$

The result for $P_n^{(-1/2, \delta)}$ is similar. The inequality

$$|P_n^{(-1/2, -1/2)}(\cos \theta)| \leqq \frac{(\frac{1}{2}, n)}{n!} e^{n|\mathrm{Im}\,\theta|} \tag{5.15}$$

is obtained by taking the limit of either (5.14) or (5.4). It is equivalent to the elementary inequality $|\cos n\theta| \leqq \exp(n|\mathrm{Im}\,\theta|)$.

If α or δ is $-\frac{3}{2}, -\frac{5}{2}, \cdots$, the right side of (5.13) may be infinite, but the following theorem gives inequalities of type (1.2), thereby completing the proof of Theorem 1.1.

THEOREM 5.5. *Let* $n \in \mathbb{N}$, $(\alpha, \delta) \in \mathbb{C}^2$, $\theta \in \mathbb{C}$, $r - 1 \in \mathbb{N}$, $r' - 1 \in \mathbb{N}$. *Assume* $r < n$ *and* $r' < n$. *Then* (5.12) *remains valid in each of three cases*:

(i) *If* $|1 + 2\alpha + 2r| < 1$, *replace*

$$\frac{1}{2^{2n}|(\frac{1}{2} + \alpha, n)|} \quad by \quad \frac{|(1 + \alpha, n)|}{2|(1 + 2\alpha, 2r)(2 + 2\alpha + 2r, 2n - 2r - 1)|} \tag{5.16}$$

on the right side of (5.13) *if* $(\frac{1}{2} + \delta, n) \neq 0$ *or* (5.14) *if* $\delta = -\frac{1}{2}$.

(ii) *If* $|1 + 2\delta + 2r'| < 1$, *replace*

$$\frac{1}{2^{2n}|(\frac{1}{2} + \delta, n)|} \quad by \quad \frac{|(1 + \delta, n)|}{2|(1 + 2\delta, 2r')(2 + 2\delta + 2r', 2n - 2r' - 1)|} \tag{5.17}$$

on the right side of (5.13) *if* $(\frac{1}{2} + \alpha, n) \neq 0$ *or the analogue of* (5.14) *if* $\alpha = -\frac{1}{2}$.

(iii) *If* $|1 + 2\alpha + 2r| < 1$ *and* $|1 + 2\delta + 2r'| < 1$, *make both replacements* (5.16) *and* (5.17) *on the right side of* (5.13).

Proof. Apply Theorem 3.4 to (5.10).

We conclude with a variant of Theorem 1.1 which will be used in a subsequent paper on Jacobi series [4].

THEOREM 5.6. *Let* $(\alpha, \delta) \in \mathbb{C}^2$, $(x, y) \in \mathbb{C}^2$, $|x + y| \geqq |x - y|$. *There exists a sequence of positive numbers* $f_n(\alpha, \delta)$ *such that*

$$\begin{aligned} &|(1 + \alpha + \delta + n, n) R_n(-\alpha - n, -\delta - n; x^2, y^2)| \\ &\qquad \leqq n!\, f_n(\alpha, \delta)|x + y|^{2n}, \qquad n \in \mathbb{N}, \end{aligned} \tag{5.18a}$$

$$\lim_{n \to \infty} [f_n(\alpha, \delta)]^{1/n} = 1. \tag{5.18b}$$

Proof. Apply (3.10) and Theorem 3.4 to (5.9), which completes the proof.

The numbers $f_n(\alpha, \delta)$ are the same ones determined in Theorems 5.4 and 5.5. By (5.2), Theorem 1.1 is the case $x^2 = \cos^2(\theta/2)$, $y^2 = -\sin^2(\theta/2)$.

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AN INITIAL VALUE PROBLEM FROM SEMICONDUCTOR DEVICE THEORY*

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Abstract. A system of three quasi-linear partial differential equations is considered, as a simplified model of the transport of mobile carriers in a semiconductor device. Assuming a convenient form of the boundary conditions, it is shown that the initial value problem is well-posed, and that the steady state solution is unique and stable. A finite difference approximation preserving reasonable bounds on the numerical solutions is also described.

1. Introduction. Let D be an open bounded connected region in R^n, with smooth boundary ∂D; we discuss classical solutions of the initial value problem

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\nabla\psi) - R(u, v), \tag{1.1a}$$

$$\frac{\partial v}{\partial t} = \Delta v + \nabla \cdot (v\nabla\psi) - R(u, v), \tag{1.1b}$$

$$\kappa\Delta\psi = u - v - N, \qquad (x, t) \in \Omega = D \times (0, T]; \tag{1.2}$$

$$\nu \cdot \nabla u = \nu \cdot \nabla v = \nu \cdot \nabla\psi = 0, \qquad (x, t) \in \partial D \times [0, T], \tag{1.3}$$

for the three functions u, v, ψ defined in Ω, where T is a specified positive constant. In (1.1)–(1.3), κ is a positive constant, $x = (x_1, \cdots, x_n)$, N is a specified Hölder continuous (exponent α) function of $x \in D$, and ν is the unit normal vector at each point in ∂D; in the following, we take

$$R(u, v) = \frac{uv - 1}{\tau(u + v + 2)}, \tag{1.4}$$

where τ is a positive constant, although this particular form is not essential to our results. The prescribed initial data,

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in D, \tag{1.5}$$

is assumed to be twice continuously differentiable in x, to be strictly positive in D, to satisfy the boundary condition (1.3) and also the compatibility condition

$$\int_D (u_0(x) - v_0(x) - N(x))\, dx = 0. \tag{1.6}$$

By a solution, we mean a set of three functions u, v, ψ of $(x, t) \in \Omega$, twice continuously differentiable in x, and continuously differentiable in t, satisfying (1.1)–(1.5), with u, v strictly positive in Ω, and the requirement

$$\int_D \psi(x, t)\, dx = 0, \qquad t \in [0, T], \tag{1.7}$$

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which is imposed to remove the arbitrary additive constant in the solution of (1.2).

The system of equations (1.1), (1.2) is a simplified model of carrier transport in a semiconductor device [12], in which the functions u, v, ψ represent the electron and hole densities and the electrostatic potential, respectively. In obtaining (1.1)–(1.2), several simplifying assumptions have been made; these include the applicability of Boltzmann statistics, and constant and equal carrier mobilities. A system of units is employed in which the Boltzmann voltage, the electronic charge, the intrinsic carrier density and the common carrier mobility all have magnitude unity. The recombination term, given by (1.4), is a standard Hall–Shockley–Read expression [10].

Several numerical investigations of problems of this form have been reported [1], [3], [4] based on finite difference methods. In this paper we obtain some analytical results, which are intended to corroborate previous computations and possibly to aid in the development of improved numerical techniques.

In § 2 of this paper, we obtain some a priori bounds on solutions, one of which is essential in proving the existence of a solution in the large. The proofs of existence and uniqueness of a solution, and its continuous dependence on initial data, are carried out in § 3, using a continuity argument. The linearized problem near the stationary state is discussed in § 4. A specific finite difference approximation is introduced in § 5, and the applicability of our results to actual computations is discussed in § 6.

Below we present some notation. Except in § 6 where stated otherwise, all constants and functions are understood to be real-valued. However it is convenient to introduce a complex scalar product $(\cdot,\cdot)$ for functions defined in D, as

$$(1.8)\qquad (f,g)=\begin{cases}\displaystyle\int_D \bar{f}(x)g(x)\,dx, & f,g \text{ scalar functions},\\ \displaystyle\sum_{i=1}^{n}\int_D \bar{f}_i(x)g_i(x)\,dx, & f,g\ n\text{-vector-valued functions},\end{cases}$$

$f(x)=(f_1(x),\cdots,f_n(x)),\ g(x)=(g_1(x),\cdots,g_n(x))$.

We also introduce the norms

$$\|f\|^2=(f,f),\quad \|f\|_g^2=(f,gf),\quad \|f\|_1^2=(\nabla f,\nabla f),\quad \|f\|_{1,g}^2=(\nabla f,g\nabla f),$$

$$(1.9)\qquad |f|=\sup_{x\in D}|f(x)| \quad\text{or}\quad \sum_{i=1}^{n}\sup_{x\in D}|f_i(x)|$$

if f is vector-valued.

In the following, we use the symbols C, c to denote large and small positive constants depending on the domain D. When it is desired to refer to a particular such constant, a subscript is added.

2. A priori bounds. In this section we obtain a priori bounds on solutions of (1.1)–(1.6), which depend strongly on the assumed form of the boundary condition (1.3). For strictly positive initial data, it follows from the maximum principle applied to the functions $ue^{-\psi}$ and ve^{ψ}, and (1.1)–(1.3), that solution functions u, v are strictly positive in Ω. We also have the following estimates.

LEMMA 1. *As a function of t, $\|\psi\|_1$ is bounded by a constant depending on the initial data and N but not on T.*

Proof. Differentiating (1.2) with respect to t, and substituting (1.2) for $\partial u/\partial t$, $\partial v/\partial t$, we obtain

$$\kappa\nabla\frac{\partial\psi}{\partial t} = \nabla\cdot(\nabla u - u\nabla\psi) - \nabla\cdot(\nabla v + v\nabla\psi) = \nabla\cdot[\nabla(u - v) - (u + v)\nabla\psi] \tag{2.1}$$

which is a well-known equation expressing conservation of total current. Taking the scalar product of (2.1) with ψ, we integrate by parts and use (1.2), (1.3) to obtain

$$\begin{aligned}\frac{\kappa}{2}\frac{d}{dt}\|\psi\|_1^2 &= -(u + v, |\Delta\psi|^2) - (\Delta\psi, u - v)\\ &= -(u + v, |\nabla\psi|^2) - \kappa(\Delta\psi, \Delta\psi) + (\Delta\psi, N)\\ &\leqq -\|\psi\|_{1,u+v}^2 - \frac{\kappa}{2}\|\Delta\psi\|^2 + \frac{1}{2\kappa}\|N\|^2\\ &\leqq -c_0\|\psi\|_1^2 + \frac{1}{2\kappa}\|N\|^2,\end{aligned} \tag{2.2}$$

since $u + v$ is strictly positive in Ω. Thus

$$\|\psi\|_1(t) \leqq \max(\|\psi\|_1(0), \|N\|(2\kappa c_0)^{-1/2}). \tag{2.3}$$

LEMMA 2. *As functions of t, the L_p-norms (in D) of u, v, and the first and second space derivatives of ψ are less than or equal to $C(p)(1 + t)$, for all finite p.*

Proof. We define

$$a_p(t) = \frac{1}{p}\int_D (u^p(x, t) + v^p(x, t))\,dx, \quad p \geqq 1; \quad a_0(t) = 1. \tag{2.4}$$

Differentiating (2.4) with respect to time and using (1.1), we have for all $p \geqq 2$,

$$\begin{aligned}\frac{da_p(t)}{dt} &= (u^{p-1}, \Delta u - \nabla u\cdot\nabla\psi - u\Delta\psi) + (v^{p-1}, \Delta v + \nabla v\cdot\nabla\psi + v\Delta\psi)\\ &\quad - (u^{p-1} + v^{p-1}, R(u, v))\\ &\leqq -(p - 1)(u^{p-2}, |\nabla u|^2) - \frac{1}{p}(\nabla(u^p), \nabla\psi) - (u^p, \Delta\psi)\\ &\quad - (p - 1)(v^{p-2}, |\nabla v|^2) + \frac{1}{p}(\nabla(v^p), \nabla\psi) + (v^p, \Delta\psi) + pa_{p-2}/\tau,\end{aligned} \tag{2.5}$$

where we have performed integrations by parts using (1.3), and have estimated the recombination term rather crudely using (1.4) and the positivity of u, v. Collecting terms in (2.5) and using (1.2) we obtain

$$\begin{aligned}\frac{da_p(t)}{dt} &+ (p - 1)((u^{p-2}, |\nabla u|^2) + (v^{p-2}, |\nabla v|^2)) - \frac{pa_{p-2}}{\tau}\\ &\leqq -\frac{1}{p}(\nabla(u^p - v^p), \nabla\psi) - (u^p - v^p, \Delta\psi) \qquad (cont.)\end{aligned}$$

$$(2.6)\qquad = \left(1 - \frac{1}{p}\right)(u^p - v^p, \Delta\psi) = -\frac{p-1}{\kappa p}(u^p - v^p, u - v - N)$$

$$= -\frac{p-1}{\kappa p}(u^{p-1} + u^{p-2}v + \cdots + v^{p-1}, (u-v)^2 - (u-v)N])$$

$$\leqq \frac{p(p-1)}{4\kappa}|N|^2 a_{p-1}(t) \equiv b_p a_{p-1}(t).$$

For $p \geqq 2$, there exist positive constants $c(p)$, $C(p)$ such that for any positive function u satisfying (1.3),

$$(2.7)\qquad (p-1)(u^{p-2}, |\nabla u|^2) \geqq \frac{c(p)}{p}\int_D u^p\,dx - C(p)\left(\int_D u\,dx\right)^p.$$

Combining (2.6), (2.7) we obtain

$$(2.8)\qquad \frac{da_p(t)}{dt} \leqq -c(p)a_p(t) + b_p a_{p-1}(t) + \frac{p}{\tau}a_{p-2}(t) + C(p)a_1^p(t).$$

From (2.8), we can infer bounds on $a_p(t)$ inductively from a bound on $a_1(t)$. From (2.4), using (1.1), (1.3), (1.4) we have

$$(2.9)\qquad a_1(t) \leqq a_1(0) + t/\tau,$$

so that

$$(2.10)\qquad a_p(t) \leqq C(p)(1 + t^p)$$

which is our desired result. We remark that in the case of no recombination, $R(u, v) = 0$, we have $a_1(t) = a_1(0)$, so that the $a_p(t)$ are bounded independently of t.

3. Existence–uniqueness theory. In this section we prove the existence and uniqueness of a solution of (1.1)–(1.7), and its continuous dependence on the initial data. The proofs are based on a continuity argument, and depend on Lemma 2, for a sufficiently large value of p depending on the number of space variables n. We also use an estimate for the Green's function $G(x, t; y, s)$ of the heat operator in Ω, with respect to the boundary condition (1.3) (see [2, p. 134])

$$(3.1)\quad |G(x,t;y,s)| + \sum_{i=1}^{n}\left|\frac{\partial}{\partial x_i}G(x,t;y,s)\right| \leqq C|t-s|^{-\mu}|x-y|^{-(n+1-2\mu)},\ \tfrac{1}{2} < \mu < 1,$$

and the following lemmas on the equivalence of solutions of (1.1)–(1.7) and a corresponding system of integral equations, and on the magnitude of the minor terms in (1.1).

LEMMA 3. *Suppose there are three functions u, v, ψ, defined in $\bar{\Omega}$, with their first space derivatives continuous in x, and uniformly bounded in Ω; suppose u, v, ψ satisfy* (1.3), (1.7) *and the relations*

$$(3.2a)\qquad u(x,t) = -\int_{D\times(0,t)} G(x,t;y,s)r_1(y,s)\,dy\,ds + \int_D G(x,t;y,0)f(y)\,dy,$$

$$(3.2b)\qquad v(x,t) = -\int_{D\times(0,t)} G(x,t;y,s)r_2(y,s)\,dy\,ds + \int_D G(x,t;y,0)g(y)\,dy,$$

(3.3) $$\kappa\Delta\psi = u - v - N, \qquad (x,t)\in D \times (0,T],$$

where

(3.4) $$r_1 = \nabla\cdot(u\nabla\psi) + R(u,v), \qquad r_2 = -\nabla\cdot(v\nabla\psi) + R(u,v),$$

and f, g satisfy the requirements given above on the initial data. Then u, v, ψ are a solution of (1.1)–(1.7) *with initial data f, g.*

Proof. We apply well-known estimates for the solutions of linear elliptic and parabolic problems. From (3.3), we have the first and second space derivatives of $\psi(\cdot,t)$ Hölder continuous (exponent α) in x [5, p. 136], uniformly in t, and thus from (3.4) that r_1, r_2 are uniformly bounded in Ω. Then from (3.2), using (3.1), we have that u, v are Hölder continuous (exponent $<\frac{1}{2}$) in t, uniformly in x. Then from (3.3), the first and second space derivatives of ψ are continuous in t, so that r_1, r_2 are continuous in Ω. It then follows [2, p. 148], [6, p. 342] that u, v and their first space derivatives are Hölder continuous (exponents β, $\beta/2$, $\beta < 1$) in x, t respectively. Thus r_1, r_2 are Hölder continuous (exponent α) in x, uniformly for $t\in(0,T]$. Using the requirements on f, g, the Hölder conditions on ψ, and hence on r_1, r_2, can be extended to $t = 0$. It then follows [2, pp. 144–147] that u, v satisfy (1.1), and thus that u, v, ψ form a solution as described above.

At this point it is necessary to introduce some additional notation. Let ρ denote a time interval $(\rho_a, \rho_b]$, with $|\rho| = \rho_b - \rho_a$. In what follows, we shall repeatedly divide the interval $(0, T]$ into sufficiently small such intervals. Let f be a function of x, t defined in $D \times \rho$. Then

(3.5) $$\begin{aligned} |f(\cdot,t)| &= \sup_{x\in D}|f(x,t)|, \qquad \langle f(\cdot,t)\rangle = |f(\cdot,t)| + \sum_{i=1}^{n}\left|\frac{\partial f(\cdot,t)}{\partial x_i}\right|, \\ |f|_{\rho'} &= \sup_{\substack{x\in D\\ t\in\rho'}}|f(x,t)|, \qquad \langle\!\langle f\rangle\!\rangle_{\rho'} = |f|_{\rho'} + \sum_{i=1}^{n}\left|\frac{\partial f}{\partial x_i}\right|_{\rho'}, \end{aligned}$$

where ρ' is a time interval contained in ρ. The bilinear nature of (1.1) is reflected in the following lemma.

LEMMA 4. *Let ρ be a time interval in $[0, T]$, and suppose for $j = 1, 2$ the functions $u_j, v_j, \psi_j, f_j, g_j$ and their first space derivatives are continuous in $D \times \rho$ and satisfy*

(3.6a) $$\begin{aligned} u_j(x,t) = &-\int_{D\times(\rho_a,t)} G(x,t;y,s)[\nabla\cdot(f_j\nabla\psi_j) + R(f_j,g_j)](y,s)\,dy\,ds \\ &+ \int_D G(x,t;y,\rho_a)u_j(y,\rho_a)\,dy, \end{aligned}$$

(3.6b) $$\begin{aligned} v_j(x,t) = &-\int_{D\times(\rho_a,t)} G(x,t;y,s)[-\nabla\cdot(g_j\nabla\psi_j) + R(f_j,g_j)](y,s)\,dy\,ds \\ &+ \int_D G(x,t;y,\rho_a)v_j(y,\rho_a)\,dy, \end{aligned}$$

(3.7) $$\kappa\Delta\psi_j = f_j - g_j - N, \quad x\in D, \quad \nu\cdot\nabla\psi_j = 0, \quad x\in\partial D,$$

for all $(x,t)\in D\times\rho$. Then for any positive $\beta < \frac{1}{2}$ there exists a positive constant

$C_1(\beta)$ *such that for all* $t \in \rho$,

$$\begin{aligned}
(3.8)\qquad &\langle\!\langle u_1 - u_2 \rangle\!\rangle_{[\rho_a,t]} + \langle\!\langle v_1 - v_2 \rangle\!\rangle_{[\rho_a,t]} \leqq \langle u_1(\cdot,\rho_a) - u_2(\cdot,\rho_a)\rangle \\
&\quad + \langle v_1(\cdot,\rho_a) - v_2(\cdot,\rho_a)\rangle + C_1(\beta)(t-\rho_a)^\beta(\langle\!\langle f_1 - f_2 \rangle\!\rangle_{[\rho_a,t]} \\
&\quad + \langle\!\langle g_1 - g_2 \rangle\!\rangle_{[\rho_a,t]})[1 + \langle\!\langle f_1 \rangle\!\rangle_{[\rho_a,t]} + \langle\!\langle f_2 \rangle\!\rangle_{[\rho_a,t]} + \langle\!\langle g_1 \rangle\!\rangle_{[\rho_a,t]}] \\
&\quad + \langle\!\langle g_2 \rangle\!\rangle_{[\rho_a,t]}.
\end{aligned}$$

Proof. Subtracting (3.6), (3.7) with $j = 2$ from the equations with $j = 1$, it is sufficient to estimate the differences of the factors in brackets in (3.6), in the maximum norm; the result (3.8) then follows immediately from (3.1). The recombination terms are easily estimated, using (1.4); for the other terms, we have for all $x, t \in D \times \rho$,

$$\begin{aligned}
|\nabla\cdot(f_1\nabla\psi_1) - \nabla\cdot(f_2\nabla\psi_2)| &\leqq |\nabla f_1\cdot\nabla\psi_1 - \nabla f_2\cdot\nabla\psi_2| + |f_1\Delta\psi_1 - f_2\Delta\psi_2| \\
&\leqq |\nabla f_1||\nabla(\psi_1 - \psi_2)| + |\nabla(f_1 - f_2)||\nabla\psi_2| \\
(3.9)\qquad &\quad + |f_1 - f_2||\Delta\psi_1| + |f_2||\Delta(\psi_1 - \psi_2)| \\
&\leqq C[\langle f_1(\cdot,t)\rangle + \langle f_2(\cdot,t)\rangle + \langle g_1(\cdot,t)\rangle + \langle g_2(\cdot,t)\rangle] \\
&\quad \cdot[\langle f_1(\cdot,t) - f_2(\cdot,t)\rangle + \langle g_1(\cdot,t) - g_2(\cdot,t)\rangle],
\end{aligned}$$

where we have also used the Schauder estimates for the derivatives of $\psi_1 - \psi_2$. An entirely similar result holds for the terms from (3.6b) and establishes the desired result.

Our existence–uniqueness results are described by the following two theorems.

THEOREM 1. *For* $j = 1, 2$, *let* u_j, v_j, ψ_j *be two solutions of* (1.1)–(1.7) *with initial data* $u_{j,0}, v_{j,0}$. *Then there exists a constant C, depending on T and the initial data, such that*

$$(3.10)\qquad \langle\!\langle u_1 - u_2 \rangle\!\rangle_{[0,T]} + \langle\!\langle v_1 - v_2 \rangle\!\rangle_{[0,T]} \leqq C[\langle u_{1,0} - u_{2,0}\rangle + \langle v_{1,0} - v_{2,0}\rangle].$$

In particular, a solution corresponding to specified initial data is unique.

Proof. We divide the interval $(0, T]$ into intervals of length $|\rho|$, and apply Lemma 4 in each interval, with β fixed. Since u_j, v_j, ψ_j are solutions of (1.1)–(1.3) we have for all $t \in \rho$, from (3.8),

$$\begin{aligned}
&\langle u_1(\cdot,t) - u_2(\cdot,t)\rangle + \langle v_1(\cdot,t) - v_2(\cdot,t)\rangle \\
&\quad \leqq \langle\!\langle u_1 - u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 - v_2 \rangle\!\rangle_\rho \\
&\quad \leqq \langle u_1(\cdot,\rho_a) - u_2(\cdot,\rho_a)\rangle + \langle v_1(\cdot,\rho_a) - v_2(\cdot,\rho_a)\rangle \\
(3.11)\qquad &\qquad + C_1(t-\rho_a)^\beta[\langle\!\langle u_1 - u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 + v_2 \rangle\!\rangle_\rho] \\
&\qquad \cdot[1 + \langle\!\langle u_1 \rangle\!\rangle_\rho + \langle\!\langle u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 \rangle\!\rangle_\rho + \langle\!\langle v_2 \rangle\!\rangle_\rho] \\
&\quad \leqq \langle u_1(\cdot,\rho_a) - u_2(\cdot,\rho_a)\rangle + \langle v_1(\cdot,\rho_a) - v_2(\cdot,\rho_a)\rangle \\
&\qquad + C_1|\rho|^\beta[\langle\!\langle u_1 - u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 - v_2 \rangle\!\rangle_\rho] \\
&\qquad \cdot[1 + 2\langle\!\langle u_1 \rangle\!\rangle_\rho + 2\langle\!\langle v_1 \rangle\!\rangle_\rho + \langle\!\langle u_1 - u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 - v_2 \rangle\!\rangle_\rho].
\end{aligned}$$

Let $1 + \langle\!\langle u_1 \rangle\!\rangle_{[0,T]} + \langle\!\langle v_1 \rangle\!\rangle_{[0,T]} = C_2$, and pick $|\rho|$ so that $2C_1C_2|\rho|^\beta = \frac{1}{2}$. Then if

$$\langle\!\langle u_1 - u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 - v_2 \rangle\!\rangle_\rho \leqq 1, \tag{3.12}$$

we can infer from (3.11) that

$$\begin{aligned} \langle\!\langle u_1 - u_2 \rangle\!\rangle_\rho + \langle\!\langle v_1 - v_2 \rangle\!\rangle_\rho \leqq {} & 2\langle u_1(\cdot, \rho_a) - u_2(\cdot, \rho_a)\rangle \\ & + 2\langle v_1(\cdot, \rho_a) - v_2(\cdot, \rho_a)\rangle, \end{aligned} \tag{3.13}$$

since $\langle u_1(\cdot, t) - u_2(\cdot, t)\rangle + \langle v_1(\cdot, t) - v_2(\cdot, t)\rangle$ is a continuous function of t. We choose $\langle u_{1,0} - u_{2,0}\rangle + \langle v_{1,0} - v_{2,0}\rangle \leqq 2^{-T/|\rho|}$; then applying (3.13) successively to each time interval, noting that (3.12) remains satisfied, we obtain

$$\langle\!\langle u_1 - u_2 \rangle\!\rangle_{[0,T]} + \langle\!\langle v_1 - v_2 \rangle\!\rangle_{[0,T]} \leqq 2^{T/|\rho|}(\langle u_{1,0} - u_{2,0}\rangle + \langle v_{1,0} - v_{2,0}\rangle) \tag{3.14}$$

which is our desired result.

THEOREM 2. *The problem* (1.1)–(1.7) *as described above possesses a solution.*

Proof. Let z be the function of x in $\bar{D}$ satisfying

$$\kappa\Delta z = e^z - e^{-z} - N, \quad x \in D; \quad \nu \cdot \nabla z = 0, \quad x \in \partial D. \tag{3.15}$$

It is trivially verified that $u(x, t) = e^{z(x)}$, $v(x, t) = e^{-z(x)}$, $\psi(x, t) = z(x) + C$ satisfy (1.1)–(1.4), i.e., z defines a steady state solution. Let H be the subset of $[0, 1]$ such that for all $h \in H$, the problem (1.1)–(1.4) possesses a solution with initial data

$$u_h(\cdot, 0) = (1 - h)\, e^z + hu_0, \qquad v_h(\cdot, 0) = (1 - h)\, e^{-z} + hv_0, \tag{3.16}$$

where u_0, v_0 are the prescribed initial data. From (1.6), (3.15), (3.16) it follows that for all $h \in [0, 1]$ the initial data u_h, v_h are strictly positive in D and satisfy the compatibility condition

$$\int_D (u_h(x, 0) - v_h(x, 0) - N(x))\, dx = 0. \tag{3.17}$$

The set H is not empty, containing $h = 0$.

We next show that H is open. Given a solution u_h, v_h, ψ_h, we show the existence of a solution in $u_{h+\delta}, v_{h+\delta}, \psi_{h+\delta}$ for sufficiently small positive δ. It suffices to construct such a solution in a small time interval $\rho = (\rho_a, \rho_a + |\rho|]$ provided $|\rho|$ is bounded away from zero for all $t \in (0, T]$. Let $u_{h+\delta}, v_{h+\delta}, \psi_{h+\delta}$ be the limit of a sequence $\{u^{(m)}, v^{(m)}, \psi^{(m)}\}$, defined by

$$\begin{aligned} u^{(m+1)}(x, t) = {} & -\int_{D \times (\rho_a, t)} G(x, t; y, s)[\nabla \cdot (u^{(m)}\nabla\psi^{(m)}) + R(u^{(m)}, v^{(m)})]\, dy\, ds \\ & + \int_D G(x, t; y, 0) u_{h+\delta}(y, \rho_a)\, dy, \end{aligned} \tag{3.18}$$

$$\begin{aligned} v^{(m+1)}(x, t) = {} & -\int_{D \times (\rho_a, t)} G(x, t; y, s)[-\nabla \cdot (v^{(m)}\nabla\psi^{(m)}) + R(u^{(m)}, v^{(m)})]\, dy\, ds \\ & + \int_D G(x, t; y, 0) v_{h+\delta}(y, \rho_a)\, dy, \end{aligned} \tag{3.18b}$$

$$\kappa\Delta\psi^{(m)} = u^{(m)} - v^{(m)} - N, \quad x \in D, \quad \nu \cdot \nabla\psi^{(m)} = 0, \quad x \in \partial D, \tag{3.19}$$

for all $x, t \in D \times \rho$; with $u^{(0)} = u_h, v^{(0)} = v_h$. Let $\langle\!\langle u_h \rangle\!\rangle_{[0,T]} + \langle\!\langle v_h \rangle\!\rangle_{[0,T]} + 1 = C_3$. From Lemma 4, we have

$$
(3.20)\qquad
\begin{aligned}
&\langle\!\langle u^{(m+1)} - u^{(m)} \rangle\!\rangle_\rho + \langle\!\langle v^{(m+1)} - v^{(m)} \rangle\!\rangle_\rho \\
&\quad < C_1|\rho|^\beta(\langle\!\langle u^{(m)} - u^{(m-1)} \rangle\!\rangle_\rho + \langle\!\langle v^{(m)} - v^{(m-1)} \rangle\!\rangle_\rho) \\
&\quad \cdot 2\left[C_3 + \sum_{i=1}^{m} (\langle\!\langle u^{(i)} - u^{(i-1)} \rangle\!\rangle_\rho + \langle\!\langle v^{(i)} - v^{(i-1)} \rangle\!\rangle_\rho)\right], \qquad m \geqq 1, \\
&\langle\!\langle u^{(1)} - u^{(0)} \rangle\!\rangle_\rho + \langle\!\langle v^{(1)} - v^{(0)} \rangle\!\rangle_\rho \leqq \langle u_{h+\delta}(\cdot, \rho_a) - u_h(\cdot, \rho_a) \rangle \\
&\qquad\qquad + \langle v_{h+\delta}(\cdot, \rho_a) - v_h(\cdot, \rho_a) \rangle.
\end{aligned}
$$

By the same argument used to prove Theorem 1, it follows from (3.20) that if we take $|\rho|$, δ sufficiently small that

$$
(3.21)\qquad
\begin{aligned}
&2C_1(C_3 + 1)|\rho|^\beta \leqq \tfrac{1}{2}, \\
&\langle u_{h+\delta}(\cdot, 0) - u_h(\cdot, 0) \rangle + \langle v_{h+\delta}(\cdot, 0) - v_h(\cdot, 0) \rangle \leqq 2^{-(1+T/|\rho|)},
\end{aligned}
$$

then the iteration (3.19) converges in $\langle\!\langle \cdot \rangle\!\rangle_\rho$ in each interval and hence in $\langle\!\langle \cdot \rangle\!\rangle_{[0,T]}$. Thus the limit functions $u_{h+\delta}, v_{h+\delta}$ and their first space derivatives are continuous in Ω; a limit function $\psi_{h+\delta}$ exists, and $u_{h+\delta}, v_{h+\delta}, \psi_{h+\delta}$ satisfy (3.2)–(3.4). It then follows from Lemma 3 that $u_{h+\delta}, v_{h+\delta}, \psi_{h+\delta}$ are the defined solution of (1.1)–(1.4).

The proof of Theorem 2 is completed by showing that H is closed.

LEMMA 5. *Let* $\{u^{(j)}, v^{(j)}, \psi^{(j)}\}$ *be a sequence of solutions of* (1.1)–(1.4) *whose initial data converge in* $\langle \cdot \rangle$. *Then* $u^{(j)}, v^{(j)}, \psi^{(j)}$ *converge in* $\langle\!\langle \cdot \rangle\!\rangle_{[0,T]}$ *to limit functions* u, v, ψ *satisfying* (1.1)–(1.4).

Proof. Again it suffices to consider a small time interval, denoted by ρ. From (1.1)–(1.3), we have, performing an integration by parts,

$$
(3.22)\qquad
\begin{aligned}
u^{(j)}(x,t) - u^{(k)}(x,t) = &\int_{D\times(\rho_a,t)} \nabla_y G(x,t;y,s)[u^{(j)}\nabla\psi^{(j)} - u^{(k)}\nabla\psi^{(k)}](y,s)\,dy\,ds \\
&- \int_{D\times(\rho_a,t)} G(x,t;y,s)[R(u^{(j)}, v^{(j)}) - R(u^{(k)}, v^{(k)})](y,s)\,dy\,ds \\
&+ \int_D G(x,t;y,\rho_a)(u^{(j)}(y,\rho_a) - u^{(k)}(y,\rho_a))\,dy,
\end{aligned}
$$

similarly for v, and

$$
(3.23)\qquad
\begin{aligned}
&\kappa\Delta(\psi^{(j)} - \psi^{(k)}) = u^{(j)} - u^{(k)} - v^{(j)} + v^{(k)}, \quad && x \in D, \\
&\nu \cdot \nabla(\psi^{(j)} - \psi^{(k)}) = 0, && x \in \partial D.
\end{aligned}
$$

From (3.22), using (1.4), (3.1), (3.5) we have ($\beta < \frac{1}{2}$)

$$
\begin{aligned}
|u^{(j)} - u^{(k)}|_\rho &\leqq |u^{(j)}(\cdot, \rho_a) - u^{(k)}(\cdot, \rho_a)| + C|\rho|^\beta(|u^{(j)} - u^{(k)}|_\rho + |v^{(j)} - v^{(k)}|_\rho) \\
&\quad + \int_{D\times\rho} |\nabla_y G(x,t;y,s)|\,[|u^{(j)}(y,s)|\,|\nabla(\psi^{(j)} - \psi^{(k)})(y,s)| \\
&\qquad + |(u^{(j)} - u^{(k)})(y,s)|\,|\nabla\psi^{(k)}(y,s)|]\,dy\,ds \\
&\leqq |u^{(j)}(\cdot, \rho_a) - u^{(k)}(\cdot, \rho_a)| + C|\rho|^\beta(|u^{(j)} - u^{(k)}|_\rho + |v^{(j)} - v^{(k)}|_\rho) \\
&\quad + C(|u^{(j)} - u^{(k)}|_\rho + |v^{(j)} - v^{(k)}|_\rho) \int_{D\times\rho} |\nabla_y G(x,t;y,s)|(|u^{(j)}(y,s)| \\
&\qquad + |\nabla\psi^{(k)}(y,s)|)\,dy\,ds.
\end{aligned}
\tag{3.24}
$$

We now estimate the y-integral in the last term in (3.24) by Hölder's inequality, using Lemma 2 for L_p-bounds of $|u^{(j)}|$ and $|\nabla\psi^{(k)}|$ for sufficiently large p; such bounds depend on T and the initial data (h) but not on j, k. The s-integral is then absolutely convergent, and we obtain, using (3.1),

$$
\begin{aligned}
|u^{(j)} - u^{(k)}|_\rho &\leqq |u^{(j)}(\cdot, \rho_a) - u^{(k)}(\cdot, \rho_a)| \\
&\quad + C_4(T, h, \beta)(|u^{(j)} - u^{(k)}|_\rho + |v^{(j)} - v^{(k)}|_\rho)|\rho|^\beta
\end{aligned}
\tag{3.25}
$$

and a similar expression for $|v^{(j)} - v^{(k)}|_\rho$. Picking $C_4|\rho|^\beta = \frac{1}{2}$, we find that the sequences $\{u^{(j)}\}, \{v^{(j)}\}$ are uniformly convergent in $D \times \rho$ and hence in Ω. The limit functions u, v are thus continuous in x and in t, in Ω. Then $u^{(j)}, v^{(j)}$, and the first space derivatives of $\psi^{(j)}$ are uniformly bounded in Ω, independent of j.

Let ∂ denote differentiation with respect to an arbitrary space variable, and let ρ' denote another time interval. From (1.1), (1.2) we have

$$
\begin{aligned}
|\partial u^{(j)}|_{\rho'} &\leqq |\partial u^{(j)}(\cdot, \rho'_a)| + \left|\int_{D\times(\rho'_a,t)} \partial_x G(x,t;y,s) R(u^{(j)}, v^{(j)})\,dy\,ds\right|_{\rho'} \\
&\quad + \left|\int_{D\times(\rho'_a,t)} \partial_x G(x,t;y,s)\left(\nabla u^{(j)}\cdot\nabla\psi^{(j)} + u^{(j)}\,\frac{(u^{(j)} - v^{(j)} - N)}{\kappa}\right)dy\,ds\right|_{\rho'} \\
&\leqq |\partial u^{(j)}(\cdot, \rho'_a)| + C|\rho'|^\beta + C|\rho'|^\beta \langle\!\langle u^{(j)} \rangle\!\rangle_{\rho'}
\end{aligned}
\tag{3.26}
$$

for $\beta < \frac{1}{2}$, where the C's do not depend on j or ρ'_a for $\rho'_a < T$. Taking $|\rho'|$ sufficiently small, we may infer a bound for $\langle\!\langle u^{(j)} \rangle\!\rangle_{[0,T]}$ independent of j, and $\langle\!\langle v^{(j)} \rangle\!\rangle_{[0,T]}$ similarly.

Then from (1.1)–(1.3) we have in any time interval ρ,

$$
\begin{aligned}
\partial(u^{(j)} - u^{(k)})(x,t) = &-\int_{D\times(\rho_a,t)} \partial_x G(x,t;y,s)[\nabla u^{(j)}\cdot\nabla\psi^{(j)} - \nabla u^{(k)}\cdot\nabla\psi^{(k)} \\
&+ \frac{u^{(j)}}{\kappa}(u^{(j)} - v^{(j)} - N) - \frac{u^{(k)}}{\kappa}(u^{(k)} - v^{(k)} - N) \\
&- R(u^{(j)}, v^{(j)}) + R(u^{(k)}, v^{(k)})](y,s)\,dy\,ds \\
&+ \int_D \partial_x G(x,t;y,\rho_a)(u^{(j)}(y,\rho_a) - u^{(k)}(y,\rho_a))\,dy;
\end{aligned}
\tag{3.27}
$$

separating and collecting terms, we readily obtain

$$\begin{aligned}\langle\!\langle u^{(j)} - u^{(k)} \rangle\!\rangle_\rho &\leqq \langle u^{(j)}(\cdot, \rho_a) - u^{(k)}(\cdot, \rho_a)\rangle \\ &\quad + C_5(T, \beta, \langle\!\langle u^{(\cdot)} \rangle\!\rangle_\rho, \langle\!\langle v^{(\cdot)} \rangle\!\rangle_\rho)|\rho|^\beta(\langle\!\langle u^{(j)} - u^{(k)} \rangle\!\rangle_\rho + \langle\!\langle v^{(j)} - v^{(k)} \rangle\!\rangle_\rho)\end{aligned} \tag{3.28}$$

for $\beta < \frac{1}{2}$, where C_5 does not depend on j, k. A similar relation exists for $\langle\!\langle v^{(j)} - v^{(k)} \rangle\!\rangle$. Taking $|\rho|$ sufficiently small, we have that the sequences $\{u^{(j)}\}$, $\{v^{(j)}\}$ converge in the norm $\langle\!\langle \cdot \rangle\!\rangle_{[0,T]}$.

Thus the first space derivatives of the limit functions exist and are continuous in Ω; hence a limit function ψ exists, and the limit functions u, v, ψ satisfy (3.2)–(3.4). By Lemma 3, it follows that u, v, ψ are a solution of the problem (1.1)–(1.7). This concludes the proof of Theorem 2.

4. The linearized problem. The proof of Theorem 2 introduced a steady state solution of the system (1.1)–(1.4); in this section, we discuss some of its properties. We first show the following theorem.

THEOREM 3. *The stationary solutions of* (1.1)–(1.4) *are all of the form* $u = e^z$, $v = e^{-z}$, $\psi = z + c$, *where* z *is determined from* (3.15); *in particular, if* (1.7) *is also imposed, the stationary solution is unique. This remains true if* (1.4) *is replaced by*

$$R(u, v) = (uv - 1)d, \tag{4.1}$$

where $d = d(u, v, x)$ *is a strictly positive bounded function in* D.

Proof. Dropping the time derivative terms, we rewrite (1.1) as

$$\nabla \cdot (e^\psi \nabla(e^{-\psi}u)) - (uv - 1)d = 0, \tag{4.2}$$

$$\nabla \cdot (e^{-\psi} \nabla(e^{\psi}v)) - (uv - 1)d = 0. \tag{4.3}$$

Taking the scalar product of (4.2) with $e^{-\psi}(u - 1/v)$, we obtain, integrating by parts, and using (4.3),

$$\begin{aligned}(e^{-\psi}d/v, (u - 1/v)^2) &= -(e^\psi, |\nabla(e^{-\psi}u)|^2) - (e^{-\psi}/v, \nabla \cdot (e^\psi \nabla(e^{-\psi}u))) \\ &= -(e^\psi, |\nabla(e^{-\psi}u)|^2) - (e^{-\psi}/v, \nabla \cdot (e^{-\psi} \nabla(e^{\psi}v))) \\ &= -(e^\psi, |\nabla(e^{-\psi}u)|^2) - (e^{-3\psi}v^{-2}, |\nabla(e^{\psi}v)|^2)\end{aligned} \tag{4.4}$$

which is clearly impossible unless $e^{-\psi}u$ and $e^\psi v$ are constants, and $uv = 1$. Setting $u = e^{\psi + c}$, $v = e^{-\psi - c}$ in (1.2) gives (3.15) with $\psi + c$ replaced by z. Thus if (1.7) is imposed, the arbitrary additive constant is removed, and the solution is unique.

We next linearize the system (1.1)–(1.3) about the stationary solution. Let z be the stationary electrostatic potential function satisfying (3.15); we set

$$u(x, t) = \exp(z(x) + (\omega(x) - \theta(x))\, e^{\lambda t}), \tag{4.5a}$$

$$v(x, t) = \exp(-z(x) + (\phi(x) - \omega(x))\, e^{\lambda t}), \tag{4.5b}$$

$$\psi(x, t) = z(x) + \omega(x)\, e^{\lambda t}, \tag{4.5c}$$

where the complex functions θ, ϕ, ω are small perturbations, and λ is a complex eigenvalue to be determined. Retaining only the first order terms in the small quantities, the system (1.1)–(1.2) becomes

$$\nabla \cdot (u \nabla \theta) - (\theta - \phi)d = \lambda u(\theta - \omega), \tag{4.6}$$

$$\nabla \cdot (v\nabla\phi) + (\theta - \phi)d = \lambda v(\phi - \omega), \tag{4.7}$$

$$\kappa\Delta\omega = u(\omega - \theta) + v(\omega - \phi), \qquad x \in D, \tag{4.8}$$

and the boundary conditions (1.3) become

$$\nu \cdot \nabla\theta = \nu \cdot \nabla\phi = \nu \cdot \nabla\omega = 0, \qquad x \in \partial D; \tag{4.9}$$

except as noted below, we also impose the requirement

$$\int_D \omega(x)\, dx = 0 \tag{4.10}$$

to remove arbitrary additive constants from θ, ϕ, ω. In (4.6)–(4.8), $u = e^z$, $v = e^{-z}$, and $d = d(e^z, e^{-z}, x)$ are understood to be evaluated at the stationary state. We next show the following theorem.

THEOREM 4. *The eigenvalues* (λ) *obtained from* (4.6)–(4.10) *are real and non-positive.*

Proof. Taking the scalar products of (4.6) with θ, (4.7) with ϕ, respectively, integrating by parts and adding the results, we obtain

$$\begin{aligned} -\|\theta\|_{1,u}^2 - \|\phi\|_{1,v}^2 - \|\theta - \phi\|_d^2 &= \lambda(\theta u, \theta - \omega) + \lambda(\phi v, \phi - \omega) \\ &= \lambda(\|\theta\|_u^2 + \|\phi\|_v^2 - (\theta u + \phi v, \omega)). \end{aligned} \tag{4.11}$$

The scalar product of (4.8) with ω gives similarly,

$$\begin{aligned} \kappa\|\omega\|_1^2 + \|\omega\|_u^2 + \|\omega\|_v^2 &= (\theta u + \phi v, \omega) \\ &\leqq \tfrac{1}{2}(\|\theta\|_u^2 + \|\phi\|_v^2 + \|\omega\|_u^2 + \|\omega\|_v^2), \end{aligned} \tag{4.12}$$

so that (4.11) becomes

$$\begin{aligned} -(\|\theta\|_{1,u}^2 + \|\phi\|_{1,v}^2 + \|\theta - \phi\|_d^2)/\lambda &= \|\theta\|_u^2 + \|\phi\|_v^2 - \kappa\|\omega\|_1^2 - \|\omega\|_u^2 - \|\omega\|_v^2 \\ &\geqq \kappa\|\omega\|_1^2. \end{aligned} \tag{4.13}$$

Thus λ is real. Unless $\|\omega\|_1$ is zero, (4.13) may be rewritten

$$\lambda \leqq \frac{-\|\theta\|_{1,u}^2 - \|\phi\|_{1,v}^2 - \|\theta - \phi\|_d^2}{\kappa\|\omega\|_1^2} \tag{4.14}$$

and the result follows. If $\|\omega\|_1$ is zero, it follows from (4.10) that $\omega \equiv 0$ in D. Then (4.8) gives $u\theta = -v\phi$, and (4.6)–(4.7) becomes

$$\nabla \cdot (u\nabla\theta) - (1 + u^2)\, d\theta = \lambda u\theta, \tag{4.15}$$

$$\nabla \cdot (v\nabla\phi) - (1 + v^2)\, d\phi = \lambda v\phi, \tag{4.16}$$

either of which implies real negative values of λ.

We next show that the eigenvalues λ are bounded away from zero.

THEOREM 5. *The eigenvalues obtained from* (4.6)–(4.9) *are less than or equal to* $-c(D, z, d)$.

Proof. In the case $\|\omega\|_1 = 0$, the result is obvious from (4.15). For $\|\omega\|_1 \neq 0$, we have from (4.12),

$$\kappa\|\omega\|_1^2 + \|\omega\|_u^2 + \|\omega\|_v^2 \leqq \tfrac{1}{4}\|\theta\|_u^2 + \tfrac{1}{4}\|\phi\|_v^2 + \|\omega\|_u^2 + \|\omega\|_v^2; \tag{4.17}$$

then (4.14) becomes

$$\lambda \leqq -4\frac{\|\theta\|_{1,u}^2 + \|\phi\|_{1,v}^2 + \|\theta - \phi\|_d^2}{\|\theta\|_u^2 + \|\phi\|_v^2}. \tag{4.18}$$

We now write $\theta = \theta' + a + b$, $\phi = \phi' + a - b$, where a, b are constants, and θ', ϕ' satisfy

$$\int_D \theta' d\, dx = \int_D \phi' d\, dx = 0; \tag{4.19}$$

instead of imposing (4.10), we fix the arbitrary additive constant in θ, ϕ, ω by setting $a = 0$.

Substituting in (4.18), we obtain

$$\begin{aligned} \lambda &\leqq -4\frac{\|\theta'\|_{1,u}^2 + \|\phi'\|_{1,v}^2 + \|\theta' - \phi'\|_d^2 + 4b^2 \int_D d\, dx}{\|\theta'\|_u^2 + \|\phi'\|_v^2 + 2b\int_D (\theta' u - \phi' v)\, dx + b^2 \int_D (u + v)\, dx} \\ &\leqq -2\frac{\|\theta'\|_{1,u}^2 + \|\phi'\|_{1,v}^2 + 4b^2 \int_D d\, dx}{\|\theta'\|_u^2 + \|\phi'\|_v^2 + b^2 \int_D (u + v)\, dx}. \end{aligned} \tag{4.20}$$

It follows from (4.19) and the assumptions on d that there exists a constant C_6, depending on D, d, such that

$$\|\theta'\|_u^2 + \|\phi'\|_v^2 \leqq C_6(\|\theta'\|_{1,u}^2 + \|\phi'\|_{1,v}^2); \tag{4.21}$$

substituting (4.21) into (4.20) we have finally

$$\begin{aligned} \lambda &\leqq -2\frac{\|\theta'\|_{1,u}^2 + \|\phi'\|_{1,v}^2 + 4b^2 \int_D d\, dx}{C_6(\|\theta'\|_{1,u}^2 + \|\phi'\|_{1,v}^2) + b^2 \int_D (u + v)\, dx} \\ &\leqq \max\left(-\frac{2}{C_6}, -8\frac{\int_D d\, dx}{\int_D (u + v)\, dx}\right), \end{aligned} \tag{4.22}$$

which is the desired estimate.

In the special case where the function z satisfying (3.15) is simply a constant, the eigenvalues of the system (4.6)–(4.10) can be described by two self-adjoint second order equations. In this case, (4.15), (4.16), and (4.8) with $\omega = 0$ are consistent, and the eigenvalues of

$$\Delta\theta - 2\, d\theta \cosh z = \lambda\theta, \quad x \in D; \qquad \nu \cdot \nabla\theta = 0, \quad x \in \partial D, \tag{4.23}$$

are eigenvalues of (4.6)–(4.10). The eigenvalues corresponding to nonzero ω are obtained by differentiating (4.8), and using (4.6), (4.7) to obtain

$$\begin{aligned} 0 &= (\lambda - \Delta)(\kappa\Delta\omega + u(\theta - \omega) + v(\phi - \omega)) \\ &= \lambda\kappa\Delta\omega - \kappa\Delta^2\omega + (u + v)\Delta\omega + \lambda[u(\theta - \omega) + v(\phi - \omega)] - (u\Delta\theta + v\Delta\phi) \\ &= \lambda\kappa(\Delta\omega) - \kappa\Delta(\Delta\omega) + (u + v)(\Delta\omega), \qquad x \in D. \end{aligned} \tag{4.24}$$

From (4.8), (4.9), we obtain

$$\nu \cdot \nabla(\Delta\omega) = 0, \quad x \in \partial D; \tag{4.25}$$

equations (4.24), (4.25) define a second order eigenvalue problem with $\Delta\omega$ as the dependent variable.

We note that the eigenvalues of (4.24), (4.25) do not depend on the recombination factor d, but do depend on the "dielectric relaxation time" τ_0, given by

$$\tau_0 = \kappa/(u + v). \tag{4.26}$$

5. A finite difference scheme. In this section we present a simple finite difference scheme for the approximate solution of the system (1.1)–(1.4). We treat the space derivatives in (1.1)–(1.2) by a method which has been successfully applied to the steady state problem [8], [9], [13] and the time derivatives in (1.1) by simple backward differencing. The scheme so obtained is accurate only to order $\Delta t + (\Delta x)^2$, but assures strictly positive computed values of u, v, and preserves Lemma 1 and Lemma 2 (with $p = 1$ only) in the difference equations.

For simplicity, we write out the difference equations assuming one space dimension and equally spaced mesh points; the generalizations to higher dimensions and to unequally spaced points are immediate. We use the notation

$$\begin{aligned} &u_j^k = u(j\Delta x, k\Delta t), \qquad j = 1, 2, \cdots, M, \quad k = 0, 1, \cdots, T/\Delta t,\\ &u^k = (u_1^k, u_2^k, \cdots, u_M^k) \end{aligned} \tag{5.1}$$

with similar notations for v, ψ. We assume that the boundary conditions (1.3), (1.7) are treated by relations of the form

$$\begin{aligned} &u_0^k = u_1^k, \qquad u_{M+1}^k = u_M^k,\\ &v_0^k = v_1^k, \qquad v_{M+1}^k = v_M^k,\\ &\psi_0^k = \psi_1^k, \quad \psi_{M+1}^k = \psi_M^k, \quad \sum_{j=1}^{M} \psi_j^k = 0 \end{aligned} \tag{5.2}$$

for all values of k. The difference equations we consider are the following:

$$\begin{aligned} \frac{u_j^k - u_j^{k-1}}{\Delta t} = \frac{1}{(\Delta x)^2}\Bigg[&\left(\frac{\psi_j^k - \psi_{j+1}^k}{e^{-\psi_{j+1}^k} - e^{-\psi_j^k}}\right)(u_{j+1}^k e^{-\psi_{j+1}^k} - u_j^k e^{-\psi_j^k})\\ &-\left(\frac{\psi_{j-1}^k - \psi_j^k}{e^{-\psi_j^k} - e^{-\psi_{j-1}^k}}\right)(u_j^k e^{-\psi_j^k} - u_{j-1}^k e^{-\psi_{j-1}^k})\Bigg] - R(u_j^k, v_j^k), \end{aligned} \tag{5.3a}$$

$$\begin{aligned} \frac{v_j^k - v_j^{k-1}}{\Delta t} = \frac{1}{(\Delta x)^2}\Bigg[&\left(\frac{\psi_{j+1}^k - \psi_j^k}{e^{\psi_{j+1}^k} - e^{\psi_j}}\right)(v_{j+1}^k e^{\psi_{j+1}^k} - v_j^k e^{\psi_j^k})\\ &-\left(\frac{\psi_j^k - \psi_{j-1}^k}{e^{\psi_j^k} - e^{\psi_{j-1}^k}}\right)(v_j^k e^{\psi_j^k} - v_{j-1}^k e^{\psi_{j-1}^k})\Bigg] - R(u_j^k, v_j^k), \end{aligned} \tag{5.3b}$$

$$\kappa\frac{\psi_{j+1}^k - 2\psi_j^k + \psi_{j-1}^k}{(\Delta x)^2} = u_j^k - v_j^k - N_j, \qquad N_j = N(j\Delta x). \tag{5.4}$$

In (5.3), the factors in brackets are defined continuously at $\psi_j^k = \psi_{j\pm 1}^k$. By inspection, equations (5.3) satisfy a maximum principle, so that the positivity of u^k, v^k follows from that of u^{k-1}, v^{k-1}. Summing (5.3a) with respect to j, we obtain

$$\sum_{j=1}^{M} u_j^k - \sum_{j=1}^{M} u_j^{k-1} = -\Delta t \sum_{j=1}^{M} R(u_j^k, v_j^k) \leqq \frac{M\,\Delta x\,\Delta t}{2}, \tag{5.5}$$

using (1.4), with a similar expression for v. Hence Lemma 2 for $p = 1$ is preserved by the difference equations. The most important stability property of this difference scheme is described by the following lemma.

LEMMA 6. *For all positive integers k, the quantity*

$$E^k = \frac{\kappa}{2}(\Delta x) \sum_{j=2}^{M} \left(\frac{\psi_j^k - \psi_{j-1}^k}{\Delta x} \right)^2 \tag{5.6}$$

is bounded, by a constant depending on the initial data and $(M\,\Delta x)$, *but not on* Δx *or* Δt.

Proof. Differencing (5.4) with respect to time, multiplying by ψ_j^k and summing with respect to j by parts, using (5.2), we obtain

$$\begin{aligned} &\kappa \sum_{j=2}^{M} \left(\frac{\psi_j^k - \psi_{j-1}^k}{\Delta x} \right) \left(\frac{\psi_j^k - \psi_{j-1}^k - \psi_j^{k-1} + \psi_{j-1}^{k-1}}{\Delta x\, \Delta t} \right) \\ &\qquad = - \sum_{j=1}^{M} \psi_j^k \left(\frac{u_j^k - u_j^{k-1} - v_j^k + v_j^{k-1}}{\Delta t} \right); \end{aligned} \tag{5.7}$$

inserting (5.3) and using (5.6), we see that (5.7) becomes, performing another summation by parts,

$$\begin{aligned} E^k &\leqq E^{k-1} + \frac{\Delta t}{\Delta x} \sum_{j=2}^{M} (\psi_j^k - \psi_{j-1}^k) \Bigg[\left(\frac{\psi_{j-1}^k - \psi_j^k}{e^{-\psi_j^k} - e^{-\psi_{j-1}^k}} \right) (u_j^k e^{-\psi_j^k} - u_{j-1}^k e^{-\psi_{j-1}^k}) \\ &\qquad - \left(\frac{\psi_j^k - \psi_{j-1}^k}{e^{\psi_j^k} - e^{\psi_{j-1}^k}} \right) (v_j^k e^{\psi_j^k} - v_{j-1}^k e^{\psi_{j-1}^k}) \Bigg] \\ &= E^{k-1} + \frac{\Delta t}{\Delta x} \sum_{j=2}^{M} (\psi_j^k - \psi_{j-1}^k)(u_j^k - u_{j-1}^k - v_j^k + v_{j-1}^k) + \frac{\Delta t}{\Delta x} \sum_{j=2}^{M} (\psi_j^k - \psi_{j-1}^k) \\ &\quad \cdot \Bigg[\left(\frac{\psi_{j-1}^k - \psi_j^k}{e^{-\psi_j^k} - e^{-\psi_j^{k-1}}} \right) \left(\frac{u_j^k e^{-\psi_j^k} a(\psi_j^k - \psi_{j-1}^k) + u_{j-1}^k e^{-\psi_j^{k-1}} a(\psi_{j-1}^k - \psi_j^k)}{\psi_{j-1}^k - \psi_j^k} \right) \\ &\quad - \left(\frac{\psi_j^k - \psi_{j-1}^k}{e^{\psi_{j-1}^k} - e^{\psi_{j-1}^k}} \right) \left(\frac{v_j^k e^{\psi_j^k} a(\psi_{j-1}^k - \psi_j^k) + v_{j-1}^k e^{\psi_{j-1}^k} a(\psi_j^k - \psi_{j-1}^k)}{\psi_j^k - \psi_{j-1}^k} \right) \Bigg], \end{aligned} \tag{5.8}$$

where $a(\varepsilon) = e^\varepsilon - \varepsilon - 1$ is positive semidefinite and $O(\varepsilon^2)$ as $\varepsilon \to 0$. Setting $\gamma_j = \psi_j^k - \psi_{j-1}^k$, summing the second term on the right side of (5.8) by parts and using (5.4), we obtain

$$\begin{aligned} E^k &\leqq E^{k-1} - (\Delta t)(\Delta x) \sum_{j=1}^{M} \left(\frac{\psi_{j+1}^k - 2\psi_j^k + \psi_{j-1}^k}{(\Delta x)^2} \right) \left(\kappa \frac{\psi_{j+1}^k - 2\psi_j^k + \psi_{j-1}^k}{(\Delta x)^2} + N_j \right) \\ &\quad - \frac{\Delta t}{\Delta x} \sum_{j=2}^{M} \left[\frac{\gamma_j a(\gamma_j)}{e^{\gamma_j} - 1} u_j^k + \frac{\gamma_j a(-\gamma_j)}{1 - e^{-\gamma_j}} u_{j-1}^k + \frac{\gamma_j a(-\gamma_j)}{1 - e^{-\gamma_j}} v_j^k + \frac{\gamma_j a(\gamma_j)}{e^{\gamma_j} - 1} v_{j-1}^k \right], \end{aligned} \tag{5.9}$$

which is the difference equation analogue of (2.2). Since u_j^k, v_j^k are positive, the last term on the right side of (5.9) is negative semidefinite, and we may estimate (5.9), using (5.2), as

$$E^k \leqq E^{k-1} - c\Delta t E^k + \frac{(\Delta t)(\Delta x)}{2\kappa} \sum_{j=1}^{M} N_j^2, \tag{5.10}$$

which implies the desired bound for E^k.

Lemma 6 is sufficient to show the existence, but not the uniqueness, of a solution of the nonlinear system of difference equations (5.2)–(5.4).

THEOREM 6. *Suppose u_j^{k-1}, v_j^{k-1} are positive for $1 \leqq j \leqq M$ and satisfy*

$$\sum_{j=1}^{M} (u_j^{k-1} - v_j^{k-1} - N_j) = 0; \tag{5.11}$$

then there exists a solution to the system (5.2)–(5.4) *with u_j^k, v_j^k positive, for any value of Δt.*

Proof. We let Δt vary from zero to its desired value, and apply elementary degree theory. For $\Delta t = 0$, the existence of a unique solution with $u_j^k = u_j^{k-1}$, $v_j^k = v_j^{k-1}$ follows from (5.11). It is thus sufficient to show the existence of a constant, which may depend on Δx or M, bounding all possible solutions; we use for convenience the norm

$$\|u^k\| + \|v^k\| + \|\psi^k\| = \sup_{1 \leqq j \leqq M} |u_j^k| + \sup_{1 \leqq j \leqq M} |v_j^k| + \sup_{1 \leqq j \leqq M} |\psi_j^k|. \tag{5.12}$$

From Lemma 6, the existence of a bound on $\|\psi^k\|$, depending on M and on u^{k-1}, v^{k-1} but not on Δt, follows; then bounds on $\|u^k\|$ and $\|v^k\|$ may be obtained by applying a maximum principle argument to (5.3), using (1.4), or for finite Δt by appeal to (5.5).

6. Discussion and summary. In spite of obvious limitations, some of the results obtained above may be of practical value in the construction of suitable numerical methods for problems of this type. The most important such limitation is the assumed form of the boundary conditions. The a priori estimates of § 2, on which the existence–uniqueness theory is based, are dependent, in the case of two mobile species, on the boundary conditions as adopted above. Since the steady state problem is much more complicated when more interesting boundary conditions are adopted [7], it appears unlikely that our results can easily be generalized in this respect.

With this assumed form of the boundary conditions, our results on the asymptotic behavior of the solutions are probably not sharp. One expects intuitively that with these boundary conditions, the solution should approach the steady state for large t, independently of the initial data. Our results are not strong enough to claim this. If, however, we had established the completeness of the eigenfunctions of the linearized problem in § 4, and if the convergence result of Lemma 5 were obtained independently of T, then the asymptotic approach of all solutions to the steady state would follow by a continuity argument, using Theorem 1.

Numerous difference schemes are known for parabolic equations [11, pp. 189–191]; the difference scheme described in § 5, and in particular the backward time differences in (5.3), are motivated by the result of Lemma 6. From (2.2), or from (4.23)–(4.25), we expect that the smallest time scale τ_1 in a particular difference approximation will be approximately

$$\tau_1^{-1} \approx \tau_0^{-1} + O(\Delta x^{-2}), \tag{6.1}$$

where the dielectric relaxation time τ_0 is obtained from (4.26); in many problems of practical interest, this term strongly dominates the right side of (6.1). We infer from Lemma 6, and in particular from the positive definite nature of the terms in brackets on the right side of (5.9), that time steps of order τ_0 are not necessary for the stability of this difference scheme. We note, however, that if the Poisson equation (5.4) is oriented differently in time, so that it may be solved independently of the continuity equations (5.3), then results of the form of Lemma 6 are not obtained, and it appears likely that such schemes will require $\Delta t \leqq O(\tau_0)$ for stability.

Note added in proof. Additional results on the asymptotic behavior of solutions satisfying these boundary conditions have been obtained, and will be published separately. L_p-estimates of u, v, independent of time, have been obtained. In addition, it has been shown that the solution decays exponentially to the steady state solution, for large time, if the function N in (1.2) is simply a constant in D, or if the initial data is sufficiently close to the steady state solution, in L_2.

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TWO-TIMING METHODS VALID ON EXPANDING INTERVALS*

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Abstract. The problem

$$y'' + \varepsilon f(y, y') + y = 0, \qquad y(0) = a, \quad y'(0) = b,$$

with f a polynomial is considered. A two-timing method is described which yields series expansions in ε for the solution. These expansions are then shown to be generalized asymptotic expansions, uniformly valid on t intervals of the form $[0, k/\varepsilon]$. The method of proof also yields the existence of solutions to the above problem on such intervals for k appropriately chosen.

1. Introduction. Initial value problems of the form

$$y'' + \varepsilon f(y, y') + y = 0, \qquad y(0) = a, \quad y'(0) = b \tag{1.1}$$

describe autonomous nonlinear oscillations. Van der pol's equation, $y'' + \varepsilon(1 - y^2)y' + y = 0$, the unforced Duffing equation, $y'' + \varepsilon(y')^3 + y = 0$, and equations arising in satellite problems fall into this category. Such problems are discussed at some length by Kevorkian [3], who uses two variable expansion procedures to construct formal expansions for solutions. In his work Kevorkian points out the need of a theory to establish the asymptotic validity of such expansions. This paper aims to supply such a theoretical foundation.

As is pointed out by Morrison [5] and Perko [7], two-timing methods are closely related to the method of averaging as introduced by Krylov and Bogoliubov [4]. Theoretical work supporting the method of averaging has been presented by Bogoliubov and Mitropolsky [1] and by Perko [7], who also proves the validity of a two-timing method on $[0, k/\varepsilon]$ for certain first order systems of the form $y' = \varepsilon f(t, y, \varepsilon)$. The equations considered here are not of this form, and the proofs in this paper do not rely on relating two-timing methods to the method of averaging.

The methods used in this paper generalize arguments presented by Reiss in [8] to yield a two-timing method applicable to certain nonlinear oscillation problems. Reiss' expository paper discussed linear problems with constant coefficients. Methods similar to those used here in the proof of asymptotic validity have been used by O'Malley [6] and Erdélyi [2] to prove asymptotic validity of certain perturbation methods as applied to singular perturbation problems.

2. Preliminary definitions and a basic inequality. In what follows we will be concerned with functions of ε and t which as functions of t are uniformly bounded on intervals of length inversely proportional to ε. The following definitions are introduced to provide a proper setting for the ideas to follow.

DEFINITION 1. A function $f(\varepsilon, t)$ will be said to be in $C^0_{k,\varepsilon}$ if it is continuous on $(0, 1] \times [0, k/\varepsilon]$ and if there exists M independent of ε such that $|f(\varepsilon, t)| \leqq M$ for all $(\varepsilon, t) \in (0, 1] \times [0, k/\varepsilon]$.

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DEFINITION 2. For $f(\varepsilon, t) \in C^0_{k,\varepsilon}$, $\|f\|^0_{k,\varepsilon}$ will be used to denote

$$\sup_{(\varepsilon,t)\in(0,1]\times[0,k/\varepsilon]} |f(\varepsilon, t)|.$$

For example, if g is continuous on $[0, k]$, then $f(\varepsilon, t) = g(\varepsilon t)\sin t$ is in $C^0_{k,\varepsilon}$, and $\|f\|^0_{k,\varepsilon} \leqq \max_{x\in[0,k]} |g(x)|$.

DEFINITION 3. A function $f(\varepsilon, t)$ will be said to be in $C^1_{k,\varepsilon}$ if $f(\varepsilon, t)$ and $f_t(\varepsilon, t)$ are both continuous on $(0, 1] \times [0, k/\varepsilon]$, and if there exists M such that both $|f(\varepsilon, t)| \leqq M$ and $|f_t(\varepsilon, t)| \leqq M$ for all (ε, t) in $(0, 1] \times [0, k/\varepsilon]$.

DEFINITION 4. For $f(\varepsilon, t)$ in $C^1_{k,\varepsilon}$, $\|f\|^1_{k,\varepsilon}$ will denote

$$\max\left\{\sup_{(\varepsilon,t)\in(0,1]\times[0,k/\varepsilon]} |f(\varepsilon, t)|, \sup_{(\varepsilon,t)\in(0,1]\times[0,k/\varepsilon]} |f_t(\varepsilon, t)|\right\}.$$

For example, if g is in $C^1[0, k]$, then $f(\varepsilon, t) = g(\varepsilon t)\sin t$ is in $C^1_{k,\varepsilon}$ since $(\partial/\partial t)[g(\varepsilon t)\sin t] = \varepsilon g'(\varepsilon t)\sin t + g(\varepsilon t)\cos t$ and $\|f\|^1_{k,\varepsilon} \leqq 2\|g\|_1$, where $\|g\|_1 = \max\{\max_{x\in[0,k]}|g(x)|, \max_{x\in[0,k]}|g'(x)|\}$. This example illustrates the importance of the restriction of ε to a bounded interval. Actually, the above definitions could be stated in terms such as "for ε sufficiently small," but little would be gained in using this slightly more inclusive definition.

Note that a function of t alone belongs to $C^1_{k,\varepsilon}$ if and only if $f(t)$ and $f'(t)$ are continuous and bounded on $[0, \infty)$.

The theorem and corollary which follow are to be used in remainder estimates to establish asymptotic validity of expansions presented later in the paper.

THEOREM 1. *Let $g(t), h(t)$, and $F(t)$ be in $C^0_{k,\varepsilon}$ and $L[y(t)] = y''(t) + \varepsilon g(t)y'(t) + [1 + \varepsilon h(t)]y(t)$. Then there exist positive constants α, β, and γ such that any solution $y(t)$ of the equation*

$$L[y(t)] = F(t) \tag{2.1}$$

satisfies the inequality

$$\|y(t)\|^1_{k,\varepsilon} \leqq \alpha|y(0)| + \beta|y'(0)| + \frac{\gamma\|F(t)\|^0_{k,\varepsilon}}{\varepsilon}. \tag{2.2}$$

The following corollary is also established by the proof of the theorem.

COROLLARY 1. *Let $G(t, s)$ be the Green's function for the operator $L[y]$. Then both $G(t, s)$ and $G_t(t, s)$ are bounded for $0 \leqq s \leqq t \leqq k/\varepsilon$. ($G_t(t, s)$ denotes the partial derivative of $G(t, s)$ with respect to t.)*

Proof. Let y_1 and y_2 be the fundamental set of solutions for the homogeneous equation

$$L[y] = 0 \tag{2.1h}$$

satisfying the initial conditions $y_1(0) = \delta_{i1}, y_i'(0) = \delta_{i2}$, where δ_{ij} is the Kronecker delta. Then the solution to (2.1) can be written

$$y(t) = y(0)y_1(t) + y'(0)y_2(t) + \int_0^t G(t, s)F(s)\,ds,$$

while

$$y'(t) = y(0)y_1'(t) + y'(0)y_2'(t) + \int_0^t G_t(t, s)F(s)\,ds.$$

Showing that $y_1(t)$ and $y_2(t)$ are in $C^1_{k,\varepsilon}$, and that $G(t, s)$ and $G_t(t, s)$ are both bounded for $0 \leqq s \leqq t \leqq k/\varepsilon$ will establish (2.2).

Let $y(t)$ be any solution of (2.1h) and set

$$y'(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t). \tag{2.3}$$

Equation (2.1h) is then equivalent to the system

$$\theta'(t) = \cos^2\theta(t) + \varepsilon g(t)\cos\theta(t)\sin\theta(t) + [1 + \varepsilon h(t)]\sin^2\theta(t),$$

$$r'(t) = -r(t)[\varepsilon g(t)\cos^2\theta(t) + \varepsilon h(t)\cos\theta(t)\sin\theta(t)].$$

Clearly,

$$r(t) = C\exp\left\{-\int_0^t [\varepsilon g(s)\cos^2\theta(s) + \varepsilon h(s)\cos\theta(s)\sin\theta(s)]\,ds\right\} \tag{2.4}$$

and there exists K such that $\|r(t)\|_{k,\varepsilon}^0 \leqq K$. By (2.3), $\|y(t)\|_{k,\varepsilon}^1 \leqq K$. Then both y_1 and y_2 are in $C_{k,\varepsilon}^1$.

$$G(t, s) = \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{w(s)},$$

and

$$G_t(t, s) = \frac{y_1'(t)y_2(s) - y_1(s)y_2'(t)}{w(s)},$$

and $w(s)$ satisfies Abel's equation

$$w'(s) + \varepsilon g(s)w(s) = 0, \qquad w(0) = 1.$$

Therefore $w(s) = \exp\{-\varepsilon\int_0^s g(u)\,du\}$ is bounded away from 0 for $s \in [0, k/\varepsilon]$, so $G(t, s)$ and $G_t(t, s)$ are bounded for $0 \leqq s \leqq t \leqq k/\varepsilon$ and the theorem and corollary are proved.

The above proof remains valid if g, h, and F depend on ε as well as on t as long as $g(\varepsilon, t)$, $h(\varepsilon, t)$, and $F(\varepsilon, t)$ are in $C_{k,\varepsilon}^0$.

Perhaps it should also be mentioned that the theorem can be proved more directly by making the Prufer substitution (2.3) directly in (2.1) rather than in (2.1h) and using Gronwall's inequality. This, however, would have the disadvantage of not proving the corollary simultaneously.

In order to facilitate the reading of the later sections, a brief list of the symbols used is presented here:

$y^j(t, \tau)$ represents the coefficient of ε^j in the asymptotic expansions; $Z^N = \sum_{j=0}^N y^j(t, \tau)\varepsilon^j$; $\{\cdot\}_i$ denotes the coefficient of ε^i in the expansion of the quantity in brackets; and O-notation refers to a limiting process as $\varepsilon \to 0^+$.

3. A two-timing method for linear problems with periodic damping. The linear equation

$$y''(t) + \varepsilon g(t)y'(t) + y(t) = 0; \qquad y(0) = a, \quad y'(0) = b \tag{3.1}$$

with $g(t)$ continuous and periodic with period 2π has a solution valid for $t \geqq 0$. In this section a method is developed for generating an expansion for this solution and this expansion is shown to be asymptotically valid on intervals having length inversely proportional to ε.

The formal asymptotic expansion. Let $\tau = \varepsilon t$ and assume equation (3.1) allows an asymptotic solution $\sum_{j\geqq 0} y^j(t, \tau)\varepsilon^j$. The functions y^j will then be determined by formally inserting this expansion into equation (3.1) and equating the coefficients of ε^j to zero. Proceeding in this way, one finds that $y^j(t, \tau)$ must satisfy

$$y^j_{tt} + y^j = -y^{j-2}_{\tau\tau} - g(t)y^{j-2}_{\tau} - 2y^{j-1}_{t\tau} - g(t)y^{j-1}_{t}, \tag{3.2}$$

$$y^j(0,0) = a\delta_{j0}, \quad y^j_t(0,0) = b\delta_{j0} - y^{j-1}_{\tau}(0,0), \tag{3.3}$$

where $y^{-1} \equiv 0$, $y^{-2} \equiv 0$, and δ_{jl} is the Kronecker symbol.

Setting $j = 0$ in (3.2) yields $y^0(t, \tau) = A_0(\tau) \sin t + B_0(\tau) \cos t$, where $A_0(\tau)$ and $B_0(\tau)$ are arbitrary functions of τ. The initial conditions $y^0(0, 0) = a$, $y^0_t(0, 0) = b$ lead to the choices:

$$A_0(0) = b, \quad B_0(0) = a. \tag{3.4}$$

We now wish to choose $A_0(\tau)$ and $B_0(\tau)$ in such a way that $\|y - y^0(t, \tau)\|^1_{k,\varepsilon} = O(\varepsilon)$. As will be shown later, this will be the case if $y^1(t, \tau) \in C^1_{k,\varepsilon}$, and this can be assured by choosing $A_0(\tau)$ and $B_0(\tau)$ carefully.

More explicitly, having $y^0(t, \tau) = A_0(\tau) \sin t + B_0(\tau) \cos t$, (3.2) shows that $y^1(t, \tau)$ must satisfy

$$y^1_{tt} + y^1 = -2A_0'(\tau) \cos t + 2B_0'(\tau) \sin t - g(t)[A_0(\tau) \cos t - B_0(\tau) \sin t]. \tag{3.5}$$

The right-hand side of equation (3.5) is periodic with period 2π in the variable t, and can be expanded in a Fourier series whose coefficients are functions of τ. If the coefficients of $\sin t$ and $\cos t$ in this expansion are zero, the solution to (3.5) will be in $C^1_{k,\varepsilon}$. Equating these coefficients to zero yields a linear first order system for A_0 and B_0 which together with the initial conditions (3.4) determines $A_0(\tau)$ and $B_0(\tau)$ uniquely.

Having now determined $y^0(t, \tau)$ completely, we solve (3.5) for $y^1(t, \tau)$. The solution will involve two arbitrary functions $A_1(\tau)$ and $B_1(\tau)$, multiplying $\sin t$ and $\cos t$ respectively, which are then determined by expanding the right-hand side of the equation $y^2_{tt} + y^2 = -y^0_{\tau\tau} - g(t)y^0_{\tau} - 2y^1_{t\tau} - g(t)y^1_t$ in a Fourier series and setting the coefficients of $\sin t$ and $\cos t$ equal to zero. This again yields a first order linear system of equations which can be used together with the initial conditions for $A_1(\tau)$ and $B_1(\tau)$ to determine these two functions. This procedure can be repeated to any desired order.

An example will illustrate the method. The proof of the asymptotic validity of this expansion procedure follows the example.

Example. Consider the equation

$$y''(t) + \varepsilon \sin t y'(t) + y(t) = 0; \qquad y(0) = a, \quad y'(0) = b. \tag{3.6}$$

According to the above procedure,

$$y^0(t, \tau) = A_0(\tau) \sin t + B_0(\tau) \cos t,$$

so $y^1(t, \tau)$ satisfies

$$y^1_{tt} + y^1 = -2A_0'(\tau) \cos t - A_0(\tau) \sin t \cos t + 2B_0'(\tau) \sin t + B_0(\tau) \sin^2 t. \tag{3.7}$$

The Fourier series for the right-hand side of (3.7) is

$$\frac{B_0(\tau)}{2} - 2A_0'(\tau) \cos t - \frac{B_0(\tau)}{2} \cos 2t + 2B_0'(\tau) \sin t - \frac{A_0(\tau)}{2} \sin 2t.$$

We therefore require that

$$A_0'(\tau) \equiv 0, \quad A_0(0) = b, \quad B_0'(\tau) \equiv 0, \quad B_0(0) = a$$

yielding $A_0(\tau) \equiv b$, $B_0(\tau) \equiv a$. This choice of A_0 and B_0 reduces equation (3.7) to

$$y_{tt}^1 + y^1 = \frac{a}{2} - \frac{a}{2}\cos 2t - \frac{b}{2}\sin 2t$$

so

$$y^1(t,\tau) = \frac{a}{2} + \frac{a}{6}\cos 2t + \frac{b}{6}\sin 2t + A_1(\tau)\sin t + B_1(\tau)\cos t$$

while $y^1(0,0) = 0$, $y_t^1(0,0) = -y_\tau^0(0,0)$ yield

(3.8) $$B_1(0) = -(2/3)a, \quad A_1(0) = -b/3.$$

Continuing, y^2 satisfies

$$y_{tt}^2 + y^2 = \frac{B_1(\tau)}{2} + \left(-2A_1' + \frac{a}{6}\right)\cos t + \left(2B_1' + \frac{b}{6}\right)\sin t - \frac{B_1(\tau)}{2}\cos 2t$$
$$- \frac{A_1(\tau)}{2}\sin 2t - \frac{a}{6}\cos 3t - \frac{b}{6}\sin 3t.$$

We therefore require that

$$A_1' - \frac{a}{12} = 0, \quad B_1' + \frac{b}{12} = 0$$

which together with (3.8) implies that

$$A_1(\tau) = \frac{a}{12}\tau - \frac{b}{3}, \quad B_1(\tau) = -\frac{b}{12}\tau - \frac{2}{3}a.$$

The expansion to first order for the solution to (3.6) is therefore

$$Z^1(t,\tau) = (b\sin t + a\cos t) + \varepsilon\left[\frac{a}{2} + \left(-\frac{b}{12}\tau - \frac{2}{3}a\right)\cos t + \left(\frac{a}{12}\tau - \frac{b}{3}\right)\sin t \right.$$
$$\left. + \frac{a}{6}\cos 2t + \frac{b}{6}\sin 2t\right].$$

By the results of the next section we conclude that the solution $y(\varepsilon, t)$ of (3.6) satisfies $\|y(\varepsilon, t) - Z^1(\varepsilon, t)\|_{k,\varepsilon}^1 = O(\varepsilon^2)$.

Proof of asymptotic correctness. To prove that expansions derived in this way are asymptotically valid on intervals of length inversely proportional to ε, write

(3.9) $$y = \sum_{j=0}^{N} y^j(t,\tau)\varepsilon^j + R_{N+1}.$$

Formal substitution of (3.9) into equation (3.1) and use of (3.2) yields that R_{N+1} must satisfy

$$R_{N+1}'' + \varepsilon g(t)R_{N+1}' + R_{N+1} = (-y_{\tau\tau}^{N-1} - g(t)y_\tau^{N-1} - 2y_{t\tau}^N - g(t)y_\tau^N)\varepsilon^{N+1}$$

(3.10) $$+ (-y_{\tau\tau}^N - g(t)y_\tau^N)\varepsilon^{N+2},$$

$$R_{N+1}(0) = 0, \quad R_{N+1}'(0) = -y_\tau^N(0,0)\varepsilon^{N+1}.$$

The expansion procedure used to determine y^j insures the boundedness of the coefficients of ε^{N+1} and ε^{N+2} on the right-hand side of (3.10) for τ in any compact interval $[0, k]$.

Theorem 1 can be applied directly to (3.10) to yield $\|R_{N+1}(t)\|_{k,\varepsilon}^1 \leqq M\varepsilon^N$. To see that $\|R_{N+1}(t)\|_{k,\varepsilon}^1 = O(\varepsilon^{N+1})$, note that $y(\varepsilon, t) = \sum_{j=0}^N y^j(t, \tau)\varepsilon^j + R_{N+1}$ and also $y(\varepsilon, t) = \sum_{j=0}^N y^j(t, \tau)\varepsilon^j + y^{N+1}(t, \tau)\varepsilon^{N+1} + R_{N+2}$, where the functions $y^0, y^1, \cdots, y^N$ are unchanged. Thus $R_{N+1}(t) = y^{N+1}\varepsilon^{N+1} + R_{N+2}$, and since $y^{N+1}(t, \tau)$ and $(d/dt)y^{N+1}(t, \tau)$ are bounded for $(t, \tau) \in [0, k/\varepsilon] \times [0, k]$ and $\|R_{N+2}\|_{k,\varepsilon}^1$ is $O(\varepsilon^{N+1})$, we conclude that $\|R_{N+1}\|_{k,\varepsilon}^1$ is $O(\varepsilon^{N+1})$.

We have therefore proved the following theorem.

THEOREM 2. *Let $g(t)$ be continuous and periodic with period 2π on $[0, \infty)$. Then the solution to the initial value problem*

$$y''(t) + \varepsilon g(t)y'(t) + y(t) = 0; \qquad y(0) = a, \quad y'(0) = b$$

satisfies

$$\left\| y - \sum_{j=0}^{N} y^j(t, \tau)\varepsilon^j \right\|_{k,\varepsilon}^1 = O(\varepsilon^{N+1}),$$

where $y^j(t, \tau)$ are the functions defined above.

4. A two-timing method for autonomous nonlinear oscillations. The autonomous nonlinear initial value problem

$$y'' + \varepsilon f(y, y') + y = 0; \qquad y(0) = a, \quad y'(0) = b \tag{4.1}$$

is discussed at some length by Kevorkian [3]. He presents a method for computing a series expansion for the solution to (4.1) and states without proof that the series is asymptotically correct on intervals of length k/ε. In this section a modification of his method is presented together with a proof that the resulting expansion is uniformly asymptotically valid on intervals of length inversely proportional to ε. For sufficiently small ε, the existence of solutions to equation (4.1) on expanding intervals is also established as a direct consequence of the constructive nature of the proof of asymptotic validity.

The formal asymptotic expansion. We again seek to express the solution to (4.1) (assuming its existence) by means of a series

$$y = \sum_{j \geqq 0} y^j(t, \tau)\varepsilon^j. \tag{4.2}$$

We assume that f is a polynomial.

By substituting (4.2) into (4.1) and equating the coefficient of ε^0 to zero, we get

$$y_{tt}^0 + y^0 = 0; \qquad y^0(0, 0) = a, \quad y_t^0(0, 0) = b,$$

which has the solution

$$y^0(t, \tau) = A_0(\tau) \sin t + B_0(\tau) \cos t; \qquad A_0(0) = b, \quad B_0(0) = a. \tag{4.3}$$

Using this function as a first approximation to y, $y^1(t, \tau)$ must then satisfy

$$y_{tt}^1 + y^1 = -2y_{t\tau}^0 - \{\varepsilon f(y^0(t, \tau), y_t^0(t, \tau) + \varepsilon y_\tau^0(t, \tau))\}_1, \tag{4.4}$$

(4.5) $$y^1(0,0) = 0, \quad y_t^1(0,0) = -y_\tau^0(0,0),$$

where the notation $\{\cdot\}_i$ is used to denote the coefficient of ε^i in the expansion of the quantity in brackets. As in the linear case the quantity on the right-hand side of (4.4) is periodic with period 2π in t and can be expanded as a finite Fourier series with coefficients dependent on τ. The coefficients of $\sin t$ and $\cos t$ are then set equal to zero, eliminating resonance in equation (4.4). This yields a first order system of equations (generally nonlinear) for $A_0(\tau)$ and $B_0(\tau)$ which together with the initial conditions (4.3) determine $A_0(\tau)$ and $B_0(\tau)$ on some τ interval $[0, k]$. (k depends on the problem since only local existence is guaranteed for solutions of the nonlinear system determining A_0 and B_0.) Having determined A_0 and B_0, we solve equation (4.4) and find that $y^1(t, \tau)$ has the form $y_p^1(t, \tau) + A_1(\tau)\sin t + B_1(\tau)\cos t$, with $y_p^1(t, \tau)$ bounded on $[0, k/\varepsilon] \times [0, k]$. (4.5) yields initial conditions for A_1 and B_1. $A_1(\tau)$ and $B_1(\tau)$ are evaluated by setting the coefficients of $\sin t$ and $\cos t$ in the expansion of the right-hand side of

$$y_{tt}^2 + y^2 = -2y_{t\tau}^1 - y_{\tau\tau}^0 - \{\varepsilon f(y^0(t,\tau) + \varepsilon y^1(t,\tau), y_t^0 + \varepsilon y_t^1 + \varepsilon y_\tau^0 + \varepsilon^2 y_\tau^1)\}_2$$

equal to zero and solving an initial value problem, which is linear.

More generally, the method proceeds as follows: Substitution of (4.2) in (4.1) and equating the coefficient of ε^j to zero yields

(4.6) $$y_{tt}^j + y^j = -2y_{t\tau}^{j-1} - y_{\tau\tau}^{j-2} - \{\varepsilon f(Z^{j-1}, Z^{j-1\prime})\}_j,$$

(4.7) $$y^j(0,0) = a\delta_{j0}, \quad y_t^j(0,0) = b\delta_{j0} - y_\tau^{j-1}(0,0),$$

where Z^j denotes $\sum_{i=0}^j y^i\varepsilon^i$. The right-hand side of (4.6) is expanded in a Fourier series and the coefficients of $\sin t$ and $\cos t$ are set equal to zero, yielding a system of equations for the functions A_{j-1} and B_{j-1} multiplying $\sin t$ and $\cos t$ respectively in y^{j-1}. Once A_{j-1} and B_{j-1} are determined, equation (4.6) is solved. Then y_{tt}^j is of the form $y_p^j + A_j(\tau)\sin t + B_j(\tau)\cos t$, and the procedure is repeated.

It is important to note that the system of equations determining $A_j(\tau)$ and $B_j(\tau)$ will be linear for $j > 0$.

Proof of asymptotic validity. Suppose that the method outlined above is applied to equation (4.1) to obtain the expansion $\sum_{j=0}^N y^j(t,\tau)\varepsilon^j$. As mentioned earlier this quantity is denoted by Z^N. Supposing y to be a solution to problem (4.1), denote $y - Z^N$ by $\varepsilon^{N+1}R_{N+1}$, that is,

$$y = Z^N + \varepsilon^{N+1}R_{N+1}.$$

Direct substitution of this into (4.1) and use of equations (4.6) yields the following equation for R_{N+1}:

$$\varepsilon^{N+1}(R_{N+1}'' + R_{N+1}) = \sum_{i>N}\{-\varepsilon f(Z^N, Z^{N\prime})\}_i\varepsilon^i - 2\varepsilon^{N+1}y_{t\tau}^N - \varepsilon^{N+1}y_{\tau\tau}^{N-1}$$

(4.8) $$-\varepsilon^{N+2}y_{\tau\tau}^N - \varepsilon[f(y, y') - f(Z^N, Z^{N\prime})],$$

$$R_{N+1}(0) = 0, \quad R_{N+1}'(0) = -y_\tau^N(0,0).$$

The series on the right-hand side of (4.8) has only a finite number of terms since f is a polynomial. If we let

$$h(t) = \sum_{i>N}\{-\varepsilon f(Z^N, Z^{N\prime})\}_i\varepsilon^i - 2\varepsilon^{N+1}y_{t\tau}^N - \varepsilon^{N+1}y_{\tau\tau}^{N-1} - \varepsilon^{N+2}y_{\tau\tau}^N$$

and

$$\mathscr{F}(R_{N+1}) = \frac{f(y, y') - f(Z^N, Z^{N'})}{\varepsilon^{N+1}},$$

equation (4.8) can be rewritten as

$$\begin{aligned} &R''_{N+1} + \varepsilon f_2(Z^N, Z^{N'})R'_{N+1} + (1 + \varepsilon f_1(Z^N, Z^{N'}))R_{N+1} \\ &\quad = \frac{h(t)}{\varepsilon^{N+1}} - \varepsilon[\mathscr{F}(R_{N+1}) - f_1(Z^N, Z^{N'})R_{N+1} - f_2(Z^N, Z^{N'})R'_{N+1}], \\ &R_{N+1}(0) = 0, \quad R'_{N+1}(0) = -y^N_\tau(0,0), \end{aligned} \tag{4.9}$$

where f_i denotes the partial derivative of f with respect to the ith variable, $i = 1, 2$. Note that the quantity $h(t)/\varepsilon^{N+1}$ occurring in (4.9) is in $C^0_{k,\varepsilon}$.

Now write $R_{N+1} = P + Q$ and let P be the solution to

$$\begin{aligned} &P'' + \varepsilon f_2(Z^N, Z^{N'})P' + [1 + \varepsilon f_1(Z^N, Z^{N'})]P = \frac{h(t)}{\varepsilon^{N+1}}, \\ &P(0) = 0, \quad P'(0) = -y^N_\tau(0,0). \end{aligned} \tag{4.10}$$

By Theorem 1, εP is in $C^1_{k,\varepsilon}$, that is, both P and P' are $O(1/\varepsilon)$ on $[0, k/\varepsilon]$. Then Q is a solution of

$$\begin{aligned} &Q'' + \varepsilon f_2(Z^N, Z^{N'})Q' + [1 + \varepsilon f_1(Z^N, Z^{N'})]Q \\ &\quad = -\varepsilon[\mathscr{F}(R_{N+1}) - f_1(Z^N, Z^{N'})(P + Q) - f_2(Z^N, Z^{N'})(P' + Q')], \\ &Q(0) = 0, \quad Q'(0) = 0. \end{aligned} \tag{4.11}$$

Letting $G(t, s)$ be the Green's function for the linear operator $L(y) = y'' + \varepsilon f_2(Z^N, Z^{N'})y' + (1 + \varepsilon f_1(Z^N, Z^{N'}))y$, equation (4.11) is equivalent to

$$\begin{aligned} Q(t) &= -\varepsilon \int_0^t G(t, s)[\mathscr{F}(R_{N+1}) - f_1(Z^N, Z^{N'})(P + Q) - f_2(Z^N, Z^{N'})(P' + Q')]\, ds, \\ Q'(t) &= -\varepsilon \int_0^t G_t(t, s)[\mathscr{F}(R_{N+1}) - f_1(Z^N, Z^{N'})(P + Q) - f_2(Z^N, Z^{N'})(P' + Q')]\, ds, \end{aligned} \tag{4.12}$$

while $G(t, s)$ and $G_t(t, s)$ are both bounded on $[0, k/\varepsilon]$ by Corollary 1.

For $N \geqq 1$, Taylor's theorem for functions of two variables can be used to show that if S is any function such that εS is in $C^1_{k,\varepsilon}$, then

$$\mathscr{F}(P + S) - f_1(Z^N, Z^{N'})(P + S) - f_2(Z^N, Z^{N'})(P' + S') = O(\varepsilon^{N-1}) \quad \text{on } [0, k/\varepsilon] \tag{4.13}$$

with

$$\mathscr{F}(P + S) = \frac{f\{Z^N + \varepsilon^{N+1}(P + S), Z^{N'} + (P' + S')\} - f(Z^N, Z^{N'})}{\varepsilon^{N+1}}.$$

Using the method of successive approximations, we can then show that the system of equations (4.12) has a solution in $C^1_{k,\varepsilon}$, as follows.

Let $Q_0(t) \equiv 0$, and define recursively

$$
(4.14)\qquad
\begin{aligned}
Q_{n+1}(t) &= -\varepsilon \int_0^t G(t,s)[\mathscr{F}(P+Q_n) - f_1(Z^N, Z^{N'})(P+Q_n) \\
&\qquad\qquad - f_2(Z^N, Z^{N'})(P'+Q_n')]\,ds, \\
Q_{n+1}'(t) &= -\varepsilon \int_0^t G_t(t,s)[\mathscr{F}(P+Q_n) - f_1(Z^N, Z^{N'})(P+Q_n) \\
&\qquad\qquad - f_2(Z^N, Z^{N'})(P'+Q_n')]\,ds.
\end{aligned}
$$

By (4.13) and Corollary 1, $Q_1(t)/\varepsilon^{N-1} \in C^1_{k,\varepsilon}$.

Furthermore it can be established (see concluding paragraph of proof) that

$$
(4.15)\qquad \|Q_{n+1} - Q_n\|^1_{k,\varepsilon} \leqq [M\varepsilon\|Q_n - Q_{n-1}\|^1_{k,\varepsilon} + K]\|Q_n - Q_{n-1}\|^1_{k,\varepsilon}\varepsilon^N.
$$

Since for $N \geqq 1$, $\|Q_1\|^1_{k,\varepsilon}$ is bounded independent of ε, we find

$$
\|Q_2 - Q_1\|^1_{k,\varepsilon} \leqq C\|Q_1\|^1_{k,\varepsilon}\varepsilon^N < \|Q_1\|^1_{k,\varepsilon}
$$

for ε sufficiently small (take $C = M\|Q_1\|^1_{k,\varepsilon} + K$). Then

$$
\|Q_n - Q_{n-1}\|^1_{k,\varepsilon} < (C\varepsilon^N)^{n-1}\|Q_1\|^1_{k,\varepsilon},
$$

so the series $\sum_{n\geqq 1}\|Q_n - Q_{n-1}\|^1_{k,\varepsilon}$ is convergent, and the sequence $\{Q_n(t)\}$ converges uniformly to a function Q in $C^1_{k,\varepsilon}$ which is a solution to (4.12).

Since $\varepsilon P \in C^1_{k,\varepsilon}$ and $Q \in C^1_{k,\varepsilon}$, it is seen that $\|y - Z^N\|^1_{k,\varepsilon} = O(\varepsilon^N)$. But arguing as in the linear case, we conclude that in fact, $\|y - Z^N\|^1_{k,\varepsilon} = O(\varepsilon^{N+1})$ and the formal expansion is asymptotically valid on expanding intervals.

Equation (4.15) will now be established:

$$
\begin{aligned}
Q_{n+1} - Q_n = -\varepsilon \int_0^t G(t,s)[&\mathscr{F}(P+Q_n) - \mathscr{F}(P+Q_{n-1}) - f_1(Z^N, Z^{N'})(Q_n - Q_{n-1}) \\
&- f_2(Z^N, Z^{N'})(Q_n' - Q_{n-1}')]\,ds
\end{aligned}
$$

and the quantity multiplying $G(t,s)$ in the integrand can be simplified to

$$
(4.16)\qquad
\frac{\begin{aligned}
&f(Z^N + \varepsilon^{N+1}(P+Q_n), Z^{N'} + \varepsilon^{N+1}(P'+Q_n')) \\
&- f(Z^N + \varepsilon^{N+1}(P+Q_{n-1}), Z^{N'} + \varepsilon^{N+1}(P'+Q_{n-1}')) \\
&- f_1(Z^N, Z^{N'})\varepsilon^{N+1}(Q_n - Q_{n-1}) - f_2(Z^N, Z^{N-1})\varepsilon^{N+1}(Q_n' - Q_{n-1}')
\end{aligned}}{\varepsilon^{N+1}}.
$$

The following general argument completes the proof.

Let g be a twice continuously differentiable function of two variables. Then

$$
\begin{aligned}
&g(A+B+D) - g(A+B+C) - \nabla g(A)\cdot(D-C) \\
&\quad = [\nabla g(A+B+C+\theta_1(D-C)) - \nabla g(A)]\cdot(D-C) \\
&\quad = [\nabla g_1(A+\theta_2(B+C+\theta_1(D-C)))\cdot(B+C+\theta_1(D-C)), \\
&\qquad\quad \nabla g_2(A+\theta_3(B+C+\theta_1(D-C)))\cdot(B+C+\theta_1(D-C))]\cdot(D-C),
\end{aligned}
$$

where the capital letters denote two-dimensional vectors and the thetas are elements of $(0, 1)$.

Applying this result to (4.16) with $g = f$, $A = (Z^N, Z^{N'})$, $B = \varepsilon^{N+1}(P, P')$, $D = \varepsilon^{N+1}(Q_n, Q_n')$ and $C = \varepsilon^{N+1}(Q_{n-1}, Q_{n-1}')$, and using Schwarz' inequality yields (4.15).

THEOREM 3. *Let f be a polynomial and consider the initial value problem*

$$y'' + \varepsilon f(y, y') + y = 0; \qquad y(0) = a, \quad y'(0) = b.$$

For ε sufficiently small, there exists $k > 0$ such that this problem has a unique solution on $[0, k/\varepsilon]$. Furthermore, letting $y^j(t, \tau)$ be the functions defined above,

$$\left\| y - \sum_{j=0}^{N} y^j(t, \tau)\varepsilon^j \right\|_{k,\varepsilon}^{1} = O(\varepsilon^{N+1}).$$

Proof. Only the existence remains to be proved. It is an immediate consequence of the existence of Z^N satisfying the iterative process and the fact that P and Q satisfy (4.10) and (4.12) respectively. Then $y = Z^N + \varepsilon^{N+1}(P + Q)$ is the solution.

5. Example. In this section an example is presented to illustrate the methods described earlier. This example has also been treated by Kevorkian's method in [3]. Kevorkian works with the initial conditions $y(0) = a$, $y'(0) = 0$, claiming that any problem can be reduced to this. While in principle this is accurate, one cannot then retrieve the solution to the original problem. For this reason, the computations here are carried out with initial conditions $y(0) = a$, $y'(0) = b$. This generality leads to coupled systems of equations for the functions of τ. These equations, though much harder to solve than the uncoupled systems arising for the simplified initial conditions, yield explicit information about the dependence of the solutions on the initial conditions.

Example 1. Consider the initial value problem

$$y'' + \varepsilon(y')^3 + y = 0; \qquad y(0) = a, \quad y'(0) = b. \tag{5.1}$$

For this example,

$$\{\varepsilon f(y, y')\}_1 = (y_t^0)^3, \tag{5.2a}$$

$$\{\varepsilon f(y, y')\}_2 = 3(y_t^0)^2(y_\tau^0 + y_t^1). \tag{5.2b}$$

As always,

$$y^0(t, \tau) = A_0(\tau)\sin t + B_0(\tau)\cos t \tag{5.3}$$

and $A_0(\tau)$ and $B_0(\tau)$ must satisfy

$$A_0(0) = b, \quad B_0(0) = a. \tag{5.4}$$

Inserting (5.2a) and (5.3) into (4.4) and expanding the right-hand side in a Fourier series yields

$$\begin{aligned} y_{tt}^1 + y^1 &= [-2A_0' - \tfrac{3}{4}A_0^3 - \tfrac{3}{4}A_0B_0^2]\cos t \\ &\quad + [2B_0' + \tfrac{3}{4}A_0^2B_0 + \tfrac{3}{4}B_0^3]\sin t \\ &\quad + \left[-\frac{A_0^3}{4} + \frac{3}{4}A_0B_0^2\right]\cos 3t \\ &\quad + \left[\frac{3}{4}A_0^2B_0 - \frac{B_0^3}{4}\right]\sin 3t. \end{aligned} \tag{5.5}$$

Solving

$$2A_0' + \tfrac{3}{4}A_0^3 + \tfrac{3}{4}A_0B_0^2 = 0,$$
$$2B_0' + \tfrac{3}{4}A_0^2B_0 + \tfrac{3}{4}B_0^3 = 0$$

with initial conditions (5.4) yields

$$A_0(\tau) = 2b[3(a^2 + b^2)\tau + 4]^{-1/2},$$
$$B_0(\tau) = 2a[3(a^2 + b^2)\tau + 4]^{-1/2},$$

so

$$y^0(t, \tau) = [3(a^2 + b^2)\tau + 4]^{-1/2}(2b \sin t + 2a \cos t). \tag{5.6}$$

With this choice of $y^0(t, \tau)$, (5.5) reduces to

$$y_{tt}^1 + y^1 = \left[\frac{3A_0B_0^2}{4} - \frac{A_0}{4}\right]\cos 3t + \left[\frac{3}{4}A_0^2B_0 - \frac{B_0^3}{4}\right]\sin 3t,$$

and upon solving and substituting the known values of A_0 and B_0,

$$\begin{aligned} y^1(t, \tau) = {} & \left[\frac{b^3 - 3ba^2}{4[3(a^2 + b^2)\tau + 4]^{3/2}}\right]\cos 3t + \left[\frac{a^3 - 3ab^2}{4[3(a^2 + b^2)\tau + 4]^{3/2}}\right]\sin 3t \\ & + A_1(\tau)\sin t + B_1(\tau)\cos t. \end{aligned} \tag{5.7}$$

Equations (4.5) then yield

$$A_1(0) = \frac{9a^3 + 21ab^2}{32}, \qquad B_1(0) = \frac{3ba^2 - b^3}{32}. \tag{5.8}$$

Use of (5.2b), (5.6), and (5.7) in (4.6) yields

$$\begin{aligned} y_{tt}^2 + y^2 = {} & \left[-2A_1' - \frac{[9b^2 + 3a^2]A_1}{[3(a^2 + b^2)\tau + 4]} - \frac{6abB_1}{[3(a^2 + b^2)\tau + 4]}\right. \\ & \left. - \frac{9a(a^2 + b^2)^2}{4[3(a^2 + b^2)\tau + 4]^{3/2}}\right]\cos t \\ & + \left[2B_1' + \frac{6abA_1}{[3(a^2 + b^2)\tau + 4]} + \frac{(3b^2 + 9a^2)B_1}{[3(a^2 + b^2)\tau + 4]}\right. \\ & \left. - \frac{9b(a^2 + b^2)^2}{4[3(a^2 + b^2)\tau + 4]^{3/2}}\right]\sin t \\ & + \text{terms involving } \cos 3t, \sin 3t, \cos 5t, \text{ and } \sin 5t. \end{aligned}$$

We therefore require that $A_1(\tau)$ and $B_1(\tau)$ be solutions to the coupled system of equations

$$\begin{aligned} 2A_1'(\tau) + \frac{[9b^2 + 3a^2]A_1}{[3(a^2 + b^2)\tau + 4]} + \frac{6abB_1}{[3(a^2 + b^2)\tau + 4]} + \frac{9}{4}\frac{a(a^2 + b^2)^2}{[3(a^2 + b^2)\tau + 4]^{3/2}} = 0, \\ 2B_1'(\tau) + \frac{6abA_1}{[3(a^2 + b^2)\tau + 4]} + \frac{[9a^2 + 3b^2]B_1}{[3(a^2 + b^2)\tau + 4]} - \frac{9}{4}\frac{b(a^2 + b^2)^2}{[3(a^2 + b^2)\tau + 4]^{3/2}} = 0 \end{aligned} \tag{5.9}$$

with initial conditions (5.8). This problem is equidimensional, and the change of variable

$$e^s = 3(a^2 + b^2)\tau + 4 \tag{5.10}$$

reduces this system to

$$\begin{aligned} 6(a^2 + b^2)A_1' + (3a^2 + 9b^2)A_1 + 6abB_1 &= -\tfrac{9}{4}a(a^2 + b^2)^2 e^{-(3/2)s}, \\ 6(a^2 + b^2)B_1' + 6abA_1 + (9a^2 + 3b^2)B_1 &= \tfrac{9}{4}b(a^2 + b^2) e^{-(3/2)s}, \end{aligned} \tag{5.11}$$

where prime now represents differentiation with respect to s. The general solution of (5.11) is

$$\begin{aligned} A_1 &= \tfrac{3}{8}a(a^2 + b^2) e^{-(3/2)s} + c_1(a e^{-(3/2)s}) + c_2(b e^{-(3/2)s}), \\ B_1 &= -\tfrac{3}{8}b(a^2 + b^2) e^{-(3/2)s} + c_1(-b e^{-(3/2)s}) + c_2(a e^{-(3/2)s}). \end{aligned} \tag{5.12}$$

Equations (5.12), (5.10), and (5.8) together yield

$$A_1(\tau) = \frac{3a^5 + 30a^3b^2 + 67ab^4}{8(a^2 + b^2)[3(a^2 + b^2)\tau + 4]^{3/2}} + \frac{15a^5 + 30a^3b^2 - ab^4}{32[3(a^2 + b^2)\tau + 4]^{1/2}(a^2 + b^2)},$$

$$B_1(\tau) = \frac{21a^4b + 34a^2b^3 - 3b^5}{8(a^2 + b^2)[3(a^2 + b^2)\tau + 4]^{3/2}} + \frac{b^5 - 30a^2b^3 - 15a^4b}{32[3(a^2 + b^2)\tau + 4]^{1/2}(a^2 + b^2)}.$$

We therefore conclude that the solution to (5.1) correct on $[0, k/\varepsilon]$ is

$$\begin{aligned} y = {} & [3(a^1 + b^2)\tau + 4]^{-1/2}[2b \sin t + 2a \cos t] \\ & + \varepsilon \left[\begin{aligned} & \left(\frac{3a^5 + 30a^3b^2 + 67ab^4}{8(a^2 + b^2)[3(a^2 + b^2)\tau + 4]^{3/2}} + \frac{15a^5 + 30a^3b^2 - ab^4}{32[3(a^2 + b^2)\tau + 4]^{1/2}(a^2 + b^2)} \right) \sin t \\ & + \left(\frac{21a^4b + 34a^2b^3 + 3b^5}{8(a^2 + b^2)[3(a^2 + b^2)\tau + 4]^{3/2}} + \frac{b^5 - 30a^2b^3 - 15a^4b}{32[3(a^2 + b^2)\tau + 4]^{1/2}(a^2 + b^2)} \right) \cos t \\ & + \frac{(b^3 - 3ba^2)}{4[3(a^2 + b^2)\tau + 4]^{3/2}} \cos 3t + \frac{(a^3 - 3b^2a)}{4[3(a^2 + b^2)\tau + 4]^{3/2}} \sin 3t \end{aligned} \right] \\ & + O(\varepsilon^2). \end{aligned}$$

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LIE THEORY AND SEPARATION OF VARIABLES. 1: PARABOLIC CYLINDER COORDINATES*

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Abstract. Winternitz and co-workers have characterized the parabolic cylinder function solutions of the reduced wave equation in two variables as eigenfunctions of a quadratic operator $E = MP_2 + P_2M$ in the enveloping algebra of the Lie algebra of the Euclidean group in the plane. Here we study the representation theory of the Euclidean and pseudo-Euclidean groups in an E-basis and use the results to derive a number of new addition and expansion theorems for products of parabolic cylinder functions.

Introduction. In the papers [1] and [2], Winternitz and Friš and Winternitz, Lukăc and Smorodinskiĭ have introduced a group theoretic method for the description of separation of variables in the principal partial differential equations of mathematical physics. In this paper we apply their idea to the reduced wave equation in two variables and the separation of this equation in parabolic cylinder coordinates.

The relevant group is $E(2)$, the Euclidean group in the plane. Its Lie algebra $\mathscr{E}(2)$ is three-dimensional with basis P_1, P_2, M and commutation relations

$$[M, P_1] = P_2, \quad [M, P_2] = -P_1, \quad [P_1, P_2] = 0.$$

A two-variable model of this Lie algebra is

$$P_1 = \partial_x, \quad P_2 = \partial_y, \quad M = y\partial_x - x\partial_y$$

and the reduced wave equation is

$$(*) \qquad (P_1^2 + P_2^2)f(x, y) = -\omega^2 f(x, y),$$

where ω is a nonzero constant. In [1] and [2] the authors characterize solutions f of $(*)$ by requiring in addition that f be an eigenfunction of a quadratic operator L in the universal enveloping algebra $\mathscr{U}$ of $\mathscr{E}(2)$: $Lf = \lambda f$.

More precisely, let $\mathscr{S}$ be the space of symmetric quadratic elements in $\mathscr{U}$ and let $\mathscr{C}$ be the center of $\mathscr{U}$. The group $E(2)$ acts on $\mathscr{S}$ and $\mathscr{C}$ via the adjoint representation and leaves these vector spaces invariant. Hence the vector space $\mathscr{T} = \mathscr{S}/\mathscr{S} \cap \mathscr{C}$ is also invariant under the adjoint representation. (In this case $\mathscr{S} \cap \mathscr{C} = \{\alpha(P_1^2 + P_2^2)\}$.) Thus $\mathscr{T}$ is decomposed into orbits under the group action. In [1] it is shown that there are exactly four orbits, up to multiplication by a scalar, and that these orbits correspond exactly to the four coordinate systems in which $(*)$ is separable. If L belongs to one of these orbits the corresponding solutions of $(*)$ in which variables separate are determined by $Lf = \lambda f$. The parameter λ corresponds to a separation constant.

In this paper we will be concerned with the orbit associated with parabolic cylinder coordinates: $x = (\xi^2 - \eta^2)/2, y = \xi\eta$. For L we will choose the operator $E = MP_2 + P_2M$ on the orbit.

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We study the spectral resolution of E corresponding to irreducible representations of $E(2)$ and of the closely related groups $CE(2)$, the complex Euclidean group, and $PE(2)$, the pseudo-Euclidean group in the plane. We also determine the matrix elements of the representation operators with respect to an E-basis. Past treatments of the representation theory of these groups have used P- and M-bases and have led to addition theorems for Bessel functions. Here we obtain addition theorems for products of parabolic cylinder functions. We also construct two-variable models of these irreducible representations which lead us naturally to solutions of (*) in parabolic cylinder coordinates.

Finally we show how to decompose the quasi-regular representation of $E(2)$ in terms of an E-basis.

Some special cases of the addition theorem in § 2 were discovered by Epstein [3] and most of the plane wave and cylindrical wave expansion theorems can be found in Buchholz [4]. However, the general addition theorems appear to be new as does the explicit group theoretic and functional analytic significance of the results.

1. The complex Euclidean group $CE(2)$. Let $\mathscr{CE}(2)$ be the Lie algebra of the complex Euclidean group in the plane. There is a basis for $\mathscr{CE}(2)$ such that

$$[M, Q_1] = Q_2, \quad [M, Q_2] = Q_1, \quad [Q_1, Q_2] = 0. \tag{1.1}$$

The complex Euclidean group $CE(2)$ is the matrix group with elements

$$\begin{pmatrix} 1 & 0 & 0 & \theta \\ 0 & \cosh\theta & \sinh\theta & a \\ 0 & \sinh\theta & \cosh\theta & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \theta, a, b \in \mathbb{C}. \tag{1.2}$$

The Lie algebra $\mathscr{CE}(2)$ can be related to $CE(2)$ via the exponential mapping and the formula

$$g(\theta, a, b) = \exp(aQ_1 + bQ_2)\exp(\theta M). \tag{1.3}$$

As is well known [5], corresponding to constants $\omega, \alpha \in \mathbb{C}$ such that $\omega \neq 0$, $0 \leqq \operatorname{Re}\alpha < 1$, there exists an algebraically irreducible representation $\rho(\omega, \alpha)$ of $\mathscr{CE}(2)$ such that

$$\begin{gathered} Mf_m = mf_m, \quad P^+f_m = \omega f_{m+1}, \quad P^-f_m = \omega f_{m-1}, \\ m = \alpha + n, \quad n = 0, \pm 1, \pm 2, \cdots. \end{gathered} \tag{1.4}$$

Here $P^\pm = Q_1 \pm Q_2$, $\{f_m\}$ is a basis for the representation space $\mathscr{V}$ of $\rho(\omega, \alpha)$ and M, Q_1, Q_2 are considered as linear operators on $\mathscr{V}$.

A simple one-variable model of $\rho(\omega, \alpha)$ is given by the assignment [5, p. 50],

$$M = z\partial_z + \alpha, \quad P^+ = \omega z, \quad P^- = \omega/z, \quad f_{m_0+n}(z) = z^n, \tag{1.5}$$

where $\mathscr{V}$ is the space of functions $f(z)$ analytic in a deleted neighborhood of $z = 0$.

Using this model we study the eigenvalue problem

$$\begin{aligned} &Eh = 2\lambda\omega h, \qquad h \in \mathscr{V}, \lambda \in \mathbb{C}, \\ (1.6) \qquad &E = MQ_2 + Q_2M = 2Q_2M + Q_1, \\ \text{i.e.,} \qquad &(z^2-1)\frac{dh}{dz} + [(\alpha+\tfrac{1}{2})z - (\alpha - \tfrac{1}{2})z^{-1}]h = 2\lambda h. \end{aligned}$$

For given $\lambda \in \mathbb{C}$ the solution is $h_\lambda(z) = z^{1/2-\alpha}(1-z)^{\lambda-1/2}(1+z)^{-\lambda-1/2}$, unique to within a multiplicative constant. Restricting z to the domain $0 < |z| < 1$ we find $h \in \mathscr{V}$ if and only if $\alpha = \frac{1}{2}$. Thus eigenfunctions of E exist only for the representations $\rho(\omega, \frac{1}{2})$ and are given by

$$\begin{aligned} &h_\lambda(z) = (1-z)^{\lambda-1/2}(1+z)^{-\lambda-1/2} = \sum_{n=0}^{\infty} c_n(\lambda) f_{n+1/2}(z), \\ (1.7) \qquad &c_n(\lambda) = \sum_{k=0}^{\infty}(-1)^k \binom{\lambda-\frac{1}{2}}{k}\binom{-\lambda-\frac{1}{2}}{n-k} = \binom{-\lambda-\frac{1}{2}}{n} {}_2F_1\left(\begin{matrix}\frac{1}{2}-\lambda, -n \\ \frac{1}{2}-\lambda-n\end{matrix}\,\middle|\, -1\right) \end{aligned}$$

for $\lambda \in \mathbb{C}$. (However, expansion (1.7) makes sense formally for all α.)

Consider the linear transformation S from the space $z^{1/2-\alpha}\mathscr{W}$ to the space $\mathscr{W}$ of all functions analytic in $0 < |z| < 1$, defined by

$$f'(z) = Sf(z) = z^{\alpha-1/2}(1-z^2)^{1/2}f(z) \in \mathscr{W}$$

for $f \in z^{1/2-\alpha}\mathscr{W}$. Defining operators $M' = SMS^{-1}, (P^\pm)' = SP^\pm S^{-1}$ where M, $P^\pm$ are given by (1.5), we find

$$\begin{aligned} &M' = \tfrac{1}{4}(\xi + \xi^{-1}) + \tfrac{1}{2}(\xi^2 - 1)\frac{d}{d\xi}, \qquad P^{+\prime} = \omega\left(\frac{1-\xi}{1+\xi}\right), \qquad P^{-\prime} = \omega\left(\frac{1+\xi}{1-\xi}\right), \\ (1.8) \qquad &\xi = \frac{1-z}{1+z}, \qquad h'_\lambda(\xi) = \xi^\lambda. \end{aligned}$$

Let $\beta \in \mathbb{C}$ with $0 \leqq \operatorname{Re}\beta < 1$ and consider the space $\mathscr{W}_\beta$ consisting of all functions $h(\xi) = \sum_{n=-\infty}^{\infty} k_n \xi^{\beta+n}$, where the Laurent series converges in a deleted neighborhood of $\xi = 0$. Clearly, the operators (1.8) define a representation of $\mathscr{C}\mathscr{E}(2)$ on $\mathscr{W}_\beta$ which we denote $\mu(\omega, \beta)$. The functions

$$(1.9) \qquad j_{\beta+s}(\xi) = \xi^{\beta+s}, \qquad s = 0, \pm 1, \pm 2, \cdots,$$

define a basis for $\mathscr{W}_\beta$ and it is easy to show that (for $\beta \neq \frac{1}{2}$) the action of the operators (1.8) on this basis determines a representation not equivalent to any $\rho(\omega, \alpha)$. For $\beta = \frac{1}{2}$ this representation is reducible. The action of the operators (1.8) on the basis $j_\lambda(\xi) = \xi^\lambda$, $\lambda = \beta + s, s = 0, \pm 1, \pm 2, \cdots$, is

$$\begin{aligned} &M'j_\lambda = \tfrac{1}{4}(1+2\lambda)j_{\lambda+1} + \tfrac{1}{4}(1-2\lambda)j_{\lambda-1}, \\ (1.10) \qquad &\omega^{-1}P^{+\prime}j_\lambda = j_\lambda + 2\sum_{k=1}^{\infty}(-1)^k j_{\lambda+k}, \\ &\omega^{-1}P^{-\prime}j_\lambda = j_\lambda + 2\sum_{k=1}^{\infty} j_{\lambda+k}. \end{aligned}$$

The operators (1.8) acting on $\mathscr{W}_\beta$ induce a local multiplier representation $\mathbf{R}(\theta, a, b)$ of $CE(2)$ given by

$$\mathbf{R}(\theta, a, b) = \exp(aQ_1' + bQ_2')\exp(\theta M').$$

A straightforward computation [5] yields

$$[\mathbf{R}(\theta, a, b)j](\xi) = \left[\left(\cosh\frac{\theta}{2} - \xi\sinh\frac{\theta}{2}\right)\left(\cosh\frac{\theta}{2} - \xi^{-1}\sinh\frac{\theta}{2}\right)\right]^{-1/2}$$
$$\cdot\exp\left\{\frac{\omega}{1-\xi^2}[b(1+\xi^2) - a\xi]\right\} j\left(\frac{\xi\cosh(\theta/2) - \sinh(\theta/2)}{\cosh(\theta/2) - \xi\sinh(\theta/2)}\right).$$

We define the matrix elements $R(\theta, a, b)_{mn}$ of $\mathbf{R}$ with respect to the basis $\{j_{\beta+n} : n = 0, \pm 1, \pm 2, \cdots\}$ by

$$\mathbf{R}(\theta, a, b)j_{\beta+n} = \sum_{m=-\infty}^{\infty} R(\theta, a, b)_{mn} j_{\beta+m} \tag{1.11}$$

or

$$\begin{aligned}&\exp\left\{\frac{\omega}{1-\xi^2}[b(1+\xi^2) - 2a\xi]\right\}\left(\cosh\frac{\theta}{2} - \xi^{-1}\sinh\frac{\theta}{2}\right)^{\beta+n-1/2}\\ &\cdot\left(\cosh\frac{\theta}{2} - \xi\sinh\frac{\theta}{2}\right)^{-\beta-n-1/2} = \sum_{m=-\infty}^{\infty} R(\theta, a, b)_{mn}\xi^{m-n}.\end{aligned} \tag{1.12}$$

Clearly, the matrix elements satisfy the group property

$$\begin{aligned}&R(\theta + \theta', a + a'\cosh\theta + b'\sinh\theta, b + a'\sinh\theta + b'\cosh\theta)_{mn}\\ &\quad = \sum_{k=-\infty}^{\infty} R(\theta, a, b)_{mk} R(\theta', a', b')_{kn}.\end{aligned} \tag{1.13}$$

Using well-known generating functions for Laguerre and hypergeometric functions we obtain the following explicit expressions:

$$\begin{aligned}
R(0, a, a)_{mn} &= \begin{cases} 0 & \text{if } m < n,\\ e^{\omega a}(-1)^{m-n}L_{m-n}^{(-1)}(-2\omega a) & \text{if } m \geqq n,\end{cases}\\
R(0, -b, b)_{mn} &= \begin{cases} 0 & \text{if } m < n,\\ e^{\omega b}L_{m-n}^{(-1)}(-2\omega b) & \text{if } m \geqq n,\end{cases}\\
R(\theta, 0, 0)_{mn} &= \left(\cosh\frac{\theta}{2}\right)^{m-n-1}\left(-\sinh\frac{\theta}{2}\right)^{m-n}\frac{\Gamma(\beta + n + \frac{1}{2})}{\Gamma(\beta + m + \frac{1}{2})\Gamma(m - n + 1)}\\
&\quad\cdot {}_2F_1\left(\begin{matrix}-\beta - n + \frac{1}{2},\ \beta + m + \frac{1}{2}\\ m - n + 1\end{matrix}\,\middle|\, \tanh^2\frac{\theta}{2}\right),\\
&\qquad\qquad\left|\tanh\frac{\theta}{2}\right| < 1 < \left|\coth\frac{\theta}{2}\right|,\\
R(0, a, b)_{mn} &= \begin{cases} 0 & \text{if } m < n,\\ e^{\omega b}\displaystyle\sum_{l=n}^{m}(-1)^{m-l}L_{m-l}^{(-1)}(-\omega[a+b])L_{l-n}^{(-1)}(\omega[a-b]) & \text{if } m \geqq n.\end{cases}
\end{aligned} \tag{1.14}$$

Here, $L_m^{(-1)}(x)$ is a generalized Laguerre polynomial and ${}_2F_1$ is a hypergeometric function [6]. An alternative expression for the last matrix element is

$$R(0, a, b)_{mn} = e^{\omega b} \sum_{k=0}^{[(m-n)/2]} (-1)^k 2^{2k+n-m} \binom{1/2}{k} H_{m-n-2k}(\eta) H_{m-n-2k}(\xi),$$

where $m \geqq n$ and $\eta\xi = \omega a$, $\eta^2 + \xi^2 = -2b\omega$ and $H_n(z)$ is a Hermite polynomial; see [3].

2. A two-variable model for $CE(2)$. We now construct a model of (1.11) in which M', $P^{\pm\prime}$ are differential operators in two complex variables. There is only one such model [5]:

$$M_3 = y\partial_x - x\partial_y, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \tag{2.1}$$

where x and y are complex variables. Here

$$[M_3, P_1] = P_2, \quad [M_3, P_2] = -P_1, \quad [P_1, P_2] = 0.$$

We can satisfy relations (1.1) by setting

$$M = iM_3, \quad Q_1 = P_1, \quad Q_2 = iP_2, \quad P^\pm = P_1 \pm iP_2.$$

To compute the eigenfunctions $f(x, y)$ of E corresponding to a model of $\rho(\omega, \alpha)$ we must solve the equations

$$(P_1^2 + P_2^2)f = -\omega^2 f, \qquad (2P_2M_3 + P_1)f = -2i\lambda\omega f, \tag{2.2a}$$

or

$$(\partial_x^2 + \partial_y^2 + \omega^2)f = 0, \qquad (-2x\,\partial_y^2 + 2y\,\partial_{xy} + \partial_x)f = -2i\lambda\omega f. \tag{2.2b}$$

These equations have solutions which are products of parabolic cylinder functions expressed in terms of parabolic cylinder coordinates. Rather than verify this we will construct the basis functions j_λ satisfying (1.11) directly.

Note that the functions $h(x, y) = \exp[i\omega(ax + by)]$, $a^2 + b^2 = 1$, satisfy

$$(P_1^2 + P_2^2)h = -\omega^2 h, \quad P_1 h = i\omega a h, \quad P_2 h = i\omega b h.$$

They are the simultaneous eigenfunctions of P_1 and P_2. We shall look for basis functions of the form

$$\begin{aligned} f(x, y) &= \int_{C'} F(e^\theta) \exp[\omega(x\cosh\theta - iy\sinh\theta)] e^\theta\, d\theta \\ &= \int_C F(z) \exp\left[\frac{\omega}{2}(z(x - iy) + z^{-1}(x + iy))\right] dz \\ &= I(F), \end{aligned}$$

i.e., integrals over the functions h. Here, C is a simple path on the Reimann surface associated with the analytic function $F(z)$ such that either C is closed or the integrand vanishes faster than any power of z at the endpoints. We also assume that the contour is chosen such that $I(F)$ converges absolutely and arbitrary differentiation in x and y is permitted under the integral sign. It follows easily that

$$\begin{aligned} &i(y\partial_x - x\partial_y)f = I((z\partial_z + 1)F), \\ &(\partial_x \pm i\partial_y)f = I(\omega z^{\pm 1} f). \end{aligned}$$

Thus, $(P_1^2 + P_2^2)f = -\omega^2 f$ and the action of $\mathscr{E}(2)$ on f corresponds exactly to the action of the operators (1.5), with $\alpha = 1$, on F.

We conclude that the functions

$$f_m(x, y) = \int_C z^{m-1} \exp\left[\frac{\omega}{2}(z(x - iy) + z^{-1}(x + iy))\right] dz, \tag{2.3}$$
$$m_0 \in \mathbb{C}, \quad m = m_0 + n, \quad n = 0, \pm 1, \pm 2, \cdots,$$

satisfy equations (1.4) and define a basis for a model of the irreducible representation $\rho(\omega, m_0)$. If C is the contour in Fig. 1 and $\mathrm{Re}\,[\omega(x + iy)] < 0$ it is straightforward to show that

$$f_m(x, y) = 2\pi i\, e^{im(\theta - \pi/2)} J_m(-i\omega r), \qquad x = r\cos\theta, \quad y = r\sin\theta, \tag{2.4}$$

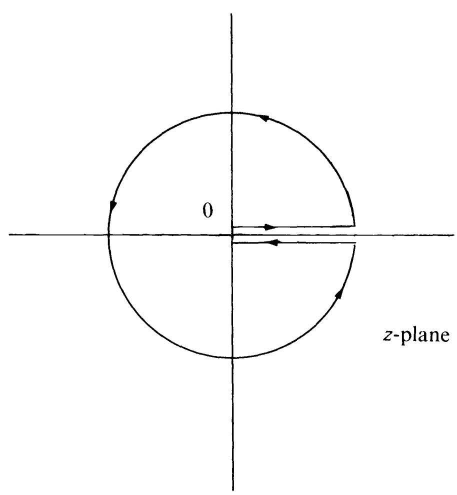

FIG. 1

where $J_m(z)$ is a Bessel function [7]. Similarly, the functions

$$\begin{aligned} j_\lambda(x, y) &= \int_{C'} z^{-1/2}(1 - z^2)^{-1/2}\left(\frac{1 - z}{1 + z}\right)^\lambda \exp\left[\frac{\omega}{2}(z(x - iy) + z^{-1}(x + iy))\right] dz \\ &= \int_{C''} \frac{t^{\lambda - 1/2}}{\sqrt{1 - t^2}} \exp\left\{\frac{\omega}{2}\left[\left(\frac{1 - t}{1 + t}\right)(x - iy) + \left(\frac{1 + t}{1 - t}\right)(x + iy)\right]\right\} dt, \end{aligned} \tag{2.5}$$

where $z = (1 - t)/(1 + t)$, satisfy the equations

$$Ej_\lambda = 2\lambda\omega j_\lambda \tag{2.6}$$

and (1.11). Here C'' is the contour in Fig. 2 in the t-plane and $\mathrm{Re}\,[\omega(x + iy)] < 0$. (This results from a comparison of the integrand of (2.5) and the functions $h_\lambda(z)$ following (1.6).) In terms of parabolic cylinder coordinates ξ, η,

$$2x = \xi^2 - \eta^2, \quad y = \xi\eta, \tag{2.7}$$

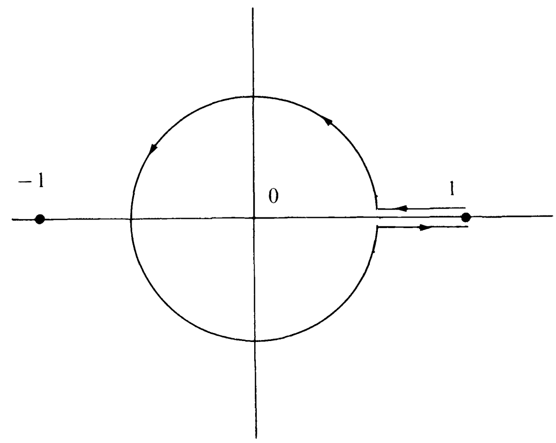

FIG. 2

the basis functions are

$$j_\lambda[\xi, i\eta] = \frac{2\pi i e^{i\pi(\lambda - 3/2)}}{\Gamma(\frac{1}{2} - \lambda)} D_{-\lambda-1/2}(\sqrt{-2\omega}\xi) D_{-\lambda-1/2}(\sqrt{-2\omega}i\eta), \tag{2.8}$$

where $D_\nu(z)$ is a parabolic cylinder function [7], and $\Gamma(z)$ is a gamma function. (We do not give the straightforward details of the verification of (2.8) but merely mention that the result follows from the facts that $j_\lambda[\xi, i\eta]$ is symmetric in ξ and $i\eta$ and that $j_\lambda[\xi, i\eta]$ is a solution of the parabolic cylinder equation in the variable $\sqrt{-2\omega}\xi$ as can be checked by differentiation under the integral sign.) In the special case $\lambda = -n - \frac{1}{2}$, n an integer,

$$j_{-n-1/2}[\xi, i\eta] = \begin{cases} \dfrac{2\pi i(-1)^n}{n!\,2^n} e^{(\omega/2)(\xi^2 - \eta^2)} H_n(\sqrt{-\omega}\xi) H_n(\sqrt{-\omega}i\eta) & \text{if } n \geqq 0, \\ 0 \quad \text{if } n < 0, \end{cases} \tag{2.9}$$

where $H_n(z)$ is a Hermite polynomial. Other choices of integration contours lead to new sets of basis functions.

As we have seen, the $j_\lambda[\xi, i\eta]$ form bases for representations $\mu(\omega, \beta)$ of $\mathscr{CE}(2)$ which are not equivalent to the better-known representations $\rho(\omega, \alpha)$. The $\mu(\omega, \beta)$ extend to local Lie representations of $CE(2)$ with matrix elements (1.14). Indeed, from (1.11), (1.12) and (2.5) we find

$$\{\mathbf{R}(\theta, a, b) j_{\beta+n}\}[\xi, i\eta] = \sum_{m=-\infty}^{\infty} R(\theta, a, b)_{mn} j_{\beta+m}[\xi, i\eta], \qquad 0 \leqq \operatorname{Re}\beta < 1,$$

for θ, a, b in a sufficiently small neighborhood of zero. This is the addition theorem for the solutions (2.8) of the reduced wave equation under the action of $CE(2)$. Here the operator $\mathbf{R}(\theta, a, b)$ is defined on functions of x, y by

$$\begin{aligned} [\mathbf{R}(\theta, a, b) f](x, y) = f(&(x + a)\cosh\theta + i(u + ib)\sinh\theta, \\ &-i(x + a)\sinh\theta + (y + ib)\cosh\theta). \end{aligned} \tag{2.10}$$

Using (2.7) one can easily compute the group action in coordinates ξ, η. Some special cases are of interest. For $\theta = 0$, $b = a$, we find

$$j_\lambda[\xi', i\eta'] = e^{\omega a} \sum_{l=0}^{\infty} (-1)^l L_l^{(-1)}(-2\omega a) j_{\lambda+l}[\xi, i\eta],$$

$$\xi'^2 - \eta'^2 = \xi^2 - \eta^2 + 2a, \qquad \xi'\eta' = \xi\eta + ia.$$

For $\theta = 0, a = -b$ and $|b| < |\xi + i\eta|^2$ we find

$$j_\lambda[\xi', i\eta'] = e^{\omega b} \sum_{l=0}^{\infty} (-1)^l L_l^{(-1)}(-2\omega b) j_{\lambda+l}[\xi, i\eta],$$

$$\xi'^2 - \eta'^2 = \xi^2 - \eta^2 - 2b, \qquad \xi'\eta' = \xi\eta + ib.$$

If $\lambda = -n - \frac{1}{2}, n = 0, \pm 1, \cdots$, the restrictions on $|b|$ can be omitted and the sums become finite (since $j_{-n-1/2} \equiv 0$ for $n < 0$). Finally, for $a = b = 0$ we find

$$j_\lambda[\xi', i\eta'] = \sum_{l=-\infty}^{\infty} \left[\cosh\frac{\theta}{2}\right]^{l-1} \left[-\sinh\frac{\theta}{2}\right]^{-l} \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + l + \frac{1}{2})}$$

$$\cdot \frac{1}{\Gamma(l)} {}_2F_1\left(\begin{matrix} -\lambda + \frac{1}{2}, & \lambda + l + \frac{1}{2} \\ & l \end{matrix} \middle| \tanh^2\frac{\theta}{2}\right) j_{\lambda+l}[\xi, i\eta],$$

$$\xi' = \xi \cosh\frac{\theta}{2} + i\eta \sinh\frac{\theta}{2}, \qquad \eta' = -i\xi \sinh\frac{\theta}{2} + \eta \cosh\frac{\theta}{2}$$

valid for $|\tanh(\theta/2)| < 1 < |\coth(\theta/2)|$.

The formal relation (1.7) between basis vectors in different representation spaces can sometimes be made meaningful in our two-variable model. For example, consider the function $j'_\lambda(x, y)$ given by (2.5) with contour C' (Fig. 1) in the z-plane and $\mathrm{Re}\,[\omega(x + iy)] < 0$. Here,

$$(P_1^2 + P_2^2) j'_\lambda = -\omega^2 j'_\lambda, \quad E j'_\lambda = 2\lambda\omega j'_\lambda.$$

Using (1.7) and (2.3) we obtain the expansion

$$j'_\lambda(x, y) = 2\pi i \sum_{n=0}^{\infty} \binom{-\lambda - \frac{1}{2}}{n} {}_2F_1\left(\begin{matrix} \frac{1}{2} - \lambda, & -n \\ \frac{1}{2} - \lambda - n & \end{matrix} \middle| -1\right) e^{-im(\theta - \pi/2)} J_{n+1/2}(-i\omega r),$$

$$x = r\cos\theta, \quad y = r\sin\theta,$$

of j'_λ in terms of eigenfunctions of M. However, direct evaluation of the contour integral in the coordinates ξ, η yields

$$j'_\lambda[\xi, i\eta] = 2\sqrt{-\omega\pi}\, e^{\omega(\xi^2 - \eta^2)/2} \left[i\eta\, {}_1F_1\left(\begin{matrix} \lambda/2 + \frac{1}{4} \\ \frac{1}{2} \end{matrix} \middle| -\omega\xi^2\right)\right.$$

$$\left. \cdot\, {}_1F_1\left(\begin{matrix} \lambda/2 + \frac{3}{4} \\ \frac{3}{2} \end{matrix} \middle| \omega\eta^2\right) + \xi\, {}_1F_1\left(\begin{matrix} \lambda/2 + \frac{1}{4} \\ \frac{1}{2} \end{matrix} \middle| \omega\eta^2\right) {}_1F_1\left(\begin{matrix} \lambda/2 + \frac{3}{4} \\ \frac{3}{2} \end{matrix} \middle| -\omega\xi^2\right)\right].$$

Other expansions of eigenfunctions of E in terms of eigenfunctions of M (Bessel functions) can be derived via Weisner's method [5], [10].

3. Representations of the pseudo-Euclidean group. Let $PE(2)$ be the group of all matrices (1.2) with real parameters θ, a, b, i.e., the group of motions of the pseudo-

Euclidean plane [8]. As is well known, the irreducible faithful unitary representations of $PE(2)$ are defined by operators $\mathbf{T}(\theta, a, b)$,

$$\mathbf{T}(\theta, a, b)f(x) = \exp\left[i\gamma(a\cosh x + b\sinh x)\right]f(x+\theta) \tag{3.1}$$

acting on the Hilbert space $L_2(R)$ of Lebesgue square integrable functions $f(x)$ on the real line. The inner product is

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx.$$

Here γ is a nonzero real number.

The operators Q_1, Q_2, M related to $PE(2)$ by (1.4) are easily shown to be

$$Q_1 = i\gamma\cosh x,\quad Q_2 = i\gamma\sinh x,\quad M = \partial_x, \tag{3.2a}$$

or

$$P^+ = i\gamma\, e^x,\quad P^- = i\gamma\, e^{-x},\quad M = \partial_x. \tag{3.2b}$$

We could now study the operator E, (1.6), acting on $L_2(R)$. Instead we will study the eigenvalue equation $\mathscr{E}f = -2\gamma\,\delta f$ for the related operator

$$\mathscr{E} = MQ_1 + Q_1M = 2Q_1M + Q_2, \tag{3.3a}$$

or

$$\mathscr{E} = 2i\gamma\cosh x\,\partial_x + i\gamma\sinh_x. \tag{3.3b}$$

As given this operator is not well-defined. Initially we define $\mathscr{E}$ by (3.3) with domain the subspace of C^∞-functions with compact support. It is easily seen that $\mathscr{E}$ is a symmetric operator on this domain with deficiency indices (1, 1). Thus, $\mathscr{E}$ has a one-parameter family of self-adjoint extensions [9]. A straightforward computation yields the self-adjoint operators $\{\mathscr{E}_\alpha\}$ where $0 \leqq \alpha < 2$. Each $\mathscr{E}_\alpha$ is defined by (3.3) with domain

$$\mathscr{D}_\alpha = \{f \in L_2(R) : f \text{ abs. cont.}, \mathscr{E}^*f \in L_2(R), B_\alpha f = 0\},$$

where

$$B_\alpha f = e^{i\alpha\pi}\lim_{x\to-\infty}\sqrt{\cosh x}\,f(x) - \lim_{x\to\infty}\sqrt{\cosh x}\,f(x).$$

Each $\mathscr{E}_\alpha$ has discrete spectrum $\delta - \alpha = 0, \pm 2, \pm 4, \cdots$ and normalized eigenfunctions

$$j_\delta(x) = \sqrt{\frac{2}{\pi}}\,e^{x/2}(1 + i\,e^x)^{\delta - 1/2}(1 - i\,e^x)^{-\delta - 1/2}. \tag{3.4}$$

Eigenfunctions of $\mathscr{E}_\alpha$ and $\mathscr{E}_{\alpha'}$ are related by

$$j_{\alpha'+2n} = \sum_{m=-\infty}^{\infty}\langle j_{\alpha'+2n}, j_{\alpha+2m}\rangle j_{\alpha+2m},$$

$$\langle j_{\alpha'+2n}, j_{\alpha+2m}\rangle = \frac{e^{i\beta\pi} - 1}{i\beta\pi},\qquad \beta = \alpha' - \alpha + 2(n-m).$$

From now on we fix α and concentrate on a single operator $\mathscr{E}_\alpha$.

In the basis $\{j_{\alpha+2n}\}$ the matrix elements of the operators $\mathbf{T}(\theta, a, b)$ are given by

$$T_{mn}(\theta, a, b) = \langle \mathbf{T}(\theta, a, b)j_{\alpha+2n}, j_{\alpha+2m}\rangle.$$

The $\{T_{mn}(\theta, a, b)\}$ form a unitary matrix representation of $PE(2)$. The following special cases are of interest:

$$(3.5)\quad T_{mn}(0, a, a) = \frac{2}{\pi}\int_0^\infty e^{i\gamma at}(1 + it)^{2n-2m-1}(1 - it)^{2m-2n-1}\,dt,$$

$$T_{mn}(0, a, a) = \frac{2}{\pi}\sum_{k=0}^{2n-2m-1} \frac{(2n - 2m - 1)!}{(2n - 2m - k - 1)!}(-1)^{m-n} i^k (-\gamma a)^{(2n-2m-k-1)/2} \cdot e^{\gamma a/2} W_{(2m-2n-k-1)/2,(2m-2n+k)/2}(\gamma a)$$

$$(3.6)\quad = \frac{i}{\pi(n - m)}\Phi_1(1, 2n - 2m + 1, 2n - 2m + 1, -1, -\gamma a) \cdot -\frac{i}{\pi}(n - m - 1)!\, W_{m-n,-1/2}(2\gamma a) \quad \text{if } n > m,$$

$$T_{mn}(0, a, a) = \overline{T_{nm}}(0, -a, -a) \quad \text{if } m \geqq n,$$

$$T_{mn}(0, a, a) = \frac{i}{\pi}\left[\frac{e^{\gamma a/2}}{\sqrt{\gamma a}} W_{-1/2,0}(\gamma a) - \frac{e^{-\gamma a/2}}{\sqrt{-\gamma a}} W_{-1/2,0}(-\gamma a)\right].$$

The right-hand sides of (3.6) are defined by continuity from the domain $\operatorname{Im} \gamma a > 0$. Here $W_{\gamma,\mu}(z)$ is a Whittaker function [4], [6], and Φ_1 is a generalized hypergeometric function defined by [6]

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n} m!\, n!} x^m y^n, \qquad |x| < 1,$$

$$(\beta)_m = \beta(\beta + 1) \cdots (\beta + m - 1)$$

with analytic continuation via the transformation formula

$$\Phi_1(\alpha, \beta, \gamma, x, y) = e^y(1 - x)^{-\beta}\Phi_1\left(\gamma - \alpha, \beta, \gamma, \frac{x}{x - 1}, -y\right).$$

Additional matrix elements are given by

$$(3.7)\quad T_{mn}(0, a, -a) = T_{nm}(0, a, a),$$

$$T_{mn}(\theta, 0, 0) = \frac{2e^{\theta/2}}{\pi}\int_0^\infty (1 + it\, e^\theta)^{\alpha+2n-1/2}(1 - it\, e^\theta)^{-a-2n-1/2} \cdot (1 - it)^{\alpha+2m-1/2}(1 + it)^{-\alpha-2m-1/2}\,dt$$

$$(3.8)\quad = \frac{2e^{\theta/2}}{\pi} F_A(1, -\alpha - 2n + \tfrac{1}{2}, \alpha + 2n + \tfrac{1}{2}, -\alpha - 2m + \tfrac{1}{2}, \alpha + 2m + \tfrac{1}{2}, 1; 1 - ie^\theta, 1 + ie^\theta, 1 + i, 1 - i),$$

where $F_A(\alpha, \beta_1, \cdots, \beta_4, \gamma; x_1, \cdots, x_4)$ is a Lauricella function [6]. We will later derive another expression for (3.8).

The action of M on a basis is given by

$$(3.9)\quad Mj_{\alpha+2n} = \sum_{m=-\infty}^{\infty} \frac{-2i}{\pi}\,\frac{\alpha + m + n}{4(n - m)^2 - 1} j_{\alpha+2m}.$$

The basis vectors do not lie in the domains of Q_1 or Q_2.

Vilenkin [8, Chap. 5] has studied the unitary representations of $PE(2)$ in terms of the spectral resolution for M. To obtain this resolution we map the space $L_2(R)$ onto $L_2(R')$ using the Fourier transform:

$$\mathscr{F}h(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\, e^{-ixy}\, dx \in L_2(R'), \qquad h \in L_2(R),$$

$$\mathscr{F}^{-1}k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(y)\, e^{ixy}\, dy \in L_2(R), \qquad k \in L_2(R').$$

On $L_2(R')$ the action of the operators

$$\mathbf{T}'(\theta, a, b) = \mathscr{F}\mathbf{T}(\theta, a, b)\mathscr{F}^{-1}$$

is

$$\mathbf{T}'(\theta, 0, 0)k(y) = e^{i\theta y}k(y),$$

$$\mathbf{T}'(0, a, b)k(y) = \int_{-\infty}^{\infty} K(z - y; a, b)k(z)\, dz, \tag{3.10}$$

$$K(z - y; a, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[i\gamma(a \cosh x + b \sinh x) + ix(z - y)\right] dx,$$

where the kernel $K(z - y; a, b)$ can be expressed in terms of Macdonald functions. Note that

$$M'k(y) = iyk(y), \quad M' = \mathscr{F}M\mathscr{F}^{-1}.$$

To determine the relationship between the E-basis and the M-basis we compute the ON-basis $\{\mathscr{F}j_{\alpha+2n}\}$ for $L_2(R')$.

$$\begin{aligned} j'_{\alpha+2n}(y) &= \mathscr{F}j_{\alpha+2n}(y) \\ &= e^{(\pi/2)(y+i/2)} \frac{\Gamma(\delta + \frac{1}{2})}{\pi} \left[\frac{\Gamma(-iy + \frac{1}{2})}{\Gamma(-iy + \delta + 1)}\, {}_2F_1\left(\begin{matrix} \delta + \frac{1}{2}, & -iy + \frac{1}{2} \\ & -iy + \delta + 1 \end{matrix} \middle| -1\right) \right. \\ &\quad \left. + e^{i\pi(\delta - 1/2)} \frac{\Gamma(iy + \frac{1}{2})}{\Gamma(iy + \delta + 1)}\, {}_2F_1\left(\begin{matrix} \delta + \frac{1}{2}, & iy + \frac{1}{2} \\ & iy + \delta + 1 \end{matrix} \middle| -1\right) \right] \\ &\qquad\qquad \text{if } \delta = \alpha + 2n > 0, \\ j'_{-\delta}(y) &= \bar{j}'_{\delta}(-y). \end{aligned} \tag{3.11}$$

Since

$$T_{mn}(\theta, a, b) = \langle \mathbf{T}'(\theta, a, b) j'_{\alpha+2n}, j'_{\alpha+2m} \rangle,$$

we can derive new expressions for the matrix elements. For example,

$$T_{mn}(\theta, 0, 0) = \int_{-\infty}^{\infty} e^{i\theta y} j'_{\alpha+2n}(y) \bar{j}'_{\alpha+2m}(y)\, dy.$$

4. A two-variable model for $PE(2)$**.** We use a method analogous to that of § 2 to construct two-variable models of the unitary representations of $PE(2)$.

Consider the functions

$$h_{s,t}(x) = \exp[i\gamma(s \cosh x + t \sinh x)], \qquad s, t \in \mathbb{C},$$

which belong to $L_2(R)$ for $\operatorname{Im} \gamma(s \pm t) > 0$. Then the functions

$$H_{\alpha+2n}(s,t) = \langle j_{\alpha+2n}, \bar{h}_{s,t} \rangle = \langle h_{s,t}, \bar{j}_{\alpha+2n} \rangle = \langle h_{s,t}, j_{-\alpha-2n} \rangle \tag{4.1}$$

satisfy the equations

$$\begin{aligned} [\mathbf{T}(\theta, a, b) H_{\alpha+2n}](s,t) &= \langle \mathbf{T}(\theta, a, b) j_{\alpha+2n}, \bar{h}_{s,t} \rangle \\ &= \sum_{m=-\infty}^{\infty} T_{mn}(\theta, a, b) H_{\alpha+2m}(s,t) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} [\mathbf{T}(\theta, a, b) H_{\alpha+2n}](s,t) &= \langle j_{\alpha+2n}, \mathbf{T}^{-1}(\theta, a, b) \bar{h}_{s,t} \rangle \\ &= H_{\alpha+2n}((s+a)\cosh\theta - (b+t)\sinh\theta, \\ &\qquad (t+b)\cosh\theta - (a+s)\sinh\theta). \end{aligned} \tag{4.3}$$

We see that the $H_{\alpha+2n}(s,t)$ transform under $PE(2)$ exactly as the basis vectors $j_{\alpha+2n}(x)$. The Lie algebra action is determined by the operators

$$Q_1 = \partial_s, \quad Q_2 = \partial_t, \quad M = -t\partial_s - s\partial_t.$$

It follows easily from (4.1) that

$$(Q_2^2 - Q_1^2) H_{\alpha+2n} = \gamma^2 H_{\alpha+2n}, \qquad \mathscr{E} H_{\alpha+2n} = -2\gamma(\alpha+2n) H_{\alpha+2n}.$$

In terms of the new coordinates

$$s = i\xi\eta, \quad t = \frac{\eta^2 - \xi^2}{2},$$

we find

$$H_\delta(s,t) = H_\delta[\xi, \eta] = 2e^{3i\pi(\delta-1)/2} D_{-\delta-1/2}(\sqrt{-2\gamma}\,\xi) D_{\delta-1/2}(\sqrt{-2\gamma}\,\eta). \tag{4.4}$$

Note also the relation

$$h_{s,t}(x) = \sum_{n=-\infty}^{\infty} j_{-\alpha-2n}(x) H_{\alpha+2n}(s,t), \tag{4.5}$$

where the right-hand side converges in $L_2(R)$ and also pointwise. We can consider (4.5) as a generating function for the H_δ.

It is of interest to study these relations in the Fourier transform space $L_2(R')$, i.e., in terms of an M-basis. We have

$$h'_{s,t}(y) = \mathscr{F} h_{s,t}(y) = \sqrt{\frac{2}{\pi}} \left(\frac{s-t}{s+t} \right)^{-iy/2} K_{iy}(i\gamma\sqrt{s^2 - t^2}), \tag{4.6}$$

where $K_\nu(z)$ is a Macdonald function [7]. In $L_2(R')$ relation (4.5) becomes

$$h'_{s,t}(y) = \sum_{n=-\infty}^{\infty} j'_{-\alpha-2n}(y) H_{\alpha+2n}(s,t), \tag{4.7}$$

where $j'_\delta(y)$ is given by (3.1). Furthermore,

$$H_\delta(s,t) = \langle h_{s,t}, j_{-\delta}\rangle = \int_{-\infty}^{\infty} h'_{s,t}(y)\overline{j'_{-\delta}(y)}\, dy. \tag{4.8}$$

Note that $h'_{s,t}(y)$ satisfies

$$(Q_2^2 - Q_1^2)h'_{s,t}(y) = \gamma^2 h'_{s,t}(y), \qquad Mh'_{s,t}(y) = -iyh'_{s,t}(y),$$

so expressions (4.7), (4.8) yield relationships between $\mathscr{E}$-basis and M-basis solutions of the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) f(s,t) = \gamma^2 f(s,t).$$

For $\operatorname{Im}\gamma(s_j \pm t_j) > 0, j = 1, 2$, we have

$$\begin{aligned}\langle h_{s_1,t_1}, \overline{h_{s_2,t_2}}\rangle &= \int_{-\infty}^{+\infty} \exp\left[i\gamma((s_1+s_2)\cosh x + (t_1+t_2)\sinh x)\right] dx \\ &= 2K_0(i\gamma[(s_1+s_2)^2 - (t_1+t_2)^2]^{1/2}).\end{aligned} \tag{4.9}$$

On the other hand, computing in $L_2(R')$ we find

$$\begin{aligned}\langle h_{s_1,t_1}, \overline{h_{s_2,t_2}}\rangle &= \int_{-\infty}^{\infty} (\mathscr{F}h_{s_1,t_1})(\overline{\mathscr{F}h_{-\bar{s}_2,-\bar{t}_2}})\, dy \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{s_1 - t_1}{s_1 + t_1}\right)^{-iy/2} \left(\frac{s_2 - t_2}{s_2 + t_2}\right)^{iy/2} \\ &\quad \cdot K_{iy}(i\gamma\sqrt{s_1^2 - t_1^2})K_{iy}(i\gamma\sqrt{s_2^2 - t_2^2})\, dy,\end{aligned} \tag{4.10}$$

which is a special case of Crum's formula [7, p. 55]. (The general case follows from a study of $h_{s,t}(x) = \exp[i\gamma(s\cosh x + t\sinh x) + \beta x]$.) Finally, using the $\mathscr{E}$-basis in $L_2(R)$ we find

$$\begin{aligned}\langle h_{s_1,t_1}, \overline{h_{s_2,t_2}}\rangle &= \sum_{n=-\infty}^{\infty} \langle h_{s_1,t_1}, j_{\alpha+2n}\rangle\langle j_{\alpha+2n}, \overline{h_{s_2,t_2}}\rangle \\ &= \sum_{n=-\infty}^{\infty} H_{-\alpha-2n}(s_1,t_1)H_{\alpha+2n}(s_2,t_2).\end{aligned} \tag{4.11}$$

Comparison of (4.9) and (4.11) yields a bilateral generating function for the $H_\delta(s,t)$.

Taking the inner product of $h_{s,t}(x)$ with $j_{\alpha'+2n}$ and using (3.4) we find the relation

$$H_{\alpha'+2n}(s,t) = \sum_{m=-\infty}^{\infty} \frac{(\exp[i\pi(\alpha' - \alpha + 2n - 2m)] - 1)}{\pi i(\alpha' - \alpha + 2n - 2m)} H_{\alpha+2m}(s,t).$$

In [8, Chap. 5], Vilenkin decomposes the quasi-regular representation of $PE(2)$ into a direct integral of irreducible representations. He expands an arbitrary function $f(s,t)$ such that

$$\int_{R^2} |f(s,t)|^2\, ds\, dt < \infty$$

in terms of an M-basis, i.e., Macdonald functions. A very similar analysis allows one to expand $f(s, t)$ in terms of an $\mathscr{E}$-basis of the form (4.4) for s, t real. We omit the straightforward computation.

5. The real Euclidean group. The real Euclidean group $E(2)$ is the multiplicative matrix group with elements

$$g(\theta, a, b) = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad \theta, a, b \in R. \tag{5.1}$$

The Lie algebra $\mathscr{E}(2)$ can be associated with $E(2)$ via the exponential mapping and the formula

$$g(\theta, a, b) = \exp(aP_1 + bP_2)\exp(\theta M). \tag{5.2}$$

The commutation relations are

$$[M, P_1] = P_2, \quad [M, P_2] = -P_1, \quad [P_1, P_2] = 0.$$

The faithful unitary irreducible representations of $E(2)$ are defined by operators [8]

$$\mathbf{T}(\theta, a, b)f(\varphi) = \exp[i\gamma(a\cos\varphi - b\sin\varphi)]f(\varphi + \theta) \tag{5.3}$$

acting on the Hilbert space $L_2[-\pi, \pi]$ of Lebesgue square integrable functions $f(\varphi)$ on the interval $[-\pi, \pi]$, with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(\varphi)\overline{g(\varphi)}\, d\varphi.$$

In (5.3) we assume f is defined on the whole real axis by the periodicity condition $f(\varphi) = f(\varphi + 2\pi)$. Here γ is a nonzero real number. The induced Lie algebra representation is defined by operators

$$P_1 = i\gamma\cos\varphi, \quad P_2 = -i\gamma\sin\varphi, \quad M = \partial_\varphi. \tag{5.4}$$

The operator E on $L_2[-\pi, \pi]$ is defined formally by

$$E = MP_2 + P_2M = 2P_2M - P_1 = -2i\sin\varphi\,\partial_\varphi - i\cos\varphi. \tag{5.5}$$

As given this operator is not well-defined. To be definite we define E by (5.5) with domain the space of all C^∞-functions on $[-\pi, \pi]$ which vanish in neighborhoods of $\varphi = 0$ and $\pm\pi$. Then E is symmetric on this domain and essentially self-adjoint. To compute the self-adjoint extension we define a unitary mapping $\mathbf{U}$ from $L_2[-\pi, \pi]$ onto $L_2(R) \oplus L_2(R)$ by

$$\mathbf{U}f(v) = \mathbf{F}(v) = \begin{pmatrix} F_+(v) \\ F_-(v) \end{pmatrix} = [1 - \cos^2\varphi]^{1/4}\begin{pmatrix} f_+(\cos\varphi) \\ f_-(\cos\varphi) \end{pmatrix}, \tag{5.6}$$

$$\cos\varphi = \tanh v.$$

Here,

$$f_-(\cos\varphi) = f(\varphi), \qquad -\pi \leqq \varphi < 0,$$
$$f_+(\cos\varphi) = f(\varphi), \qquad 0 < \varphi \leqq \pi.$$

It follows that

$$(5.7)\qquad \int_{-\pi}^{\pi} f(\varphi)\overline{g(\varphi)}\,d\varphi = \int_{-\infty}^{\infty} F_+(\nu)\overline{G_+(\nu)}\,d\nu + \int_{-\infty}^{\infty} F_-(\nu)\overline{G_-(\nu)}\,d\nu$$

and $E\mathbf{F}(\nu) = 2i\partial_\nu \mathbf{F}(\nu)$. Now we take the Fourier transform

$$(5.8)\qquad \mathscr{F}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathbf{F}(\nu)\,e^{i\nu\lambda}\,d\nu$$

with inverse

$$(5.9)\qquad \mathbf{F}(\nu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathscr{F}(\lambda)\,e^{-i\nu\lambda}\,d\lambda.$$

Then setting

$$(5.10)\qquad (\mathscr{F},\mathscr{G}) = \int_{-\infty}^{\infty} \mathscr{F}_+(\lambda)\overline{\mathscr{G}_+(\lambda)}\,d\lambda + \int_{-\infty}^{\infty} \mathscr{F}_-(\lambda)\overline{\mathscr{G}_-(\lambda)}\,d\lambda$$

we find

$$\langle f, g\rangle = (\mathscr{F},\mathscr{G}),$$

and $E\mathscr{F}(\lambda) = 2\lambda\mathscr{F}(\lambda)$. Thus E can be extended to a unique self-adjoint operator with continuous spectrum of multiplicity two covering the whole real axis.

The ON M-basis for representations (5.3) is given by functions

$$(5.11)\qquad f_n(\varphi) = \frac{e^{in\varphi}}{\sqrt{2\pi}}, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

To find the relationship between the M- and E-bases we compute the vector-valued functions $\mathscr{F}^n(\lambda)$:

$$(5.12)\qquad \begin{aligned} \mathscr{F}^n_+(\lambda) &= \frac{1}{2\pi}\int_0^{\pi} e^{in\varphi}(1+\cos\varphi)^{i\lambda/2-1/4}(1-\cos\varphi)^{-i\lambda/2-1/4}\,d\varphi \\ &= \frac{e^{(\pi/2)(i/2-\lambda)}}{\pi\sqrt{2}}\Gamma(-n+\tfrac12)\left[\frac{(-1)^n\Gamma(i\lambda+\frac12)}{\Gamma(i\lambda-n+\frac12)}\,{}_2F_1\left(\begin{matrix} i\lambda+\frac12, & -n+\frac12 \\ & i\lambda-n+1\end{matrix}\middle| -1\right)\right. \\ &\qquad \left. -\frac{i\Gamma(-i\lambda+\frac12)}{\Gamma(-i\lambda-n+1}\,{}_2F_1\left(\begin{matrix} -i\lambda+\frac12, & -n+\frac12 \\ & -i\lambda-n+1\end{matrix}\middle| -1\right)\right], \\ \mathscr{F}^n_-(\lambda) &= \mathscr{F}^{-n}_+(\lambda). \end{aligned}$$

Note that the $\mathscr{F}^n(\lambda)$ form an ON-basis for $L_2(R)\oplus L_2(R)$.

Consider the functions

$$h_{x,y}(\varphi) = \exp\,[i\gamma(x\cos\varphi + y\sin\varphi)] \in L_2[-\pi,\pi],$$

for $x, y\in\mathbb{C}$. Computing the expansion coefficients of $h_{x,y}$ with respect to the M-basis we find

$$(5.13)\qquad H_n(x,y) = \langle h_{x,y}, f_n\rangle = \sqrt{2\pi}\,i^n\,e^{-in\theta}J_n(\gamma r),$$

where $x = r\cos\theta$, $y = r\sin\theta$, $r \geqq 0$, and $J_n(z)$ is a Bessel function of integral order.

On the other hand, in the E-basis we have

$$\mathscr{H}^{x,y}(\lambda) = \begin{pmatrix} \mathscr{H}_{+}^{x,y}(\lambda) \\ \mathscr{H}_{-}^{x,y}(\lambda) \end{pmatrix},$$

where

$$\mathscr{H}_{+}^{x,y}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iv\lambda} (\operatorname{sech} v)^{1/2} \exp\left[i\gamma(x \tanh v + y \operatorname{sech} v)\right]$$

$$(5.14) \quad = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{t^{-i\lambda - 1/2}}{\sqrt{1+t^2}} \exp\left[i\gamma\left\{ x\left(\frac{1-t^2}{1+t^2}\right) + \frac{2yt}{1+t^2} \right\}\right] dt$$

$$= \frac{1}{\sqrt{2}\cos(i\lambda\pi)} \left[D_{-i\lambda-1/2}(\sigma\xi) D_{i\lambda-1/2}(\sigma\eta) + D_{-i\lambda-1/2}(-\sigma\xi) D_{i\lambda-1/2}(-\sigma\eta)\right],$$

where

$$\sigma = e^{i\pi/4}\sqrt{2\gamma}, \quad x = \frac{\xi^2 - \eta^2}{2}, \quad y = \xi\eta.$$

Similarly,

$$\mathscr{H}_{-}^{x,y}(\lambda) = \mathscr{H}_{+}^{x,-y}(\lambda).$$

Expanding $h_{x,y}(\varphi)$ in the E-basis we find

$$(5.15) \quad \begin{aligned} \exp\left[i\gamma(x\cos\varphi + y \sin\varphi)\right] = \frac{(\sin\varphi)^{-1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\cot\frac{\varphi}{2}\right)^{-i\lambda} \\ \cdot \mathscr{H}_{+}^{x,y}(\lambda)\, d\lambda, \qquad 0 < \varphi < \pi, \end{aligned}$$

with a similar result for $-\pi < \varphi < 0$. This is a well-known expansion formula for a plane wave in terms of parabolic cylinder functions [7, p. 126]. In exact analogy with the computation following (4.3) we can show that $\mathscr{H}_{\pm}^{x,y}(\lambda)$ are solutions of the reduced wave equation

$$(\partial_x^2 + \partial_y^2) f = -\gamma^2 f$$

and eigenfunctions of the operator $E = MP_2 + P_2M$ with eigenvalue 2λ, where

$$P_1 = -\partial_x, \quad P_2 = \partial_y, \quad M = x\partial_y - y\partial_x.$$

A straightforward computation yields

$$(5.16) \qquad \langle h_{x,y}, h_{x',y'} \rangle = 2\pi J_0(\gamma\sqrt{(x-x')^2 + (y-y')^2}).$$

On the other hand,

$$\langle h_{x,y}, h_{x',y'} \rangle = \sum_{n=-\infty}^{\infty} \langle h_{x,y}, f_n \rangle \langle f_n, h_{x',y'} \rangle$$

$$(5.17) \qquad = 2\pi \sum_{n=-\infty}^{\infty} e^{in(\theta'-\theta)} J_n(\gamma r) J_n(\gamma r'),$$

$$x' = r'\cos\theta', \quad y' = r'\sin\theta',$$

and

$$\langle h_{x,y}, h_{x',y'}\rangle = \int_{-\infty}^{\infty} \mathscr{H}_{+}^{x,y}(\lambda)\overline{\mathscr{H}}_{+}^{x',y'}(\lambda)\,d\lambda + \int_{-\infty}^{\infty} \mathscr{H}_{-}^{x,y}(\lambda)\overline{\mathscr{H}}_{-}^{x',y'}(\lambda)\,d\lambda. \tag{5.18}$$

This last expression can be considered as a continuous bilinear generating function for the E-basis.

In Chapter 5 of [8], Vilenkin explicitly decomposes the quasi-regular representation of $E(2)$ as a direct integral of irreducible representations. He expresses his results in terms of the M-basis (5.13). A very similar computation yields the decomposition in the E-basis (5.14). Of course, the two bases are related by (5.12):

$$H_n(x,y) = \langle h_{x,y}, f_n\rangle = \int_{-\infty}^{\infty} \mathscr{H}_{+}^{x,y}(\lambda)\overline{\mathscr{F}}_{+}^{n}(\lambda)\,d\lambda + \int_{-\infty}^{\infty} \mathscr{H}_{-}^{x,y}(\lambda)\overline{\mathscr{F}}_{-}^{n}(\lambda)\,d\lambda. \tag{5.19}$$

We describe the decomposition of the quasi-regular representation in the E-basis. Let $\mathscr{L}_2(R_2)$ be the Hilbert space of Lebesgue square integrable functions $f(x_1, x_2)$ on the plane:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_1, x_2)|^2\,dx_1\,dx_2 < \infty.$$

Then

$$f(x_1, x_2) = \frac{1}{2\pi}\int_0^{\infty} r\,dr \int_{-\infty}^{\infty} d\lambda[\mathscr{H}_{+}^{-x_1,-x_2}(-\lambda)\mathscr{F}_{+}^{r}(\lambda) + \mathscr{H}_{-}^{-x_1,-x_2}(-\lambda)\mathscr{F}_{-}^{r}(\lambda)],$$

where

$$\mathscr{F}_{\pm}^{r}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_1\,dx_2 \mathscr{H}_{\pm}^{x_1,x_2}(\lambda) f(x_1, x_2)$$

and $\mathscr{H}_{\pm}^{x_1,x_2}(\lambda)$ are given by (5.14) with $\gamma = r$.

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A NOTE ON A NONEXISTENCE THEOREM FOR NONLINEAR WAVE EQUATIONS*

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Abstract. Let $u:[0, T) \to D$ be a C^2 Hilbert-space-valued function satisfying an "abstract" wave equation of the form $Pu_{tt} = -Au + \mathscr{F}(u(t))$, where P and A are symmetric linear operators defined on D (a dense linear subspace of a Hilbert space H) and where $\mathscr{F}:D \to H$ is a gradient operator with potential $\mathscr{G}:D \to R$. Let $\mathscr{G}$ be almost homogeneous of degree $4\alpha + 2$ for some $\alpha > 0$ ($(4\alpha + 2)\mathscr{G}(x) \leqq (x, \mathscr{F}(x))$ for all $x \in D$) and let $u(0) = u_0$, $u_t(0) = v_0$. We extend the results of [1] to the case $E_\Sigma(0) \equiv \frac{1}{2}(u_0, Pu_0) + \frac{1}{2}(v_0, Pv_0) - \mathscr{G}(u_0) = 0$ and $(u_0, Pv_0) \leqq 0$. We prove the following.

If $E_\Sigma(0) = 0$, $(u_0, Pv_0) = 0$ and $u_0 \neq 0$, then u cannot exist on $[0, \infty)$ in the sense that there exists $T, 0 < T < \infty$, such that $\lim_{t \to T^-} (u(t), Pu(t)) = +\infty$. If $E_\Sigma(0) = 0$ and $(u_0, Pv_0) < 0$, then either u exists on $[0, \infty)$ and $(u(t), Pu(t)) < (u_0, Pu_0)$ for all $t > 0$ or $\lim_{t \to T^-} (u(t), Pu(t)) = +\infty$ for some $T < \infty$. Moreover, we show by example that both situations can occur.

In this paper, we conclude our study of abstract wave equations of the form $Pu_{tt} = -Au + \mathscr{F}(u(t))$ begun in [1] and continued in [2].

Let H be a real Hilbert space and $D \subseteq H$ be a dense linear subspace. Denote by $(\cdot, \cdot)$ the scalar product on H and by $\|\cdot\|$, the corresponding norm. Suppose that D is also a Hilbert space and the injection $i:D \to H$ is continuous as a mapping of Hilbert spaces. (That is, there exists $c > 0$ such that $\|x\| \leqq c\|x\|_D$ for $x \in D$.) Let P and A be symmetric (not necessarily bounded) linear operators mapping D into H such that $(x, Px) > 0$ for all $x \in D$, $x \neq 0$ and $(x, Ax) \geqq 0$ for all $x \in D$. Let $\mathscr{F}:D \to H$ be a gradient operator, that is, $\mathscr{F}$ is the Fréchet derivative in the D norm of a scalar-valued function $\mathscr{G}:D \to R$ called a potential associated with $\mathscr{F}$. It is well known that if $\mathscr{F}$ is Fréchet differentiable, then $\mathscr{G}$ exists if and only if $\mathscr{F}_x$ is symmetric at each $x \in D$. Assume that there exists a constant $\alpha > 0$ such that for all $x \in D$,

$$(*) \qquad 2(2\alpha + 1)\mathscr{G}(x) \leqq (x, \mathscr{F}(x)).$$

Consider the initial value problem

$$(1) \qquad \begin{gathered} P\frac{d^2u}{dt^2} = -Au + \mathscr{F}(u(t)) \quad \text{in } [0, T), \\ u(0) = u_0, \\ u_t(0) = v_0, \end{gathered}$$

where $u:[0, T) \to D$ is a "classical" solution in the sense defined precisely in [1]. We shall assume, for simplicity, that $\mathscr{G}(0) = 0$. Since we are not interested in the regularity question here, we shall omit all further reference to it and assume that our solutions have the necessary regularity needed in order to justify our calculations. Moreover, we shall assume that such a solution to (1) always exists locally, that is, near 0.

Let u satisfy (1) and define the total energy at time t, $E_\Sigma(t)$, by

$$(2) \qquad E_\Sigma(t) = \tfrac{1}{2}[(u(t), Au(t)) + (u_t(t), Pu_t(t))] - \mathscr{G}(u(t)) \equiv E(t) - \mathscr{G}(u(t)).$$

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Then one knows that $E_\Sigma(t) = E_\Sigma(0) = E(0) - \mathscr{G}(u_0)$. In [1] the following statements were proved:

I. If $\mathscr{G}(u_0) > E(0)$, i.e., if $E_\Sigma(0) < 0$, or if $\mathscr{G}(u_0) = E(0)$ and $(u_0, Pv_0) > 0$ then there exists T, $0 < T < \infty$, such that

$$\lim_{t \to T^-} (u(t), Pu(t)) = +\infty.$$

II. If $0 < E_\Sigma(0) < \alpha(u_0, Pv_0)^2/4(2\alpha + 1)(u_0, Pu_0)$, $(u_0, Pv_0) > 0$, *and* u *exists on* $[0, \infty)$, then there is $\gamma > 0$ such that

$$\liminf_{t \to +\infty} e^{-\gamma t}(u, Pu) > 0.$$

III. If $(u_0, Pv_0) > 0$,

$$\alpha(u_0, Pv_0)^2/4(2\alpha + 1)(u_0, Pu_0) \leqq E_\Sigma(0) < \tfrac{1}{2}(u_0, Pv_0)^2/(u_0, Pu_0),$$

and if u *exists on* $[0, \infty)$, then

$$\liminf_{t \to +\infty} (u, Pu)t^{-2} > 0.$$

In this note we wish to examine the situation in which $(u_0, Pv_0) \leqq 0$ and $E_\Sigma(0) = E(0) - \mathscr{G}(u_0) = 0$. This case is of interest, because, as was pointed out in [1], if $E_\Sigma(0) = E(0) - \mathscr{G}(u_0) < 0$, then the larger the escape time T, the less negative was this difference and the "more likely" (u_0, Pv_0) was to be nonpositive. That is, the following more precise version of I holds:

I′. Let $0 < r(u_0) = \sqrt{2}[\mathscr{G}(u_0) - \tfrac{1}{2}(u_0, Au_0)]^{1/2}$. Let

$$S_{u_0} = \{v_0 \in D | (v_0, Pv_0) < r^2(u_0)\}$$

and for each $T > 0$,

$$S_{T,u_0} = \{v_0 \in D | (v_0 - u_0/\alpha T, P(v_0 - u_0/\alpha T)) < r^2(u_0)\}.$$

Then if $v_0 \in S_{u_0}$ (that is, if $E_\Sigma(0) < 0$), $v_0 \in S_{u_0} - S_{T,u_0}$ for some $T > 0$ and $\lim_{t \to T^-}(u(t), Pu(t)) = +\infty$.

Thus, since the sets $B_T \equiv S_{u_0} - S_{T,u_0}$ decrease with increasing $T (B_{T_1} \subset B_{T_2}$ if $T_2 < T_1)$, we might expect that if $v_0 \in \bar{B}_T$ for all $T > 0$, that is, if $v_0 \in \{v_0 \in D | (v_0, Pv_0) = r^2(u_0)$ and $(u_0, Pv_0) \leqq 0\}$, then either the solution "blows up" in infinite time or it remains bounded in the sense that (u, Pu) is bounded on $[0, \infty)$. However, we show that if u exists for all time, (u, Pu) remains bounded. Otherwise, (u, Pu) becomes unbounded in a finite time. This is the content of the following theorem.

THEOREM. *Assume* $r(u_0) > 0$. *If* $v_0 \in \{v_0 \in D | (u_0, Pv_0) = 0$ *and* $(v_0, Pv_0) = r^2(u_0)(E_\Sigma(0) = 0)\}$, *then there exists* $T, 0 < T < \infty$, *such that* $\lim_{t \to T^-} (u, Pu) = +\infty$. *If, however,* $v_0 \in \{v_0 \in D | (v_0, Pv_0) < 0$ *and* $(v_0, Pv_0) = r^2(u_0)\}$, either (i) u *exists on* $[0, \infty)$ *and, for all* $t > 0$,

$$(u_0, Pu_0) > (u(t), Pu(t)) \geqq (u_0, Pu_0)\{1 - 2\alpha t(u_0, Pv_0)/(u_0, Pu_0)\}^{-1/\alpha} \tag{3}$$

or (ii) *there exists* $T, 0 < T < \infty$, *such that* $\lim_{t \to T^-} (u, Pu) = +\infty$. *Moreover, both situations* (i) *and* (ii) *can occur.*

Proof. Let $F(t) = (u, Pu) + Q^2$, where $Q > 0$ is a constant, be defined for all $t \geqq 0$. Then, as in [1], one can show that

$$FF'' - (\alpha + 1)(F')^2 \geqq -2(2\alpha + 1)F(t) \cdot E_\Sigma(0), \tag{**}$$

where

$$F'(t) = 2(u, Pu_t)$$

and

$$F''(t) = 2(u, Pu_{tt}) + 2(u_t, Pu_t).$$

Suppose $E_\Sigma(0) = 0$ and $F'(0) = 2(u_0, Pv_0) = 0$. Then $(F^{-\alpha})'' \leqq 0$ on the existence interval. Since $F'' \geqq 0$ for all t in this interval, $F' \geqq 0$ there. If $F''(\tau) > 0$ at some point $\tau > 0$, then $F'(\tau + \delta) > 0$ for some $\delta > 0$ and hence $F^{-\alpha} = G$ must have a zero to the right of $\tau + \delta$ since

$$F^{-\alpha}(t) \leqq F^{-\alpha}(\tau + \delta) + (t - (\tau + \delta))(F^{-\alpha})'(\tau + \delta) \quad \text{for } t \geqq \tau + \delta.$$

Thus $\lim_{t \to T^-} F(t) = +\infty$ for some $T < \infty$. The only other possibility is that $F''(t) \equiv 0$. However, we then have

$$0 = -2(u, Au) + 2(u_t, Pu_t) + 2(u, \mathscr{F}(u)).$$

Since $E_\Sigma(t) = E_\Sigma(0) = 0$,

$$\mathscr{G}(u(t)) = \tfrac{1}{2}(u, Au) + \tfrac{1}{2}(u_t, Pu_t),$$

so that, from (*),

$$\begin{aligned} 2(u_t, Pu_t) &= 2\mathscr{G}(u) - (u, \mathscr{F}(u)) \\ &\leqq -4\alpha\mathscr{G}(u) \\ &\leqq -2\alpha[(u, Au) + (u_t, Pu_t)]. \end{aligned}$$

Consequently $(1 + \alpha)(u_t, Pu_t) \leqq 0$ and hence $u_t \equiv 0$. Thus $u_t(0) = v_0 = 0$ which contradicts the fact that $E_\Sigma(0) = 0$.

For the second statement, assume again that u exists on $[0, \infty)$. Since $F'' \geqq 0$, F' is increasing and there are two cases. If $F'(0) = 2(u_0, Pv_0) \leqq F'(t) \leqq 0$ for all t, then $F(t) = \int_0^t F'(\eta)\, d\eta + F(0) < F(0)$ and since $F^{-\alpha}$ is concave, $F^{-\alpha}(t) \leqq F^{-\alpha}(0) - \alpha t F^{-\alpha-1}(0)F'(0)$. Rearranging this latter inequality and letting $Q \to 0$, we obtain (3). If, on the other hand, $F'(\tau) > 0$ at some point, then, we again see that $F^{-\alpha}$ has a zero to the right of τ and hence $(u, Pu) \to +\infty$ as $t \to T$ from below for some finite T.

Remark. A version of this theorem can be proved for weak solutions to (1) taken in the sense of the definition in [2], provided one postulates separately that $E_\Sigma(t) \leqq E_\Sigma(0)$ for all $t \geqq 0$. Very informally, by a weak solution to (1) we mean a function $u:[0, T) \to D$ such that, in $[0, T)$,

$$\begin{aligned} &(P^{1/2}\varphi, P^{1/2}u_t) + \int_0^t (A^{1/2}\varphi(\eta), A^{1/2}u(\eta))\, d\eta \\ &= (P^{1/2}\varphi_0, P^{1/2}v_0) + \int_0^t (P^{1/2}\varphi_\eta, P^{1/2}u_\eta)\, d\eta + \int_0^t (\varphi(\eta), \mathscr{F}(u(\eta)))\, d\eta \end{aligned}$$

for all "smooth enough" functions $\varphi:[0, t) \to D$. The argument used to establish (**) in this case is given in [2].

As an example to show that both situations (i) and (ii) are possible, let A have eigenvalues λ_1, λ_2 with $0 = \lambda_1 < \lambda_2$. Let $P = I$ and let x_1 and x_2 be orthonormal eigenvectors corresponding to λ_1 and λ_2 respectively. Consider the nonlinear functional $\mathscr{G}(x) \equiv (1/3)(x, x_1)^3$. Then $\mathscr{F}(x) = \mathscr{G}'(x) = (x, x_1)^2 x_1$ as a routine calculation shows.

We consider

$$\begin{aligned} u_{tt} &= -Au + (u, x_1)^2 x_1, \\ u(0) &= a_1 x_1 + a_2 x_2 \equiv u_0, \qquad (4)\\ u_t(0) &= b_1 x_1 + b_2 x_2 \equiv v_0, \end{aligned}$$

and look for solutions of the form $u(t) = g(t)x_1 + h(t)x_2$. This leads to two equivalent problems in ordinary differential equations, namely,

$$g'' = g^2,$$
$$g(0) = a_1, \quad g'(0) = b_1$$

and

$$h'' + \lambda_2 h = 0,$$
$$h(0) = a_2, \quad h'(0) = b_2.$$

The energy balance

$$\tfrac{1}{2}(\lambda_2 a_2^2 + b_2^2 + b_1^2) = \tfrac{1}{3}a_1^3 \qquad (\lambda_1 = 0) \tag{5}$$

must hold while we want to have

$$a_1 b_1 + a_2 b_2 < 0 \qquad (F(t) = g(t)^2 + h(t)^2). \tag{6}$$

If we choose $b_1 < 0$, $a_1 > 0$ such that $\frac{1}{3}a_1^3 = \frac{1}{2}b_1^2$ and $a_2 = b_2 = 0$, then $u(t) = a_1(1 - b_1 t/2a_1)^{-2}x_1$ satisfies (4) in such a way that (5) and (6) are also satisfied and $u(t)$ exists and is bounded on $[0, \infty)$. (In fact, (3) holds with equality on the right-hand side.)

If we now choose $a_1 > 0$ and $b_1 > 0$ such that $a_1^3 > \frac{3}{2}b_1^2$ and $a_2 = -b_2 = (\frac{2}{3}a_1^3 - b_1^2)^{1/2}(1 + \lambda_2)^{-1/2}$, then (5) and (6) hold provided a_1 is chosen so large that $\frac{2}{3}a_1^3 > b_1^2 + (1 + \lambda_2)a_1 b_1$. Moreover, in this case,

$$u(t) = a_1(1 - b_1 t/2a_1)^{-2}x_1 + [a_2 \cos\sqrt{\lambda_2}t + (b_2/\sqrt{\lambda_2})\sin\sqrt{\lambda_2}t]x_2$$

clearly becomes unbounded in norm in a finite time and solves (4).

As an example from partial differential equations we can take $A = -d^2/dx^2$ on $(0, \pi)$ with $D = D_A = \{f \in C^2(0, \pi) \mid f'(0) = f'(\pi) = 0\}$ and consider the following realization of (4) as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\pi^2}\left(\int_0^\pi u(y, t)\,dy\right)^2,$$
$$u(x, 0) = u_0(x), \qquad 0 < x < \pi,$$
$$u_t(x, 0) = v_0(x),$$
$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \qquad t \geqq 0.$$

Note added in proof. Brian Straughan of Heriot-Watt University, Edinburgh, Scotland, has shown that statements II and III can be combined and improved.

THEOREM (Straughan). (A) *If* $u:[0, T] \to D$ *solves* (1) *and the initial data satisfy*

(i) $0 < E_\Sigma(0) < \frac{1}{2}(u_0, Pv_0)^2/(u_0, Pu_0)$,

(ii) $(u_0, Pv_0) > 0$,

then

$$\lim_{t\to T}(u(t), Pu(t)) = \infty \quad \textit{for some } T < \infty.$$

(B) *If* u *exists on* $(0, \infty)$, $(u_0, Pv_0) > 0$ *and*

(ii′) $E_\Sigma(0) = \frac{1}{2}(u_0, Pv_0)^2/(u_0, Pu_0)$,

then

$$\liminf_{t\to\infty} t^{-2}(u, Pu) > 0.$$

These results and their proofs will appear elsewhere.

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AN ELEMENTARY TAUBERIAN THEOREM FOR ABSOLUTELY CONTINUOUS FUNCTIONS AND FOR SERIES*

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Abstract. A general nonlinear condition on f' is found which, when coupled with the assumption that either $f(t) = O(t^\alpha)$ or $f = o(t^\alpha)$, implies respectively that $f'(t) = O(t^{\alpha-1})$ or $f'(t) = o(t^{\alpha-1})$. Here α is a constant and $t \to \infty$ or $t \to 0+$. Function f is either absolutely continuous on compact intervals of $(0, \infty)$ or monotonically nondecreasing on the right half-line. Since f' is required to exist only almost everywhere, the nonlinear condition holds not only for functions, but also for series. Here a nonlinear Tauberian condition on the terms a_k, when coupled with limiting estimates of the partial sums $\sum^n a_k$ or $\sum_n^\infty a_k$, gives limiting estimates for the terms a_k as $k \to \infty$.

1. Introduction. Elementary Tauberian theorems are usually considered to be those Tauberian theorems whose proofs do not depend upon Fourier transform methods. This class of theorems treats the problems of differentiating an inequality or asymptotic relationship. For example, what conditions should be imposed upon the function f, in addition to requiring that $f = O(t^2)$, to ensure that $f'(t) = O(t)$ as $t \to \infty$ (or $t \to 0$)? If $f(t) \sim t^2$, when is it true that $f'(t) \sim 2t$? The example $f(t) = t^2 + \cos(t^8)$ shows that without additional conditions neither of these statements is, in general, true.

Theorems about the asymptotic behavior of derivatives have proved to be of interest and value in several subjects, including the study of the Laplace transform [13] and the n-body problem of celestial mechanics [7], [8], [10], [14, pp. 428–429]. (A brief history of their role in celestial mechanics can be found in [10].) The purpose of this communication is to offer a general Tauberian theorem of this type which generalizes several of the results found in the literature. In addition to relaxing the Tauberian condition, the present statement admits a wider class of functions. Whereas previous theorems usually require the functions to be in $C^2(0, \infty)$ or $C^1(0, \infty)$, the result given here holds equally well for functions which are monotonically increasing or absolutely continuous on bounded intervals.

An attractive feature of this theorem is that the central idea of the proof is very simple. In order to illustrate this fact, we initially require stronger smoothness conditions on the class of functions than are imposed later. In § 4, we state the theorem in the case where the functions are either absolutely continuous on bounded intervals (that is, they can be expressed as the indefinite integral of locally integrable functions), or monotonically increasing. Finally, in § 5, we show how this theorem holds for the case of series and summability. Here, the Tauberian condition reduces to R. Schmidt's condition for slowly increasing functions.

The central idea of this paper finds its origin in [9], where it is shown that the limiting estimate $f(t) = O(t)$ coupled with a nonlinear Tauberian condition on f'' results in the conclusion that $f'(t) = O(1)$. The role of this Tauberian condition is to control the rate of decrease (increase) of function f'. In this paper we find a more general nonlinear Tauberian condition on f' which plays the same role.

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Throughout this paper we restrict our attention to real-valued functions, constants, and sequences. The independent variable t will be described as "time".

2. Continuously differentiable case.

THEOREM 1. *Suppose that $f \in C^1(0, \infty)$ and that after some time $|f(t)| \leqq Ct^\alpha$, where α and C are real constants, C being positive. Suppose there exist functions $W(y)$ defined on $(-\infty, \infty)$ and $\varphi(y)$ defined on $(1, \infty)$ such that for all $\lambda > 1$,*

(i) $\lim_{t\to\infty} \sup \{\sup_{t \leqq t_1 \leqq \lambda_t} (W(f'(t)/t^{\alpha-1}) - W(f'(t_1)/t_1^{\alpha-1}))\} < \varphi(\lambda)$

and

(ii) *there exist constants $M > |\alpha|$ and $\beta > 1$ such that $|W(MC\beta) - W(MC)| \geqq \varphi(\lambda_1)$, where $\lambda_1 = e^{2/M}$ if $\alpha = 0$ and $\lambda_1^{|\alpha|} = (M + |\alpha|)/(M - |\alpha|)$ if $\alpha \neq 0$.*

Then for all values of t after some time, $f'(t) < MC\beta t^{\alpha-1}$.

Notice that if W is unbounded for increasing y, then condition (ii) can always be satisfied. If W is unbounded as y goes to both positive and negative infinity, then it follows immediately that $f'(t) = O(t^{\alpha-1})$ as $t \to \infty$.

We defer until §3 the discussion on how this result generalizes other Tauberian theorems. We simply mention at this point that this theorem generalizes Theorem 3 of [9] in the following respects. Reference [9] holds only for $\alpha = 1$ and requires that $f \in C^2(0, \infty)$. Furthermore, it has a condition on the second derivative of f which yields a continuous and increasing W. Finally, the resulting $\varphi(\lambda)$ is $\ln \lambda$.

Before we prove the theorem, we need the following lemma.

LEMMA 1. *Suppose that $f \in C^1(0, \infty)$ and that after some time, say for $t > t_0$, that $|f(t)| \leqq Ct^\alpha$, where α and C are constants, C being positive. If there exists an interval (t_1, t_2), $t_1 > t_0$, such that for all t in this interval $f'(t) > MCt^{\alpha-1}$, $M > |\alpha|$, then there exists some constant λ such that $t_2 \leqq \lambda t_1$, where $1 < \lambda < ((M + |\alpha|)/(M - |\alpha|))^{1/|\alpha|}$ if $\alpha \neq 0$, and $1 < \lambda < e^{2/M}$ if $\alpha = 0$.*

Proof of the lemma. Clearly, only the upper bound on λ needs to be determined. According to the hypothesis of the lemma, if $\alpha \neq 0$, then

$$MC(\lambda^\alpha - 1)t^\alpha/\alpha < \int_t^{\lambda t} f'(s)\, ds = f(\lambda t) - f(t) \leqq C(\lambda^\alpha + 1)t^\alpha. \tag{2.1}$$

The extreme ends of this inequality yield the desired upper bound on λ. If $\alpha = 0$, then the left-hand side of (2.1) is $MC \ln \lambda$, and the right-hand side is $2C$. Again, this inequality gives the stated result.

Proof of Theorem 1. According to condition (i), for all values of t after some time we have $\sup(W(f'(t)/t^{\alpha-1}) - W(f'(t_1)/t_1^{\alpha-1})) < \varphi(\lambda_1)$, where $t \leqq t_1 \leqq \lambda_1 t$ and λ_1 is given above. (Assume in condition (ii) that $W(MC\beta) > W(MC)$.)

Assume the conclusion of the theorem to be false. Then sufficiently large values of t can be found such that $t^{1-\alpha}f'(t) \geqq MC\beta$. According to Lemma 1, the continuity of function $t^{1-\alpha}f'(t)$, and the intermediate value property for continuous functions, sufficiently large values of t can be chosen so that they satisfy the inequality in the preceding paragraph and they satisfy the relationship $t^{1-\alpha}f'(t) = MC\beta$. For these values of t it now follows from the lemma that there exists $\xi \in (t, \lambda_1 t)$ such that $f'(\xi) \leqq MC\xi^{\alpha-1}$. Again, using the continuity of $t^{1-\alpha}f'(t)$ and the intermediate value property for continuous functions, we have that there exists $t_1 \in (t, \xi]$ such that $f'(t_1) = MCt_1^{\alpha-1}$. Combining these facts with condition (ii), we obtain

$$\begin{aligned}\varphi(\lambda_1) &> W(f(t)t^{1-\alpha}) - W(f(t_1)t_1^{1-\alpha}) \\ &= |W(MC\beta) - W(MC)| \geqq \varphi(\lambda_1).\end{aligned} \tag{2.2}$$

This contradiction proves the theorem in the case where $W(MC\beta) > W(MC)$. In the case where $W(MC) > W(MC\beta)$, we let t_1 be the point such that $f'(t_1) = MC\beta t_1^{\alpha-1}$. Again from the lemma and the continuity of $f'(t)/t^{\alpha-1}$, there exists t such that $f'(t) = MCt^{\alpha-1}$ and $t_1 \in (t, \lambda_1 t]$. The contradictory inequality (2.2) remains unchanged, and the theorem is proved.

We now state and prove the "small o" version of this theorem.

THEOREM 2. *Suppose for some constant* α *that* $f(t) = o(t^\alpha)$ *as* $t \to \infty$ *where* $f \in C^1(0, \infty)$. *Let* $W(y)$ *be a function defined on* $(-\infty, \infty)$ *which is not equal to a constant in any neighborhood of zero* (*in any interval* $(0, \delta)$). *Let* $\varphi(y)$ *be a function defined on the interval* $[1, 2]$ *such that it is continuous at* $y = 1$ *and* $\varphi(1) = 0$. *If*

$$\limsup_{t\to\infty} \{ \sup_{t \leqq t_1 \leqq \lambda t} (W(f'(t)/t^{\alpha-1}) - W(f'(t_1)/t_1^{\alpha-1}))\} < \varphi(\lambda), \tag{2.3}$$

where $\lambda \in [1, 2]$, *then* $f'(t) = o(t^{\alpha-1})$ *as* $t \to \infty$. (*Then* $\limsup t^{1-\alpha} f'(t) \leqq 0$.)

COROLLARY 1. *If the hypothesis of Theorem* 2 *holds with the exceptions that for constant* A, $f(t) \sim At^\alpha$ *and that relation* (2.3) *reads*

$$\limsup \{\sup (W((f'(t) - \alpha A)/t^{\alpha-1}) - W((f'(t_1) - \alpha A)/t_1^{\alpha-1}))\} < \varphi(\lambda),$$

then $f'(t) \sim \alpha A t^{\alpha-1}$ *as* $t \to \infty$. (*Then* $\limsup t^{1-\alpha} f'(t) \leqq \alpha A$.)

Proof of the corollary. Let $g(t) = f(t) - At^\alpha$. Function $g(t)$ satisfies the hypothesis of Theorem 2. Consequently, $g'(t) = f'(t) - \alpha A t^{\alpha-1} = o(t^{\alpha-1})$. But this is what was to be proved.

Before we prove the theorem, some of the differences in the hypotheses of the two theorems will be pointed out. It will be clear from the proof that the second theorem is actually a corollary of the first. It turns out that condition (ii) of Theorem 1 is satisfied in Theorem 2 by the combination of the continuity condition on φ and the hypothesis that $f(t) = o(t^\alpha)$. However, in Theorem 1 it takes both conditions (i) and (ii) to rule out the case where W is a constant. Since condition (ii) is omitted in Theorem 2, the fact that W cannot be a constant must be explicitly stated.

Proof of Theorem 2. Let ε be an arbitrarily small positive constant. Let ξ be a positive number less than ε such that $|W(\varepsilon) - W(\xi)| = \eta > 0$. Let M be so large that $\varphi(\lambda_1) < \eta$, where λ_1 is defined in the statement of Theorem 1. This can be done because of the continuity requirements on φ and since both $(M + |\alpha|)/(M - |\alpha|)$ and $e^{2/M}$ approach the value one as $M \to \infty$. Thus we have that $|W(\varepsilon) - W(\xi)| > \varphi(\lambda_1)$.

Define $\varepsilon_1 = \xi/M$. Since $f(t) = o(t^\alpha)$, after some time $|f(t)| \leqq \varepsilon_1 t^\alpha$. According to Theorem 1, where $C = \varepsilon_1$, $\beta = \varepsilon/\xi$, we have for all values of t after some time that $f'(t) \leqq \varepsilon t^{\alpha-1}$. A similar argument shows for all sufficiently large values of t that $f'(t) \geqq -\varepsilon t^{\alpha-1}$. Since ε is arbitrary, we have $f'(t) = o(t^{\alpha-1})$ as $t \to \infty$, and the proof is completed.

Notice in the proofs of both theorems that the continuity requirement on f' was used only to obtain the intermediate value property. However, by a theorem of Darboux [5, p. 16], *if f is continuous and f' is fined at all points of interval $[a, b]$ where $f'(a) \neq f'(b)$, then for any value c between the values $f'(a)$ and $f'(b)$, there exists a $\xi \in (a, b)$ such that $f'(\xi) = c$.* Consequently, this property can be used to replace the continuity requirement imposed on f'. Also, by use of these intermediate value statements, it turns out that W need only be defined almost everywhere on $(-\infty, \infty)$.

This is because in the proof we need only the values $W(MC\beta)$ and $W(MC)$. If W is defined almost everywhere, only minor modifications in the choices of M and β would be needed so that function W is well-defined at the points $MC\beta$ and MC. The assumptions that f' satisfies an intermediate value property and violates the inequality $t^{1-\alpha}f'(t) < MC\beta$ give us that there exist values of t, t_1 such that $t^{1-\alpha}f'(t) = MC\beta$ and $t_1^{1-\alpha}f'(t_1) = MC$. However, the limiting statements for $W(t^{1-\alpha}f'(t))$ must be interpreted as being over those values of t such that the function is well-defined.

THEOREM 3. *In Theorems* 1 *and* 2 *and in Corollary* 1, *the condition that $f \in C^1(0, \infty)$ can be replaced by the condition that $f \in C(0, \infty)$ and f' is defined at each point on $(0, \infty)$. Also, W need only be defined almost everywhere on $(-\infty, \infty)$.*

The extension of Theorem 2 contained in Theorem 3 improves a result of Karamata [4] by extending it to values of $\alpha \neq 1$ and by removing the continuity conditions on f', W, and φ. Also, Karamata requires W to be monotonically increasing. (Karamata's theorem generalized a result due to R. P. Boas [1]. Boas's theorem was motivated by the machinery developed by K. Sundman [12] in his study of the collisions in the three body problem [14, p. 429].)

The corresponding "large O" and "small o" theorems for $t \to 0$ also hold. The proofs are given most simply by modifying the reasoning rather than a change of variables argument.

3. Applications. In the last section we showed how the theorems given here generalize an earlier result of ours and papers by Karamata [4] and Boas [1]. In this section we show how this theorem generalizes some of the other theorems found in the literature and how it generates some new ones. By no means is the study found in this section intended to be exhaustive or particularly sharp. The prime purpose of this section is to show how Theorems 1 and 2 can be used. In all cases we will be assuming that $t \to \infty$. The corresponding results where $t \to 0$ hold with similar arguments.

COROLLARY 2 [13, p. 193]. *Suppose for $f \in C^2(0, \infty)$ that $f \sim At^\alpha$ $(=O(t^\alpha))$ as $t \to \infty$, where α and A are constants. If there exists a constant C such that after some time $f''(t) < Ct^{\alpha-2}$, then $f'(t) \sim \alpha At^{\alpha-1}$ $(=O(t^{\alpha-1}))$ as $t \to \infty$.*

Proof. According to the argument given in the proof of Corollary 1, we can assume without loss of generality that $A = 0$. From the condition on the second derivative, we have $f''(t)/t^{\alpha-1} < Ct^{-1}$. For all values of α, the choice of functions $W(y)$ and $\varphi(\lambda)$ and the fact that they satisfy condition (i) will be obtained by integrating this inequality. It will turn out that in all cases, $W(y) = y$. If $\alpha = 1$, we define $W(y) = y$ and $\varphi(\lambda) = 2 \ln \lambda$. Integrating the above inequality from t_1 to t_2, $t_2 \leqq \lambda t_1$, we see that $W(f'(t_2)) - W(f'(t_1)) \leqq C \ln (t_2/t_1) < 2C \ln \lambda$.

If α does not equal unity, then by integrating the above inequality by parts we obtain

$$
\begin{aligned}
C \ln (t_2/t_1) &\geqq \int_{t_1}^{t_2} s^{1-\alpha} f''(s)\, ds \\
&= s^{1-\alpha} f'(s) - (1-\alpha) s^{-\alpha} f(s) \Big|_{t_1}^{t_2} \\
&\quad + (\alpha - 1)\alpha \cdot \int_{t_1}^{t_2} s^{-\alpha-1} f(s)\, ds.
\end{aligned} \tag{3.1}
$$

In the case where $f(t) = o(t^\alpha)$ the second term on the right-hand side is $o(1)$, and the integral is $o(\ln (t_2/t_1))$. Condition (i) will now be satisfied if we define $W(y) = y$ and $\varphi(\lambda) = 3|C| \ln \lambda$. In the case where $f(t) = O(t^\alpha)$ there exists a constant D such that for all t after some time $|f(t)t^{-\alpha}| \leqq D$. This implies that the integral on the right-hand side of (3.1) is bounded above by $|(1-\alpha)\alpha| 3D \ln (t_2/t_1)$. Hence we define $W(y) = y$ and $\varphi(\lambda) = 3D|1-\alpha| + (2|C| + 3|(1-\alpha)\alpha|D) \ln \lambda$. Since W is unbounded the stated result follows in all cases from either Theorem 1 or Theorem 2.

COROLLARY 3. *Suppose for constant $\alpha \neq 1$ and $f(t) \in C^2(0, \infty)$ that $f(t) \sim At^\alpha$. If $f''(t) \leqq B|f'(t)|^b t^\gamma$, where $b = (2 - \alpha + \gamma)/(1-\alpha)$, then $f'(t) \sim \alpha A t^{\alpha-1}$.*

This is an extension of a result due to H. Pollard [6]. He required α to be positive and did not include the t^γ term.

Proof. Again we assume without loss of generality that $A = 0$. Now,

$$
\begin{aligned}
(d/dt)(t^{1-\alpha} f'(t)) &= t^{1-\alpha} f''(t) + (1-\alpha) t^{-\alpha} f'(t) \\
&\leqq t^{1-\alpha} B|f'(t)|^b t^\gamma + |(1-\alpha) t^{1-\alpha} f'(t)| t^{-1} \\
&< (1 + B|t^{1-\alpha} f'(t)|^b + |(1-\alpha) t^{1-\alpha} f'(t)|) t^{-1}.
\end{aligned}
$$

Hence we can define $W(y) = \int^y (1 + B|s|^b + |1-\alpha|\,|s|)^{-1}\, ds$ and $\varphi(\lambda) = 2 \ln \lambda$. The theorem now follows from Theorem 2.

Actually, the choice of the value of b is such that the above turns out to be an immediate corollary of the following.

COROLLARY 4. *Suppose for constant α and $f(t) \in C^2(0, \infty)$ that $f(t) \sim At^\alpha$. If there exists a positive measurable function $\omega(y)$ such that $f''(t) \leqq \omega(t^{1-\alpha} f'(t)) t^{\alpha-2}$, then $f'(t) \sim \alpha A t^{\alpha-1}$ as $t \to \infty$.*

With only minor modifications, the proof of this corollary is the same as the proof of Corollary 3.

The "large O" theorems have an additional condition that needs to be satisfied, and hence the statement of these results is slightly more delicate than the above. What we give here is the "large O" analogue of Corollary 4. A "large O" analogue of Corollary 3 can then be found in a straightforward fashion.

COROLLARY 5. *Let $f \in C^2(0, \infty)$. Suppose there exist constants $C > 0$ and α such that after some time $|f(t)| \leqq Ct^\alpha$. Furthermore, suppose there exists a positive measurable function $\omega(y)$ such that*

$$
\frac{d}{dt}(t^{1-\alpha} f'(t)) \leqq \omega(t^{1-\alpha} f'(t)) t^{-1}. \tag{3.2}
$$

Suppose either $\int^y \omega^{-1}(s)\,ds \to \infty$ *as* $y \to \infty$, *or for some constant* $\beta > 1$

$$\lim_{M\to\infty} (\omega^{-1}(MC) - \beta\omega^{-1}(MC\beta))(M^2 - \alpha^2) \tag{3.3}$$

exists and is greater than $2C^{-1}$. *Then* $f'(t)$ *is bounded above as* $t \to \infty$.

Proof. Our choices for $W(y)$ and $\varphi(\lambda)$ are $W(y) = \int^y \omega^{-1}(s)\,ds$ and $\varphi(\lambda) = (1 + \varepsilon)\ln\lambda$, where ε is an arbitrarily small positive constant which will be specified later. (If there is a convergence problem for the integral as $y \to 0$, then define $W(y) = \int^y (\varepsilon + \omega(s))^{-1}\,ds$.) That condition (i) is satisfied for this choice of W and φ follows by dividing both sides of inequality (3.2) by $\omega(t^{1-\alpha}f'(t))$, and integrating both sides of the new inequality. If $W(y)$ is unbounded as $y \to \infty$, then the result follows immediately. If $W(y)$ is bounded, then more care must be taken to show that condition (ii) of Theorem 1 is satisfied. Since ω is positive, the assumption that W is bounded implies that $W(MC\beta) - W(MC) \to 0$ as $M \to \infty$. We also have that $\varphi(\lambda_1) \to 0$ as $M \to \infty$. Thus, from L'Hospital's rule, a sufficient condition that values M and β can be found to satisfy condition (ii) is that the limiting condition (3.3) be satisfied. Since the limit is required to be strictly greater than $2C^{-1}$, a sufficiently small value of ε can be determined to conform with the definition of W and φ and still satisfy condition (ii). This completes the proof.

There is some difficulty in determining what should be the "natural" conditions to impose upon a function $f(t) \sim \ln t$ to obtain $f'(t) \sim t^{-1}$. Differentiating the function $\ln t$ twice and comparing the first and second derivatives suggests that possibly either the restriction $f''(t) < Ct^{-1}$ or the restriction $f'' < |f'|^2$ would suffice. However, we saw in Corollary 2, and we can easily derive from Corollary 5, that these exact same conditions imposed upon f, where $|f(t)| < 3/2$, yield simply that tf' is bounded, rather than being asymptotic to some constant. Indeed, counterexamples to such a conjecture are very plentiful. One simply adds a bounded function with the above properties, such as $\cos(\ln t) + \sin(\ln t)$, to the function $\ln t$.

One approach to the solution of this problem would be to follow the reasoning leading to Theorem 1 for the function $\ln t$ rather than t^α. However, the following is in some sense more general and seems to be sufficient for most applications. Here we impose stronger restrictions on the asymptotic relation rather than on the Tauberian conditions. The idea is that if the average of $h(t)/g(t)$ is essentially a constant and does not suffer rapid oscillations, then $h \sim g$. To apply this result to a problem such as $f(t) \sim t^\alpha(\ln t)^n \ln(\ln t)$, we simply let $h = f'$ and define g to be the derivative of the right-hand side of this asymptotic relation.

THEOREM 4. *Suppose for* f *and positive* g *in* $C^1(0, \infty)$ *that* $t^{-1}\int_0^t f(s)g^{-1}(s)\,ds \to A$, *where* A *is some constant. If there exists a positive measurable function* ω *such that*

$$\frac{d}{dt}(f(t)g^{-1}(t)) \leqq \omega(f(t)g^{-1}(t))t^{-1}, \tag{3.4}$$

then $f(t) \sim Ag(t)$ *as* $t \to \infty$.

Proof. Let $G(t) = \int_0^t f(s)g^{-1}(s)\,ds$. The stated result now follows from Corollary 4 where $\alpha = 1$.

Notice that if $g(t) = (\ln t)^{n-1}/t$, where n is a positive integer, then Tauberian condition (3.4) contains as a special case the condition

$$f' \leqq |f|^{\delta}, \quad \text{where } \delta = 2 \text{ if } n = 1$$

and $\delta > 2$ if $n > 1$. Notice how this compares with the statements made in the preamble to the theorem.

In a similar fashion Corollary 5 can be used to obtain the corresponding result in the case where $t^{-1} \int_0^t h(s)g^{-1}(s)\,ds = O(1)$ as $t \to \infty$.

4. Absolutely continuous case. In this section we relax the smoothness restrictions on function f to be that either f is monotonically increasing or f is absolutely continuous on bounded intervals. In both settings f' is defined almost everywhere, but it no longer has the intermediate value property. We compensate for this by requiring function W to be defined at each point of the real line and to be monotonically nondecreasing (or nonincreasing). Also, instead of considering points on the real line, we now must consider sets of positive measure contained in "small" intervals. To reflect this change of emphasis from points to sets, in this section "sup" will mean the essential supremum.

THEOREM 5. *Suppose that function f is either monotonically nondecreasing or absolutely continuous on bounded intervals of* $(0, \infty)$. *Assume for constant α and for positive constant C that after some time $|f(t)| \leqq Ct^{\alpha}$. Suppose there exist a nondecreasing function $W(y)$ defined on the real line and a function $\varphi(y)$ defined on* $[1, \infty)$ *such that for $\lambda > 1$,*

(i) $\limsup_{t\to\infty} \{\sup_{t \leqq t_1 \leqq \lambda t} (W(f'(t)t^{1-\alpha}) - W(f'(t_1)t_1^{1-\alpha}))\} < \varphi(\lambda)$,

and

(ii) *there exist constants $M > |\alpha|$ and $\beta > 1$ such that $W(MC\beta) - W(MC) \geqq \varphi(\lambda_1)$, where $\lambda_1 = e^{2/M} + \varepsilon$ if $\alpha = 0$ and $(\lambda_1 - \varepsilon)^{|\alpha|} = (M + |\alpha|)/(M - |\alpha|)$ if $\alpha \neq 0$. (Here, ε is an arbitrarily small positive constant.)*

Then after some time we have for almost all values of t that $f'(t) \leqq MC\beta t^{\alpha-1}$.

Proof. Since the proof of this theorem follows quite closely that of Theorem 1, we simply provide a sketch of the proof where we emphasize what changes are needed.

The first alteration is in the proof of Lemma 1. If f is monotonically nondecreasing, the sign of equality in relation (2.1) becomes the sign "$\leqq$". This is due to the well-known fact that if f is a monotonically nondecreasing function, then $\int_a^b f'(t)\,dt \leqq f(b) - f(a)$.

Assume the conclusion of the theorem to be false. That is, assume that $\limsup f'(t)t^{1-\alpha} > MC\beta$ as $t \to \infty$. Hence, after any value of time, there can be found a set A of positive measure with the properties that

(a) A is contained in an interval of length $\varepsilon/2$ and

(b) if $t \in A$, then $f'(t)t^{1-\alpha} > MC\beta$.

Define point $t^* = \inf\{t | t \in A\}$. According to Lemma 1 it follows that a set B of positive measure can be found such that (a) $B \subset [t^*, \lambda_1 t^*]$ and (b) if $t \in B$, then $f'(t)t^{1-\alpha} \leqq MC$. (To prove the existence of set B, we use the fact that λ_1 is at least ε units larger than the upper bound given in the lemma for the values of λ.)

Combining the properties of sets A and B with the fact that W is nondecreasing yields

$$(4.1)\quad \sup_{t\in A}\{\sup_{t_1\in[t,\lambda_1 t]}(W(f'(t)t^{1-\alpha})-W(f'(t_1)t_1^{1-\alpha}))\}\geqq W(MC\beta)-W(MC).$$

Condition (i) of Theorem 5 states that after some time $t = t_0$,

$$\sup_{t>t_0}\{\sup_{t_1\in[t,\lambda_1 t]}(W(f'(t)t^{1-\alpha})-W(f'(t_1)t_1^{1-\alpha}))\}<\varphi(\lambda_1).$$

This fact combined with inequality (4.1) and condition (b) yields an inequality similar to (2.2). This inequality proves the theorem.

In the same fashion as before, the "small o" version follows from the "large O" theorem.

THEOREM 6. *Suppose for constant α that $f(t) = o(t^\alpha)$ as $t \to \infty$, where f is either monotonically nondecreasing or f is absolutely continuous on bounded intervals of* $(0, \infty)$. *Suppose that function $W(y)$ is defined on the real line and is an increasing function in some neighborhood of zero (in some interval* $(0, \delta)$). *Let $\varphi(y)$ be a function defined on the interval* $[1, 2]$ *such that it is continuous at $y = 1$ and $\varphi(1) = 0$. If*

$$\limsup_{t\to\infty}\{\sup_{t_1\in[t,\lambda t]}(W(f'(t)t^{1-\alpha})-W(f'(t_1)t_1^{1-\alpha}))\}<\varphi(\lambda)$$

where $\lambda\in[1,2]$, then $f'(t) = o(t^{\alpha-1})$ for almost all t as $t\to\infty$ (then $\limsup t^{1-\alpha}f'(t) = 0$).

One immediate application of this increased flexibility in the choice of the function f is to obtain more general statements for the results found in § 3. Another advantage will be explored in the next section.

Theorem 6 generalizes a result due to N. G. de Bruijn [2, pp. 139–140]. For constant $\alpha \geqq 1$, he considers an absolutely continuous function on bounded intervals, $f(t)$, with the property $f(t) \sim t^\alpha$ as $t \to \infty$. He shows that the Tauberian condition "f' is nondecreasing" implies that $f'(t) \sim \alpha t^{\alpha-1}$ as $t \to \infty$. His result follows from Theorem 6 by defining $W(y) = y$ and $\varphi(\lambda) = 2\ln\lambda$, and it can be generalized to the "large O" case by the use of Theorem 5.

The corresponding "large O" and "small o" theorems for $t \to 0$ also hold. Here we would require f to be absolutely continuous on compact subsets of $(0, 1]$. The proofs seem to be given most simply by modifying the reasoning rather than by a change of variable argument.

5. Series. Since the proofs of Theorems 5 and 6 do not rely on the continuity of f', they can be applied immediately to series.

THEOREM 7. *For constant α, positive constant C, and sufficiently large integers n, suppose either that $|\sum_1^n a_m| \leqq Cn^\alpha$, where α is arbitrary, or that $|\sum_n^\infty a_m| \leqq Cn^\alpha$, where α is negative. Suppose there exist a nondecreasing function $W(y)$ defined on the real line and a function $\varphi(y)$ defined on* $[1, \infty)$ *such that*

(i) $\limsup_{k\to\infty}\{\max_j(W(a_k k^{1-\alpha}) - W(a_j j^{1-\alpha}))\} < \varphi(\lambda)$, *where $j\in[k,\lambda k]$ and $\lambda > 1$,*

and (ii) *there exist constants $M > |\alpha|$ and $\beta > 1$ such that $W(MC\beta) - W(MC) > \varphi(\lambda_1)$, where $\lambda_1 = e^{2/M} + \varepsilon$ if $\alpha = 0$ and $(\lambda_1 - \varepsilon)^{|\alpha|} = (M + |\alpha|)/(M - |\alpha|)$ if $\alpha \neq 0$.*

Then for all n greater than some integer, $a_n < MC\beta n^{\alpha-1}$. Here ε is some positive constant.

Proof. The proof can be obtained either by mimicking the ideas found in the proof of Theorem 1, or by defining $f'(t) = a_k$ for $t \in (k-1, k)$, $k = 1, 2, \cdots$, and $f(0) = 0$. In the latter case, the conclusion of Theorem 7 follows from Theorem 5. If α is negative, then the condition $|\sum^n a_n| \leqq Cn^\alpha$ implies that the series converges to a limit. Hence we may be interested in the hypothesis $|\sum_n^\infty a_n| \leqq Cn^\alpha$. In this case the theorem is proved by defining $f'(t)$ as given above and setting $f(0) = -\sum_0^\infty a_k$.

The "small o" version of Theorem 7 follows directly from Theorem 6. The hypothesis on the series will be $\sum^n a_k = o(n^\alpha)$, with the optional hypothesis for negative values of α being $\sum_n^\infty a_k = o(n^\alpha)$. It is clear that in certain cases this includes the hypothesis that the nth partial sum is asymptotic to An^α. (For example, if α is negative, we replace $\sum_n^\infty a_k$ with $\sum_n^\infty (a_k - \alpha A k^{\alpha-1})$.) Here, the conclusion would be $a_n \sim \alpha A n^{\alpha-1}$.

If we identify the operations of differentiation and integration of functions with the operations of evaluating differences and sums of sequences, then the statements and proofs of the results in § 3 suggest the proofs of analogous corollaries for series. We list some of them here. Since the purpose is simply to give a flavor of the types of statements which follow from Theorems 1, 5 and 7, we follow the spirit of § 3 and make no attempt to obtain sharp results or to provide an exhaustive survey.

COROLLARY 6. *Suppose for constant α that $\sum^n a_k = O(n^\alpha)$ $(=o(n^\alpha))$ as $n \to \infty$. If there exists a positive constant C such that $a_{k+1} - a_k \leqq Ck^{\alpha-2}$, $k = 1, 2, \cdots$, then $a_k = O(k^{\alpha-1})$ $(=o(k^{\alpha-1}))$ as $k \to \infty$. If α is negative, the series $\sum^n a_k$ could be replaced with the series $\sum_n^\infty a_k$.*

Notice that cases of convergent series are given not only by negative values of α, but also by $\alpha = 0$ and the "small o" hypothesis. Both the statement and proof of this corollary are motivated by Corollary 2.

Proof. If $\alpha = 1$, then $a_{k+1} - a_k \leqq Ck^{-1}$. Notice that

$$\begin{aligned} a_m - a_n &= (a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n) \\ &\leqq C \sum_{k=n}^{m} k^{-1} < 2C \ln(m/n). \end{aligned}$$

Hence, if $W(y) = y$ and $\varphi(\lambda) = 2C \ln \lambda$, then the hypothesis of both Theorem 7 and its "small o" version are satisfied.

Let $s_n = \sum^n a_n$, $a_0 = 0$, and $\Delta b_k = b_{k+1} - b_k$. For arbitrary $\alpha \neq 1$, we have from the hypothesis that $(k-1)^{1-\alpha}(a_k - a_{k-1}) < Ck^{-1}$. Using the technique of summing a series by parts, we obtain

$$\begin{aligned} 2C\ln(m/n) > C\sum_{n+1}^{m} k^{-1} &\geqq \sum_{k=n+1}^{m} (k-1)^{1-\alpha}(a_k - a_{k-1}) \\ &= \sum_{n+1}^{m} (k-1)^{1-\alpha}\Delta a_{k-1} = m^{1-\alpha}a_m - n^{1-\alpha}a_n - \sum_{n+1}^{m} a_k \Delta(k-1)^{1-\alpha} \\ &= m^{1-\alpha}a_m - n^{1-\alpha}a_n - \sum_{n+1}^{m} \Delta s_{k-1}\Delta(k-1)^{1-\alpha} \\ &= m^{1-\alpha}a_m - n^{1-\alpha}a_n - [s_m \Delta m^{(1-\alpha)} - s_n \Delta n^{1-\alpha}] + \sum_{n+1}^{m} s_k \Delta(\Delta(k-1)^{1-\alpha}). \end{aligned}$$

Applying the mean value theorem to $\Delta k^{(1-\alpha)}$ and using the hypothesis on the growth of s_k, we have for the "large O" hypothesis that the term in the brackets is bounded in magnitude by some positive constant D, and for the "small o" hypothesis that the term in the brackets goes to zero as $m, n \to \infty$. Applying the mean value theorem twice to the term $\Delta(\Delta(k-1)^{1-\alpha})$, we have that there exists a positive constant E such that the summation on the right-hand side is in magnitude bounded above by $E \ln(m/n)$ as $m, n \to \infty$. (Note, if $\alpha = 0$, then $E = 0$.) We then define $W(y) = y$. To satisfy the "large O" hypothesis of Theorem 7, we define $\varphi(\lambda) = 2D + (2C + E)\ln \lambda$. To satisfy the "small o" hypothesis, we define $\varphi(\lambda) = 2(C + E)\ln \lambda$. In either case the statement is proved.

If for negative values of α we use the hypothesis $\sum_n^\infty a_k = O(n^\alpha)\ (= o(n^\alpha))$, then the above proof still holds by defining $s_n = -\sum_n^\infty a_k$. This completes the proof of the corollary.

Although the above proof followed very closely the proof of Corollary 2, this will not always be the case. Some difficulties may arise in determining the definition of function $W(y)$ and satisfying the condition that $W(y)$ must be defined everywhere (except possibly at zero). In certain problems a natural definition for $W(y)$ involves integration. In order to compare the integral favorably with any summations resulting from the Tauberian condition, the proof may require either additional arguments or stricter Tauberian restrictions. We illustrate this in the proof of the following corollary which is motivated by Corollary 4. It shows that nonlinear Tauberian conditions similar to those which were applied to functions also hold for series.

COROLLARY 7. *Suppose for constant α that either $\sum^n a_k = o(n^\alpha)$ or (if $\alpha < 0$) $\sum_n^\infty a_k = o(n^\alpha)$. If any of the following three conditions are satisfied for some positive measurable function $\omega(y)$, then $a_k = o(k^{\alpha-1})$ as $k \to \infty$.*

(i) *There exists a positive constant D such that $\omega(y) + Dy$ is an increasing function for positive values of y and $|\Delta a_k| = |a_{k+1} - a_k| \leqq \omega(|k^{1-\alpha}a_k|)k^{\alpha-2}$.*

(ii) *Function $\omega(y)$ has the properties that $\omega(y) + (1-\alpha)y$ is a nonincreasing function for positive values of y and $|\Delta a_{k-1}| < \omega(|k^{1-\alpha}a_k|)k^{\alpha-2}$.*

(iii) *Function $\omega(y)$ is a nonincreasing function for positive values of y and*

$$|\Delta a_{k-1}|, |\Delta a_k| \leqq \omega(|k^{1-\alpha}a_k|)k^{\alpha-2}.$$

Proof. According to the mean value theorem, if $\alpha \neq 0, 1$, then $\Delta k^{1-\alpha} \sim (1-\alpha)k^{-\alpha}$. If $\alpha = 0$, then $\Delta k = 1 = (1-0)k^{-0}$. If $\alpha = 1$, then $\Delta k^0 = 0 = (1-1)k^{-1}$. Thus, in all cases $\Delta k^{1-\alpha} \sim (1-\alpha)k^{-\alpha}$ as $k \to \infty$.

We first prove the theorem under the first hypothesis. According to the definition of the terms involved and the hypothesis, we have for sufficiently large values of k that

$$\begin{aligned}
\Delta(|k^{1-\alpha}a_k|) &= (k+1)^{1-\alpha}\Delta|a_k| + |a_k|\Delta k^{1-\alpha} \\
&\leqq (k+1)^{1-\alpha}\omega(|k^{1-\alpha}a_k|)k^{\alpha-2} + |a_k|\Delta k^{1-\alpha} \\
&< 2\omega(|k^{1-\alpha}a_k|)k^{-1} + (2-\alpha)k^{-\alpha}|a_k| \\
&\leqq (2\omega(|k^{1-\alpha}a_k|) + D_1|k^{1-\alpha}a_k|)k^{-1}.
\end{aligned} \tag{5.1}$$

Here constant $D_1 \geqq \max(2D, |2-\alpha|)$. Define $g(s) = (1 + 2\omega(|s|) + D_1|s|)^{-1}$. Then $g(s) > 0$ is a nonincreasing function for positive s.

Let $W(y) = \int_0^y g(s)\,ds$, and let $\varphi(\lambda) = 2\ln\lambda$. Notice that $\Delta W(|k^{1-\alpha}a_k|) = g(\xi)\Delta(|k^{1-\alpha}a_k|)$, where ξ is some value between $|k^{1-\alpha}a_k|$ and $|(k+1)^{1-\alpha}a_{k+1}|$.

Since g is a nonincreasing function, we have for sufficiently large integers m and n, $m > n$, that

$$
\begin{aligned}
W(|m^{1-\alpha}a_m|) - W(|n^{1-\alpha}a_n|) &= \sum_{k=n}^{m-1} \Delta W(|k^{1-\alpha}a_k|) \\
&= \sum_{n}^{m-1} g(\xi_k)\Delta(|k^{1-\alpha}a_k|) \qquad (5.2)\\
&\leqq \textstyle\sum' g(\xi_k)\Delta(|k^{1-\alpha}a_k|) \\
&\leqq \textstyle\sum' g(|k^{1-\alpha}a_k|)\Delta(|k^{1-\alpha}a_k|),
\end{aligned}
$$

where $\sum'$ denotes the summation over those values of k such that $\Delta(|k^{1-\alpha}a_k|) \geqq 0$. According to inequality (5.1), the right-hand side of (5.2) is bounded above by $\sum_n^{m-1} k^{-1} < 2\ln(m/n)$. This shows that the hypothesis of the "small o" version of Theorem 7 (or Theorem 6) is satisfied for $\varphi(\lambda) = 2\ln\lambda$, and the proof is completed.

Notice that the second hypothesis forces the value of α to be greater than or equal to unity. If α were less than one, then $(1-\alpha)y \to \infty$ as $y \to \infty$. Since $\omega(y)$ is positive, this would contradict the hypothesis that $\omega(y) + (1-\alpha)y$ is nonincreasing. Using similar reasoning, we have that either there exists a value y_0 such that $\omega(y_0) \neq (1-\alpha)y_0 = 0$ or $\omega(y) \sim (\alpha - 1)y$ as $y \to \infty$.

Finally, since $\alpha \geqq 1$, we have from the mean value theorem that $\Delta(k+1)^{1-\alpha} = (1-\alpha)\xi^{-\alpha} \leqq (1-\alpha)(k+1)^{-\alpha}$, where $k+1 < \xi < k+2$.

According to the above, the second hypothesis implies for large values of k that

$$
\begin{aligned}
\Delta(|k^{1-\alpha}a_k|) &= |a_{k+1}|\Delta(k+1)^{1-\alpha} + k^{1-\alpha}\Delta|a_k| \\
&\leqq (1-\alpha)(k+1)^{-\alpha}|a_{k+1}| + k^{1-\alpha}\Delta|a_k| \qquad (5.3)\\
&< [\omega(|(k+1)^{1-\alpha}a_{k+1}|) + (1-\alpha)|(k+1)^{1-\alpha}a_{k+1}|](k+1)^{-1}.
\end{aligned}
$$

If there exists a value y_0 such that $\omega(y_0) = (\alpha - 1)y_0$, then define

$$g(s) = \begin{cases} [1 + \omega(s) + (1-\alpha)\,s]^{-1}, & 0 < s < y_0, \\ 1, & s \geqq y_0. \end{cases}$$

If such a value y_0 does not exist, then define

$$g(s) = [1 + \omega(s) + (1-\alpha)s]^{-1} \quad \text{for } s > 0.$$

Define $g(0) = \lim_{s\to 0} g(s)$. Notice that $g(s)$ is a nondecreasing function for $s \geqq 0$.

Also, notice that

$$\Delta(|k^{1-\alpha}a_k|) < g^{-1}(|(k+1)^{1-\alpha}a_{k+1}|)(k+1)^{-1}. \tag{5.4}$$

Now define

$$W(y) = \int_0^y g(s)\,ds.$$

For sufficiently large integers m and n, $m > n$, we have that

$$\begin{aligned} W(|m^{1-\alpha}a_m|) - W(|n^{1-\alpha}a_n|) &= \sum_{k=n}^{m-1} \Delta W(|k^{1-\alpha}a_k|) \\ &= \sum_{n}^{m-1} g(\xi_k)\Delta(|k^{1-\alpha}a_k|) \\ &\leqq \sum{}' g(\xi_k)\Delta(|k^{1-\alpha}a_k|) \\ &\leqq \sum{}' g(|(k+1)^{1-\alpha}a_{k+1}|)\Delta(|k^{1-\alpha}a_k|) \\ &< \sum_{n}^{m-1} (k+1)^{-1} < 2\ln(m/n), \end{aligned}$$

where $\sum'$ denotes the summation over those values of k such that $\Delta(|k^{1-\alpha}a_k|) \geqq 0$. The second to last summation on the right-hand side follows from the fact that $g(s)$ is nondecreasing and $|k^{1-\alpha}a_k| \leqq \xi_n \leqq |(k+1)^{1-\alpha}a_{k+1}|$. The last summation on the right-hand side follows from inequality (5.4). Hence, we have that if $\varphi(\lambda) = 2\ln\lambda$, then the conclusion follows from Theorem 6 or Theorem 7.

Let D_1 be a positive constant greater than $|1-\alpha|$. It follows under the third hypothesis that for positive $\varepsilon > 0$ there exists a positive value y_0 such that $\omega(y_0) + D_1y_0 - \varepsilon$ serves as a lower bound for the function $\omega(y) + D_1y$ over positive values of y. Define

$$g^{-1}(s) = \begin{cases} \omega(s) + D_1y_0 + \varepsilon & \text{for } 0 < s < y_0, \\ D_1(s + y_0) + \varepsilon & \text{for } s \geqq y_0. \end{cases}$$

Notice that $g(s)$ is increasing for $0 < s < y_0$ and decreasing for $s > y_0$. Define $g(0) = \lim_{s\to 0} g(s)$.

Using the third hypothesis and following the reasoning used to obtain inequalities (5.1) and (5.3), we find that

$$\Delta(|k^{1-\alpha}a_k|) < \min\{g^{-1}(|k^{1-\alpha}a_k|)k^{-1}, g^{-1}(|(k+1)^{1-\alpha}a_{k+1}|)(k+1)^{-1}\}. \tag{5.5}$$

Let $W(y) = \int_0^y g(s)\,ds$. In the same fashion that inequality (5.2) was derived we see for m and n sufficiently large, $m > n$, that

$$W(|m^{1-\alpha}a_m|) - W(|n^{1-\alpha}a_n|) \leqq \sum{}' g(\xi_k)\,\Delta(|k^{1-\alpha}a_k|).$$

It follows from the definition of g that

$$g(\xi_k) \leqq \max(g(|k^{1-\alpha}a_k|), g(|(k+1)^{1-\alpha}a_{k+1}|)).$$

Replacing $g(\xi_k)$ by the right-hand side of the above inequality, using inequality (5.5), and observing that $(k+1)^{-1} < k^{-1}$, we find that

$$W(|m^{1-\alpha}a_m|) - W(|n^{1-\alpha}a_n|) \leqq \sum_{n}^{m-1} k^{-1} < 2\ln(m/n).$$

The proof is completed by defining $\varphi(\lambda) = 2\ln\lambda$. This completes the proof of the corollary.

A similar type of analysis yields the "large O" analogue of Corollary 7.

The following corollary is an immediate consequence of Corollary 7. Here $\omega(y) = B|y|^b$, where the value of constant b is given below.

COROLLARY 8. *Suppose* $\alpha \neq 1$ *and either* $\sum^n a_k = o(n^\alpha)$ *or* (*if* α *is negative*) $\sum_n^\infty a_k = o(n^\alpha)$. *Let* B *be a positive constant, let* γ *be a constant, and define* $b = (2 - \alpha + \gamma)/(1 - \alpha)$. *Then* $a_k = o(k^{\alpha - 1})$ *as* $k \to \infty$ *if any of the following three conditions are satisfied*;

(i) $b \geqq 0$ *and* $|\Delta a_k| \leqq B|a_k|^b(k^\gamma)$,

(ii) $b < 0, \alpha > 1$ *and* $|\Delta a_{k-1}| \leqq B|a_k|^b(k^\gamma)$,

(iii) $b < 0, \alpha < 1$ *and* $|\Delta a_{k-1}|, |\Delta a_k| \leqq B|a_k|^b(k^\gamma)$.

Comparing the statement of Corollary 8 with that of Corollary 3 might prompt the conjecture that it is not necessary to take the absolute value of the term Δa_k in the Tauberian condition. The following is a counterexample to such a conjecture. Let $\alpha = 1/2$, $\gamma = 0$ and $b = 3$. The Tauberian condition of the conjecture would be $\Delta a_k \leqq |a_k|^3$. Define

$$a_k = \begin{cases} -k^{1/4} & \text{if } k = 10^{10^m} \text{ for } m = 1, 2, \cdots, \\ k^{-3} & \text{for all other values of } k. \end{cases}$$

Clearly, $\sum^n a_k = O(n^{1/4}) = o(n^{1/2})$. Also, we have that $\Delta a_k \leqq 0$ unless $k = 10^{10^m}$. In this case $\Delta a_k = (k + 1)^{-3} + k^{1/4}$. But this is clearly less than $(k^{1/4})^b = k^{3/4}$. Thus the conjectured Tauberian condition would be satisfied. However, $\limsup k^{1/2}|a_k| = \infty$, not zero, so the conclusion would not hold.

Results for series which are parallel to those obtained for functions in Theorem 4 can take several directions. One would be to impose the appropriate Tauberian condition on the term $\Delta(a_k/b_k)$ so that $\sum^n a_k/b_k \sim An^\alpha$ ($\sum_n^\infty a_k/b_k \sim An^\alpha$) implies that $a_k/b_k \sim \alpha A k^{\alpha - 1}$. Another direction suggested by Theorem 4 yields the following statement which was motivated by work of G. H. Hardy and proved by Szász [3, pp. 124–145].

COROLLARY 9. *Suppose for constants* μ_k *that* $0 \leqq \mu_1 < \mu_2 < \cdots$, $\mu_n \to \infty$, *and* $\mu_{n+1}/\mu_n \to 1$ *as* $n \to \infty$. *Furthermore, suppose that*

$$t^{-1} \int_0^t A(s)\, ds = t^{-1} \int_0^t \left(\sum_{\mu_k \leqq s} a_k \right) ds \to L \ (is\ O(1))$$

as $t \to \infty$, *and for positive constant* C *that*

$$a_k \geqq -C(\mu_k - \mu_{k-1})/\mu_k.$$

Then $\sum^n a_k$ *converges to* L (*is bounded*) *as* $n \to \infty$.

Here again, the linear Tauberian condition could be generalized to a nonlinear Tauberian condition.

Proof. Without loss of generality we assume that $L = 0$. Define $W(y) = y$, and $G(t) = \int_0^t A(s)\, ds$. Then for constant $\lambda > 1$ and sufficiently large values of t, we have

$$\sup_{t < t_1 \leqq \lambda t} [W(A(t)) - W(A(t_1))]$$

$$\leqq \sup_{\mu_n < \mu_k \leqq \lambda^2 \mu_n} \sum (-a_k) \leqq C\left(\frac{\mu_{n+1} - \mu_n}{\mu_{n+1}} + \cdots + \frac{\mu_{n+r} - \mu_{n+r-1}}{\mu_{n+r}} \right)$$

$$< C\left(\frac{\mu_{n+r} - \mu_n}{\mu_n} \right) \leqq C\left(\frac{\lambda^2 \mu_n - \mu_n}{\mu_n} \right) = C(\lambda^2 - 1),$$

where μ_n is the term such that $\mu_n \leqq t$, but $\mu_{n+1} > t$. Here the hypothesis $\mu_{n+1}/\mu_n \to 1$ is used to guarantee the existence of at least one index k which satisfies the above conditions. If we define $\phi(\lambda) = 2C(\lambda^2 - 1)$, then the conclusion of the corollary follows from Theorems 5 and 6.

Finally, we show in a simple case how Theorem 5 is related to the classical problem of determining what conditions on a series, besides its $(C, 1)$ summability, imply that it converges. The following approach is suggested by a direct proof of Corollary 6 for the case $\alpha = 1$ which was given by H. Pollard in a personal correspondence.

Let $s_n = \sum_1^n b_k$.

COROLLARY 10. *Let function $W(y)$ be defined on the real line and increasing in some neighborhood of zero. Let function $\varphi(y)$ be defined on $[1,2]$ and have the properties that $\varphi(1) = 0$ and φ is continuous at $y = 1$. Suppose that $\sum^\infty b_k = s(C, 1)$. If*

$$\limsup_{m} \{\max_{n} (W(s - s_m) - W(s - s_n))\} < \varphi(\lambda)$$

as $m \to \infty$, where $m \leqq n \leqq \lambda m$ and $\lambda \in (1, 2]$, then $\sum_1^\infty b_k = s$.

Notice that if $W(y) = y$, then the above becomes R. Schmidt's slowly increasing sequence condition [11, p. 136].

Proof. Define $a_1 = s$ and $a_{n+1} = a_n - b_n$. Then the statement $\lim \sum^N (1 - k/N)b_k = s$ as $N \to \infty$ is the same as the statement $\sum^N a_k = o(N)$ as $N \to \infty$. The W condition becomes $\limsup \{\max (W(-a_m) - W(-a_n))\} < \varphi(\lambda)$. Thus we now have from the "small o" version of Theorem 7 that $a_n = o(1)$ as $n \to \infty$. But since $s_n - s = -a_{n+1} = o(1)$, we have that $\sum^\infty b_k = s$.

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CONSTRUCTIVE EXISTENCE FOR SEMILINEAR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS*

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Abstract. Estimates of the form $|u|_2 \leqq C|Lu|_0$ are established for an elliptic operator with discontinuous coefficients, and applications are made to show convergence of an iteration procedure to solutions of semilinear equations with discontinuous right-hand side.

Introduction. In the study of existence problems for elliptic partial differential equations, a fundamental task is the establishment of a priori estimates for possible solutions. This work is devoted to deriving such estimates for a linear elliptic operator with bounded, measurable coefficients, and the application of these estimates to solutions of semilinear equations. In this application, precise information on the dependence of the estimates on the coefficients of the linear elliptic operator is used to show that a certain iteration procedure converges. This iteration is a function space version of a standard method used to solve nonlinear systems of algebraic equations. The explicit information obtained is paid for with restrictions on the domain in which solutions are sought and on the coefficients of the operator. The most severe of these is that the coefficients of the principal part are smooth enough for the equation to be written in divergence form, and that the operator be sufficiently close to a self-adjoint operator. Nevertheless, the results of § 1 may be thought of as a contribution to the theory of linear equations with measurable coefficients (cf. remarks at the end of that section).

1. Establishment of estimates. We consider the second order elliptic operator

$$Lu = a_{ij}u_{x_ix_j} + a_iu_{x_i} - au$$

(the summation convention on repeated indices is employed here) defined for functions u that have square summable second (distribution) derivatives on an n-dimensional domain Ω. In particular we will be interested in the action of L on the class $W^2_{2,0}(\Omega)$, which may be defined as the closure of functions in $C_2(\bar{\Omega})$ that vanish on $\partial\Omega$, this closure being with respect to the topology defined by the inner product

$$\langle f, g\rangle = \int_\Omega fg\,dx + \int_\Omega \nabla f\cdot\nabla g\,dx + \int_\Omega \sum D^2 f D^2 g\,dx.$$

(This last sum is extended over all second derivatives.) We assume immediately that a_i and a are bounded, measurable functions, that $\{a_{ij}\}$ is a symmetric matrix of measurable functions satisfying an inequality

$$\nu^2|\xi|^2 \leqq a_{ij}\xi_i\xi_j$$

for some positive constant ν, all n-dimensional vectors ξ and all x in Ω, and that the functions a_{ij} are sufficiently regular to ensure validity of the identity

$$\int_\Omega u(a_{ij}u_{x_j})_{x_i}\,dx = -\int_\Omega a_{ij}u_{x_i}u_{x_j}\,dx \tag{1}$$

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for u in $W^2_{2,0}(\Omega)$. If each a_{ij} is bounded and has square summable first derivatives, this last identity will be valid [1, Chap. 4]. We will impose a stronger condition on a_{ij}.

With the above assumptions L may be thought of as a linear operator mapping $W^2_{2,0}(\Omega)$ into $L_2(\Omega)$. Suppose the norms in these spaces are denoted, respectively, $|\cdot|_2$ and $|\cdot|_0$. Our first goal is then to establish an inequality of the form

$$|u|_2 \leqq C|Lu|_0$$

for u in $W^2_{2,0}(\Omega)$ and to obtain information about the constant C in terms of the coefficients of L. In order to obtain the desired results we make the following, more specialized assumptions:

A1. The boundary of Ω is piecewise smooth and has nonnegative mean curvature everywhere.

A2. The distribution derivatives of a_{ij} are bounded measurable functions.

It is perhaps worth pointing out at the outset that polyhedra, and smoothly bounded convex regions are classes of regions that satisfy A1.

This assumption is not essential for the validity of an estimate of the above type, but is of great utility in studying the dependence of C on the coefficients of L. The assumption A2 implies that L has divergence structure.

In order to facilitate statements to be made later we define

$$S = \sup |a_i - (a_{ij})_{x_j}|,$$

the supremum taken over Ω and all values of the index.

We will establish our result in a series of lemmas in which preliminary estimates are obtained.

For the moment we assume that

$$a_0 = \inf a > S^2/4\nu^2,$$

the infimum taken over Ω.

LEMMA 1. *For $\varepsilon > 0$ and $\alpha > 0$ the inequality*

$$\int_\Omega (Lu)^2\,dx \geqq 2\varepsilon\nu^2(1-\alpha)\int_\Omega |\nabla u|^2\,dx + \left(a_0 - \frac{\varepsilon}{2} - \frac{S^2}{4\nu^2\alpha}\right)2\varepsilon\int_\Omega u^2\,dx$$

is valid for u in $W^2_{2,0}(\Omega)$.

Proof. Since the techniques involved are standard the proof will only be sketched. Observing the identity (1) we have

$$-\int_\Omega uLu\,dx = \int_\Omega a_{ij}u_{x_i}u_{x_j}\,dx + \int_\Omega au^2\,dx + \int_\Omega [(a_{ij})_{x_j} - a_i]u_{x_i}u\,dx,$$

and using the inequality

$$2bc \leqq \varepsilon b^2 + \varepsilon^{-1}c^2 \tag{2}$$

(valid for positive ε) we have

$$-\int_\Omega uLu\,dx \leqq \frac{1}{2\varepsilon}\int_\Omega (Lu)^2\,dx + \frac{\varepsilon}{2}\int_\Omega u^2\,dx.$$

To complete the proof we combine the latter, make use of our assumptions about the coefficients of L, apply (2) once more with $\varepsilon = \delta$, and set δ equal to $S/2v^2\alpha$.

LEMMA 2. *The inequality*

$$\int_\Omega (Lu)^2\,dx \geqq \left(a_0 - \frac{S^2}{4v^2}\right)^2 \int_\Omega u^2\,dx$$

is valid for u in $W^2_{2,0}(\Omega)$.

Proof. Set $\alpha = 1$ and $\varepsilon = a_0 - S^2/4v^2$ in the inequality given in Lemma 1.

LEMMA 3. *The inequality*

$$\int_\Omega (Lu)^2\,dx \geqq (4a_0v^2 - 4a_0v^2S)\int_\Omega |\nabla u|^2\,dx$$

is valid for u in $W^2_{2,0}(\Omega)$.

Proof. Set $\varepsilon = 2(a_0 - S^2/4v^2\alpha)$ and $\alpha = S^2/4v^2a_0$. In order to simplify succeeding statements we define

$$Pu = a_{ij}u_{x_ix_j},$$

and

$$B_{ijl} = (a_{ij}a_{kl} - a_{ik}a_{jl})_{x_k}.$$

LEMMA 4. *The inequality*

$$\int_\Omega (Pu)^2\,dx \geqq v^2\int_\Omega \sum |D^2u|^2\,dx + \int_\Omega B_{ijl}u_{x_i}u_{x_jx_l}\,dx$$

holds for u in $W^2_{2,0}(\Omega)$.

Proof. This result is essentially established in [2, Chap. 2, § 8]. We need only apply A1 to the identity established there.

In the case $n = 2$, the following alternative to Lemma 4 is available. A proof is given in [3].

LEMMA 4′. *If $n = 2$, and $a_{ij}\xi_i\xi_j \leqq \mu^2|\xi|^2$ for all n-vectors ξ, the inequality*

$$\int_\Omega (Pu)^2\,dx \geqq \frac{v^8}{2\mu^4}\int_\Omega \sum |D^2u|^2\,dx$$

is valid.

Returning to the general case, we denote by B the supremum of $|B_{ijl}|$ over Ω and all values of the indices.

LEMMA 5. *The inequality*

$$\frac{v^2}{2}\int_\Omega \sum |D^2u|^2\,dx \leqq \int_\Omega (Pu)^2\,dx + \frac{n^4B^2}{2v^2}\int_\Omega |\nabla u|^2\,dx$$

holds.

Proof. The Cauchy–Schwarz inequality implies

$$\left|\int_\Omega B_{ijl}u_{x_i}u_{x_jx_l}\,dx\right| \leqq n^2B\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega \sum |D^2u|^2\,dx\right)^{1/2}.$$

The result is obtained by combining this with Lemma 4, and applying (2) with $\varepsilon = v^2/n^2B$.

We denote by A, a_1 the supremum of a_i^2, a over Ω, respectively.

LEMMA 6. *Under the assumption* $a_0 > S^2/4\nu^2$ *the inequality*

$$|u|_2 \leqq C|Lu|_0$$

is valid with

$$C^2 = \frac{4}{\nu^2} + \frac{16A\nu^2 + n^4B^2 + \nu^4}{\nu^4(4a_0\nu^2 - 2\nu\sqrt{a_0}\,S)} + \frac{256a_1^2\nu^2 + 16\nu^4}{(4a_0\nu^2 - S^2)^2}.$$

Proof. Since $Lu = Pu + a_iu_{x_i} - au$, several applications of (2) yield

$$\int_\Omega (Lu)^2\,dx \geqq (1 - \varepsilon_1 - \varepsilon_2)\int_\Omega (Pu)^2\,dx + (1 - \varepsilon_1^{-1} - \varepsilon_3^{-1})\int_\Omega (a_iu_{x_i})^2\,dx$$
$$+ (1 - \varepsilon_2^{-1} - \varepsilon_3^{-1})\int_\Omega a^2u^2\,dx$$

for $\varepsilon_1, \varepsilon_2, \varepsilon_3$ positive. Setting $\varepsilon_1 = \varepsilon_2 = \frac{1}{4}, \varepsilon_3 = 1$, and applying Lemma 5 implies

$$(3)\quad \int_\Omega \sum|D^2u|^2\,dx \leqq \frac{4}{\nu^2}\int_\Omega (Lu)^2\,dx + \left(\frac{16A}{\nu^2} + \frac{n^4B^2}{\nu^4}\right)\int_\Omega |\nabla u|^2\,dx + \frac{16a_1^2}{\nu^2}\int_\Omega u^2\,dx.$$

To complete the proof we apply Lemma 2 and Lemma 3.

We have established the required inequality for sufficiently large values of a_0. The lower range of a_0 will be dealt with now. We will make use of the lowest eigenvalue λ for the Laplacian with homogeneous boundary conditions on Ω, which is given by

$$(4)\qquad \lambda = \inf\frac{\int_\Omega|\nabla u|^2\,dx}{\int_\Omega u^2\,dx},$$

the infimum taken over u in $\dot{W}_2^1(\Omega)$.

LEMMA 7. *If* a_0 *satisfies*

$$(5)\qquad \sqrt{\lambda}\,S - \lambda\nu^2 < a_0 < \lambda\nu^2,$$

the inequality

$$(6)\qquad \int_\Omega u^2\,dx \leqq \left[\frac{2(\lambda\nu^2 + a_0)}{(\lambda\nu^2 + a_0)^2 - \lambda S^2}\right]^2\int_\Omega (Lu)^2\,dx$$

holds.

Proof. Using Lemma 1 and the inequality implied by (4) we have

$$\int_\Omega (Lu)^2\,dx \geqq f(\varepsilon,\alpha)\int_\Omega u^2\,dx,$$

where

$$f(\varepsilon,\alpha) = \varepsilon\left(2\nu^2(1 - \alpha)\lambda + 2a_0 - \frac{S^2}{2\nu^2\alpha} - \varepsilon\right),$$

for ε, α positive. If the optimum choice for α(ε fixed) is taken and the requirement that $f(\varepsilon, \alpha)$ be positive is imposed, the theorem follows.

It must be remarked here that this result is vacuous if S is not smaller than $2\sqrt{\lambda}\,\nu^2$. Of course, if L is self-adjoint, that is, $S = 0$, this condition is satisfied.

LEMMA 8. *Suppose* a_0 *satisfies* (5). *Then if*

$$a_0 \leqq S^2/2\nu^2 + 1/2,$$

the inequality

$$\int_\Omega |\nabla u|^2\,dx \leqq \nu^{-2}\left\{1 + \left[\frac{2(\lambda\nu^2 + a_0)}{(\lambda\nu^2 + a_0)^2 - \lambda S^2}\right]^2\left(1 + \frac{S^2}{\nu^2} - 2a_0\right)\right\}\int_\Omega (Lu)^2\,dx$$

holds. On the other hand if

$$a_0 > S^2/2\nu^2,$$

we have

$$\int_\Omega |\nabla u|^2\,dx \leqq (2\nu^2 a_0 - S^2)^{-1}\int_\Omega (Lu)^2\,dx.$$

Proof. Set α equal to 1/2 in Lemma 1, invoke Lemma 7, and optimize the constant factor in each of the above cases.

For brevity we denote by ξ the constant in the inequality

$$\int_\Omega |\nabla u|^2\,dx \leqq \xi\int_\Omega (Lu)^2\,dx \tag{7}$$

guaranteed by Lemma 8.

It is clear that Lemmas 7 and 8 combined with (3) yield an inequality of the desired type for a_0 satisfying (5). The next step in our procedure is to combine the results for the two ranges of a_0 that we have considered. Here we impose a restriction on the size of S, this being the "closeness" of L to a self-adjoint operator alluded to in the introductory remarks.

A3. $S < \sqrt{\lambda}\,\nu^2$.

An inspection of the inequalities involved shows that this condition guarantees that

$$\lambda\nu^2 > S^2/4\nu^2,$$

and that

$$\sqrt{\lambda}S - \lambda\nu^2 < 0.$$

Therefore the two ranges of a_0 intersect and their union contains all nonnegative a_0 (properly). We now have the following theorem.

THEOREM 1. *Assuming* A1–A3, *there is a constant* $C(a_0)$ *such that*

$$|u|_2 \leqq C(a_0)|Lu|_0, \qquad u \in W^2_{2,0}(\Omega);$$

the constant $C(a_0)$ *may be taken to be a continuous function of* a_0 *for* a_0 *in the interval* $(\sqrt{\lambda}\,S - \lambda\nu^2, \infty)$.

$C(a_0)$ is given explicitly by "matching" the expression given in Lemma 6 with that implied by (3), (6) and (7) for a_0 satisfying (5).

Remarks. 1. If L is self-adjoint, the above simplifies to a certain extent. In particular, we have a bound for $\|L^{-1}\|$ valid for $-\lambda\nu^2 < a_0 < \infty$ which is given by a continuous function of a_0.

2. In the two-dimensional case Lemma 4′ can be used to eliminate the dependence on B.

3. The dependence of C on a_1, as a_1 becomes large, is *linear* as is shown by the expression given for C in Lemma 6. This fact will be of crucial importance in the applications to follow.

4. The special case $L = \Delta - a$ is of some interest. Routine modification of the above reasoning yields the following bounds for $\|L^{-1}\|$ for $a_0 \in (-\lambda, \infty)$:

$$C^2 = \begin{cases} 2 + \dfrac{4a_1^2 - 1}{a_0^2} + \dfrac{1}{4a_0}, & 0 < a_0, \\[2ex] 3 + \dfrac{8(1 - 2a_1^2) + 5}{(\lambda + a_0)^2}, & -\lambda < a_0 < \min\{\lambda, 1/2\}. \end{cases}$$

If a vanishes, the value $3 + 13\lambda^{-2}$ is obtained as an upper bound for $\|\Delta^{-1}\|^2$, which may be compared with the value $1 + \lambda^{-1} + \lambda^{-2}$ derived in [4]. The latter value has been shown to be sharp in the case of a convex polyhedron [5].

5. The condition that the boundary of Ω have nonnegative mean curvature has been used previously in the study of elliptic equations with discontinuous coefficients by Talenti [7] and Chicco [8]. In particular, Talenti derived an explicit bound, similar to that obtained above, for an equation without lower order terms under the assumption of "Cordes conditions" [9] on the coefficients a_{ij}. Whereas no such restriction has been used here, we have imposed a certain amount of smoothness on the a_{ij} (A2), as well as divergence structure of the equation.

Nonnegativity of the mean curvature of the boundary of Ω, that is of the sum of the principal curvatures, is used to ensure that the boundary integral that arises in transforming the integral of $(Pu)^2$ in Lemma 4 can be neglected (cf. [2, p. 178]). It is possible to relax this hypothesis to the assumption that the mean curvature be bounded below by a negative constant, and the most important aspects of the results obtained above would remain intact under this weaker hypothesis.

6. The proof given above is elementary in that the imbedding theorems have not been invoked. In addition, no smallness restriction has been placed on the region Ω. In connection with this feature it is of interest to compare the above results with some recent work of Sharovskii [6]. He derives an inequality of the type given above under the assumption that $n - 1$ columns of $\{a_{ij}\}$ consist of continuous functions with possible discontinuities in the remaining column. Also a certain smallness of the region, related to the modulus of continuity of the a_{ij}, is required, whereas the equations he studies need not have divergence structure.

It is of particular interest to compare the above results with some recent work of Chicco [10]. He studies the problem of unique solvability of the problem $Lu = f$, $f \in L_2(\Omega)$ for $u \in W^2_{2,0}(\Omega)$. It is well known that this problem is uniquely solvable if $n = 2$, a_{ij} is bounded, and a is suitably restricted, but that if $n \geqq 3$ some additional restriction is required on $\{a_{ij}\}$. (In particular, an example is given in [11] in which a_{ij} is bounded, $a \equiv 0$, and uniqueness is violated.) In this work a condition on $\{a_{ij}\}$ is used which might be interpreted as "closeness" of L to divergence structure. Under this assumption the existence of a positive constant α_0 such that $a \geqq \alpha_0$ implies unique solvability is proved. In several ways the situation dealt with there is more general than that discussed above, but it is instructive to observe that if $\partial\Omega$ is such that $L(W_{2,0}^{2}(\Omega))$ is dense in $L_2(\Omega)$ (e.g., $\partial\Omega$ smooth) then our Theorem 1 says that $a_0 \geqq \sqrt{\lambda}\, S - \lambda v^2$ implies unique solvability of this problem. Therefore, we are able to obtain a stronger result, but under more stringent hypotheses.

The explicit information obtained about C has been paid for with somewhat restrictive hypotheses on the coefficients of L. In what follows we will attempt to justify this expenditure by making an application which requires this information.

2. An application to nonlinear equations. We will study the Dirichlet problem for the equation

$$L_0 u = f(x, u), \tag{8}$$

where

$$L_0 u = a_{ij} u_{x_i x_j} + b_i u_{x_i}$$

is an elliptic operator satisfying the assumptions of the first paragraph. All of our results are for Ω either two- or three-dimensional, as the proofs given make use of the imbedding of $W_2^2(\Omega)$ in $C(\bar{\Omega})$. Some generations to higher dimensions are possible under more stringent hypotheses on f, but these are not dealt with here.

The two existence theorems to be presented are based on certain "continuation" results for the Newton–Kantorovich method in Hilbert spaces. In particular, suppose P is a twice continuously differentiable mapping of X into Y, X and Y Hilbert spaces, and that

$$\|[P'(x)]^{-1}\| \leqq \alpha\|x\| + \beta, \tag{9}$$

where α, β are constants. Suppose also that P'' is locally bounded, that is,

$$\|P''(x)\| \leqq K = K(r)$$

for x in $S(0, r)$. Then it is known that P is a homeomorphism of X onto Y, and, furthermore, an iteration process can be given which converges to the unique solution of $P(x) = 0$, independently of the initial guess [12]. This process is of the form

$$\begin{aligned} x_{n+1} &= \Phi(x_n, h), & n &= 1, \cdots, N-1, \\ x_{n+1} &= x_n - [P'(x_n)]^{-1} P(x_n), & n &= N, \cdots, \end{aligned} \tag{10}$$

where $h = 1/N$, and the first N steps arise from replacing the differential equation

$$\dot{x}(t) = -[P'(x)]^{-1} P(x), \qquad t \in [0, 1],$$

by a difference equation with discretization error of order h^p ($p \geqq 1$). In [12] an explicit expression is given for a constant h_0, determined by α, β and C, where $\|x(1) - x_N\| \leqq Ch^p$ is a discretization error bound, such that $h < h_0$ implies this process converges. The task to be undertaken here is to show that under fairly general assumptions on f the inequality (9) holds for the mapping $P(u) = L_0 u - f(x, u)$ of $X = W_{2,0}^2(\Omega)$ into $Y = L_2(\Omega)$. The main results needed in order to accomplish this are those of paragraph one.

In order to proceed we need to recall a standard concept [13].

DEFINITION. $g(x, u)$ defined on $\Omega \times R$ is an N-function if
(i) g is continuous in u for almost all x,
(ii) g is measurable in x for all u.
We will utilize a certain class of N-functions, $N(\Omega)$.

DEFINITION. $g \in N(\Omega)$ if g is an N-function and g is bounded for bounded values of u, i.e., for any $M > 0$, $g(\Omega \times [-M, M])$ is a bounded set.

An essential step in our discussion is consideration of the operator G defined on $W_2^2(\Omega)$ by $G(u) = g(x, u)$.

LEMMA 9. *Suppose that $g \in N(\Omega)$, and that for almost all x, g has three derivatives with respect to u, and $g_u, g_{uu}, g_{uuu} \in N(\Omega)$.*

Then G maps $W_2^2(\Omega)$ into $L_2(\Omega)$ continuously, G has two continuous derivatives, and G'' is locally bounded.

Proof. Repeated use will be made of the imbedding of $W_2^2(\Omega)$ in $C(\bar{\Omega})$. This implies that $G(u)$ is a bounded measurable function if $u \in W_2^2(\Omega)$, and that

$$G_u'(\phi) = g_u(x, u(x))\phi(x)$$

is a bounded linear operator with

$$\|G_u'\| = \text{ess. max}\,|g_u(x, u(x))|.$$

To see that G is differentiable at u (and hence continuous) the "remainder"

$$R(u, \phi) = G(u + \phi) - G(u) - G_u'(\phi) = \int_0^1 [g_u(x, u + t\phi) - g_u(x, u)]\phi \, dt$$

must be investigated. We have

$$\|R(u, \phi)\|_0 \leqq \|G_u'(u + t\phi) - G_u'(u)\| \, \|\phi\|_0,$$

so it suffices to show that the left-hand term goes to zero as t does. (This will also show that $u \to G_u'$ is continuous.) This follows from

$$G_u'(u + t\phi) - G_u'(u) = \left[\int_0^1 g_{uu}(x, u + st\phi)\, ds\right] t\phi$$

since $g_{uu} \in N(\Omega)$. Existence, continuity and local boundedness is proven similarly, using the hypothesis that $g_{uuu} \in N(\Omega)$.

Example. Suppose that $a_i(x)$ is bounded, measurable on Ω, and $g_i(x, u)$ is continuous and has three continuous derivatives with respect to u on $\Omega \times R$. Then

$$g(x, u) = \sum_{i=1}^{n} a_i(x) g_i(x, u)$$

satisfies the hypotheses of the above lemma.

THEOREM 2. *Suppose that f satisfies the hypotheses of Lemma 9 and that*

$$\sqrt{\lambda} S - \lambda v^2 + \varepsilon \leqq f_u, \qquad f_u = O(u),$$

where ε is a positive number. Then the iteration (10) *converges to a unique solution of* (8).

Proof. We need only observe that

$$P'(u) = L_0 - f_u,$$

and make use of the estimate of Theorem 1.

It is desirable to eliminate the growth rate on f_u required in Theorem 2. We will make use of the following maximum principle in doing this. The proof is essentially taken from [14, p. 426], but as generalized solutions are not considered there it will be given for completeness.

THEOREM 3. *Suppose that* $f(x, u) \geqq 0$ *for all* $u > M$. *Then if* $P(u) = 0$,

$$u \leqq M.$$

Proof. Set $v = u - M$, and let k be the supremum of v over Ω. We suppose k positive and seek a contradiction. Since v is continuous there are points P where $v(P) = k$, and since $v < k$ near $\partial\Omega$ we may choose P and a neighborhood N of P so that $0 < v \leqq k$, $v \not\equiv k$ in N. Our hypotheses imply that

$$L_0(M) - f(x, M + b) \leqq 0 \quad \text{for all } b > 0.$$

This is also true if b is replaced by any positive function $b(x)$. Therefore, we may set $b = v$ if we restrict our attention to N, that is,

$$L_0(M) - f(x, u) \leqq 0 \quad \text{in } N.$$

Then, since

$$P(u) = L_0 u - f(x, u) = 0 \quad \text{in } N,$$

we have $L_0 v \geqq 0$ in N. Before completing the proof we need to observe that L_0 has divergence structure and that $L_0 v \geqq 0$ implies that v is a (positive) subsolution for L_0 in N. Therefore, the results of [15] imply that v is constant in N, and this is the contradiction that we were seeking.

In an obvious way we also obtain the following.

THEOREM 3′. *Suppose that* $f(x, u) \leqq 0$ *for* $u < -M$. *Then if* $P(u) = 0$,

$$u \geqq -M.$$

We conclude with a theorem which follows from Theorems 1, 3 and 3′.

THEOREM 4. *Suppose that f satisfies the hypotheses of Lemma* 9, *and that*

(i) $f_u \geqq \sqrt{\lambda}\, S - \lambda v^2 + \varepsilon, \varepsilon > 0$,

(ii) $uf(x, u) \geqq 0$ *if* $|u| > M$.

Then the iteration (10) *converges to a unique solution of* (8).

We need only replace $f(x, u)$ by a bounded function $\tilde{f}(x, u)$ coinciding with f if $|u| \leqq M$, and satisfying the other requirements on f. The iteration converges to a solution of $L_0 u - \tilde{f}(x, u) = 0$ such that $|u| \leqq M$ and this function is a solution of (8).

The essential feature common to Theorems 2 and 4 is the use of the a priori bound of paragraph one to deduce the validity of the inequality (9) for the operator P. In the case of Theorem 2 this follows directly from the fact that the constant C of Lemma 6 has a linear growth rate with respect to a_1, for large values of a_1, since, in the context of this paragraph,

$$a_1 = \sup |f_u(x, u(x))| = O(\sup |u(x)|) = O(|u|_2),$$

the last equality being a consequence of the Sobolev imbedding theorem. In the case of Theorem 4 we use an a priori estimate for the *nonlinear* equation to replace the problem with an equivalent one in which f_u is bounded. It then follows from the estimate of paragraph one that $\|[P'(u)]^{-1}\|$ is bounded.

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CALCULATION OF AN ADIABATIC INVARIANT BY TURNING POINT THEORY*

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Abstract. Let u be a solution of the real differential equation $\varepsilon^2\ddot{u} + \phi^2(t)u = 0$. The quantity $r^2(t, \varepsilon) = \phi(t)u^2(t, \varepsilon) + \varepsilon^2\phi^{-1}(t)\dot{u}^2(t, \varepsilon)$ is the ratio of the local energy to the local frequency for the oscillator described by the differential equation. The total change, $r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon)$, of this "adiabatic invariant" is very small for small $\varepsilon > 0$. It is shown that under certain conditions the asymptotically leading term, for this total change, as $\varepsilon \to 0+$, has the form $b(\varepsilon)\, e^{-c/\varepsilon}$, where c is a positive constant and $b(\varepsilon)$ a bounded function, both of which can be calculated by quadratures. The most important assumption is that the function $\phi^2(t)$ is positive for real t and holomorphic in a neighborhood of the real axis that contains a simple zero of $\phi^2(t)$. The proof is based on the theory of simple turning points.

1. Introduction. It is known that if the length of a simple pendulum is changed slowly the ratio of the energy to the frequency is very nearly the same at the beginning and at the end of the process. This is the simplest example of a so-called "adiabatic invariant."

In [7] I have studied this invariant in a mathematical formulation due to Littlewood [2] and obtained results somewhat more complete than Littlewood's. At the end of [7] I announced an explicit asymptotic formula for the adiabatic invariant and gave a brief sketch of the arguments leading to it. The purpose of the present paper is to supply a detailed statement and proof of that result, which is exhibited in Theorem 8.2 of this article.

I have tried to make this paper self-contained in a narrow sense of the word, but numerous facts are quoted from [7] without repeating proofs or giving motivation. Some acquaintance with [7] is therefore desirable.

In purely mathematical terms the problem is defined as follows: Let $u = u(t, \varepsilon)$ be the solution of the initial value problem

$$\begin{aligned} &\varepsilon^2\ddot{u} + \phi^2(t)u = 0, \\ &u(0, \varepsilon) = u_0, \quad \varepsilon\dot{u}(0, \varepsilon) = u_1. \end{aligned} \tag{1.1}$$

Here $\dot{u} = du/dt$, and u_0 and u_1 are taken independent of the parameter ε. Define $r^2 = r^2(t, \varepsilon)$ by

$$r^2 = \phi u^2 + \varepsilon^2\phi^{-1}\dot{u}^2. \tag{1.2}$$

One wishes to calculate the total change of the "adiabatic invariant" r^2:

$$\Delta r^2(\varepsilon) = r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon) \tag{1.3}$$

asymptotically as $\varepsilon \to +0$, under some reasonable conditions on the coefficient $\phi^2(t)$. The existence of the limit (1.3) is, of course, part of what has to be established.

In [7] and in [2] the function ϕ was subjected to the hypotheses below.

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Hypotheses (H).

(i) $\phi(t) > 0$ in $-\infty < t < \infty$.

(ii) The numbers $\lim_{t\to\pm\infty} \phi(t)$ exist and are positive.

(iii) $\dot{\phi}$ and all its derivatives exist and are in $L_1(-\infty, \infty)$. (A function with this property was called *gentle* in [7].)

Littlewood's result in [1], reproved in [7] by a different method, was the relation

$$\Delta r^2(\varepsilon) = O(\varepsilon^n) \quad \text{for all } n, \text{ as } \varepsilon \to +0. \tag{1.4}$$

In the method of [7] the Riccati equation

$$\varepsilon\dot{\pi} = 2i\phi\pi + \psi - \varepsilon^2\psi\pi^2 \tag{1.5}$$

for $\pi = \pi(t, \varepsilon)$, where

$$\psi = \dot{\phi}/2\phi, \tag{1.6}$$

plays a decisive role. Its particular solution $\pi = p(t, \varepsilon)$, characterized by the initial condition

$$p(-\infty, \varepsilon) = 0, \tag{1.7}$$

is especially important. This solution is unique, as was proved in [7]. Formula (1.8), below, which represents the total change of r^2 in terms of $p(t, \varepsilon)$, is the starting point of the present paper. It follows immediately from formulas (4.10) and (6.1) of [7]:

$$\begin{aligned}\Delta r^2(\varepsilon) = 2\,\mathrm{Re}&\left\{[\phi^{1/2}(0)u_0 + i\phi^{-1/2}(0)u_1]^2 \int_{-\infty}^{\infty} e^{-(2i/\varepsilon)\Phi(t)}\psi(t)[1 - \varepsilon^2 p^2(t, \varepsilon)]\,dt\right\}\\ &\cdot(1 + O(\varepsilon)).\end{aligned} \tag{1.8}$$

Here,

$$\Phi(t) = \int_0^t \phi(s)\,ds. \tag{1.9}$$

In [7], an asymptotic series

$$p(t, \varepsilon) \sim \sum_{j=0}^{\infty} p_j(t)\varepsilon^j, \quad \text{as } \varepsilon \to 0+, \tag{1.10}$$

uniformly valid in $-\infty < t < \infty$, was constructed, and Littlewood's relation (1.4) was proved from this expansion. Formula (1.10) is, however, not sufficient to derive more explicit results on Δr^2 from (1.8). In fact, it is likely that nothing more precise can be said without additional restrictions on ϕ^2. In the present paper ϕ^2 will be assumed to be *analytic*. It is then possible to shift the path of integration in (1.8) into the complex plane and thereby to improve the asymptotic information on the adiabatic invariant r^2, provided the asymptotic evaluation of $p(t, \varepsilon)$ can be extended into the complex domain. This is the program of the present article.

The hypothesis that the change of the oscillator is infinitely differentiable is essential for the validity of (1.4). If $\phi^2(t)$ is continuous and only *piecewise* differentiable, it is known that $r^2(\varepsilon) = O(\varepsilon)$, but no better.

By different, but related methods R. E. Meyer has independently and simultaneously obtained results that overlap with those of this paper. In [4] he proves that—under certain analyticity hypotheses—the total change of r^2 is exponentially small, as $\varepsilon \to 0+$. He has also derived more explicit formulas which will be published soon. Weaker results in this direction can also be found in a paper by G. Knorr and D. Pfirsch [1].

2. Assumptions and preparations. In accordance with the program just stated we add to Hypothesis (H) the assumption that $\phi^2(t)$ is holomorphic on the real t-axis and hence in a simply connected complex region containing the real axis. If the path of integration is deformed into the complex plane one may eventually meet some points where the integrand is no longer holomorphic. The nature of our problem depends decisively on the type of the singularities thus encountered.

In the present paper we specialize the situation by requiring that the neighborhood of the real axis in which ϕ^2 is holomorphic can be chosen so as to contain exactly *one* zero $t = t_0$ of $\phi^2(t)$, and that the zero be *simple*. Then $\phi(t)$ (which on the real axis is defined as the positive square root of the positive function $\phi^2(t)$) can also be uniquely continued, as long as one remains in a simply connected subdomain that does not contain t_0. At t_0, the function $\phi(t)$ has a simple branchpoint.

It is likely that the decisive exponential factor $\exp(-2i\zeta_0/\varepsilon)$ in our final result, Theorem 8.2, will also appear if the restriction to a first order zero of $\phi^2(t)$ is relaxed. The precise asymptotic expression for the total change of the adiabatic invariant will, however, vary a great deal with the order of the zero. A similar remark applies if $\phi^2(t)$ has a pole.

The function $\Phi(t)$ defined in (1.9) is likewise holomorphic in the same region as $\phi(t)$, provided the path of integration in (1.9) remains in that region.

In the terminology of the asymptotic theory of differential equations, t_0 *is a simple turning point* of the differential equation (1.1). The function $\Phi(t)$ plays an important role in the theory of such turning points. It has a branch point at $t = t_0$ with the property that the condition

$$\operatorname{Im} \Phi(t) = \operatorname{Im} \Phi(t_0) \tag{2.1}$$

defines three curves in the neighborhood of t_0 that meet at t_0 forming equal angles there. These so-called *Stokes curves* for the turning point t_0 bound three *Stokes sectors* with vertices at t_0.

Since $\Phi(t)$ is real on the real axis its branchpoints are symmetric to the real axis. Without losing generality we may therefore assume that t_0 is on the side of the real axis where $\Phi(t)$ is at first negative. Then

$$\operatorname{Im} \Phi(t_0) \leqq 0. \tag{2.2}$$

In §9 the theory of this paper is illustrated by the example

$$\phi^2(t) = 1 + \frac{1}{1 + 2e^{-t}}. \tag{2.3}$$

This function is holomorphic for $|\operatorname{Im} t| < \pi$ (here, $\pi = 3.14 \cdots$) and $\phi^2(t)$ has simple zeros at $t = \pm i\pi$. A look at the geometry of that example may facilitate the reading of the general description in this section.

The global nature of $\Phi(t)$ and of the Stokes curves depends on global properties of $\phi^2(t)$ in the complex plane, which will now be discussed.

The equation

$$\zeta = \Phi(t) \tag{2.4}$$

defines a mapping from the t-plane into the plane of the complex variable

$$\zeta = \xi + i\eta. \tag{2.5}$$

We set

$$\zeta_0 = \xi_0 + i\eta_0 = \Phi(t_0).$$

The images of the Stokes curves at t_0 under the mapping (2.4) lie in the line $\eta = \eta_0$ of the ζ-plane.

The three hypotheses below simplify the global behavior of the mapping (2.4). They are not difficult to confirm for the function (2.3) and other examples.

Hypothesis (K_1). Two of the three Stokes curves at t_0, say S_1 and S_2, extend to infinity as analytic curves, as do their images under the mapping $\zeta = \Phi(t)$. The curvilinear sector of central angle $2\pi/3$ bounded by S_1 and S_2 contains the real t-axis.

Hypothesis (K_2). Let G be the region of the t-plane bounded by S_1, S_2 and the real axis. The function ϕ is holomorphic, and has no zeros, in $\overline{G} - \{t_0\}$. ($\overline{G}$ is the closure of G.)

Hypothesis (K_3).

$$\operatorname{Im} \Phi(t_0) \neq 0. \tag{2.6}$$

It can furthermore be proved that the whole set $\overline{G}$ is homeomorphic, under the mapping (2.4), to the strip

$$\zeta(\overline{G}) = \overline{\Gamma} = \{\zeta | \eta_0 \leqq \eta \leqq 0\}.$$

The proof can be found in Appendix A.

In particular, S_1 and S_2 are mapped onto

$$\zeta(S_1) = \Sigma_1 = \{\zeta | \eta = \eta_0, \xi < \xi_0\}, \tag{2.7}$$

$$\zeta(S_2) = \Sigma_2 = \{\zeta | \eta = \eta_0, \xi > \xi_0\}. \tag{2.8}$$

The transformation (2.4) takes the differential equation (1.5) into

$$\varepsilon \frac{d\pi}{d\zeta} = 2i\pi + \chi - \varepsilon^2 \chi \pi^2, \tag{2.9}$$

where

$$\chi = \psi/\phi = -\frac{d^2t/d\zeta^2}{2dt/d\zeta}. \tag{2.10}$$

As long as we operate in the domains $\overline{G}$ or $\overline{\Gamma}$, where the relation between t and ζ is one-to-one, the use of the same letter, e.g., f, for a function of ζ with values $f(\zeta)$ and for the corresponding function of t with values $f(\Phi(t))$ will not cause confusion.

The quantity χ as a function of ζ is holomorphic in $\overline{\Gamma}$, except at $\zeta = \zeta_0$, where it becomes infinite and has a branchpoint. In fact, from the property of ϕ^2 of having a simple zero at $t = t_0$ and from formulas (1.6), (1.9) and (2.10) one derives readily the formula

$$\chi(\zeta) = \frac{1}{6(\zeta - \zeta_0)}\{1 + (\zeta - \zeta_0)^{2/3}\tilde{\chi}(\zeta - \zeta_0)\}, \tag{2.11}$$

where $\tilde{\chi}(\zeta - \zeta_0)$ is a holomorphic function of $(\zeta - \zeta_0)^{2/3}$ in $\overline{\Gamma}$.

As a solution of an analytic differential equation the function p in formula (1.8) is analytic. It is not immediately clear, however, where its singularities lie and what its asymptotic behavior is for small ε. For the purpose of this paper we need such information in the whole strip $\overline{\Gamma}$. The calculations that follow are complicated by our desire not to introduce hypotheses on the data outside the domain $\overline{G}$. This limits our freedom of operation in the complex plane in comparison with related work in the literature.

The function χ of ζ is gentle on the real ζ-axis (i.e., its derivatives are all in $L_1(-\infty, \infty)$). This follows immediately from Hypotheses (H) in § 1, the definition (2.10) of χ and formula (2.4). We need a hypothesis which assures that χ has corresponding properties near infinity, *uniformly in the whole set* $\overline{\Gamma}$.

Hypothesis (K_4). For every point $\zeta \in \overline{\Gamma}$ let $C_+(\zeta)$ denote the ray from ζ to infinity parallel to the real axis in the positive direction and $C_-(\zeta)$ the corresponding ray in the negative direction. Then, for $n = 0, 1, 2$,

$$\lim_{\xi \to \pm\infty} \int_{C\pm(\zeta)} |d^n\chi/d\sigma^n|\, d\sigma = 0 \tag{2.12}$$

uniformly for $\zeta \in \overline{\Gamma}$. (Formula (2.12) combines two statements according as the "+" or "−" sign is taken.)

Hypothesis (K_4) could be reformulated in terms of integrals in the t-plane, and it is likely that in that form it implies Hypothesis (K_3). The assumptions as stated here are simpler to apply and more natural.

By integrating the derivatives of χ one sees that Hypothesis (K_4) implies

$$\lim_{\xi \to \pm\infty} [d^n\chi/d\zeta^n] = 0, \quad n = 0, 1, \tag{2.13}$$

uniformly for $\zeta \in \overline{\Gamma}$.

3. Boundedness of $p(t, \varepsilon)$ inside $\overline{G} - S_2$. Every continuous solution of the integral equation

$$\pi(\zeta, \varepsilon) = \varepsilon^{-1} \int_{\gamma(\zeta)} e^{(2i/\varepsilon)(\zeta - \sigma)}\chi(\sigma)\, d\sigma - \varepsilon \int_{\gamma(\zeta)} e^{(2i/\varepsilon)(\zeta - \sigma)}\chi(\sigma)\pi^2(\sigma)\, d\sigma, \tag{3.1}$$

where $\gamma(\zeta)$ is some path in the σ-plane ending at $\sigma = \zeta$ and starting at some fixed point, also solves the differential equation (2.9), from which it is derived by use of the variation of parameter formula. We postpone the question as to which choice of $\gamma(\zeta)$ yields the solution corresponding to the particular solution of (1.5) called $p(t, \varepsilon)$ in [7] and choose as paths of integration curves that start at infinity on Σ_1 and continue to $\sigma = \zeta$ in such a way that $\operatorname{Im} \sigma$ never decreases on $\gamma(\zeta)$. Since we must avoid the turning point ζ_0, the point ζ must be restricted to the subset Γ_δ of Γ whose distance from Σ_2 is at least δ, where δ is an arbitrarily small positive

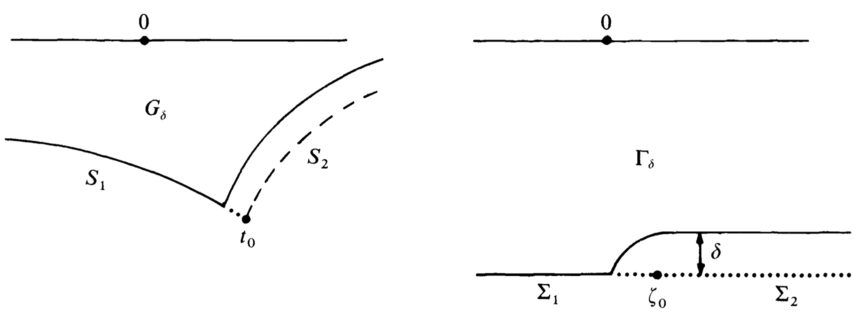

FIG. 1

constant. Let G_δ be the preimage of Γ_δ under the mapping $\zeta = \Phi(t)$. Figure 1 illustrates these two domains.

It is possible to extend the study of the Riccati equation as given in [7] from the real axis to the whole domain G_δ, and, in particular to continue the asymptotic power series for the solution $p(t, \varepsilon)$ into all of G_δ. For the purpose of the present paper it will be sufficient to know that $p(t, \varepsilon)$ is *bounded*, and only this fact will now be proved.

The term

$$b(\zeta, \varepsilon) = \frac{1}{\varepsilon}\int_{\gamma(\zeta)} e^{(2i/\varepsilon)(\zeta-\sigma)}\chi(\sigma)\,d\sigma$$

in (3.1) is bounded, say, $|b(\zeta, \varepsilon)| \leqq \tilde{b}$, for $\zeta \in \Gamma_\delta$, $0 < \varepsilon \leqq \varepsilon_0$. To see this it suffices to perform an integration by parts, which yields

$$b(\zeta, \varepsilon) = \frac{i}{2}\chi(\zeta) - \frac{i}{2}\int_{\gamma(\zeta)} e^{(2i/\varepsilon)(\zeta-\sigma)}\chi'(\sigma)\,d\sigma.$$

The right member is uniformly bounded, as claimed, because of formulas (2.11) and (2.13).

The operator T on π defined by the last term of (3.1), i.e., by

$$(Tv)(\zeta) = -\varepsilon\int_{\gamma(\zeta)} \exp\left[\frac{2i}{\varepsilon}(\zeta - \sigma)\right]\chi(\sigma)v^2(\sigma)\,d\sigma,$$

can be applied to all functions v in the Banach space of bounded continuous functions of ζ in Γ_δ with the maximum modulus norm. The function $b(\zeta, \varepsilon)$ is an element of this space for each $\varepsilon > 0$. One readily verifies that T is a contraction operator in the ball $\|v\| \leqq \tilde{b}$, provided $0 < \varepsilon \leqq \varepsilon_0$ with

$$\varepsilon_0 < \left[2\tilde{b}\sup_{\zeta\in\Gamma_\delta}\int_{\gamma(\zeta)} |\chi|\,d\sigma\right]^{-1}.$$

The supremum in the right member is finite, thanks to Hypothesis (K_4).

We conclude that the integral equation (3.1) has a unique bounded and continuous solution in Γ_δ for each ε in $0 < \varepsilon \leqq \varepsilon_0$. Since our bound, $\tilde{b}$, for $b(t, \varepsilon)$ is independent of ε, it follows that this solution, $\pi(\zeta, \varepsilon)$, is bounded uniformly in ε, for $0 < \varepsilon \leqq \varepsilon_0$.

The solution $\pi(\zeta, \varepsilon)$ of the integral equation (3.1) is also a solution of the differential equation (2.7), and $\pi(\Phi(t), \varepsilon)$ is a solution of (1.5). In fact, it is the same solution as the one called $p(t, \varepsilon)$ before and characterized by the initial condition (1.7). To show this, assume that ζ is real and take as the path of integration $\gamma(\zeta)$ in the σ-plane a ray on Σ_1 from $-\infty$ to some point σ_1 on Σ_1 with σ_1 large negative, followed by the segment $\operatorname{Re}\sigma = \sigma_1, \eta_0 \leqq \operatorname{Im}\sigma \leqq 0$ and then by the segment $\operatorname{Re}\sigma_1 < \sigma < \zeta$. Since we know already that π is bounded in Γ_δ, and because of Hypothesis (K_4), we can conclude that the contribution to the integrals in the right-hand member of (3.1) which comes from the portion of $\gamma(\zeta)$ between $-\infty$ and $\sigma = \operatorname{Re}\sigma_1$ tends to zero, as σ_1 recedes to $-\infty$ on Σ_1. The resulting modified form of (3.1) has a path of integration from $-\infty$ to ζ on the real σ-axis. Hence $\pi(\Phi(t), \varepsilon)$ is identical with the solution $p(t, \varepsilon)$ of (1.5) that satisfies the initial condition (1.7).

We have now proved the following theorem.

THEOREM 3.1. *The (unique) solution $\pi = p(t, \varepsilon)$ of the Riccati equation (1.5) that satisfies the initial condition $p(-\infty, \varepsilon) = 0$ is holomorphic and uniformly bounded for $0 < \varepsilon \leqq \varepsilon_0$ in the domain $\overline{G}_\delta$. Its bound, as well as ε_0, depends on δ.*

For later application we need two corollaries of this theorem. We recall that ϕ, Φ, ψ and χ are real on the real axis, and, hence, take on conjugate values in conjugate complex points. The proof of Theorem 3.1 extends therefore without difficulty to the larger region $\overline{G}_\delta \cup \overline{G}_\delta^*$, where G_δ^* is obtained by reflecting G_δ in the real axis. Consequently one has the following corollary.

COROLLARY 3.1. *If $p(t, \varepsilon)$ is the function defined in Theorem 3.1, then the function $\overline{p(\bar{t})}$ is likewise holomorphic and uniformly bounded in $\overline{G}_\delta$.*

The whole argument of this section can be repeated by strict analogy with the boundary condition (1.7) replaced by

$$\pi(+\infty, \varepsilon) = 0.$$

The result is formulated below.

COROLLARY 3.2. *Let Γ_δ^+ be the region obtained from Γ by removing all points having at least distance δ from Σ_1. Let G_δ^+ be the pre-image of Γ_δ^+ in the t-plane. Then the Riccati equation (1.5) has a unique solution $p^+(t, \varepsilon)$ satisfying the initial condition $p^+(\infty, \varepsilon) = 0$. This solution is holomorphic and uniformly bounded for $0 < \varepsilon \leqq \varepsilon_0$ in the domain $\overline{G}_\delta^+$.*

4. Three fundamental solutions of the linear differential equation. To carry out our program of shifting the path of integration into the complex plane, the differential equation (1.1) has to be solved asymptotically in all of $\overline{G}$. This requires knowledge of three fundamental systems of solutions, as well as of the connection formulas between them. In this section we give a description of these fundamental solutions.

(i) *The left outer solution.* In [7] a solution of the differential equation (1.1) with known asymptotic behavior on the real axis was derived. By Theorem 3.1 and Corollary 3.1 of this paper, Theorem 3.2 of [7] has been extended to $\overline{G}_\delta$, except for the asymptotic power series for $p(t, \varepsilon)$. (Actually, this power series is also valid in $\bar{G}_\delta$, but it is not essential for this paper.)

As in [7] the theorem will be stated in terms of the system

$$\varepsilon \dot{z} = \begin{bmatrix} \varepsilon\psi & \phi \\ -\phi & -\varepsilon\psi \end{bmatrix} z, \tag{4.1}$$

which is equivalent to the differential equation in (1.1) through the transformation

$$x = \phi^{1/2} u, \quad y = \varepsilon \phi^{-1/2} \dot{u}, \quad z = (x, y)^T. \tag{4.2}$$

THEOREM 4.1. *The differential equation* (4.1) *possesses a fundamental matrix solution* $Z(t, \varepsilon)$ *with the following properties*:

$$Z = SPV \tag{4.3}$$

with

$$S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \quad P = P(t, \varepsilon) = \begin{bmatrix} 1 & \varepsilon p(t, \varepsilon) \\ \varepsilon \overline{p(\bar{t}, \varepsilon)} & 1 \end{bmatrix}, \tag{4.4}$$

$$V = V(t, \varepsilon) = \exp \begin{bmatrix} \frac{i}{\varepsilon}\Phi(t) + \varepsilon \int_{-\infty}^{t} \psi(s)\overline{p(\bar{s}, \varepsilon)}\, ds & 0 \\ 0 & -\frac{i}{\varepsilon}\Phi(t) + \varepsilon \int_{-\infty}^{t} \psi(s) p(s, \varepsilon)\, ds \end{bmatrix}. \tag{4.5}$$

Here, $p(t, \varepsilon)$ *is the function described in Theorem* 3.1.

Observe that $\overline{p(\bar{t}, \varepsilon)}$ has replaced $\overline{p(t, \varepsilon)}$, wherever it occurred in [7, Thm. 3.2]. In [7] the variable t was real, therefore either notation was correct. The analytic continuation of $\overline{p(t, \varepsilon)}$ into the complex plane is, of course, $\overline{p(\bar{t}, \varepsilon)}$.

(ii) *The inner solution.* The function $p(t, \varepsilon)$ cannot be expected to preserve its analytic form at $t = t_0$, where the coefficient ψ in the Riccati equation (1.5) has a pole. Rather than to study (1.5) directly near that singularity, we base the analysis on the well-known asymptotic theory of the differential equation (1.1) near the turning point t_0. The results stated below are an extension of the work of R. Langer as developed in [5]. (See also [6].)

Let the function $\check{t}$ of t be defined by

$$\check{t} = \left(\frac{3}{2} i \int_{t_0}^{t} \phi(s)\, ds\right)^{2/3}. \tag{4.6}$$

The right member defines three distinct holomorphic functions near $t = t_0$, depending on the choice of the cube root. The three Stokes curves in the t-plane are mapped into the rays $\arg \check{t} = -\pi/3, \pi/3, \pi (\mathrm{mod}\ 2\pi)$. We make the mapping (4.6) unique by requiring that S_1 be mapped into

$$\check{t}(S_1) \subset \{\check{t} | \arg \check{t} = -\pi/3\}. \tag{4.7}$$

Since the mapping preserves orientation,

$$\check{t}(S_2) \subset \{\check{t} | \arg \check{t} = -\pi\}. \tag{4.8}$$

We leave open the question as to the global properties of the mapping (4.6). In some neighborhood $\mathcal{N}$ of $t = t_0$ the mapping is conformal with a conformal inverse, because $(d\check{t}/dt)_{t=t_0} \neq 0$ in consequence of the fact that the zero of ϕ^2 at t_0 is simple.

The image of $\mathcal{N}$ in the $\check{t}$-plane is sketched in Fig. 2.

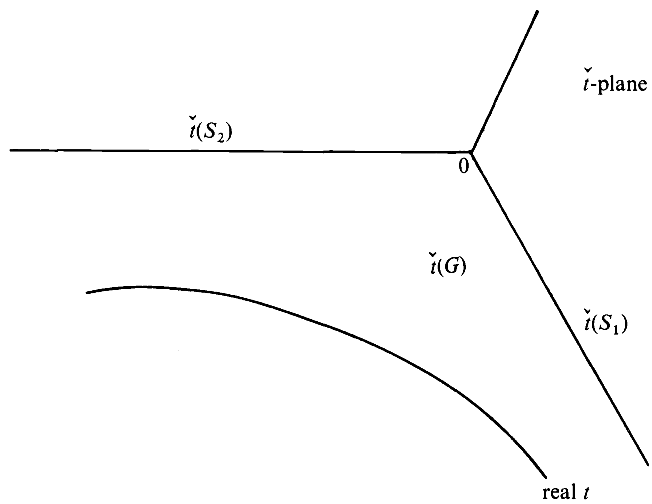

FIG. 2

The transformation

$$\check{t} = \check{t}(t), \qquad \check{z} = \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \begin{bmatrix} u \\ \varepsilon\, du/d\check{t} \end{bmatrix} \tag{4.9}$$

takes the differential equation (1.1) into the system

$$\varepsilon \frac{d\check{z}}{d\check{t}} = \begin{bmatrix} 0 & 1 \\ \check{t} & -\varepsilon \check{g}(\check{t}) \end{bmatrix} \check{z}, \tag{4.10}$$

where

$$\check{g}(\check{t}) = \frac{d^2\check{t}}{dt^2} \bigg/ \left(\frac{d\check{t}}{dt}\right)^2. \tag{4.11}$$

The function $g(\check{t})$ is holomorphic in $\check{t}(\mathcal{N})$ if $\mathcal{N}$ is small enough.

While the left outer fundamental matrix Z of Theorem 4.1 was derived in [2] by a transformation to a system with a *diagonal* coefficient matrix, the inner fundamental solution to be described in the next theorem is the result of a transformation to the system form of *Airy's equation*, of which (4.10) is a perturbation (cf. [5] and [6]).

THEOREM 4.2. *There exists a holomorphic matrix function* $\check{P}(\check{t}, \varepsilon)$ *in some neighborhood* $\check{t}(\mathcal{N})$ *of* $\check{t} = 0$ *with a uniformly asymptotic expansion*:

$$\check{P}(\check{t}, \varepsilon) \sim \sum_{r=0}^{\infty} \check{P}_r(\check{t})\varepsilon^r, \quad \text{as } \varepsilon \to 0+, \tag{4.12}$$

$$\check{P}_0(\check{t}) = \left(\frac{d\check{t}}{dt}\right)^{-1/2} I, \tag{4.13}$$

so that the transformation

$$\check{z} = \check{P}(\check{t}, \varepsilon)\check{v} \tag{4.14}$$

takes the differential equation (4.10) *into*

$$\varepsilon \frac{d\check{v}}{d\check{t}} = \begin{bmatrix} 0 & 1 \\ \check{t} & 0 \end{bmatrix} \check{v}. \tag{4.15}$$

This theorem gives us a complete asymptotic solution of equation (4.10) in $\check{t}(\mathcal{N})$, since (4.15) can be solved in terms of Airy's function Ai(z). The following particular fundamental matrix solution will be useful to us:

$$\check{V}(\check{t}, \varepsilon) = \begin{bmatrix} e^{\pi i/6}\mathrm{Ai}(\varepsilon^{-2/3}\omega\check{t}) & \mathrm{Ai}(\varepsilon^{-2/3}\check{t}) \\ \varepsilon^{1/3} e^{\pi i/6}\omega\mathrm{Ai}'(\varepsilon^{-2/3}\omega\check{t}) & \varepsilon^{1/3}\mathrm{Ai}'(\varepsilon^{-2/3}\check{t}) \end{bmatrix}. \tag{4.16}$$

Here, $\omega = e^{2\pi i/3}$, and $\mathrm{Ai}'(z) = d\mathrm{Ai}(z)/dz$.

The asymptotic form of $\check{V}(\check{t}, \varepsilon)$, as $\varepsilon^{-2/3}\check{t} \to \infty$, will be needed shortly: The standard asymptotic series expansion for Airy's function implies that

$$\check{V}(\check{t}, \varepsilon) = \frac{\varepsilon^{1/6}}{2\sqrt{\pi}} \begin{bmatrix} \check{t}^{-1/4} & 0 \\ 0 & \check{t}^{1/4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} (I + \varepsilon\check{t}^{-3/2}B_1) \begin{bmatrix} \exp\left(\frac{2}{3\varepsilon}\check{t}^{3/2}\right) & 0 \\ 0 & \exp\left(-\frac{2}{3\varepsilon}\check{t}^{3/2}\right) \end{bmatrix}, \tag{4.17}$$

where $B_1 = B_1(\varepsilon^{-2/3}\check{t})$ is a uniformly bounded holomorphic function for

$$|\varepsilon^{-2/3}\check{t}| \geqq \rho_0 > 0, \quad -\pi + \beta \leqq \arg \check{t} \leqq -\frac{\pi}{3}, \quad \rho_0 > 0, \quad \beta > 0, \text{arbitrary}. \tag{4.18}$$

The fractional powers of t in this formula are determined by the rule that $\arg(\check{t}^m) = m \arg \check{t}$.

Near the negative $\check{t}$-axis (i.e., near $\check{t}(S_2)$) formula (4.17) becomes invalid. Instead, one has

$$\begin{aligned} \check{V}(\check{t}, \varepsilon) = {} & \frac{\varepsilon^{1/6}}{2\sqrt{\pi}} \begin{bmatrix} (-\check{t})^{-1/4} & 0 \\ 0 & (-\check{t})^{1/4} \end{bmatrix} \\ & \cdot \left\{ \begin{bmatrix} \exp\left(\frac{2}{3\varepsilon}\check{t}^{3/2} + \frac{\pi i}{4}\right) & 2\cos\left(\frac{2}{3\varepsilon}(-\check{t})^{3/2} - \frac{\pi}{4}\right) \\ \exp\left(\frac{2}{3\varepsilon}\check{t}^{3/2} - \frac{\pi i}{4}\right) & 2\sin\left(\frac{2}{3\varepsilon}(-\check{t})^{3/2} - \frac{\pi}{4}\right) \end{bmatrix} + \varepsilon\check{t}^{-3/2}B_2 \right\}, \end{aligned} \tag{4.19}$$

$B_2 = B_2(\varepsilon^{-2/3}\check{t})$ being bounded and holomorphic on

$$|\varepsilon^{-2/3}\check{t}| \geqq \rho_0 > 0, \quad \arg \check{t} = -\pi. \tag{4.20}$$

In the sector $-\pi < \arg \check{t} \leqq -\pi + \beta$ the entries of B_2 are of the same order of magnitude as the corresponding entries of the leading terms in (4.19), as $\varepsilon^{-2/3}t \to \infty$.

A third fundamental solution of the differential equation in (1.1) is needed for the calculation of $p(t, \varepsilon)$ on S_2.

(iii) *The right outer solution.* This solution can be established in exact analogy to the way Theorem 4.1 of this paper was proved in [7], but with the function $p^+(t, \varepsilon)$ of Corollary 3.2 taking the place of $p(t, \varepsilon)$.

THEOREM 4.3. *The differential equation* (4.1) *possesses a fundamental matrix solution $Z^+(t, \varepsilon)$ with the following properties*:

$$Z^+ = SP^+V^+; \tag{4.21}$$

$$P^+ = P^+(t, \varepsilon) = \begin{bmatrix} 1 & \varepsilon p^+(t, \varepsilon) \\ \overline{\varepsilon p^+(\bar{t}, \varepsilon)} & 1 \end{bmatrix}, \tag{4.22}$$

$$V^+ = V^+(t, \varepsilon)$$

$$= \begin{bmatrix} \exp\left[\frac{i}{\varepsilon}\Phi(t) + \varepsilon \int_\infty^t \overline{\psi(s)p^+(\bar{s}, \varepsilon)}\, ds\right] & 0 \\ 0 & \exp\left[-\frac{i}{\varepsilon}\Phi(t) + \varepsilon \int_\infty^t \psi(s)p^+(s, \varepsilon)\, ds\right] \end{bmatrix}.$$

Here $p^+(t, \varepsilon)$ is the function described in Corollary 3.2.

5. The connection formulas. Let

$$D = D(t) = \begin{bmatrix} \phi^{1/2}(t) & 0 \\ 0 & \phi^{-1/2}(t)\dfrac{d\check{t}}{dt} \end{bmatrix} \tag{5.1}$$

and set

$$\check{Z} = \check{P}\check{V}. \tag{5.2}$$

By Theorem 4.2 and formula (4.16) the matrix $\check{Z}$ is an inner fundamental solution of the system (4.10). Formulas (4.2), (4.6) and (5.1) then show that $D\check{Z}$ is a solution matrix of the system (4.1). Since the matrix Z of Theorem 4.1 is also a fundamental solution of the same system, there must exist a nonsingular matrix $C = C(\varepsilon)$, independent of t, such that

$$Z = D\check{Z}C. \tag{5.3}$$

From (5.3), (5.2) and (4.3) it follows that

$$C = \check{V}^{-1}\check{P}^{-1}D^{-1}SPV. \tag{5.4}$$

This formula makes it possible to calculate C asymptotically. The details can be found in Appendix B. They result in the theorem below.

THEOREM 5.1. *The matrix C in the relation* (5.3), *which connects the left outer solution Z with the inner solution $D\check{Z}$ of the system* (4.1), *has the form*

$$C = C(\varepsilon) = 2\sqrt{\pi}\varepsilon^{-1/6}e^{\pi i/4}(I + \varepsilon B_3)\begin{bmatrix} e^{(i/\varepsilon)\zeta_0} & 0 \\ 0 & e^{-(i/\varepsilon)\zeta_0} \end{bmatrix}, \tag{5.5}$$

with $B_3 = B_3(\varepsilon)$ a bounded continuous function for $0 < \varepsilon \leqq \varepsilon_0$.

The asymptotic calculation of the connection matrix $C^+ = C^+(\varepsilon)$ in the formula

$$(5.6) \qquad Z^+ = D\check{Z}C^+$$

is not quite analogous to that of C, although the formula

$$(5.7) \qquad C^+ = \check{V}^{-1}\check{P}^{-1}D^{-1}SP^+V^+$$

resembles (5.4), because the asymptotic formula to be used for $\check{V}$ is now (4.19), not (4.17). Again the details of the proof are postponed to Appendix B.

THEOREM 5.2. *The matrix C^+ in the relation which connects the right outer solution Z^+ with the inner solution $D\check{Z}$ of the system* (4.1) *has the form*

$$(5.8) \qquad C^+ = C^+(\varepsilon) = 2\sqrt{\pi}\varepsilon^{-1/6}\, e^{\pi i/4}(I + \varepsilon B_4)\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{(i/\varepsilon)\zeta_0} & 0 \\ 0 & e^{-(i/\varepsilon)\zeta_0} \end{bmatrix},$$

with the matrix $B_4 = B_4(\varepsilon)$ bounded and continuous in $0 < \varepsilon \leqq \varepsilon_0$.

6. Asymptotic calculation of $p(t, \varepsilon)$ near the turning point. By Theorem 4.1 the function $p(t, \varepsilon)$ is related to the entries of the matrix $Z = \{z_{jk}\}$ through the formula

$$\frac{z_{12}}{z_{22}} = \frac{1 + \varepsilon p}{i(\varepsilon p - 1)},$$

i.e.,

$$(6.1) \qquad p = \varepsilon^{-1}\frac{z_{12} - iz_{22}}{z_{12} + iz_{22}} = \varepsilon^{-1}\frac{z_{12}/z_{22} - i}{z_{12}/z_{22} + i}.$$

The right member of (6.1) will now be asymptotically evaluated in the neighborhood $\mathcal{N}$ of t_0 by means of the relation (5.3). From Theorems 4.2 and 5.1 and formulas (5.1), (B.3) we find that

$$Z = D\check{P}\check{V}C = 2\sqrt{\pi}\varepsilon^{-1/6}\begin{bmatrix} \check{t}^{1/4} & 0 \\ 0 & i\check{t}^{-1/4} \end{bmatrix}(I + \varepsilon B_0)\check{V}(I + \varepsilon B_3)\begin{bmatrix} e^{(i/\varepsilon)\zeta_0} & 0 \\ 0 & e^{-(i/\varepsilon)\zeta_0} \end{bmatrix}.$$

(6.2)

The matrix $B_0 = B_0(\check{t}, \varepsilon)$ is defined by $B_0 = (\check{P} - \check{P}_0)\varepsilon^{-1}(d\check{t}/dt)^{1/2}$. It is bounded for $\check{t} \in \mathcal{N}, 0 < \varepsilon \leqq \varepsilon_0$, and holomorphic in $\check{t}$.

We define the matrix $M = M(t, \varepsilon)$ by the relation

$$(6.3) \qquad \check{V} + \varepsilon M = (I + \varepsilon B_0)\check{V}(I + \varepsilon B_3).$$

Its entries $\{m_{jk}\}$ are linear combinations of the entries of $\check{V}$ with coefficients that are bounded for $t \in \mathcal{N}, 0 < \varepsilon \leqq \varepsilon_0$, and holomorphic in t (or $\check{t}$). Formulas (6.2) and (6.3) imply that

$$\check{V} + \varepsilon M = \frac{\varepsilon^{1/6}}{2\sqrt{\pi}}\begin{bmatrix} \check{t}^{-1/4} & 0 \\ 0 & -i\check{t}^{1/4} \end{bmatrix} Z \begin{bmatrix} e^{-(i/\varepsilon)\zeta_0} & 0 \\ 0 & e^{(i/\varepsilon)\zeta_0} \end{bmatrix},$$

and hence,

$$\frac{\check{v}_{12} + \varepsilon m_{12}}{\check{v}_{22} + \varepsilon m_{22}} = i\check{t}^{-1/2}\frac{z_{12}}{z_{22}}.$$

Insertion into (6.1) leads to

$$p = \varepsilon^{-1}\frac{\check{t}^{1/2}v_{12} + v_{22} + \varepsilon m_1}{\check{t}^{1/2}v_{12} - v_{22} + \varepsilon m_2},$$

where

$$m_1 = \check{t}^{1/2}m_{12} + m_{22}, \quad m_2 = \check{t}^{1/2}m_{12} - m_{22}. \tag{6.4}$$

Now we refer to formula (4.16) and see that the lemma below has been proved.

LEMMA 6.1. *Let $t \in \mathcal{N} \cap \overline{G}$, where $\mathcal{N}$ is a certain neighborhood of the turning point $t = t_0$. Set*

$$\rho = \varepsilon^{-2/3}\check{t}, \qquad -\pi \leqq \arg\rho \leqq -\pi/3, \textit{ for } t \in G. \tag{6.5}$$

Then the function $p(t, \varepsilon)$ that solves equation (1.5) *subject to the initial condition* (1.7) *has in $\mathcal{N} \cap \overline{G}$ the form*

$$p(t, \varepsilon) = \varepsilon^{-1}\frac{\rho^{1/2}\mathrm{Ai}(\rho) + \mathrm{Ai}'(\rho) + \varepsilon^{2/3}m_1}{\rho^{1/2}\mathrm{Ai}(\rho) - \mathrm{Ai}'(\rho) + \varepsilon^{2/3}m_2}. \tag{6.6}$$

The functions m_1, m_2 are linear combinations of the entries v_{jk} of $\check{V}$ with coefficients that are bounded for $t \in \mathcal{N} \cap \overline{G}, 0 < \varepsilon \leqq \varepsilon_0$, and holomorphic functions of $\check{t}$.

At points where $\rho^{1/2}\mathrm{Ai}(\rho) - \mathrm{Ai}'(\rho)$ vanishes the formula (6.6) has little value. The next lemma, proved in Appendix C, is therefore important.

LEMMA 6.2. *The function $\rho^{1/2}\mathrm{Ai}(\rho) - \mathrm{Ai}'(\rho)$ has no zeros in the sector $-\pi \leqq \arg\rho \leqq -\pi/3$ (if $\arg\rho^{1/2} = (1/2)\arg\rho$).*

Our next aim is to expand (6.6) through the two first terms with respect to ε. Lemma 6.3 below gives us information about the size of the quantities in (6.6). For abbreviation we set

$$\rho^{1/2}\mathrm{Ai}(\rho) + \mathrm{Ai}'(\rho) = f_1(\rho), \qquad \rho^{1/2}\mathrm{Ai}(\rho) - \mathrm{Ai}'(\rho) = f_2(\rho). \tag{6.7}$$

LEMMA 6.3. *Let $\tilde{f}_j, \tilde{m}_j, j = 1, 2$, be defined by*

$$f_j = \tilde{f}_j|e^{-(2/3)\rho^{3/2}}|(|\rho|^{1/4} + 1), \qquad m_j = \tilde{m}_j|e^{-(2/3)\rho^{3/2}}|(|\rho|^{1/4} + 1).$$

Then there exist two constants c_1, c_2 independent of ε (but dependent on ε_0) such that for $t \in \mathcal{N} \cap \overline{G}, 0 < \varepsilon \leqq \varepsilon_0$,

$$|\tilde{m}_j| < c_1, \quad |\tilde{f}_j| < c_1, \quad |\tilde{f}_2| > c_2. \tag{6.8}$$

(Remember that $\mathrm{Re}\,\rho^{3/2} < 0$ for $t \in G$.)

Proof. The upper bounds in (6.8) are an immediate consequence of the asymptotic properties of Airy's function. The last inequality in (6.8) is true for $|\rho| < \rho_0$ (ρ_0 arbitrary) because of Lemma 6.2. For $|\rho| > \rho_0$ a short calculation based on the two asymptotic representations for $\mathrm{Ai}(\rho)$ in $-\pi + \beta \leqq \arg\rho \leqq -\pi/3$ and in $-\pi \leqq \arg\rho \leqq -\pi + \beta$ shows that, in spite of the difference between these two representations, the formula

$$f_2(\rho) = \frac{1}{\sqrt{\pi}}\rho^{1/4}\,e^{-(2/3)\rho^{3/2}}(1 + O(\rho^{-3/2})), \qquad \rho \to \infty,$$

remains valid in the combined sectors. Hence, the lower bound for $|\tilde{f}_2|$ is seen to be correct, provided ρ_0 is taken large enough.

THEOREM 6.1.

$$1 - \varepsilon^2 p^2(t, \varepsilon) = g + \varepsilon^{2/3} h + \varepsilon^{4/3} b_1, \tag{6.9}$$

where

$$g = g(\rho) = 1 - [f_1(\rho)/f_2(\rho)]^2, \tag{6.10}$$

$$h = h(\rho, \check{t}, \varepsilon) = 4 f_1(\rho) f_2^{-3}(\rho)[\check{t}^{1/2}\mathrm{Ai}'(\rho)m_{12} - \rho^{1/2}\mathrm{Ai}(\rho)m_{22}], \tag{6.11}$$

and $b_1 = b_1(t, \varepsilon)$ *is bounded for* $t \in \mathcal{N} \cap \bar{G}, 0 < \varepsilon \leqq \varepsilon_0$, *provided* ε_0 *is sufficiently small.*

Proof. Formula (6.9) is the result of a straightforward calculation based on formulas (6.4), (6.6), (6.7) and Lemma 6.3. It involves an expansion of the denominator in (6.6) by the formula for a geometric series.

The next two lemmas will be needed for estimates in § 8.

LEMMA 6.4.

$$g(\rho) = \begin{cases} O(\rho^{1/2}), & \text{as } \rho \to 0, \\ 1 + \rho^{-3}\tilde{g}(\rho), & \text{as } \rho \to \infty \text{ in } -\pi + \beta \leqq \arg \rho \leqq -\pi/3, \\ 1 + i \exp\{-\frac{8}{3} i \rho^{3/2}\}(1 + \rho^{-3}\tilde{\tilde{g}}(\rho)), & \text{as } \rho \to \infty \text{ in } -\pi \leqq \arg \rho \leqq -\pi + \beta, \end{cases}$$

with $\tilde{g}(\rho), \tilde{\tilde{g}}(\rho)$ *remaining bounded in the indicated sectors.*

The proof consists in a direct verification from the convergent and asymptotic series for Airy's function and is, therefore, omitted. This remark also applies to the next lemma.

LEMMA 6.5.

$$h = \begin{cases} O(\rho^{1/2}), & \text{as } \rho \to 0, \\ O(\rho^{-3/2}), & \text{as } \rho \to \infty \text{ in } -\pi + \beta \leqq \arg \rho \leqq -\pi/3, \end{cases}$$

$$h = \sum_{j=1}^{3} q_j(\rho, \check{t}) \exp\{-\tfrac{4}{3} j i \rho^{3/2}\} + O(\varepsilon), \quad \text{as } \rho \to \infty \text{ in } -\pi \leqq \arg \rho \leqq -\pi + \beta$$

uniformly for $t \in \mathcal{N} \cap \bar{G}, 0 < \varepsilon \leqq \varepsilon_0$. *The functions* $q_j(\rho, \check{t})$ *and their derivatives with respect to* $\check{t}$ *are uniformly bounded in the indicated region and holomorphic in* $\check{t}$.

7. Asymptotic calculation of $p(t, \varepsilon)$ **on** S_2**.** The three fundamental matrix solutions of the differential equation (4.1), namely $Z, D\check{Z}, Z^+$, are related by the formulas (5.3) and (5.6):

$$Z^+ = D\check{Z}C^+, \quad Z = D\check{Z}C. \tag{7.1}$$

Since

$$Z^+ = SP^+V^+, \quad Z = SPV \tag{7.2}$$

formulas (4.3) and (4.21), the matrix P satisfies the identity

$$PV = P^+V^+(C^+)^{-1}C, \tag{7.3}$$

which is convenient for the calculation of p on and near S_2 outside of $\mathcal{N}$. The right member of (7.3) can, in fact, be calculated there by means of Theorems 4.3, 5.1 and

5.2. If $R = \{r_{jk}\}$ is an abbreviated notation for PV in (7.3), it follows from the structures of P and V, as described in Theorem 4.1, that

$$\varepsilon p = r_{12}/r_{22}. \tag{7.4}$$

Again the details of the subsequent calculation are relegated to the Appendix (Appendix D), and only the result is stated below.

THEOREM 7.1.

$$\varepsilon p(t, \varepsilon) = -i\, e^{(2i/\varepsilon)(\zeta - \zeta_0)}(1 + \varepsilon b_2) + \varepsilon b_3, \tag{7.5}$$

where b_2 and b_3 are bounded functions for $t \in G_\delta^+, 0 < \varepsilon \leqq \varepsilon_0$.

8. Calculation of the adiabatic invariant. We finally have assembled enough information on the function $p(t, \varepsilon)$ to begin the asymptotic calculation of the integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} e^{-2i\Phi(t)/\varepsilon}\psi(t)[1 - \varepsilon^2 p^2(t, \varepsilon)]\, dt \tag{8.1}$$

in formula (1.8). The transformation (2.4) takes (8.1) into

$$I(\varepsilon) = \int_{-\infty}^{\infty} e^{-2i\zeta/\varepsilon}\chi(\zeta)[1 - \varepsilon^2 p^2(t(\zeta), \varepsilon)]\, d\zeta \tag{8.2}$$

with χ defined as in (2.10). Our investigation of p has shown that the integrand is holomorphic in $\overline{\Gamma}$ except, possibly, at $\zeta = \zeta_0$. Thanks to Hypothesis (K_3) the path of integration in (8.2) may be replaced by a path L_δ consisting of the line $\eta = \eta_0$ in the ζ-plane except that the point $\zeta = \zeta_0$ is circumvented by a semicircle of radius δ described in clockwise direction (δ may depend on ε here). It will be shown, below, that this semicircle may be shrunk to a point.

We extract the factor $e^{-(2i/\varepsilon)\zeta_0}$ from the integral and get

$$I(\varepsilon) = k(\varepsilon)\, e^{-(2i/\varepsilon)\zeta_0} \tag{8.3}$$

with

$$k(\varepsilon) = \int_{L_\delta} e^{-(2i/\varepsilon)(\zeta - \zeta_0)}\chi(\zeta)[1 - \varepsilon^2 p^2(t, \varepsilon)]\, d\zeta. \tag{8.4}$$

LEMMA 8.1. *For $\eta = \eta_0, 0 < \varepsilon \leqq \varepsilon_0$, one has*

$$\chi(\zeta)(1 - \varepsilon^2 p^2(t(\zeta), \varepsilon)) \in L_1(-\infty, \infty)$$

as a function of $\xi = \operatorname{Re} \zeta$.

Proof. Hypothesis (K_3), Theorem 3.1 and Theorem 7.1 guarantee that $\chi(1 - \varepsilon^2 p^2)$ is in $L_1(-\infty, \xi_0 - \delta)$ and in $L_1(\xi_0 + \delta, \infty)$ for all $\delta > 0$. In $-\delta \leqq \zeta - \zeta_0 \leqq \delta$ the function $1 - \varepsilon^2 p^2$ can be appraised with the help of formulas (6.1) and (6.2). Set

$$(I + \varepsilon B_0)\check{V}(I + \varepsilon B_3) = W = \{w_{jk}\}. \tag{8.5}$$

Then

$$\frac{z_{12}}{z_{22}} = -i\check{t}^{1/2}\frac{w_{12}}{w_{22}},$$

hence,

$$1 - \varepsilon^2 p^2 = -\check{t}^{1/2} \frac{4w_{12}w_{22}}{(\check{t}^{1/2}w_{12} - w_{22})^2}. \tag{8.6}$$

Now, V is bounded as long as $(\zeta - \zeta_0)\varepsilon^{-1} <$ const., and $\|W - V\| \leqq$ const. $\varepsilon\|V\|$, by (8.5), the constant depending on $\mathcal{N}$ only. On the other hand, we know from Lemma 6.2 that $\check{t}^{1/2}\check{v}_{12} - \check{v}_{22}$, which is equal to $\varepsilon^{1/3}f_2(\rho)$, does not vanish for real $\zeta - \zeta_0$. Therefore the denominator in the right member of (8.6) is also different from zero, provided $0 < \varepsilon \leqq \varepsilon_0$, with ε_0 sufficiently small, and $|\zeta - \zeta_0| < \varepsilon$, say. Thus, the expression in (8.6) is $O(\check{t}^{1/2}) = O((\zeta - \zeta_0)^{1/3})$ if we take $\delta = \varepsilon$. As $\chi(\zeta) = O((\zeta - \zeta_0)^{-1})$, because of (2.9), we have $\chi(1 - \varepsilon^2p^2) = O((\zeta - \zeta_0)^{-2/3})$ for $\zeta - \zeta_0 \to 0$, i.e., this function is in $L_1(\zeta_0 - \delta, \zeta_0 + \delta)$. This completes the proof of Lemma 8.1.

Thanks to Lemma 8.1 the path of integration in (8.4) may be replaced by the line $\eta = \eta_0$. However, as the integrability was proved in that lemma for each fixed ε only, but not uniformly in ε, the asymptotic form of $k(\varepsilon)$ is not easily recognizable from (8.4). A method of calculation that circumvents the laborious estimates that follow would be very desirable.

LEMMA 8.2. *Let $\beta > 0$ be a constant independent of ε. Denote by $k_e(\varepsilon)$ (the "e" stands for "exterior") the contribution to $k(\varepsilon)$ from the two rays $|\zeta - \zeta_0| \geqq \beta$, $\eta = \eta_0$. Then*

$$k_e(\varepsilon) = O(\varepsilon). \tag{8.7}$$

Proof. The contribution to $k_e(\varepsilon)$ from $\zeta - \zeta_0 \leqq -\beta$ is

$$\int_{-\infty}^{-\beta} e^{-2i\sigma/\varepsilon}\chi(\zeta_0 + \sigma)[1 - \varepsilon^2p^2]\,d\sigma$$

$$= \int_{-\infty}^{-\beta} e^{-2i\sigma/\varepsilon}\chi(\zeta_0 + \sigma)\,d\sigma - \varepsilon^2 \int_{-\infty}^{-\beta} e^{-2i\sigma/\varepsilon}\chi(\zeta_0 + \sigma)p^2\,d\sigma.$$

The last integral is $O(1)$, because p is bounded by Theorem 3.1 and χ is integrable. The first integral in the right member can be subjected to an integration by parts, because χ' is also integrable on $(-\infty, -\beta]$. That integral is therefore of order $O(\varepsilon)$. By Theorem 7.1, the contribution to $k_e(\varepsilon)$ from Σ_2 is

$$\int_{\beta}^{\infty} e^{-2i\sigma/\varepsilon}\chi(\zeta_0 + \sigma)(1 + e^{4i\sigma/\varepsilon} + \varepsilon b_4)\,d\sigma,$$

where b_4 is a uniformly bounded function of σ and ε. Again we conclude, with the help of an integration by parts, that the integral is of order $O(\varepsilon)$.

LEMMA 8.3. *Let*

$$k_{i1}(\varepsilon) = \int_{\zeta_0 - \alpha\varepsilon}^{\zeta_0 + \alpha\varepsilon} \exp[-2i(\zeta - \zeta_0)/\varepsilon]\chi(\zeta)g(\rho)\,d\zeta.$$

Then

$$k_{i1}(\varepsilon) = \frac{1}{6}\int_{-\alpha}^{\alpha} \frac{e^{-2is}}{s} g(\rho)\,ds + \varepsilon^{2/3}\frac{\tilde{\chi}(0)}{6}\int_{-\alpha}^{\alpha} \frac{e^{-2is}}{s^{1/3}} g(\rho)\,ds + O(\varepsilon^{5/3}). \tag{8.8}$$

In this notation the subscript "1" refers to the fact that $g(\rho)$, defined in Theorem 6.1, is the first term in the formula (6.9) for $1 - \varepsilon^2 p^2$. The "i" stands for "inner," since $\xi_0 - \alpha\varepsilon \leqq \xi \leqq \xi_0 + \alpha\varepsilon, \eta = \eta_0$ is the inner part of the path of integration for $k(\varepsilon)$.

Proof of Lemma 8.3. We set

$$s = (\zeta - \zeta_0)/\varepsilon. \tag{8.9}$$

Then

$$\rho = (\tfrac{3}{2}i)^{2/3} s^{2/3}, \tag{8.10}$$

and

$$\begin{aligned} k_{i1}(\varepsilon) &= \varepsilon \int_{-\alpha}^{\alpha} e^{-2is}\chi(\zeta_0 + s\varepsilon) g(\rho)\, ds \\ &= \frac{1}{6}\int_{-\alpha}^{\alpha} \frac{e^{-2is}}{s} g(\rho)\, ds + \frac{\varepsilon^{2/3}}{6}\int_{-\alpha}^{\alpha} \frac{e^{-2is}}{s^{1/3}}\tilde{\chi}(\varepsilon s) g(\rho)\, ds. \end{aligned} \tag{8.11}$$

The first integral exists as a Lebesgue integral, because $g(\rho) = O(s^{1/3})$, as $s \to 0$, by Lemma 6.4. $\tilde{\chi}$ was defined in formula (2.11). The proof of formula (8.8) is now immediate.

LEMMA 8.4. *Let*

$$k_{m1}^{-}(\varepsilon) = \int_{\zeta_0 - \beta}^{\zeta_0 - \alpha\varepsilon} \exp\left[-2i(\zeta - \zeta_0)/\varepsilon\right]\chi(\zeta) g(\rho)\, d\zeta.$$

Then

$$\begin{aligned} k_{m1}^{-}(\varepsilon) = {} & \frac{1}{6}\int_{-\beta/\varepsilon}^{-\alpha} \frac{e^{-2is}}{s} g(\rho)\, d\rho \\ & + \left[\frac{1}{12i} e^{2i\alpha}\alpha^{-1/3} - \frac{1}{36i}\int_{-\infty}^{-\alpha} e^{-2is} s^{-4/3}\, ds \right. \\ & \left. \quad + \frac{1}{6}\int_{-\infty}^{-\alpha} e^{-2is} s^{-1/3}\tilde{g}(\rho)\rho^{-3}\, ds\right]\tilde{\chi}(0)\varepsilon^{2/3} \\ & + o(\varepsilon^{2/3}), \end{aligned}$$

with $\tilde{g}(\rho)$ bounded, as $s \to -\infty$.

Proof. We substitute formula (2.11) and the formula for $g(\rho)$ from Lemma 6.4 into the integral for $k_{m1}^{-}(\varepsilon)$ and get, after the change of variable $\zeta_0 - \zeta = \varepsilon s$,

$$\begin{aligned} k_{m1}^{-}(\varepsilon) = {} & \frac{1}{6}\int_{-\beta/\varepsilon}^{-\alpha} \frac{e^{-2is}}{s} g(\rho)\, ds + \frac{1}{6}\varepsilon^{2/3}\int_{-\beta/\varepsilon}^{-\alpha} e^{-2is} s^{-1/3}\tilde{\chi}(\varepsilon s)\, ds \\ & + \frac{1}{6}\varepsilon^{2/3}\int_{-\beta/\varepsilon}^{-\alpha} e^{-2is} s^{-1/3}\tilde{\chi}(\varepsilon s)\rho^{-3}\tilde{g}(\rho)\, ds, \end{aligned}$$

where $\tilde{g}(\rho)$ remains bounded as $s \to -\infty$, because of Lemma 6.4. If the integrand in the last integral is continued as zero to $s = -\infty$, the theorem of bounded convergence shows that this integral tends to the last integral in (8.10), as $\varepsilon \to 0$.

The integral before the last one in (8.11) can be integrated by parts, and in the new integral one can again let ε tend to zero. In this way one obtains formula (8.10).

LEMMA 8.5. *Let*

$$k_{m1}^{+}(\varepsilon) = \int_{\zeta_0 + \alpha\varepsilon}^{\zeta_0 + \beta} \exp\left[-2i(\zeta - \zeta_0)/\varepsilon\right]\chi(\zeta)g(\rho)\, d\zeta.$$

Then

$$\begin{aligned} k_{m1}^{+}(\varepsilon) = {} & \frac{1}{6}\int_{\alpha}^{\beta/\varepsilon} \frac{e^{-2is}}{s} g(\rho)\, ds + \left\{\frac{1}{12i} e^{-2i\alpha}\alpha^{-1/3} - \frac{1}{12} e^{2i\alpha}\alpha^{-1/3} \right. \\ & + \frac{1}{36}\int_{\alpha}^{\infty} (ie^{-2is} + e^{2is})s^{-4/3}\, ds \\ & \left. + \frac{i}{6}\int_{\alpha}^{\infty} e^{2is}s^{-1/3}\tilde{\tilde{g}}(\rho)\rho^{-3}\, ds\right\}\tilde{\chi}(0)\varepsilon^{2/3} + o(\varepsilon^{2/3}), \end{aligned}$$

where $\tilde{\tilde{g}}(\rho)$ remains bounded as $s \to +\infty$.

Proof. The only difference between the proofs of Lemmas 8.4 and 8.5 is that in the latter the last part of Lemma 6.4 must be used. The details are straightforward and therefore omitted.

LEMMA 8.6. *Let*

$$k_2(\varepsilon) = \varepsilon^{2/3}\int_{\zeta_0 - \beta}^{\zeta_0 + \beta} \exp\left[-2i(\zeta - \zeta_0)/\varepsilon\right]\chi(\zeta)h\, d\zeta.$$

Then

$$k_2(\varepsilon) = O(\varepsilon^{2/3}\log\varepsilon).$$

Proof. We proceed as in Lemmas 8.3 and 8.4 with Lemma 6.5 taking the place of Lemma 6.4 in the arguments. Since the last estimate in Lemma 6.5 does not contain the convenient factor $1 + \rho^{-3}\tilde{\tilde{g}}(\rho)$, as in Lemma 6.4, the proof of Lemma 8.5 cannot be simply repeated by a straightforward analogy. We content ourselves with the crude appraisal

$$\begin{aligned} \varepsilon^{-2/3}k_{m2}^{+}(\varepsilon) &= \int_{\zeta_0 + \alpha\varepsilon}^{\zeta_0 + \beta} \exp\left[-2i(\zeta - \zeta_0)/\varepsilon\right]\chi(\zeta)h\, d\zeta \\ &= \frac{1}{6}\int_{\alpha}^{\beta/\varepsilon} \frac{e^{-2is}}{s} h\, ds + \frac{1}{6}\varepsilon^{2/3}\int_{\alpha}^{\beta/\varepsilon} \frac{e^{-2is}}{s^{1/3}}\tilde{\chi}(\varepsilon s)h\, ds \qquad (8.12) \\ &= O(\log\varepsilon), \end{aligned}$$

which proves the lemma.

Remark. A rather elaborate analysis of the function $h(\rho, \check{t}, \varepsilon)$ shows that the preceding lemma can be sharpened to the statement that

$$\begin{aligned} k_2(\varepsilon) &= \varepsilon^{2/3}\frac{1}{6}\int_{-\infty}^{\infty} \frac{e^{-is}}{s} h(\rho, \rho\varepsilon^{2/3}, 0)\, ds + O(\varepsilon^{2/3}), \\ \rho &= (\tfrac{3}{2}is)^{2/3}, \end{aligned} \qquad (8.13)$$

where the integral is uniformly bounded, as $\varepsilon \to 0$. Such a refinement will be needed if the constant $k(\varepsilon)$ is to be approximated beyond its leading term, but this will not be done in this paper.

LEMMA 8.7. *Let*

$$k_3(\varepsilon) = \varepsilon^{4/3} \int_{\zeta_0 - \beta}^{\zeta_0 + \beta} \exp[-2i(\zeta - \zeta_0)/\varepsilon]\chi(\zeta) b_1(t, \varepsilon)\, d\zeta.$$

Then

$$k_3(\varepsilon) = O(\varepsilon^{4/3} \log \varepsilon).$$

Proof. The existence of the integral is assured by Lemma 8.1 and by the existence of the contributions to $k(\varepsilon)$ that have already been calculated. For $|\zeta - \zeta_0| \leqq \varepsilon\alpha$ the path can be replaced by a semicircle. The corresponding contribution to the integral is $O(1)$. Along the remaining parts of the path the integrand is $O(\zeta^{-1})$, for $\zeta \to 0$, and the integral is therefore $O(\log \varepsilon)$.

THEOREM 8.1. *The quantity $k(\varepsilon)$ defined in formula* (8.4) *has the asymptotic expression*

$$k(\varepsilon) = \frac{1}{6} \int_{-\infty}^{\infty} \frac{e^{-2is}}{s} g(\rho)\, ds + o(1), \quad \text{as } \varepsilon \to 0+.$$

Proof. From Lemmas 8.1 to 8.7 and from formulas (6.9), (8.4) it follows that

$$\begin{aligned} k(\varepsilon) &= k_e(\varepsilon) + k_{i1}(\varepsilon) + k_{m1}^-(\varepsilon) + k_{m1}^+(\varepsilon) + k_2(\varepsilon) + k_3(\varepsilon) \\ &= \frac{1}{6} \int_{-\beta/\varepsilon}^{\beta/\varepsilon} \frac{e^{-2is}}{s} g(\rho)\, ds + O(\varepsilon^{2/3} \log \varepsilon). \end{aligned}$$

Thanks to Lemma 6.4 the last integral can be extended from $-\infty$ to ∞ with an error that is infinitesimal in ε. Thus, the theorem has been proved.

Remark. With the help of formulas (8.11) and (8.13) the following refinement of Theorem 8.1 can be proved:

$$\begin{aligned} k(\varepsilon) = \frac{1}{6} \int_{-\infty}^{\infty} \frac{e^{-2is}}{s} g(\rho)\, ds &+ \varepsilon^{2/3} \frac{1}{6} \int_{-\infty}^{\infty} \frac{e^{-2is}}{s} h(\rho, \rho\varepsilon^{2/3}, 0)\, ds \\ &+ \varepsilon^{2/3} \frac{\tilde{\chi}(0)}{6} \int_{-\infty}^{\infty} \frac{e^{-2is}}{s^{1/3}} g(\rho)\, ds + o(\varepsilon^{2/3}). \end{aligned}$$

The proof, which is not immediate, is omitted.

We have now completed the asymptotic calculation of the integral $k(\varepsilon)$ of (8.4) and, hence, through formula (8.3), the asymptotic calculation of $I(\varepsilon)$ as defined in (8.1). Returning to formula (1.8) we can summarize our results in the Main Theorem below, where $g(\rho)$ has been replaced by its explicit expression from (6.10) and (6.7).

MAIN THEOREM (Thm. 8.2). *Let $u(t, \varepsilon)$ be the solution of the initial value problem* (1.1). *Let $r^2(t, \varepsilon)$ be defined by* (1.2). *Assume that the Hypotheses* (H), (K_1), (K_2), (K_3), (K_4) *are satisfied. Let*

$$\zeta_0 = \int_0^{t_0} \phi(s)\, ds.$$

Then, as $\varepsilon \to 0+$, *the quantity* $\Delta r^2(\varepsilon) = r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon)$ *has the asymptotic representation*

$$\Delta r^2(\varepsilon) = r_0^2 \operatorname{Re}[\kappa \exp(-2i\zeta_0/\varepsilon + 2i\theta_0)] + o(e^{-2i\zeta_0/\varepsilon}).$$

Here, r_0 *and* θ_0 *are defined by*

$$r_0 e^{i\theta_0} = \phi^{1/2}(0)u_0 + i\phi^{-1/2}(0)u_1,$$

and κ *is the constant*

$$\kappa = -\frac{4}{3}\int_{-\infty}^{\infty} \frac{e^{-2is}}{s}\rho^{1/2}\frac{\operatorname{Ai}(\rho)\operatorname{Ai}'(\rho)}{[\rho^{1/2}\operatorname{Ai}(\rho) - \operatorname{Ai}'(\rho)]^2}\,ds$$

with

$$\rho = (\tfrac{3}{2}is)^{2/3}, \qquad \arg\rho = \begin{cases} -\pi/3, & s < 0, \\ -\pi, & s > 0, \end{cases}$$

$\arg\rho^{1/2} = (1/2)\arg\rho$. $\operatorname{Ai}(\rho)$ *is Airy's function and* $\operatorname{Ai}'(\rho)$ *is its derivative.*

9. An example. The function (2.3), i.e.,

$$\phi^2(t) = 1 + \frac{1}{1 + 2e^{-t}} = \frac{4}{3 - \tanh t}, \tag{9.1}$$

offers a simple illustration for the theory of this paper.

We begin by verifying Hypotheses (H). Clearly, $\phi(t) > 0$ for all real t, and the limits of ϕ at infinity are positive:

$$\lim_{t\to-\infty} \phi(t) = 1, \quad \lim_{t\to\infty} \phi(t) = \sqrt{2}.$$

It is a simple exercise to prove that

$$\frac{d^n\phi}{dt^n} = \begin{cases} O(e^{-t}) & \text{as } t \to +\infty, \\ O(e^{t}) & \text{as } t \to -\infty, \end{cases}$$

for all $n \geqq 1$, so that all these derivatives are indeed in $L_1(-\infty, \infty)$.

For complex t one finds that ϕ^2 is meromorphic with simple zeros at $t = (2n + 1)\pi i$ and simple poles at $t = \log 2 + (2n + 1)\pi i$, but no other zeros or poles.

The function $\Phi(t)$ is elementary in this example: By the change of variable

$$\phi(t) = \left(1 + \frac{1}{1 + 2e^{-t}}\right)^{1/2} = s \tag{9.2}$$

in the indefinite integral

$$F(t) = \int \left(1 + \frac{1}{1 + 2e^{-t}}\right)^{1/2} dt,$$

one finds

$$F(t) = 2\int \frac{s^2}{(s^2 - 1)(2 - s^2)}\,ds = \frac{1}{2}\log\frac{s - 1}{s + 1} - \frac{1}{\sqrt{2}}\log\frac{s - \sqrt{2}}{s + \sqrt{2}} + \text{const.}$$

Since $s = +2/\sqrt{3}$ for $t = 0$, it follows that

$$\Phi(t) = \frac{1}{2}\log\frac{s-1}{s+1} - \frac{1}{\sqrt{2}}\log\frac{s-\sqrt{2}}{s+\sqrt{2}} - \frac{1}{2}\log\frac{2-\sqrt{3}}{2+\sqrt{3}} + \frac{1}{\sqrt{2}}\log\frac{2-\sqrt{6}}{2+\sqrt{6}}. \tag{9.3}$$

The branch of the logarithm in this formula is arbitrary as long as it is such that $\Phi(0) = 0$, and provided that Φ is continued as a holomorphic function from $t = 0$. The segment from $t = 0$ to $t = -\pi i$ is mapped onto an arc that goes from $s = 2/\sqrt{3}$ to $s = 0$ in the lower half of the s-plane. From this and formula (9.3) one concludes that

$$\Phi(-\pi i) = -\frac{\pi}{2}i + \frac{1}{\sqrt{2}}\log\frac{\sqrt{6}-2}{\sqrt{6}+2} - \frac{1}{2}\log\frac{2-\sqrt{3}}{2+\sqrt{3}}. \tag{9.4}$$

Therefore $t_0 = -\pi i$ is the turning point where the inequalities (2.2) and (2.6) are satisfied.

Next, we investigate S_1 and S_2. We set $t = \mu + i\nu$. Then, for $t = \mu - i\pi$, $\mu < 0$, one has $\phi^2(t) > 0$. Hence, this ray is a Stokes line, namely S_1. The easily verifiable formula

$$\phi^2(t) = 1 + \frac{1 + 2e^{-\mu}\cos\nu}{1 + 4e^{-\mu}\cos\nu + 4e^{-2\mu}} + i\frac{2e^{-\mu}\sin\nu}{1 + 4e^{-\mu}\cos\nu + 4e^{-2\mu}} \tag{9.5}$$

shows that the complex number $\phi^2(t)$ lies in the fourth quadrant for $\mu > 0$, $0 > \nu > -\pi$. The same is then true of $\phi(t)$, which is the branch of $(\phi^2)^{1/2}$ that is positive on the real t-axis. In Appendix A the differential equations

$$\frac{d\mu}{ds} = \operatorname{Re}\phi, \quad \frac{d\nu}{ds} = -\operatorname{Im}\phi$$

are introduced. It is shown there that the orbit of their solution through a point of S_2 is the Stokes curve S_2. From our observation above about the vector field $(\operatorname{Re}\phi, \operatorname{Im}\phi)$, it follows that S_2, which is known to start with a directional angle of $+\pi/3$ at $t = -\pi i$, remains in the half strip $\mu > 0$, $0 > \nu > -\pi$, and tends to infinity with a slope that approaches zero. The curve maintains, however, a positive distance from the real axis. In fact, if $t_1 = \mu_1 + i\nu_1$, then

$$\begin{aligned}\operatorname{Im}\Phi(t_1) &= \operatorname{Im}\int_0^{t_1}\phi(t)\,dt = \operatorname{Im}\int_{\mu_1}^{t_1}\phi(t)\,dt = \operatorname{Im}\left[i\int_0^{\nu_1}\phi(\mu_1 + i\nu)\,d\nu\right]\\ &= \operatorname{Re}\int_0^{\nu_1}\left(1 + \frac{1}{1 + 2e^{-\mu_1}e^{-i\nu}}\right)^{1/2} d\nu.\end{aligned}$$

Thus, in terms of the variables μ_1, ν_1, the equation of the curve S_2 can be written

$$-\frac{\pi}{2} = \operatorname{Re}\int_0^{\nu_1}\left(1 + \frac{1}{1 + 2e^{-\mu_1}e^{-i\nu}}\right)^{1/2} d\nu, \qquad \mu_1 > 0.$$

Hence, in the limit,

$$\lim_{\mu_1 \to +\infty} \nu_1 = -\frac{\pi}{2\sqrt{2}}.$$

The limit relations

$$\lim_{\operatorname{Re} t \to +\infty} \phi(t) = \sqrt{2}, \qquad \lim_{\operatorname{Re} t \to -\infty} \phi(t) = -1 \tag{9.6}$$

are uniformly valid for all values of Im t. In view of (9.2) and (9.3) this implies that

$$\lim_{\operatorname{Re} t \to +\infty} \operatorname{Re} \Phi(t) = \infty, \qquad \lim_{\operatorname{Re} t \to -\infty} \operatorname{Re} \Phi(t) = -\infty.$$

Therefore, the images of S_1 and S_2 in the ζ-plane are the *whole* half-lines through $\zeta = \zeta_0$ parallel to the real axis.

Finally, we show that Hypothesis (K_4) is satisfied. By differentiations and elementary estimates one verifies that

$$\frac{d^n}{dt^n}(\phi^2(t)) = O(e^{-|\operatorname{Re} t|}), \quad \text{as } \operatorname{Re} t \to \pm\infty, \tag{9.7}$$

for all n, uniformly in Im t.

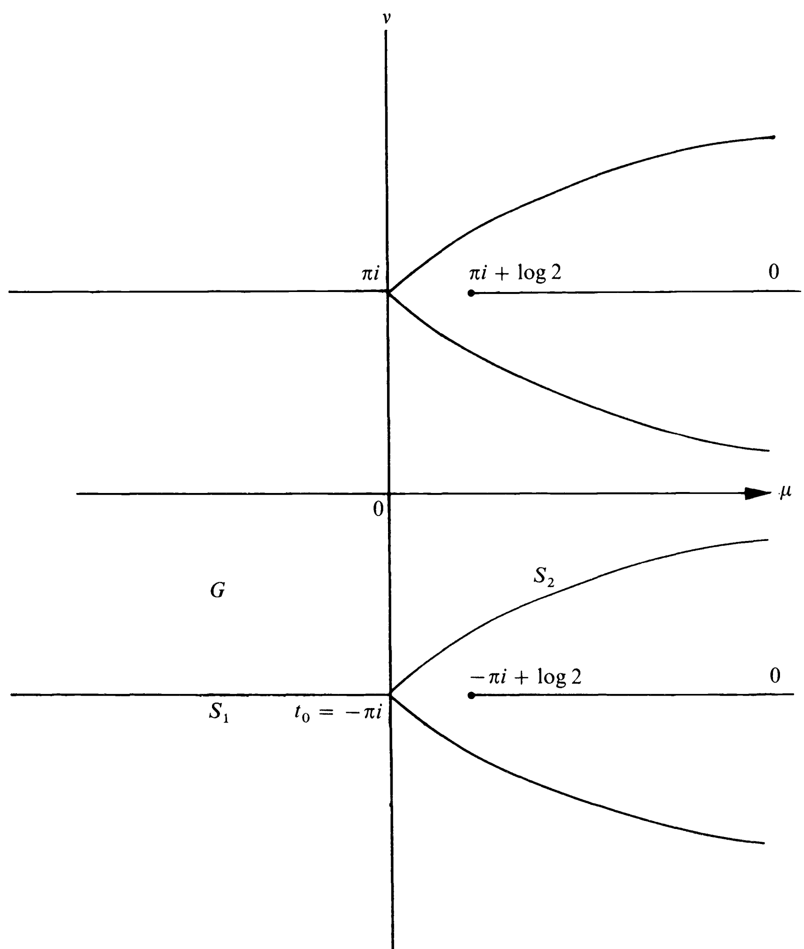

FIG. 3

If the integral in formula (2.10) is transformed back into the t-plane, the inequalities (9.6), (9.7) are seen to guarantee the validity of Hypothesis (K_4).

This completes the verification of the assumptions of this paper for the function (9.1). Figure 3 shows not only the region G but also the pattern of the other Stokes curves for this particular differential equation, i.e., of all curves $\operatorname{Im} \phi(t) = \text{const.}$ that have one endpoint at a zero or a pole of $\phi^2(t)$. Since this goes beyond what is needed for the present paper the simple arguments on which that sketch is based are omitted. The figure is to be continued with period $2\pi i$.

Appendix A.

THEOREM A.1. *The mapping $\zeta = \Phi(t)$ defines a homeomorphism between $\overline{G}$ and $\overline{\Gamma}$.*

Proof. The real axes of the t- and ζ-planes are homeomorphically related by $\zeta = \Phi(t)$, because $\phi(t)$ is positive and bounded away from zero on the real t-axis.

To prove that S_1 is mapped homeomorphically onto the ray Σ_1 we write $t = \mu + i\nu$, $\phi(t) = \phi_r + i\phi_i$ and consider the initial value problem

$$\frac{d\mu}{ds} = \phi_r(\mu, \nu), \quad \frac{d\nu}{ds} = -\phi_i(\mu, \nu), \tag{A.1}$$

$$\mu(0) = \mu_1, \quad \nu(0) = \nu_1, \tag{A.2}$$

where $t_1 = \mu_1 + i\nu_1$ is some fixed arbitrary point on S_1. The orbit of the (unique) solution of this problem lies on S_1, because

$$\frac{d}{ds} \operatorname{Im} \Phi(t) = \frac{\partial \operatorname{Im} \Phi}{\partial \mu} \frac{d\mu}{ds} + \frac{\partial \operatorname{Im} \Phi}{\partial \nu} \frac{d\nu}{ds} = \phi_i \frac{d\mu}{ds} + \phi_r \frac{d\nu}{ds} = 0,$$

in consequence of (A.1) and the Cauchy–Riemann equations. By the standard theory of ordinary differential equations, this solution can be continued as a function of s as long as the right members remain continuous. It is also known that the endpoints of the orbit, if any, are at critical points (i.e., zeros of both right members) or at infinity. There are no critical points on the curve S_1, because those are the zeros of $\phi(t)$. For the same reason the arc length

$$l(s) = \int_0^s |\phi(\mu(\sigma), \nu(\sigma))| \, d\sigma$$

on S_1 is an increasing function of s. Hence, the relation $t = \mu(s) + i\nu(s)$ establishes a homeomorphism between some open (finite or infinite) segment L of the real s-line and S_1. For the point $\zeta = \Phi(t)$ with $t = \mu(s) + i\nu(s)$ one finds

$$\frac{d}{ds} \operatorname{Re} \Phi = \frac{\partial \operatorname{Re} \Phi}{\partial \mu} \frac{d\mu}{ds} + \frac{\partial \operatorname{Re} \Phi}{\partial \nu} \frac{d\nu}{ds} = \phi_r \frac{d\mu}{ds} - \phi_i \frac{d\mu}{ds} = \phi_r^2 + \phi_i^2 = |\phi|^2.$$

This means that $\operatorname{Re} \Phi = \xi$ increases with s, i.e., the relation between s and $\Phi(t(s))$ is one-to-one. Because it is part of Hypothesis (K_1) that $\zeta = \Phi(t)$ maps S_1 *onto* Σ_1, we can conclude that $\Phi(t(s))$ is a homeomorphism between the segment L and Σ_1. Thus, S_1 and Σ_1 are each homeomorphic images of L. As points on S_1 and Σ_1 corresponding to the same value of s are related by $\zeta = \Phi(t)$, the latter function is a homeomorphism.

The same is true for S_2 and Σ_2, of course. Since $\zeta_0 = \Phi(t_0)$, it has been proved that the mapping $\zeta = \Phi(t)$, which is continuous in $\bar{G}$, establishes a *homeomorphism* between *the boundary of G and the boundary of* Γ.

The foregoing fact implies that the mapping must be a homeomorphism between $\bar{G}$ and $\bar{\Gamma}$. The proof of this is easy and will be omitted. (See also A. I. Markushevich, *Functions of a Complex Variable*, v. II, § 18.)

Appendix B.

Proof of Theorem 5.1. Insertion of the expressions (4.4), (4.5), (4.12), (4.13), (4.17) and (5.1) into (5.4) produces the lengthy formula

$$
\begin{aligned}
C = {} & \begin{bmatrix} \exp\left[-\dfrac{2}{3\varepsilon}\check{t}^{3/2}\right] & 0 \\ 0 & \exp\left[\dfrac{2}{3\varepsilon}\check{t}^{3/2}\right] \end{bmatrix} (I + \varepsilon \check{t}^{-3/2} B_4(\varepsilon^{-2/3}\check{t})) \\
& \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \check{t}^{1/4} & 0 \\ 0 & \check{t}^{-1/4} \end{bmatrix} 2\sqrt{\pi}\varepsilon^{-1/6}\left(\frac{dt}{d\check{t}}\right)^{-1/2} \\
& \cdot (I + \varepsilon B_5(\check{t}, \varepsilon)) \begin{bmatrix} \phi^{-1/2}(t) & 0 \\ 0 & \phi^{1/2}(t)\dfrac{dt}{d\check{t}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \\
& \cdot (I + \varepsilon B_6(t, \varepsilon)) \begin{bmatrix} \exp\left[\dfrac{i}{\varepsilon}\Phi(t)\right] & 0 \\ 0 & \exp\left[-\dfrac{i}{\varepsilon}\Phi(t)\right] \end{bmatrix}.
\end{aligned} \tag{B.1}
$$

The matrixes B_4 to B_6 are bounded for $t \in G_\delta \cap \mathcal{N}, 0 < \varepsilon \leqq \varepsilon_0$. Because, generally, $BN = N(N^{-1}BN)$ for any matrix B and any nonsingular matrix N, the matrices $B_j, j = 4, 5, 6$, in the product in (B.1) can be interchanged with any matrix that is nonsingular and bounded, together with its inverse, for $0 < \varepsilon \leqq \varepsilon_0$ and for the value of t at which the matrix $C = C(\varepsilon)$ is to be calculated. This will change the matrix, but not its bounded character.

We choose for t a point on $S_1 \cap \mathcal{N}$. Then $\check{t} = |\check{t}|\, e^{-\pi i/3}$ and therefore,

$$\check{t}^{3/2} = -i|\check{t}|^{3/2}. \tag{B.2}$$

This means that the first factor in the right member of (B.1) and its inverse are bounded. The same is true of all the other matrix factors with the exception of the last. Hence, the three factors of the form $(I + \varepsilon\check{t}^{-3/2}B_4), (I + \varepsilon B_5)$ and $(I + \varepsilon B_6)$ can be shifted to the beginning of the product and combined into $(I + \varepsilon B(\varepsilon))$.

Next, we observe that differentiation of the identity

$$\check{t}^{3/2} = \frac{3}{2}i\int_{t_0}^{t} \phi(s)\, ds,$$

which follows from (4.6), leads to

$$\check{t}^{1/2}\frac{d\check{t}}{dt} = i\phi(t),$$

and hence to

$$\check{t}^{1/4}\left(\frac{d\check{t}}{dt}\right)^{1/2} = \pm e^{\pi i/4}\phi^{1/2}(t). \tag{B.3}$$

Now, the factor $(dt/d\check{t})^{-1/2}$ in the product (B.1) originated from the matrix $\check{P}_0(\check{t})$ in (4.13). Since the matrix $\check{P}(\check{t}, \varepsilon)$ in Theorem 4.2 can be replaced by its negative without affecting the conclusion of that theorem, formula (B.3) may be used, with either sign, to simplify the expression in (B.1). We choose the plus sign and find that

$$C = 2\pi\varepsilon^{-1/6}\, e^{\pi i/4}(I + \varepsilon B(\varepsilon))\begin{bmatrix} \exp\left[-\frac{2}{3\varepsilon}\check{t}^{3/2} + \frac{i}{\varepsilon}\Phi(t)\right] & 0 \\ 0 & \exp\left[\frac{2}{3\varepsilon}\check{t}^{3/2} - \frac{i}{\varepsilon}\Phi(t)\right]\end{bmatrix}.$$

Formula (5.5) follows from this by reference to (4.6), (1.9) and (2.4).

Proof of Theorem 5.2. We insert (4.4), (4.12), (4.13), (4.19), (4.22) and (5.1) into (5.7) and proceed in analogy with (B.1), i.e., we calculate the product in (5.7) at some fixed, arbitrary point t on S_2.

A convenient way to calculate $\check{V}^{-1}$ is from the formula

$$\check{V}^{-1} = (\det \check{V})^{-1}\operatorname{adj}(\check{V}),$$

where $\operatorname{adj}(\check{V})$ is the adjugate of $\check{V}$, i.e., the transpose of the matrix of cofactors of $\check{V}$. This adjugate can be asymptotically calculated from (4.19). We recall that $\check{t} = |\check{t}|\, e^{-\pi i}$ for $t \in S_2$. One finds

$$\operatorname{adj}\check{V}(\check{t}, \varepsilon) = \frac{\varepsilon^{1/6}}{2\sqrt{\pi}}\begin{bmatrix} 2\sin\left(\frac{2}{3}|\rho|^{3/2} - \frac{\pi}{4}\right) & -2\cos\left(\frac{2}{3}|\rho|^{3/2} - \frac{\pi}{4}\right) \\ -\exp\left(\frac{2}{3}i|\rho|^{3/2} - \frac{\pi}{4}i\right) & \exp\left(\frac{2}{3}i|\rho|^{3/2} + \frac{\pi}{4}i\right)\end{bmatrix}$$
$$\cdot\begin{bmatrix} |t|^{-1/4} & 0 \\ 0 & |t|^{1/4}\end{bmatrix} + \varepsilon|\check{t}|^{-3/2}\tilde{B}_2(\varepsilon^{-2/3}|\check{t}|)$$

with $\tilde{B}_2(\varepsilon^{-2/3}|\check{t}|)$ bounded as $\varepsilon^{-2/3}|\check{t}| \to +\infty$. The quantity $\det \check{V}$ is independent of $\check{t}$ and is most easily computed at $\check{t} = 0$. One obtains

$$\det \check{V}(t, \varepsilon) = \det \check{V}(0, \varepsilon) = \det\begin{bmatrix} e^{\pi i/6}\mathrm{Ai}(0) & \mathrm{Ai}(0) \\ \varepsilon^{1/3}\, e^{\pi i/6}\omega\mathrm{Ai}(0) & \varepsilon^{1/3}\mathrm{Ai}'(0)\end{bmatrix}$$
$$= e^{\pi i/6}\varepsilon^{1/3}\mathrm{Ai}(0)\mathrm{Ai}'(0)(1 - \omega) = -\varepsilon^{1/3}/2\pi.$$

Therefore the analogue of formula (B.1) is

$$C(\varepsilon)^+ = -\frac{\sqrt{\pi}}{\varepsilon^{1/6}}\left\{\begin{bmatrix} 2\sin\left(\frac{2}{3}|\rho|^{3/2} - \frac{\pi}{4}\right) & -2\cos\left(\frac{2}{3}|\rho|^{3/2} - \frac{\pi}{4}\right) \\ -\exp\left(\frac{2}{3}i|\rho|^{3/2} - \frac{\pi}{4}i\right) & \exp\left(\frac{2}{3}i|\rho|^{3/2} + \frac{\pi}{4}i\right)\end{bmatrix} + \varepsilon|\check{t}|^{-3/2}\tilde{B}_2(\varepsilon^{-2/3}\check{t})\right\}$$

(*cont.*)

$$\cdot\begin{bmatrix}|\check{t}|^{1/4} & 0\\ 0 & |\check{t}|^{-1/4}\end{bmatrix}\left(\frac{dt}{d\check{t}}\right)^{-1/2}(I+\varepsilon B_7(\check{t},\varepsilon))\begin{bmatrix}\phi^{-1/2}(t) & 0\\ 0 & \phi^{1/2}(t)\left(\dfrac{dt}{d\check{t}}\right)\end{bmatrix}$$

$$\cdot\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}(I+\varepsilon B_8(t,\varepsilon))\begin{bmatrix}e^{(i/\varepsilon)\Phi(t)} & 0\\ 0 & e^{-(i/\varepsilon)\Phi(t)}\end{bmatrix}.$$

The matrices $\tilde{B}_2, B_7, B_8$ are bounded functions of ε, as $\varepsilon \to 0+$. Hence, $I + \varepsilon B_7$, $I + \varepsilon B_8$ can be replaced by factors of the same form, $I + O(\varepsilon)$, in front of the whole product, by the same argument as in the proof of Theorem 5.1. We can also use (B.3) again and write (remember that $\check{t} = |\check{t}|\, e^{-\pi i}$, for $t \in S_2$)

$$\begin{pmatrix}|\check{t}|^{1/4} & 0\\ 0 & |\check{t}|^{-1/4}\end{pmatrix}\left(\frac{dt}{d\check{t}}\right)^{-1/2}\begin{bmatrix}\phi^{-1/2}(t) & 0\\ 0 & \phi^{1/2}(t)\dfrac{dt}{d\check{t}}\end{bmatrix}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$$

$$=\begin{bmatrix}e^{\pi i/2} & 0\\ 0 & e^{-\pi i/2}\end{bmatrix}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}=\begin{bmatrix}i & i\\ 1 & -1\end{bmatrix}.$$

The product for C^+ now simplifies into

$$C^+ = -\frac{\sqrt{\pi}}{\varepsilon^{1/6}}(I+\varepsilon B^+(\varepsilon))\left\{\begin{bmatrix}2\sin\left(\dfrac{2}{3}|\rho|^{3/2}-\dfrac{\pi}{4}\right) & -2\cos\left(\dfrac{2}{3}|\rho|-\dfrac{\pi}{4}\right)\\ -\exp\left(\dfrac{2}{3}i|\rho|^{3/2}-\dfrac{\pi}{4}i\right) & \exp\left(\dfrac{2}{3}i|\rho|^{3/2}+\dfrac{\pi}{4}i\right)\end{bmatrix}+\varepsilon B_9(\varepsilon)\right\}$$

$$\cdot\begin{bmatrix}i & i\\ 1 & -1\end{bmatrix}\begin{bmatrix}\exp\left(\dfrac{i}{\varepsilon}\Phi(t)\right) & 0\\ 0 & \exp\left(-\dfrac{i}{\varepsilon}\Phi(t)\right)\end{bmatrix}$$

$$=\frac{2\sqrt{\pi}}{\varepsilon^{1/6}}(I+\varepsilon B_4(\varepsilon))\begin{bmatrix}1 & i\\ 0 & 1\end{bmatrix}\begin{bmatrix}\exp\left[-\dfrac{2}{3\varepsilon}\check{t}^{3/2}+\dfrac{i}{\varepsilon}\Phi(t)\right] & 0\\ 0 & \exp\left[\dfrac{2}{3\varepsilon}\check{t}^{3/2}-\dfrac{i}{\varepsilon}\Phi(t)\right]\end{bmatrix}.$$

The term εB_9 is $O(\varepsilon)$, and its contribution has been combined with $\varepsilon B^+(\varepsilon)$ in the last formula. From the last expression, formula (5.8) follows immediately by means of the definition of $\check{t}$ in (4.6).

Appendix C.

Proof of Lemma 6.1. This proof follows the method of E. C. Lommel in [3] as outlined in [8]. Letters used in the formulas below do not necessarily designate the same quantities as in the main part of the paper. Let

$$f(z) = z^{1/2}\mathrm{Ai}(z) - \mathrm{Ai}'(z),$$

and

$$g(z) = z^{1/4}f(z).$$

For abbreviation we set $\mathrm{Ai}(z) = y$. Using the fact that $y'' = zy$, one finds, after some calculation, that $g(z)$ is a solution of the differential equation

$$g'' - (z - \tfrac{3}{2}z^{-1/2} + \tfrac{3}{16}z^{-2})g = 0.$$

Let

$$u = g(az), \quad v = g(\bar{a}z),$$

a being a complex constant. These functions satisfy the differential equations

$$u'' - (a^3 z - \tfrac{3}{2}a^{3/2}z^{-1/2} + \tfrac{3}{16}z^{-2})u = 0,$$
$$v'' - (\bar{a}^3 z - \tfrac{3}{2}\bar{a}^{3/2}z^{-1/2} + \tfrac{3}{16}z^{-2})v = 0.$$

Now, generally, if

$$u'' = Pu, \quad v'' = Qv, \qquad P, Q \text{ continuous},$$

then

$$(P - Q)uv = \frac{d}{dz}(uv' - u'v).$$

Hence, applying this identity and integrating from zero to one, one gets

$$\text{(C.1)} \quad \int_0^1 [(a^3 - \bar{a}^3)t - \tfrac{3}{2}(a^{3/2} - \bar{a}^{3/2})t^{-1/2}]g(at)g(\bar{a}t)\,dt = g(a)g'(\bar{a}) - g(\bar{a})g'(a),$$

because $g(z)$ vanishes at zero.

If a is any zero of $f(z)$, it is also a zero of $g(z)$. We now specify the branches of the multivalued functions f and g by taking the determinations that are real on the positive real axis and continuing them into the sector $|\arg z| \leqq \pi$. (The values on $\arg z = \pi$ are, of course, different from those on $\arg z = -\pi$.) By the reflection principle this branch of $g(z)$ assumes conjugate values in conjugate points. Therefore, if $g(a)$ is zero, so is $g(\bar{a})$, i.e., the right member of (C.1) vanishes if a is a zero of $f(z)$.

Let

$$a = |a|\,e^{i\alpha}.$$

Then

$$a^3 - \bar{a}^3 = 2i|a|^3 \sin 3\alpha,$$
$$a^{3/2} - \bar{a}^{3/2} = 2i|a|^{3/2} \sin \tfrac{3}{2}\alpha.$$

As

$$\text{(C.2)} \qquad \sin \tfrac{3}{2}\alpha \cdot \sin 3\alpha < 0 \quad \text{in } \frac{\pi}{3} < \alpha < \frac{2\pi}{3}, \ \frac{2\pi}{3} < \alpha < \pi,$$

the integrand in (C.1) is a purely imaginary function of t in $0 \leqq t \leqq 1$, and its imaginary part does not change sign there, when α is in the sectors indicated in (C.2). Hence, the branch of $f(z)$ under consideration does not have zeros in these sectors or their conjugates.

For $\alpha = \pi/3$ and $\alpha = \pi$ one of the two terms in brackets in the integrand of (C.1) vanishes, but not the other one. Therefore, there cannot be any zeros of $f(z)$ on those rays, either.

There remains the ray $\alpha = 2\pi/3$. From the structure of the series in ascending powers of z for $f(z)$ one finds that for z on that ray,

$$f(|z|\, e^{i2\pi/3}) = b(|z|) + e^{\pi i/3} c(|z|),$$

where $b(|z|) > 0$, $c(|z|) > 0$ for all values of $|z|$. Hence, f has no zeros on $\arg z = \pm(2/3)\pi$, and the proof of Lemma 6.1 is complete.

Appendix D.

Proof of Theorem 7.1. From (7.3) and Theorems 5.1, 5.2, as well as Corollary 3.2 and Theorem 4.3 one finds

$$\begin{aligned} PV &= P^+V^+(C^+)^{-1}C \\ &= \{I + \varepsilon B_7\}\begin{bmatrix} e^{(i/\varepsilon)\zeta} & 0 \\ 0 & e^{-(i/\varepsilon)\zeta} \end{bmatrix}\begin{bmatrix} e^{-(i/\varepsilon)\zeta_0} & 0 \\ 0 & e^{(i/\varepsilon)\zeta_0} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}\{I + \varepsilon B_8\} \\ &\quad \cdot \begin{bmatrix} e^{(i/\varepsilon)\zeta_0} & 0 \\ 0 & e^{-(i/\varepsilon)\zeta_0} \end{bmatrix}, \end{aligned}$$

where B_7, B_8 designate functions of t and ε bounded for $t \in G_\delta^+$, $0 < \varepsilon \leqq \varepsilon_0$. If the product in the right member is expanded the quotient in the right member of (7.4) turns out to have the form below:

$$\varepsilon p = \frac{-i\, e^{(i/\varepsilon)(\zeta - 2\zeta_0)}(1 + \varepsilon b) + \varepsilon\, e^{-(i/\varepsilon)\zeta} b}{e^{-(i/\varepsilon)\zeta}(1 + \varepsilon b) + \varepsilon\, e^{(i/\varepsilon)(\zeta - 2\zeta_0)} b}.$$

Here the letter b stands for some bounded functions of t and ε. From this the formula (7.5) follows immediately.

Observe that $\operatorname{Re} i(\zeta - \zeta_0) < 0$ in the *interior* of G_δ^+. Therefore the first right-hand term in (7.5) is the leading term on S_2 only.

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AN ELEMENTARY PROOF OF THE EXISTENCE OF SOLUTIONS TO SECOND ORDER NONLINEAR BOUNDARY VALUE PROBLEMS*

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Abstract. An essentially elementary proof of the existence of a solution to the boundary value problem $x'' = f(t, x)$, $(x(0), x'(0)) \in \Lambda_0$, $(x(1), -x'(1)) \in \Lambda_1$, provided there exist functions α, β such that $\alpha''(t) \geqq f(t, \alpha(t))$ and $\beta''(t) \leqq f(t, \beta(t))$, where Λ_0 and Λ_1 are appropriate sets in R^2 depending upon α and β, is given. The boundary conditions in this case are sufficiently general to include all the previous linear and nonlinear boundary conditions known to the authors (cf. the references). A similar theorem is established for the periodic boundary value problem.

1. Introduction. A proof is given here of the existence of a solution to the scalar differential equation

$$x'' = f(t, x), \quad \text{a.e. } t \in [0, 1], \tag{1}$$

which satisfies certain boundary conditions. The technique used is elementary and employs only the Tonelli procedure for demonstrating the local existence of solutions to initial value problems and the intermediate value (preservation of connectedness) theorem for continuous functions. The discussion is also applicable to equations of the form $x'' = f(t, x', x)$ provided some restriction, such as a Nagumo condition, is imposed on the growth of the derivatives of solutions.

2. Preliminary lemmas. Suppose

$$\alpha, \beta \in AC^{(1)}[0, 1], \qquad \alpha(t) \leqq \beta(t), \tag{2}$$

and

$$\alpha'' \geqq f(t, \alpha), \quad \beta'' \leqq f(t, \beta) \quad \text{a.e. } t \in [0, 1].$$

It is assumed that $f(t, x)$ satisfies the Carathéodory conditions on the set $\{(t, x) : x \in [\alpha(t), \beta(t)], t \in [0, 1]\}$, i.e., $f(t, x)$ is measurable in t for each x, continuous in x for each t and $\lambda, \Lambda \in L^1[0, 1]$, where $\lambda(t) = \inf f(t, x)$, $\Lambda(t) = \sup f(t, x)$, $\alpha(t) \leqq x \leqq \beta(t)$. Define $\tilde{f}(t, x)$ on $[0, 1] \times R$ by

$$\tilde{f}(t, x) = \begin{cases} f(t, \beta(t)), & x \geqq \beta(t), \\ f(t, x), & \alpha(t) \leqq x \leqq \beta(t), \\ f(t, \alpha(t)), & x \leqq \alpha(t). \end{cases} \tag{3}$$

Evidently $\lambda(t) \leqq \tilde{f}(t, x) \leqq \Lambda(t)$.

LEMMA 1. *If $x'' = \tilde{f}(t, x)$ and $\alpha(i) \leqq x(i) \leqq \beta(i)$, $i = 0, 1$, then $\alpha(t) \leqq x(t) \leqq \beta(t)$, $t \in [0, 1]$, so that $x'' = f(t, x)$.*

Proof. If $[t_1, t_2] \subset [0, 1]$ and $x(t_i) = \beta(t_i)$, $i = 1, 2$, $x(t) > \beta(t)$, $t \in (t_1, t_2)$, then $z(t) = x(t) - \beta(t)$, by (1), (2) and (3), satisfies $z''(t) \geqq 0$ for a.e. $t \in [t_1, t_2]$,

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$z(t_i) = 0$, $i = 1, 2$. Hence,

$$0 < z(t) = \int_{t_1}^{t_2} g(t, s) z''(s)\, ds, \qquad t \in (t_1, t_2),$$

where

$$0 \leqq g(t, s)(t_2 - t_1) = \begin{cases} (t_2 - t)(t_1 - s), & s \in (t_1, t], \\ (t_2 - s)(t_1 - t), & s \in [t, t_2), \end{cases}$$

which yields a contradiction. Thus $x(t) \leqq \beta(t)$, $t \in [0, 1]$; similarly $x(t) \geqq \alpha(t)$, $t \in [0, 1]$.

LEMMA 2. *If $x \in AC^{(1)}[0, 1]$ satisfies $\lambda \leqq x''(t) \leqq \Lambda$ for a.e. t in $[0, 1]$ and*

$$a_0 = \alpha(1) - \int_0^1 (1 - s)\Lambda(s)\, ds, \qquad b_0 = \beta(1) - \int_0^1 (1 - s)\lambda(s)\, ds,$$

$$a_1 = \alpha(0) - \int_0^1 s\Lambda(s)\, ds, \qquad b_1 = \beta(0) - \int_0^1 s\lambda(s)\, ds,$$

then

$$x(0) + x'(0) \leqq a_0 \ (\geqq b_0) \Rightarrow x(1) \leqq \alpha(1) \ (\geqq \beta(1))$$

$$x(1) - x'(1) \geqq a_1 \ (\leqq b_1) \Leftarrow x(0) \geqq \alpha(0) \ (\leqq \beta(0)).$$

Proof. This result follows from the identities

$$x(1) = x(0) + x'(0) + \int_0^1 (1 - s)x''(s)\, ds,$$

$$x(0) = x(1) - x'(1) + \int_0^1 s x''(s)\, ds,$$

which imply

$$x(1) - \int_0^1 (1 - s)\Lambda(s)\, ds \leqq x(0) + x'(0) \leqq x(1) - \int_0^1 (1 - s)\lambda(s)\, ds,$$

$$x(0) - \int_0^1 s\Lambda(s)\, ds \leqq x(1) - x'(1) \leqq x(0) - \int_0^1 s\lambda(s)\, ds,$$

respectively.

In Lemmas 3 and 4, X_0, X_1, Y_0, Y_1 denote the line segments $\{(x, 0): 0 \leqq x \leqq 1\}$, $\{(x, 1): 0 \leqq x \leqq 1\}$, $\{(0, y): 0 \leqq y \leqq 1\}$, $\{(1, y): 0 \leqq y \leqq 1\}$ in R^2, respectively.

LEMMA 3. *Suppose Σ, Ω are continua (closed connected sets) in $J = [0, 1] \times [0, 1]$ such that $\Sigma \cap X_0$, $\Sigma \cap X_1$, $\Omega \cap Y_0$, $\Omega \cap Y_1$ are all nonempty. Then $\Sigma \cap \Omega$ is nonempty.*

Proof. In the case that Ω is a Jordan arc which, except for its endpoints, lies in the interior of J, it follows from the Jordan curve theorem that $J - \Omega$ has two components which are open (in J) and which would provide a disconnection of Σ if $\Sigma \cap \Omega = \varnothing$ (cf. [7, Thms. 11.7, 11.8, pp. 118, 119]). In general, since Ω is compact, it may be covered by a finite number of balls of radius ε with centers in Ω. Thus, for each $\varepsilon > 0$, there exists a polygonal Jordan arc $\Omega_\varepsilon \subset S(\Omega, \varepsilon)$ lying, except for its endpoints, in the interior of J with $\Omega_\varepsilon \cap Y_0 \neq \emptyset, \Omega_\varepsilon \cap Y_1 \neq \emptyset$. By the preceding remarks there exists a point $(p_\varepsilon, q_\varepsilon) \in \Sigma \cap \Omega_\varepsilon$. Evidently, since Σ and Ω are closed, the closure of $\{(p_\varepsilon, q_\varepsilon) : \varepsilon > 0\}$ contains a point $(p, q) \in \Sigma \cap \Omega$.

LEMMA 4. *Let μ be a real-valued continuous function on $J = [0, 1] \times [0, 1]$ such that $\mu(x, 0) \leqq 0$, $\mu(x, 1) \geqq 0$, $0 \leqq x \leqq 1$. Then there exists a continuum $\Sigma \subset \mu^{-1}(0)$ such that $\Sigma \cap Y_0 \neq \emptyset, \Sigma \cap Y_1 \neq \emptyset$.*

Proof. The following proof is due to J. Timourian. Since the lemma is clearly true if $\mu(x, 0) \equiv 0$ or $\mu(x, 1) \equiv 0$ we may assume without loss of generality that there exist $x_0, x_1 \in [0, 1]$ such that $\mu(x_0, 0) < -1$, $\mu(x_1, 1) > 1$. For $k = 1, 2, \cdots$, let $\varepsilon_k > 0$ be such that $\rho((x, y), \mu^{-1}(0)) < \varepsilon_k$ implies $|\mu(x, y)| < 1/k$. The compact set $\mu^{-1}(0)$ is covered by an open set U_k composed of a finite number of balls of radius ε_k with centers in $\mu^{-1}(0)$. Further, ε_{k+1} may be chosen smaller than $\rho(\mu^{-1}(0), \partial U_k)$. Clearly, $\mu(x, y) \neq 0$ if $(x, y) \in J - U_k$. It is asserted that there exist continua $\Sigma_{kj} \subset \bar{U}_k$, $j = 1, \cdots, n_k, n_k \geqq 1$, such that $\Sigma_{kj} \cap Y_0 \neq \emptyset$, $\Sigma_{kj} \cap Y_1 \neq \emptyset$, where each Σ_{kj} is the union of some of the spheres used to construct $\bar{U}_k$. If this were not the case, since there are only finitely many such spheres, it would be possible to find a Jordan arc in $J - U_k$ with $(x_0, 0)$ and $(x_1, 1)$ as endpoints so that $\mu(x_0, 0) < 0$, $\mu(x_1, 1) > 0$ and the continuity of μ contradict $\mu(x, y) \neq 0$ if $(x, y) \in J - U_k$. Each of the sets $\Sigma_{kj}, k > 1$, must be a subset of some member of the sets Σ_{1j} so that at least one of the sets Σ_{1j} contains infinitely many Σ_{kj} as subsets and it is possible to find a nested sequence of continua Σ_{kj} which intersect Y_0 and Y_1. The intersection of this sequence is a continuum $\Sigma \subset \mu^{-1}(0)$ (cf. [7, Thm. 5.3, p. 81]) and $\Sigma \cap Y_0 \neq \emptyset, \Sigma \cap Y_1 \neq \emptyset$.

3. Two-point boundary value problems. In Theorem 1 let

$$J_i = \{(x, y) : \alpha(i) \leqq x \leqq \beta(i), a_i \leqq x + y \leqq b_i\},$$

$$\Gamma_i = \{(\alpha(i), y) : y < (-1)^i \alpha'(i)\} \cup \{(\beta(i), y) : y > (-1)^i \beta'(i)\}, \qquad i = 0, 1.$$

THEOREM 1. *Suppose α, β satisfy* (2) *and, for $i = 0, 1$, let Ω_i be a continuum in the set J_i which intersects both of the lines $x + y = a_i$, $x + y = b_i$. There exists a solution $x(t)$ of* (1) *such that*

$$(x(i), (-1)^i x'(i)) \in \Omega_i - \Gamma_i, \qquad i = 0, 1, \tag{4}$$

and $\alpha(t) \leqq x(t) \leqq \beta(t)$, $t \in [0, 1]$.

Proof. Let $\varepsilon > 0$ and consider the integral equation

$$\begin{aligned} x(t) &= p + qt + \int_0^t (t - s)\tilde{f}(s, x(s - \varepsilon))\, ds, \quad t \in [0, 1], \\ x(t) &= p, \quad t \in (-\varepsilon, 0), \end{aligned} \tag{5}$$

which is equivalent to the delay differential equation

$$\begin{aligned} x''(t) &= \tilde{f}(t, x(t - \varepsilon)), \qquad t \in [0, 1], \\ x(t) &= p, \quad x'(0) = q, \qquad t \in (-\varepsilon, 0]. \end{aligned}$$

Equation (5) has a unique solution $x(t) = x(t; p, q)$ for each $(p, q) \in R^2$ and, for each $t \in [0, 1]$, $(x(t; p, q), -x'(t; p, q))$ is a continuous function from R^2 to R^2. Let

$$T_\varepsilon(p, q) = (x(1; p, q), -x'(1; p, q)).$$

Since T_ε is continuous on J_0 and Ω_0 is a continuum, $T_\varepsilon(\Omega_0)$ is a continuum. Clearly $\lambda(t) \leqq x''(t) \leqq \Lambda(t)$, so that Lemma 2 implies $a_1 \leqq x(1) - x'(1) \leqq b_1$ if $(p, q) \in \Omega_0$, and $x(1) \leqq \alpha(1)$ $(\geqq \beta(1))$ if $(p, q) \in \Omega_0$ and $p + q = a_0 (= b_0)$. Thus $T_\varepsilon(\Omega_0)$ is a continuum in the strip $a_1 \leqq x + y \leqq b_1$ which intersects the lines $x = \alpha(1)$, $x = \beta(1)$, and Ω_1 is a continuum in the strip $\alpha(1) \leqq x \leqq \beta(1)$ which intersects the lines $x + y = a_1$, $x + y = b_1$. By Lemma 3, $T_\varepsilon(\Omega_0) \cap \Omega_1 \neq \emptyset$, i.e., there exists $(p_\varepsilon, q_\varepsilon) \in \Omega_0$ such that if $x_\varepsilon(t) = x(t; p_\varepsilon, q_\varepsilon)$ then $(x_\varepsilon(1), -x_\varepsilon'(1)) \in \Omega_1$. Since $\{x_\varepsilon(t)\}$ is uniformly bounded and equicontinuous, there exists a sequence $\varepsilon(k) \to 0$ $(k \to \infty)$ such that $x_{\varepsilon(k)}(t)$ converges to a solution $x(t)$ of $x''(t) = \tilde{f}(t, x(t))$. The sets Ω_i are closed so that

$$(x_\varepsilon(i), (-1)^i x_\varepsilon'(i)) \in \Omega_i \quad \text{implies } (x(i), (-1)^i x'(i)) \in \Omega_i.$$

It follows from Lemma 1 that $\alpha(t) \leqq x(t) \leqq \beta(t)$, and hence

$$x''(t) = f(t, x(t)).$$

It is not possible that $(x(i), (-1)^i x'(1)) \in \Gamma_i$ for $i = 0$ or $i = 1$ since $\alpha(t) \leqq x(t) \leqq \beta(t)$ implies

$$(-1)^i(x'(i) - \alpha'(i)) \geqq 0 \quad \text{if } x(i) = \alpha(i), \quad i = 0 \text{ or } i = 1,$$

$$(-1)^i(x'(i) - \beta'(i)) \leqq 0 \quad \text{if } x(i) = \beta(i), \quad i = 0 \text{ or } i = 1.$$

Therefore,

$$(x(i), (-1)^i x'(i)) \in \Omega_i - \Gamma_i, \qquad i = 0, 1.$$

Generally, in a particular boundary value problem, the set Ω_i is directly generated from the ith boundary condition. In case this set fails to intersect the line $y + x = a_i$ or the line $y + x = b_i$, it may still be possible to satisfy this condition without changing the conclusion (4) by including in the Ω_i of the hypotheses an appropriate line segment from Γ_i. This fact is illustrated in the proof of the following corollary.

COROLLARY 1. *Let μ_i be a real-valued continuous function on J_i such that*

$$\mu_i(\alpha(i), y)\mu_i(\beta(i), z) \leqq 0 \quad \text{if } y \geqq \alpha_i' \text{ and } z \leqq \beta_i', \tag{6}$$

where

$$\alpha_i' = \max\{(-1)^i \alpha'(i), a_i - \alpha(i)\},$$

$$\beta_i' = \min\{(-1)^i \beta'(i), b_i - \beta(i)\},$$

for $i = 0, 1$. Then there exists a solution $x(t)$ of (1) *such that*

$$\mu_i(x(i), (-1)^i x'(i)) = 0, \qquad i = 0, 1,$$

and

$$\alpha(t) \leqq x(t) \leqq \beta(t), \qquad 0 \leqq t \leqq 1.$$

Proof. The condition (6) ensures that $\mu_i(x, y)$ is of constant and opposite sign on the line segments (or points) $\{(\alpha(i), y) : y \geqq \alpha_i'\} \cap J_i$ and $\{(\beta(i), z) : z \leqq \beta_i'\} \cap J_i$. Thus, Lemma 4 implies that there is a continuum $\Sigma_i \subset \mu_i^{-1}(0)$ which intersects $\{(x, y) : x + y = a_i\} \cup \{(\alpha(i), y) : y \leqq \alpha_i'\}$ and $\{(x, y) : x + y = b_i\} \cup \{(\beta(i), z) : z \geqq \beta_i'\}$, $i = 0, 1$. By taking, as Ω_i, the set Σ_i together with line segments from the set

$$\Gamma_i = \{(\alpha(i), y) : y < (-1)^i \alpha'(i)\} \cup \{(\beta(i), y) : y > (-1)^i \beta'(i)\},$$

if necessary (i.e., when Σ_i does not intersect one or both of the lines $x + y = a_i$, $x + y = b_i$), the conditions of Theorem 1 are satisfied. Thus there exists a solution $x(t)$ of (1) such that $(x(i), (-1)^i x'(i)) \in \Omega_i - \Gamma_i \subset \Sigma_i$, i.e.,

$$\mu_i(x(i), (-1)^i x'(i)) = 0, \qquad i = 0, 1.$$

COROLLARY 2. *Let μ_i be real-valued continuous functions on R^2 such that $\mu_i(x, y)$ is nonincreasing in y for each x. If α_i', β_i' are as in Corollary* 1 *and* $\mu_i(\alpha(i), \alpha_i') \leqq u_i \leqq \mu_i(\beta(i), \beta_i')$, $i = 0, 1$, *then there exists a solution $x(t)$ of* (1) *satisfying*

$$\mu_i(x(i), (-1)^i x'(i)) = u_i, \qquad i = 0, 1,$$

and $\alpha(t) \leqq x(t) \leqq \beta(t)$, $t \in [0, 1]$.

Proof. Corollary 2 follows from Corollary 1 by replacing $\mu_i(x, y)$ with $\mu_i(x, y) - u_i$; $\mu_i(\alpha(i), \alpha_i') - u_i \leqq 0$ implies $\mu_i(\alpha(i), y) - u_i \leqq 0$ if $y \geqq \alpha_i'$ and $\mu_i(\beta(i), \beta_i') - u_i \geqq 0$ implies $\mu_i(\beta(i), y) - u_i \geqq 0$ if $y \leqq \beta_i'$, since $\mu_i(x, y)$ is nonincreasing in y, and the conditions of Corollary 1 hold.

Theorem 1 is a generalization of results of Jackson and Klaasen [4], Bebernes and Fraker [1] and Bebernes and Wilhelmsen [2] as these results apply to equation (1). As applied in Corollaries 1 and 2, Theorem 1 also generalizes a result of Erbe [3] who essentially proved Corollary 2 with the slightly more restrictive condition

$$\mu_i(\alpha(i), (-1)^i \alpha'(i)) \leqq u_i \leqq \mu_i(\beta(i), (-1)^i \beta'(i)).$$

Erbe's result also includes as a special case a result of Kaplan, Lasota and Yorke [5] which dealt with linear boundary conditions, i.e., conditions of the form

$$ax(i) + bx'(i) = u_i.$$

For such a condition, (6) will be satisfied provided $b \neq 0$ and

$$(-1)^i [a\alpha(i) + b\alpha'(i) - u_i] b^{-1} \geqq \min(0, \alpha(i) + (-1)^i \alpha'(i) - a_i),$$
$$(-1)^i [a\beta(i) + b\beta'(i) - u_i] b^{-1} \leqq \max(0, \beta(i) + (-1)^i \beta'(i) - (-1)^i b_i),$$

or $b = 0$ and $\alpha(i) \leqq u_i a^{-1} \leqq \beta(i)$, which is a generalization of the result in [5].

If the Kneser–Fukuhara funnel theorem is assumed, then Theorem 1 may be proved directly by applying that theorem to (1) rather than working with equation (5). However the argument is basically the same and no greater brevity is brought to the proof.

4. Periodic boundary value problems. In Theorem 2 conditions are given for the existence of a solution $x(t)$ of (1) satisfying "periodic" boundary conditions

(7) $$\mu_0(x(0), x'(0)) = \mu_1(x(1), -x'(1)), \qquad \nu_0(x(0), x'(0)) = \nu_1(-x(1), x'(1)).$$

It is assumed that $\mu_i(x, y)$, $\nu_i(x, y)$ are continuously differentiable real-valued functions on R^2 such that

$$\frac{\partial \mu_i}{\partial x}, -\frac{\partial \mu_i}{\partial y}, \frac{\partial \nu_i}{\partial x}, \frac{\partial \nu_i}{\partial y} \geqq 0, \tag{8}$$

$$\left(\frac{\partial \mu_i}{\partial x}\right)^2 + \left(\frac{\partial \mu_i}{\partial y}\right)^2, \quad \left(\frac{\partial \nu_i}{\partial x}\right)^2 + \left(\frac{\partial \nu_i}{\partial y}\right)^2 > 0, \qquad i = 0, 1. \tag{9}$$

For brevity, the notation

$$\mu_i x = \mu_i(x(i), (-1)^i x'(i)), \qquad \nu_i x = \nu_i((-1)^i x(i), x'(i))$$

is used.

LEMMA 5. *Let $x, y \in C^1[0, 1]$, $x(t) \leqq y(t)$; then*

$$\mu_i x = \mu_i y \Rightarrow (-1)^i \nu_i x \leqq (-1)^i \nu_i y, \qquad i = 0 \textit{ and/or } i = 1.$$

Proof. It will suffice to show that $\mu_i x = \mu_i y$ implies $(-1)^i x'(i) \leqq (-1)^i y'(i)$, since the conclusion then follows from (8) by using the mean value theorem. Since

$$0 = \mu_i x - \mu_i y = \frac{\partial \mu_i}{\partial x}(x_i, y_i)(x(i) - y(i)) + (-1)^i \frac{\partial \mu_i}{\partial y}(x_i, y_i)(x'(i) - y'(i))$$

by the mean value theorem, and

$$\frac{\partial \mu_i}{\partial x}(x_i, y_i)(x(i) - y(i)) \leqq 0$$

by (8), we must have $(-1)^i(x'(i) - y'(i)) \leqq 0$ in the case that $(\partial \mu_i/\partial y)(x_i, y_i) < 0$; when $(\partial \mu_i/\partial y)(x_i, y_i) = 0$ then $x(i) = y(i)$ and $x(t) \leqq y(t), t \in [0, 1]$, implies $(-1)^i(x'(i) - y'(i)) \leqq 0$.

THEOREM 2. *Suppose α, β satisfy* (2) *and*

$$\mu_0\alpha - \mu_1\alpha = 0, \qquad \mu_0\beta - \mu_1\beta = 0,$$
$$\nu_0\alpha - \nu_1\alpha \geqq 0, \qquad \nu_0\beta - \nu_1\beta \leqq 0,$$

and $\mu_i\alpha \leqq \mu_i\beta$, $i = 0, 1$, where μ_i, ν_i satisfy (8), (9). *Then there exists a solution $x(t)$ to the boundary value problem* (1), (7) *such that $\alpha(t) \leqq x(t) \leqq \beta(t)$, $t \in [0, 1]$.*

Proof. Let Π denote the set of solutions $x(t)$ of (1) such that $\alpha(t) \leqq x(t) \leqq \beta(t)$, $\mu_i\alpha \leqq \mu_i x \leqq \mu_i\beta$, $i = 0, 1$, $\mu_0 x = \mu_1 x$, $\nu_0 x \geqq \nu_1 x$. Π is not empty since, by Corollary 2, there exists a solution $x(t)$ of (1) satisfying $\alpha(t) \leqq x(t) \leqq \beta(t)$, $\mu_0 x = \mu_0\alpha$, $\mu_1 x = \mu_1\alpha$ (so $\mu_0 x = \mu_1 x$ since $\mu_0\alpha = \mu_1\alpha$, by hypothesis). Also, by Lemma 5, $\nu_0 x \geqq \nu_0\alpha$, $\nu_1 x \leqq \nu_1\alpha$ and therefore $\nu_0 x \geqq \nu_1 x$ since $\nu_0\alpha \geqq \nu_1\alpha$ by hypothesis. Let

$$u^* = \sup\{\mu_0 x : x \in \Pi\} = \sup\{\mu_1 x : x \in \Pi\};$$

clearly $\mu_0\alpha = \mu_1\alpha \leqq u^* \leqq \mu_0\beta = \mu_1\beta$. Since Π is uniformly bounded and equicontinuous there exists a solution $x \in \Pi$ such that

$$\mu_0 x = \mu_1 x = u^*, \qquad \nu_0 x \geqq \nu_1 x. \tag{10}$$

It is asserted that this solution, in fact, satisfies $\nu_0 x = \nu_1 x$, i.e., $x(t)$ is a solution of the problem (1), (7). Suppose $\nu_0 x > \nu_1 x$; then $u^* < \mu_0\beta = \mu_1\beta$, since $\mu_0 x = \mu_0\beta$, $\mu_1 x = \mu_1\beta$, by Lemma 5, would imply $\nu_0 x \leqq \nu_0\beta$, $\nu_1 x \geqq \nu_1\beta$, and hence

$$\nu_0 x - \nu_1 x \leqq \nu_0\beta - \nu_1\beta \leqq 0$$

contradicting $\nu_0 x > \nu_1 x$. Replacing the pair α, β in Corollary 2 by the pair x, β it is seen that the set Π_1 of solutions $y(t)$ satisfying $x(t) \leqq y(t) \leqq \beta(t)$, $u^* < \mu_0 y = \mu_1 y \leqq \mu_0\beta = \mu_1\beta$, is not empty and each such solution must, by the definition of u^*, satisfy $\nu_0 y < \nu_1 y$. Clearly $u^* = \inf\{\mu_0 y : y \in \Pi_1\} = \inf\{\mu_1 y : y \in \Pi_1\}$ so there exists a solution $y(t)$, $x(t) \leqq y(t) \leqq \beta(t)$,

$$\mu_0 y = \mu_1 y = u^*, \qquad \nu_0 y \leqq \nu_1 y.$$

But since $\mu_0 x = \mu_1 x = u^*$, by Lemma 5,

$$\nu_0 x \leqq \nu_0 y, \qquad \nu_1 x \geqq \nu_1 y,$$

and hence $\nu_0 x \geqq \nu_1 x$ which, by (10), implies $\nu_0 x = \nu_1 x$.

COROLLARY 3. *Let* $a_i, b_i, c_i, d_i \geqq 0$, $a_i + b_i > 0$, $c_i + d_i > 0$, $i = 0, 1$, *and*

$$l_i x = a_i x(i) + (-1)^{i+1} b_i x'(i), \qquad m_i x = (-1)^i c_i x(i) + d_i x'(i).$$

Suppose

$$l_0\alpha \leqq l_0\beta, \quad l_1\alpha \leqq l_1\beta, \quad l_0\alpha = l_1\alpha, \quad l_0\beta = l_1\beta, \quad m_0\alpha \geqq m_1\alpha, \quad m_0\beta \leqq m_1\beta;$$

then there exists a solution $x(t)$ *of* (1) *such that*

$$l_0 x = l_1 x, \qquad m_0 x = m_1 x.$$

The case $a_i = d_i = 1$, $b_i = c_i = 0$ has been proved by Schmitt [8]. All of the results of the present paper may be extended to more general functional differential equations by a technique developed by the authors [6].

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A REPRESENTATION FOR A DISTRIBUTIONAL SOLUTION OF THE HEAT EQUATION*

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Abstract. A boundary-integral-boundary-sum representation for certain distributional solutions of the heat equation in a finite cylinder with n-dimensional rectangular base is derived. The method employed involves use of a duality principle at the boundary.

1. Introduction. In this paper we derive a boundary-integral-boundary-sum representation for certain distributional solutions of the heat equation in a finite cylinder with n-dimensional rectangular base. To do this we show that the boundary trace, $\operatorname{tr}(u)$, of our solution u falls in the dual $\mathscr{B}'$ of a space $\mathscr{B}$ and that the "Green's" function which we use falls in $\mathscr{B}$. Since C^∞ solutions take the form $u(x, t) = \langle \operatorname{tr}(u), \mathscr{G}(x, t, \cdot, \cdot)\rangle$, where $\langle \cdot, \cdot \rangle$ stands for the duality relation between $\mathscr{B}'$ and $\mathscr{B}$, we extend both sides of this expression by continuity and obtain our result.

In the literature there are various boundary-integral representations for solutions of Laplace's equation in R^2—for instance, Herglotz (see [7, p. 32]) and Johnson [8]. Saylor [13], [14] has found similar results for more general elliptic equations in bounded domains of R^n. He has used methods suggested by Cimmino [1] and Lions and Magenes [10]. For the heat equation treated below, the approach is similar to Saylor's. The procedure was suggested by results in another paper by Lions and Magenes [11].

In addition to providing a characterization for distributional solutions, our result is useful in that it can be employed in continuation procedures similar to those in [2], [3], [4], [13]. These procedures are numerical in nature and can be programmed. The possibility of using the representation in this way is what motivated the present work.

In a future paper the author expects to extend the result of this paper to parabolic equations with variable coefficients where the region considered is a cylinder with an analytic manifold as base.

2. Preliminaries. In this and the next two sections, we carry over some results of Lions and Magenes [11] for one dimension in the space variable to n dimensions.

This is done to provide the framework for deriving our representation. This work is included for the sake of completeness and because it is slightly different from the work in [8].

Let H be the heat operator $\Delta - \partial/\partial t$ in $R_x^n \times R_t$, the Cartesian product of the real n-dimensional and one-dimensional Euclidean spaces R_x^n and R_t. A typical point in this space is $(x, t) = (x_1, \cdots, x_n, t)$, with x representing the space coordinate, t the time. The operator H has adjoint $H^* = \Delta + \partial/\partial t$. Let Ω be the

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open subset $C \times (0, T)$, where $C = \bigtimes_{j=1}^{n}(0, L_j)$, and the L_j's and T are positive. The rectangle C has boundary ∂C and closure $\bar{C}$. We shall be interested in distributional solutions of the problem

$$\begin{cases} Hu = 0 & \text{in } \Omega, \\ u = \tilde{u} & \text{on } (\bar{C} \times \{0\}) \cup (\partial C \times (0, T)), \end{cases}$$

where the space to which $\tilde{u}$ belongs will be made more explicit later.

Let $\mathscr{D}(\Omega)$, $\mathscr{D}(\bar{\Omega})$, and $\mathscr{D}(\bar{C})$ be the usual spaces of infinitely differentiable functions provided with the usual Schwartz topologies [16]. A similar definition holds when Ω, $\bar{\Omega}$, and $\bar{C}$ are replaced by other spaces. Let $\mathscr{D}'(\bar{C})$ be the strong dual of $\mathscr{D}(\bar{C})$. By $\mathscr{D}_{-}(\mathscr{D}(\bar{C}))$ (see Schwartz [15], [17]), we mean the space of functions, infinitely differentiable in t, taking values in $\mathscr{D}(\bar{C})$ for each $t \in (0, T)$, and zero in some neighborhood of T. This space has Schwartz inductive limit topology, and by a proof similar to that in [11, p. 315], is reflexive. Let $(\mathscr{D}_{-}(\mathscr{D}(\bar{C}))'$ be the strong dual of $\mathscr{D}_{-}(\mathscr{D}(\bar{C}))$. By a proof similar to that in [11, p. 314], this dual can be seen to be the same as $\mathscr{D}'_{+}(\mathscr{D}'(\bar{C}))$ (see Schwartz [15], [17]), the space of distributions with values in $\mathscr{D}'(\bar{C})$, with support for t contained in (T_1, T) for some $T_1 > 0$. The functions and functionals of all the spaces above are complex.

We now define the spaces

$$\mathscr{H}^* = \{g | g \in \mathscr{D}_{-}(\mathscr{D}(\bar{C})), H^*g \in \mathscr{D}(\Omega), g = 0 \text{ on } \partial C \times (0, T)\}, \tag{2.1}$$

$$\mathscr{H} = \{f | f \in \mathscr{D}'(\Omega), Hf \in \mathscr{D}'_{+}(\mathscr{D}'(\bar{C}))\}. \tag{2.2}$$

The former is equipped with the weakest locally convex topology for which the mappings $g \to g$ of $\mathscr{H}^*$ into $\mathscr{D}_{-}(\mathscr{D}(\bar{C}))$, and $g \to H^*g$ of $\mathscr{H}^*$ into $\mathscr{D}(\Omega)$ are continuous, and the latter with the weakest locally convex topology which makes the maps $f \to f$ of $\mathscr{H}$ into $\mathscr{D}'(\Omega)$, and $f \to Hf$ of $\mathscr{H}$ into $\mathscr{D}'_{+}(\mathscr{D}'(\bar{C}))$ continuous. A proof similar to that on p. 311 of [11] shows that $\mathscr{H}^*$ is an LF space in the sense of Trèves [20, p. 126], i.e., a countable strict inductive limit of an increasing sequence of Fréchet spaces.

Further, let us define the linear boundary operator on $\mathscr{H}^*$ given by

$$\operatorname{bn}(g) = \left(g(x, 0), \frac{\partial g(0, x_2, \cdots, x_n, t)}{\partial x_1}, \cdots, -\frac{\partial g(x_1, \cdots, x_{n-1}, L_n, t)}{\partial x_n}\right), \quad g \in \mathscr{H}^*, \tag{2.3}$$

with values in $\mathscr{D}(\bar{C}) \times \bigtimes_{j=1}^{n}\mathscr{D}_{-}(\mathscr{D}(\bar{C}_j)) \times \bigtimes_{j=1}^{n}\mathscr{D}_{-}(\mathscr{D}(\bar{C}_j)) = \mathscr{D}$, where

$$\bar{C}_j = [0, L_1] \times \cdots \times [0, L_{j-1}] \times [0, L_{j+1}] \times \cdots \times [0, L_n].$$

This in turn leads to a linear operator bn_Q given by $\operatorname{bn}_Q(g_Q) = \operatorname{bn}(g)$, $g_Q \in \mathscr{H}^*_Q$, with g any member of g_Q, N the null space $\{g \in \mathscr{H}^* | \operatorname{bn}(g) = 0\}$, and $\mathscr{H}^*_Q$ the quotient space $\mathscr{H}^*/N$.

We now set out to define a space $\mathscr{B} \subset \mathscr{D}$, which will turn out to be the image of bn_Q in $\mathscr{D}$. Let $\mathscr{B}_0$ be the space of infinitely differentiable complex-valued functions, $g_0(x)$, on $\bar{C}$ such that there are positive constants A and B, varying with each g_0, for which

$$|\Delta^k g_0(x)| \leqq AB^k \frac{(2k)!}{k!}, \tag{2.4}$$

for $x \in \bar{C}$, $k = 0, 1, \cdots$.

Supposing B to be fixed, take $\mathscr{B}_0^B$ as the subspace of those functions which satisfy (2.4) for this B. A norm on this space is given by

$$\|g_0\|_0 = \sup_{x,k} \left\{ \frac{k!}{(2k)!\,B^k} |\Delta^k g_0(x)| \right\}, \tag{2.5}$$

which makes $\mathscr{B}_0^B$ a Banach space.

Let $\mathscr{B}_j$, $j = 1, \cdots, n$, be the space of infinitely differentiable complex-valued functions, $g_j(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n, t)$, on $\bar{C}_j \times [0, T]$, such that the g_j's are real analytic in all variables in a neighborhood of $t = 0$, are zero in a neighborhood of $t = T$, and satisfy for positive constants A and B, depending on g_j, the relation

$$\left| \left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j(x_{,j}, t) \right| \leqq AB^{2k}(2k)! \tag{2.6}$$

for $(x_{,j}, t) \in \bar{C}_j \times [0, T]$, $k = 0, 1, \cdots$, where

$$\Delta_j = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{j-1}^2} + \frac{\partial^2}{\partial x_{j+1}^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right),$$

and

$$x_{,j} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n).$$

Fixing B, we define $\mathscr{B}_j^B$ to be the space of all functions g_j in $\mathscr{B}_j$, which for this fixed B satisfy (2.6) and the additional requirements

$$|g_j^{(K(j),k)}(x_{,j}, t)| \leqq AB^{|K(j)|+k} K(j)!\,k! \tag{2.7}$$

for $(x_{,j}, t) \in \bar{C}_j \times [0, 1/B]$, all $K(j)$, $k = 0, 1, \cdots$,

and

$$g_j(x_{,j}, t) = 0 \tag{2.8}$$

on $\bar{C}_j \times (T - 1/B, T]$. The standard conventions for multi-indices are used here and in what follows. If $K = (K_1, K_2, \cdots, K_n)$ and $N = (N_1, N_2, \cdots, N_n)$ are multi-indices, then $K^N = K_1^{N_1} K_2^{N_2} \cdots K_n^{N_n}$, $K! = K_1!\,K_2! \cdots K_n!$, $|K| = K_1 + K_2 + \cdots + K_n$, and $\sum_{|K|=0}^{\infty} = \sum_{K_1=0}^{\infty} \sum_{K_2=0}^{\infty} \cdots \sum_{K_n=0}^{\infty}$. In (2.7),

$$g_j^{(K(j),k)} = \left(\frac{\partial}{\partial x_1} \right)^{K_1} \cdots \left(\frac{\partial}{\partial x_{j-1}} \right)^{K_{j-1}} \left(\frac{\partial}{\partial x_{j+1}} \right)^{K_{j+1}} \cdots \left(\frac{\partial}{\partial x_n} \right)^{K_n} \left(\frac{\partial}{\partial t} \right)^k g_j,$$

where $K(j) = (K_1, \cdots, K_{j-1}, K_{j+1}, \cdots, K_n)$. Condition (2.7) brings into account the analyticity of our functions in the vicinity of $\bar{C}_j \times \{0\}$. In condition (2.8) we have assumed T large enough so that $T - 1/B$ is positive (if this were not so we could use $T - 1/mB$ instead, with m a sufficiently large positive integer). We make $\mathscr{B}_j^B$ a Banach space with the norm

$$\|g_j\|_j = \sup_k \left\{ \sup_{(x,_j,t)\in\bar{C}_j\times[0,T]} \left[\frac{1}{(2k)!\,B^{2k}} \left| \left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j \right| \right] \right\} + \sup_{K(j),k} \left\{ \sup_{(x,_j,t)\in\bar{C}_j\times[0,1/B]} \left[\frac{1}{K(j)!\,k!\,B^{|K(j)|+k}} |g_j^{(K(j),k)}| \right] \right\}. \tag{2.9}$$

Take $\mathscr{B}$ as the subspace of $\mathscr{B}_0 \times \bigtimes_{j=1}^n (\mathscr{B}_j \times \mathscr{B}_j)$ made up of vectors $(g_0, g_1, g_1', \cdots, g_n, g_n')$ which satisfy the conditions

$$\frac{\partial^{2k} g_0(x,_j, 0)}{\partial x_j^{2k}} = \frac{\partial^{2k} g_0(x,_j, L_j)}{\partial x_j^{2k}} = 0, \qquad x,_j \in \bar{C}_j, \tag{2.10}$$

$$\frac{\partial^{2k+1} g_0(x,_j, 0)}{\partial x_j^{2k+1}} = (-1)^k \left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j(x,_j, 0), \qquad x,_j \in \bar{C}_j, \tag{2.11}$$

$$\frac{\partial^{2k+1} g_0(x,_j, L_j)}{\partial x_j^{2k+1}} = (-1)^{k+1} \left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j'(x,_j, 0), \qquad x,_j \in \bar{C}_j, \tag{2.12}$$

$$\left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j(x,_{j,l}, t) = \left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j'(x,_{j,l}, t) = 0, \qquad x,_{j,l} \in \bar{C}_{jl}, \tag{2.13}$$

$$\left(\frac{\partial}{\partial t} + \Delta_j \right)^k g_j(x,_{j,l}, t) = \left(\frac{\partial}{\partial t} + \Delta_j \right) g_j'(x'_{,j,l}, t) = 0, \qquad x'_{,j,l} \in \bar{C}_{jl}, \tag{2.14}$$

$$\frac{\partial^{2k} g_l(x,_{l,j}, t)}{\partial x_j^{2k}} = \frac{\partial^{2k} g_l'(x,_{l,j}, t)}{\partial x_j^{2k}} = 0, \qquad x,_{l,j} \in \bar{C}_{lj}, \quad l \neq j, \tag{2.15}$$

$$\frac{\partial^{2k} g_l(x'_{,l,j}, t)}{\partial x_j^{2k}} = \frac{\partial^{2k} g_l'(x'_{,l,j}, t)}{\partial x_j^{2k}} = 0, \qquad x'_{,l,j} \in \bar{C}'_{lj}, \quad l \neq j, \tag{2.16}$$

$$(-1)^k \left(\frac{\partial}{\partial t} + \Delta_j \right)^k \frac{\partial g_j(x,_{j,l}, t)}{\partial x_l} = \frac{\partial^{2k+1} g_l(x,_{l,j}, t)}{\partial x_j^{2k+1}}, \tag{2.17}$$

$$l \neq j, \quad x,_{j,l} \in \bar{C}_{jl}, \quad x,_{l,j} \in \bar{C}_{lj}, \quad (x,_{j,l})_m = (x,_{l,j})_m, \quad m = 1, \cdots, n, \quad m \neq j, l,$$

$$(-1)^k \left(\frac{\partial}{\partial t} + \Delta_j \right)^k \frac{\partial g_j(x'_{,j,l}, t)}{\partial x_l} = \frac{\partial^{2k+1} g_l'(x,_{l,j}, t)}{\partial x_j^{2k+1}}, \tag{2.18}$$

$$l \neq j, \quad x'_{,j,l} \in \bar{C}'_{jl}, \quad x,_{l,j} \in \bar{C}_{lj}, \quad (x'_{,j,l})_m = (x,_{l,j})_m, \quad m = 1, \cdots, n, \quad m \neq j, l,$$

$$(-1)^k \left(\frac{\partial}{\partial t} + \Delta_j \right)^k \frac{\partial g_j'(x,_{j,l}, t)}{\partial x_l} = \frac{\partial^{2k+1} g_l(x'_{,l,j}, t)}{\partial x_j^{2k+1}}, \tag{2.19}$$

$$l \neq j, \quad x,_{j,l} \in \bar{C}_{jl}, \quad x'_{,l,j} \in \bar{C}'_{lj}, \quad (x,_{j,l})_m = (x'_{,l,j})_m, \quad m = 1, \cdots, n, \quad m \neq j, l,$$

$$(-1)^k \left(\frac{\partial}{\partial t} + \Delta_j \right)^k \frac{\partial g_j'(x'_{,j,l}, t)}{\partial x_l} = \frac{\partial^{2k+1} g_l'(x'_{,l,j}, t)}{\partial x_j^{2k+1}}, \tag{2.20}$$

$$l \neq j, \quad x'_{,j,l} \in \bar{C}'_{jl}, \quad x'_{,l,j} \in \bar{C}'_{lj}, \quad (x'_{,j,l})_m = (x'_{,l,j})_m, \quad m = 1, \cdots, n, \quad m \neq j, l,$$

where in the above $j, l = 1, \cdots, n$, $k = 0, 1, \cdots$. Also, we have written $x = (x_1, \cdots, x_n)$ as $(x_{,j}, x_j)$ on the left side of (2.10), (2.11) and (2.12), and have taken $x_{,j,l} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{l-1}, 0, x_{l+1}, \cdots, x_n)$, $x'_{,j,l} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{l-1}, L_l, x_{l+1}, \cdots, x_n)$, $\bar{C}_{jl} = [0, L_1] \times \cdots \times [0, L_{j-1}] \times [0, L_{j+1}] \times \cdots \times [0, L_{l-1}] \times \{0\} \times [0, L_{l+1}] \times \cdots \times [0, L_n]$, and $\bar{C}'_{jl} = [0, L_1] \times \cdots \times [0, L_{j-1}] \times [0, L_{j+1}] \times \cdots \times [0, L_{l-1}] \times \{L_l\} \times [0, L_{l+1}] \times \cdots \times [0, L_n]$. The quantity $(\cdot)_m$ is the mth component of the designated vector.

Topologize $\mathscr{B}$ by considering those functions $\mathscr{B}^B$ belonging to

$$\mathscr{B}_0^B \times \bigtimes_{j=1}^{n} (\mathscr{B}_j^B \times \mathscr{B}_j^B)$$

and satisfying (2.10)–(2.20), for fixed B. These functions comprise a Banach space since they form a closed subset of $\mathscr{B}_0^B \times \bigtimes_{j=1}^{n}(\mathscr{B}_j^B \times \mathscr{B}_j^B)$ which is a Banach space when provided with the product topology. Now $\mathscr{B}^{B_1} \subset \mathscr{B}^{B_2}$ for $B_1 \leqq B_2$, and the injection from $\mathscr{B}^{B_1}$ into $\mathscr{B}^{B_2}$ is continuous. We write $\mathscr{B} = \bigcup_{B=1}^{\infty} \mathscr{B}^B$, and give $\mathscr{B}$ the inductive limit topology [20, pp. 514–515] which makes it an LF space.

The elements of $\mathscr{B}'$, the dual of $\mathscr{B}$, have no simple description. However, following the lead of [11, p. 334], we can give a representation which does not include all functionals in $\mathscr{B}'$, but which is sufficiently general to be worthwhile. Take $\dot{C}_j = [0, L_1] \times \cdots \times [0, L_{j-1}] \times \{0\} \times [0, L_{j+1}] \times \cdots \times [0, L_n]$ and $\dot{C}'_j = [0, L_1] \times \cdots \times [0, L_{j-1}] \times \{L_j\} \times [0, L_{j+1}] \times \cdots \times [0, L_n]$.

LEMMA 2.1. *Let $\{\mu_k^0\}$, $\{\mu_{kl}^j\}$, $\{\mu_{kl}'^j\}$, $\{\mu_{K(j)}\}$ and $\{\mu'_{K(j)}\}$ be sequences of Borel measures on $\bar{C}$, $\dot{C}_j \times [0, T]$, $\dot{C}'_j \times [0, T]$, $\bar{C}_j$, and $\bar{C}'_j$, respectively, with $j = 1, \cdots, n$, satisfying*

$$\operatorname{var}(\mu_k^0) = O\left(\frac{k!}{B^k(2k)!}\right) \qquad \textit{for all } B > 0,$$

$$\operatorname{var}(\mu_{kl}^j), \operatorname{var}(\mu_{kl}'^j) = O\left(\frac{1}{B^{2k}(2k)!\, e^{(T-1/B)l}}\right) \quad \textit{for all } B > 0,$$

$$\operatorname{var}(\mu_{K(j)}), \operatorname{var}(\mu'_{K(j)}) = O\left(\frac{1}{B^{|K(j)|}K(j)!}\right) \qquad \textit{for all } B > 0,$$

for all $k = 0, 1, \cdots$, all $K(j)$. Let $\{M_k^j\}$ and $\{M_k'^j\}$, $j = 1, \cdots, n$, be sequences of complex numbers which make the sums $\sum_{k=0}^{\infty} M_k^j z^k$ and $\sum_{k=0}^{\infty} M_k'^j z^k$ entire functions. If $g = (g_0, g_1, g_1', \cdots, g_n, g_n')$ is any element in $\mathscr{B}$, then L as given by

$$\begin{aligned} L(g) = {} & \sum_{k=0}^{\infty} \int_{\bar{C}} \Delta^k g_0 \, d\mu_k^0 + \sum_{j=1}^{n} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \int_{\dot{C}_j \times [0,T]} e^{lt}\left(\frac{\partial}{\partial t} + \Delta_j\right)^k g_j \, d\mu_{kl}^j \\ & + \sum_{j=1}^{n} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \int_{\dot{C}'_j \times [0,T]} e^{lt}\left(\frac{\partial}{\partial t} + \Delta_j\right)^k g_j' \, d\mu_{kl}' \\ & + \sum_{j=1}^{n} \sum_{|K(j)|=0}^{\infty} \int_{\bar{C}_j} \left[\sum_{k=0}^{\infty} \frac{M_k^j g_j^{(K(j),k)}(x_{,j}, 0)}{k!}\right] d\mu_{K(j)} \\ & + \sum_{j=1}^{n} \sum_{|K(j)|=0}^{\infty} \int_{\bar{C}_j} \sum_{k=0}^{\infty} \frac{M_k'^j g_j'^{(K(j),k)}(x_{,j}, 0)}{k!} \, d\mu'_{K(j)} \end{aligned} \tag{2.21}$$

is a continuous linear functional on $\mathscr{B}$.

The symbol j in μ^j_{kl}, M^j_k, etc., is a superscript, not an exponent. The lemma is easily verified and will not be proved here.

3. A trace theorem. Let the boundary trace, $\operatorname{tr}(u)$, of a function u in $\mathscr{D}(\bar{\Omega})$, be the map from $\mathscr{D}(\bar{\Omega})$ into $\mathscr{D}(\bar{C}) \times \bigtimes_{j=1}^n(\mathscr{D}(\bar{C}_j \times [0, T]) \times \mathscr{D}(\bar{C}_j \times [0, T]))$ given by

$$\operatorname{tr}(u) = (u(x, 0), u(x_{,1}, 0, t), u(x_{,1}, L_1, t), \cdots, u(x_{,n}, L_n, t)). \tag{3.1}$$

If duality is defined by

$$\langle \operatorname{tr}(u), g\rangle = \int_C u(x, 0)g_0(x)\,dx + \sum_{j=1}^n \int_{\bar{C}_j \times (0,T)} u(x_{,j}, 0, t)g_j\,\widehat{dx}(j)\,dt$$
$$+ \sum_{j=1}^n \int_{\bar{C}_j \times (0,T)} u(x_{,j}, L_j, t)g'_j\,\widehat{dx}(j)\,dt$$

for any $g = (g_0, g_1, g'_1, \cdots, g_n, g'_n) \in \mathscr{B}$, with $\widehat{dx}(j) = dx_1 \cdots dx_{j-1}\,dx_{j+1} \cdots dx_n$, then $\operatorname{tr}(u)$, for any $u \in \mathscr{D}(\bar{\Omega})$, is seen to be a continuous linear functional on $\mathscr{B}$.

THEOREM 3.1. *The map* tr *of* $\mathscr{D}(\bar{\Omega})$ *into* $\mathscr{D}(\bar{C}) \times \bigtimes_{j=1}^n(\mathscr{D}(\bar{C}_j \times [0, T]) \times \mathscr{D}(\bar{C}_j \times [0, T]))$ *can be extended by continuity to a continuous linear map of* $\mathscr{H}$ *into* $\mathscr{B}'$, *where* $\mathscr{H}$ *has the weak* $\sigma(\mathscr{H}, \mathscr{H}')$ *topology, and* $\mathscr{B}'$ *the weak* $\sigma(\mathscr{B}', \mathscr{B})$ *topology.*

We omit the proof of this. It is the same as the proof of Theorem 4.1, [11, p. 318]. However, we do present several results which are used in place of analogous results in [11]. For the proof of Theorem 3.1, the quantities bn_Q and $\mathscr{H}$ must be substituted in place of $\tau\cdot$ and Y which appear in [11].

LEMMA 3.1. *Let* $u_0(x)$ *satisfy*

$$|\Delta^k u_0(x)| \leqq AB^k \frac{(2k)!}{k!}, \tag{3.2}$$

$x \in D$, $k = 0, 1, \cdots$, *for positive* A *and* B, D *any compact subset of* R^n_x. *Then for some* $\eta > 0$, *there exists a unique analytic solution to the heat equation*, $Hu = 0$, *in* $D \times (-\eta, \eta)$, *which satisfies* $u(x, 0) = u_0(x)$ *on* D. *Conversely, if there exists an analytic solution* $u(x, t)$ *of the heat equation in* $D \times (-\eta, \eta)$ *for some* $\eta > 0$, *then* $u(x, 0)$ *satisfies* (3.2) *for* $x \in D$.

Proof. This is suggested by work of Goursat [5, § 541]. Pick $0 < \eta \leqq 1/(4B)$. The function

$$u(x, t) = \sum_{k=0}^{\infty} \frac{\Delta^k u_0(x)}{k!} t^k, \qquad x \in C, \quad -\eta < t < \eta, \tag{3.3}$$

is a solution of $Hu = 0$ in $D \times (-\eta, \eta)$, and satisfies $u(x, 0) = u_0(x)$ on $D \times \{0\}$. Convergence of the sum is a consequence of (2.18) and the bound on η. A C^∞, infinitely differentiable, function f is analytic in D if and only if for some positive constants A and B it satisfies the relation

$$|\Delta^k f(x)| \leqq AB^k(2k)! \tag{3.4}$$

for $x \in D$, $k = 0, 1, \cdots$ (this is easily derived with the help of Theorem 1 of Kotake and Narasimhan [9, p. 451]). By (3.2), u_0 fulfills condition (3.4) in D, and is thus analytic. This means that $u(x, t)$, as given by (3.3), is also analytic. The solution is unique by Holmgren's theorem.

Next let us assume that $u(x, t)$ is an analytic solution in $D \times (-\eta, \eta)$ for some $\eta > 0$. Then

$$|\Delta^k u(x, 0)| = \left|\frac{\partial^k}{\partial t^k} u(x, 0)\right| \leqq AB^k k! \leqq AB^k \frac{(2k)!}{k!},$$

for $x \in D$, and positive constants A and B, where the second line above results from the analyticity in t of u. The lemma is now proved.

Of the three lemmas that follow, the first two are needed only to prove the third. Proofs of these are either essentially the same (Lemmas 3.3 and 3.4), or suggested (Lemma 3.2) by proofs in [11] for the one-dimensional situation.

LEMMA 3.2. *The operator* bn_Q *maps* $\mathscr{H}^*_Q$ *onto* $\mathscr{B}$.

Proof. It suffices for the demonstration of our result to prove that bn maps $\mathscr{H}^*$ onto $\mathscr{B}$. First let us show that bn maps $\mathscr{H}^*$ onto $\mathscr{B}$.

For any $g \in \mathscr{H}^*$, we can take $g(x, T) = 0$, $x \in \bar{C}$, since g is zero in a neighborhood of T, and we also have $H^*g = g^*$ in Ω where $g^* \in \mathscr{D}(\Omega)$. Let us take $g^{*\prime}(x, t)$ as the function defined for all $x \in R^n$, which is periodic in each x_j with period $2L_j$, $j = 1, \cdots, n$, and which satisfies

$$g^{*\prime}(x, t) = g^*(x, t), \qquad x \in \bar{C},$$

$$g^{*\prime}(x, t) = -g^*(x_1, \cdots, x_{j-1}, -x_j, x_{j+1}, \cdots, x_n, t),$$

for $0 \leqq x_k \leqq L_k$, $k \neq j$, $-L_j \leqq x_j \leqq 0$, $j = 1, \cdots, n$. Consider the solution

$$(3.5)\quad g'(x, t) = -\frac{1}{\left|2\sqrt{\pi}\right|^n} \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{-|\xi - x|^2/[4(\tau - t)]\}}{(\tau - t)^{n/2}} g^{*\prime}(\xi, \tau)\, d\xi\, d\tau$$

of

$$H^*g' = g^{*\prime}, \qquad g'(x, T) = 0,$$

in $x \in R^n$, $t < T$, which is antisymmetric in x_j about $x_j = 0$, and in $L_j + x_j$ about $x_j = L_j$. It fulfills the requirement $g' = 0$ on $\partial C \times (0, T)$, and thus satisfies, by uniqueness, the relation

$$g'(x, t) = g(x, t), \qquad (x, t) \in \Omega.$$

We shall now consider the function g' in place of g. Note that g' is C^∞ since $g^{*\prime}$ is C^∞. Also, it is equal to zero in a neighborhood in t of $\bar{C} \times \{T\}$. Thus on each $\dot{C}_j \times [0, T]$ and $\dot{C}'_j \times [0, T]$, $j = 0, \cdots, n$, g' is C^∞, and it is zero in a neighborhood of $t = T$ for x in $\dot{C}_j$ and $\dot{C}'_j$.

In some neighborhood of $\dot{C}_j \times [0, T]$, $\partial g'/\partial x_j$, $j = 1, \cdots, n$, satisfies $H^*(\partial g'/\partial x_j) = 0$. Thus,

$$(3.6)\qquad \left|\left(\frac{\partial}{\partial t} + \Delta_j\right)^k \frac{\partial g'}{\partial x_j}\right| = \left|\frac{\partial^{2k}}{\partial x_j^{2k}} \frac{\partial g'}{\partial x_j}\right| \leqq AB^{2k}(2k)!,$$

for positive constants A and B, depending on $\partial g'/\partial x_j$. The inequality follows from the fact that such solutions satisfy

$$\left|\left(\frac{\partial g'}{\partial x_j}\right)^{(K,k)}\right| \leqq AB^{2k+2|K|}(2k)!$$

in $\dot{C}_j \times [0, T]$ for all K, $k = 0, 1, \cdots$, where $(2k)! = (2k_1)! \cdots (2k_n)!$ (see [19, Thm. 7.9, p. 446]). A similar result holds on $\dot{C}_j' \times [0, T], j = 1, \cdots, n$, for $-\partial g'/\partial x_j$.

In some neighborhood $R_x^n \times (-\eta, \eta)$, $\eta > 0$, of $R_x^n \times \{0\}$, g' is a solution of $H^*g' = 0$, and such a solution is analytic for all points (x, t) in the region. Thus, in respective neighborhoods of $t = 0$, $\partial g'/\partial x_j$ is analytic on $\dot{C}_j \times [0, T]$, and $-\partial g'/\partial x_j$ is analytic on $\dot{C}_j' \times [0, T]$, for $j = 1, \cdots, n$. From this and remarks in preceding paragraphs, we see that for $(x, t) \in \dot{C}_j \times [0, T]$, $\partial g'/\partial x_j$ belongs to $\mathscr{B}_j$; similarly for $-\partial g'/\partial x_j$, $(x, t) \in \dot{C}_j' \times [0, T]$.

Further, by virtue of Lemma 3.1, analyticity in $R^n \times (-\eta, \eta)$ implies that g' satisfies

$$|\Delta^k g'(x, 0)| \leqq AB^k \frac{(2k)!}{k!}, \tag{3.7}$$

for $x \in \bar{C}$, $k = 0, 1, \cdots$, and positive constants A and B. Thus, $g'(x, 0), x \in \bar{C}$, belongs to $\mathscr{B}_0$.

Next note that since g' satisfies the equation $H^*g' = 0$ in all the pertinent regions,

$$\frac{\partial^{2k} g'}{\partial x_j^{2k}} - (-1)^k \left(\frac{\partial}{\partial t} + \Delta_j\right)^k g' = 0 \quad \text{on } (\dot{C}_j \times \{0\}) \cup (\dot{C}_j' \times \{0\}), \tag{3.8}$$

$$\frac{\partial^{2k+1} g'}{\partial x_j^{2k+1}} = (-1)^{k+1} \left(\frac{\partial}{\partial t} + \Delta_j\right)^k \frac{\partial g'}{\partial x_j} \quad \text{on } \dot{C}_j \times \{0\}, \tag{3.9}$$

$$\frac{\partial^{2k+1} g'}{\partial x_j^{2k+1}} = (-1)^{k+1} \left(\frac{\partial}{\partial t} + \Delta_j\right)^k \left(-\frac{\partial g'}{\partial x_j}\right) \quad \text{on } \dot{C}_j' \times \{0\}, \tag{3.10}$$

$$\left(\frac{\partial}{\partial t} + \Delta_j\right)^k \frac{\partial g'}{\partial x_j} = (-1)^k \frac{\partial^{2k+1} g'}{\partial x_j^{2k+1}} = 0 \quad \text{on } D_{jl} \cup D_{jl}' \cup D_{lj}' \cup D_{jl}'', \quad j \neq l, \tag{3.11}$$

$$(-1)^k \left(\frac{\partial}{\partial t} + \Delta_j\right)^k \frac{\partial}{\partial x_l}\left(\frac{\partial g'}{\partial x_j}\right) = \frac{\partial^{2k+1}}{\partial x_j^{2k+1}}\left(\frac{\partial g'}{\partial x_l}\right) \quad \text{on } D_{jl} \cup D_{jl}' \cup D_{lj}' \cup D_{jl}'', j \neq l, \tag{3.12}$$

for $j, l = 1, \cdots, n$, $k = 0, 1, \cdots$, where we have taken $D_{jl} = [0, L_1] \times \cdots \times [0, L_{j-1}] \times \{0\} \times [0, L_{j+1}] \times \cdots \times [0, L_{l-1}] \times \{0\} \times [0, L_{l+1}] \times \cdots \times [0, L_n] \times [0, T]$, $D_{jl}' = [0, L_1] \times \cdots \times [0, L_{j-1}] \times \{L_j\} \times [0, L_{j+1}] \times \cdots \times [0, L_{l-1}] \times \{0\} \times [0, L_{l+1}] \times \cdots \times [0, L_n] \times [0, T]$, $D_{jl}'' = [0, L_1] \times \cdots \times [0, L_{j-1}] \times \{L_j\} \times [0, L_{j+1}] \times \cdots \times [0, L_{l-1}] \times \{L_l\} \times [0, L_{l+1}] \times \cdots \times [0, L_n] \times [0, T]$. Furthermore, since $\partial g'/\partial x_l$ is antisymmetric in x_j about $x_j = 0$, and antisymmetric in $L_j + x_j$ about $x_j = L_j$, $j \neq l$, we have that

$$\frac{\partial^{2k}}{\partial x_j^{2k}}\left(\frac{\partial g'}{\partial x_l}\right) = 0 \quad \text{on } D_{jl} \cup D_{jl}' \cup D_{lj}' \cup D_{jl}'', \qquad j \neq l, \tag{3.13}$$

for $j, l = 1, \cdots, n$.

Comparing (3.8)–(3.13) with (2.10)–(2.20), we then see that $\operatorname{bn}(g')$, and thus $\operatorname{bn}(g)$, belongs to $\mathscr{B}$. Hence, $\operatorname{bn}(\mathscr{H}^*) \subset \mathscr{B}$.

Next, we show that $\operatorname{bn}(\mathscr{H}^*) = \mathscr{B}$, or in other words, that given any $(g_0, g_1, g_1', \cdots, g_n, g_n') \in \mathscr{B}$, there is a $g(x, t) \in \mathscr{H}^*$ such that

$$g(x, 0) = g_0(x), \tag{3.14}$$

$$\frac{\partial}{\partial x_j} g(x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n, t) \tag{3.15}$$
$$= g_j(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n, t), \qquad j = 1, \cdots, n,$$

$$-\frac{\partial}{\partial x_j} g(x_1, \cdots, x_{j-1}, L_j, x_{j+1}, \cdots, x_n, t) \tag{3.16}$$
$$= g_j'(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n, t), \qquad j = 1, \cdots, n.$$

Let us construct such a $g(x, t)$. Assume each $g_j \in \mathscr{B}_j^{B_j}$, $g_j' \in \mathscr{B}_j^{B_j'}$. Set

$$G_1(x, t) = \sum_{k=0}^{\infty} (-1)^k \frac{(\partial/\partial t + \Delta_1)^k g_1(x_{,1}, t)}{(2k+1)!} x_1^{2k+1}, \tag{3.17}$$

for $(x, t) \in [0, 1/B_1) \times [0, L_2] \times \cdots \times [0, L_n] \times [0, T]$, and

$$G_2(x, t) = \sum_{k=0}^{\infty} (-1)^k \frac{(\partial/\partial t + \Delta_2)^k g_2(x_{,2}, t)}{(2k+1)!} x_2^{2k+1} \tag{3.18}$$

for $(x, t) \in [0, L_1] \times [0, 1/B_2) \times [0, L_3] \times \cdots \times [0, L_n] \times [0, T]$. These are each solutions of $H^*g = 0$ in their respective regions, with the first also satisfying $G_1(0, x_2, \cdots, x_n, t) = 0$, $(\partial G_1/\partial x_1)(0, x_2, \cdots, x_n, t) = g_1(x_{,1}, t)$, and the second satisfying $G_2(x_1, 0, x_3, \cdots, x_n, t) = 0$, $(\partial G_2/\partial x_1)(x_1, 0, x_3, \cdots, x_n, t) = g_2(x_{,2}, t)$. Note that since g_1 is analytic in some neighborhood of $t = 0$, as is g_2, there exist $\eta_1, \eta_2, 0 < \eta_1 \leqq 1/B_1, 0 < \eta_2 \leqq 1/B_2$, such that G_1 is analytic in $[0, 1/B_1) \times [0, L_2] \times \cdots \times [0, L_n] \times [0, \eta_1)$, and G_2 is analytic in $[0, L_1] \times [0, 1/B_2) \times [0, L_3] \times \cdots \times [0, L_n] \times [0, \eta_2)$.

In $[0, 1/B_1) \times \{0\} \times [0, L_3] \times \cdots \times [0, L_n] \times [0, T]$, using (2.13), we have

$$G_1(x_1, 0, x_3, \cdots, x_n, t) = 0,$$

and using (2.17) and (2.15), we have

$$\frac{\partial G_1}{\partial x_2}(x_1, 0, x_3, \cdots, x_n, t) = \sum_{k=0}^{\infty} (-1)^k \frac{(\partial/\partial t + \Delta_1)^k \partial g_1(x_{,1,2}, t)/\partial x_2}{(2k+1)!} x_1^{2k+1}$$
$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{\partial^{2k+1} g_2(x_{,2,1}, t)}{\partial x_1^{2k+1}} x_1^{2k+1} = g_2(x_{,2}, t).$$

Similarly, we have $G_2(0, x_2, x_3, \cdots, x_n, t) = 0$, $(\partial G_2/\partial x_1)(0, x_2, x_3, \cdots, x_n, t) = g_1(x_{,1}, t)$, in $\{0\} \times [0, 1/B_2) \times [0, L_3] \times \cdots \times [0, L_n] \times [0, T]$.

We can also write, using (2.10) and (2.11),

$$G_1(x, 0) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{\partial^{2k+1} g_0(x_{,1}, 0)}{\partial x_1^{2k+1}} x_1^{2k+1} = g_0(x), \tag{3.19}$$

for x in $[0, 1/B_1) \times [0, L_2] \times \cdots \times [0, L_n]$, and

$$G_2(x, 0) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{\partial^{2k+1} g_0(x_{,2}, 0)}{\partial x_2^{2k+1}} x_2^{2k+1} = g_0(x), \tag{3.20}$$

for x in $[0, L_1] = [0, 1/B_2) \times [0, L_3] \times \cdots \times [0, L_n]$. Thus, by uniqueness, G_1 and G_2 match up in $[0, 1/B_1) \times [0, 1/B_2) \times [0, L_3] \times \cdots \times [0, L_n] \times [0, T]$.

On the other sides of our domain, we construct similar solutions, $G_3, \cdots, G_n, G'_1, \cdots, G'_n$, which match up with each other in their common regions. Corresponding to these solutions are η_j's and η'_j's of the same nature as η_1 and η_2 above.

By Lemma 3.1, there also exists an analytic solution $G_0(x, t)$ in $C \times [0, \eta_0)$, for some $\eta_0 > 0$, which satisfies $G_0(x, t) = g_0(x)$. There exists an $\tilde{\eta}$ such that all the G_j and G'_j, $j = 1, \cdots, n$, are analytic for $t \in [0, \tilde{\eta})$. Recalling (3.19) and (3.20), using the uniqueness of our analytic solutions, and now defining G_0 only for $t \in [0, \hat{\eta})$, with $\hat{\eta} = \min(\eta_0, \tilde{\eta})$, we have that G_0 and G_j, G'_j match up in their common regions.

Define $\tilde{G}(x, t)$, $(x, t) \in \bar{\Omega}$, as some C^∞ function which is zero in $[\eta_1, L_1 - \eta'_1] \times \cdots \times [\eta_n, L_n - \eta'_n] \times [\hat{\eta}, T]$, and one in $\bar{\Omega} - (\eta_1/2, L_1 - \eta'_1/2) \times \cdots \times (\eta_n/2, L_n - \eta'_n/2) \times (\hat{\eta}/2, T)$. The function

$$g(x,t) = \begin{cases} \tilde{G}G_j & \text{in } [0, L_1] \times \cdots \times [0, L_{j-1}] \times [0, 1/B_j) \times [0, L_{j+1}] \\ & \qquad \times \cdots \times [0, L_n] \times [0, T], \\ \tilde{G}G'_j & \text{in } [0, L_1] \times \cdots \times [0, L_{j-1}] \times (L_j - 1/B_j, L_j] \\ & \qquad \times [0, L_{j+1}] \times \cdots \times [0, L_n] \times [0, T], \\ \tilde{G}G_0 & \text{in } C \times [0, \hat{\eta}], \\ 0, & \text{elsewhere in } \bar{\Omega}, \end{cases}$$

belongs to $\mathscr{H}^*$ and satisfies (3.14)–(3.16). Thus, the map bn is onto, and the lemma is proved.

LEMMA 3.3. *The operator* bn_Q *is continuous.*

Proof. By Theorem B.2 of [6, p. 17], the closed graph theorem holds for maps of $\mathscr{H}^*$ into $\mathscr{B}$. The map $\mathrm{bn}: \mathscr{H}^* \to C(\bar{C}) \times \bigtimes_{j=1}^n (C(\dot{C}_j \times [0, T]) \times C(\dot{C}'_j \times [0, T]))$ is continuous by the definition of $\mathscr{H}^*$. This leads to the result that the graph of bn is closed in $\mathscr{H}^* \times \mathscr{B}$, and it then follows that bn, and hence bn_Q, is continuous.

LEMMA 3.4. *The map* bn_Q^{-1} *is continuous.*

Proof. This follows directly from [6, Thm. B.1, p. 17] by virtue of the results in the last two lemmas.

In proving Theorem 3.1, we use the final result above, i.e., Lemma 3.4, and the following two additional pieces of information. The first is the fact that $\mathscr{D}(\bar{\Omega})$ is dense in $\mathscr{H}$. This follows by virtually the same proof as for [11, Prop. 4.1, p. 316]. The second is the Green's relation,

$$\begin{aligned} \int_\Omega (Hf)g\,dx\,dt - \int_\Omega f(Hg)\,dx\,dt = &-\int_C fg\,dx + \sum_{j=1}^n \int_{\dot{C}_j \times (0,T)} \frac{\partial f}{\partial x_j} g\,\widehat{dx}(j)\,dt \\ &- \sum_{j=1}^n \int_{\dot{C}'_j \times (0,T)} \frac{\partial f}{\partial x_j} g\,\widehat{dx}(j)\,dt \\ &- \sum_{j=1}^n \int_{\dot{C}_j \times (0,T)} \frac{\partial g}{\partial x_j}\,\widehat{dx}(j)\,dt \\ &+ \sum_{j=1}^n \int_{\dot{C}'_j \times (0,T)} f\frac{\partial g}{\partial x_j}\,\widehat{dx}(j)\,dt, \end{aligned} \tag{3.21}$$

where $f \in \mathscr{D}(\bar{\Omega})$, $g \in \mathscr{D}_-(\mathscr{D}(\bar{\Omega}))$.

4. The function $\mathscr{G}$. The solution $u(x, t)$ of the heat equation for Ω can be represented in the form

$$u(x,t) = \int_{\bar{C}} G\Big|_{\tau=0} u_0(\xi)\,d\xi + \sum_{j=1}^{n} \int_0^t \int_{\bar{C}_j} \frac{\partial G}{\partial \xi_j}\Big|_{\xi_j=0} u_j(\xi_{,j},\tau)\,\widehat{d\xi(j)}\,d\tau$$
$$- \sum_{j=1}^{n} \int_0^t \int_{\bar{C}_j} \frac{\partial G}{\partial \xi_j}\Big|_{\xi_j=L_j} u'_j(\xi_j,\tau)\,\widehat{d\xi(j)}\,d\tau, \tag{4.1}$$

where

$$u_0(\xi) = u(\xi, 0),$$
$$u_j(\xi_{,j},\tau) = u(\xi_1, \cdots, \xi_{j-1}, 0, \xi_{j+1}, \cdots, \xi_n, \tau),$$
$$u'_j(\xi_{,j},\tau) = u(\xi_1, \cdots, \xi_{j-1}, L_j, \xi_{j+1}, \cdots, \xi_n, \tau),$$

for u_0, u_j, u'_j continuous, and $u_0(\xi_{,j}, 0) = u_j(\xi_{,j}, 0)$, $u_0(\xi_{,j}, L_j) = u'_j(\xi_{,j}, 0)$, $u_j = u_l$ on $D_{jl} \times [0, T]$, $u_j = u'_l$ on $D'_{jl} \times [0, T]$, $u'_j = u'_l$ on D''_{jl}, $j \neq l$, $j, l = 1, \cdots, n$. (See [18, p. 528], for the discussion of this in the one-dimensional x-space case.)

In this expression, G is the Green's function

$$G(x, t; \xi, \tau) = \sum_{M_1, \cdots, M_n = 0}^{1} \sum_{K=-\infty}^{\infty} (-1)^{|M|} \Gamma(x, t; \zeta(M, K), \tau), \tag{4.2}$$

where Γ is the fundamental solution

$$\Gamma(x, t; \xi, \tau) = \frac{1}{\left|2\sqrt{\pi}\right|^n} (t - \tau)^{-n/2} \exp\left[-\frac{\sum_{j=1}^{n} (x_j - \xi_j)^2}{4(t - \tau)} \right], \tag{4.3}$$

and $\zeta(M, K) = (\zeta_1(M_1, K_1), \cdots, \zeta_n(M_n, K_n))$, for $M_j = 0, 1, K_j = -\infty, \cdots, +\infty$, and $\zeta_j(0, K_j) = 2K_jL_j + \xi_j$, $\xi_j(1, K_j) = 2K_jL_j - \xi_j$, $j = 1, \cdots, n$. The series defining G is seen to be convergent, when one uses the inequality

$$e^{-x} < p!/x^p, \tag{4.4}$$

$p = 0, 1, 2, \cdots, x > 0$, with $p = 1$. Indication of how this is done appears in the course of remarks below.

Let us make several comments about G. The first is that for fixed $(x, t) \in \Omega$, G is a solution, as a function of ξ and τ, of $H^*g = 0$, for $\xi \in R^n$, $\tau < t$. To see this, we note that it can be verified by direct substitution that each of the $\Gamma(x, t; \zeta(M, K), \tau)$ is such a solution for all M and K. We define

$$A(\xi_0, \eta, t_0) = \{(\xi, \tau) \| \xi_l - (\xi_0)_l| < \eta, l = 1, \cdots, n, \tau < t_0 < t\},$$

where $0 < \eta \leqq \min_l L_l$, and ξ_0 is any point in R^n with $(\xi_0)_l$ its lth component. Using (4.4) with $p = 1$, each $\Gamma(x, t; \zeta(M, K), \tau)$ can be seen to have bound $\alpha/K'(M)^2$. The quantity $K'(M)$ is the multi-index which has components $K'_l = K_l$, except when $K_l = 0$, or when K_l is an integer equaling $(x_l - \xi_l)/(2L_l)$ if $M_l = 0$, or equaling $(x_l + \xi_l)/(2L_l)$ if $M_l = 1$, for some ξ_l in our region (at most, one such K_l will exist for each of the values of M_l). In these latter cases, we take $K'_l = 1$. By $K'(M)^2$ we mean $(K'_1 \cdot K'_2 \cdots K'_n)^2$. Also in the above, α is a constant which depends on η and t_0. Because of the above bounds, the sum for G converges uniformly in any of the open sets $A(\xi_0, \eta, t_0)$. Similarly, the sums of each of the derivatives $\partial/\partial\xi_l$, $\partial^2/\partial\xi_l^2$, $l = 1, \cdots, n$, $\partial/\partial t$ of $\Gamma(x, t; \zeta(M, K), \tau)$ using (4.4) with appropriate choices of p, can be seen to converge uniformly in these same regions. Thus, G is a solution of $H^*g = 0$ in any $A(\xi_0, \eta, t_0)$, and hence in all of $\{(\xi, \tau) | \tau < t\}$.

This solution is obviously C^∞, and clearly all of its derivatives $(\partial/\partial\xi)^K(\partial/\partial\tau)^k G = (\partial/\partial\xi_1)^{K_1} \cdots (\partial/\partial\xi_n)^{K_n}(\partial/\partial\tau)^k G$, all K, $k = 0, 1, \cdots$, are also solutions, and C^∞. Furthermore, these derivatives are represented by the sums of the respective derivatives of the terms of the series (4.2), since they can be shown to be uniformly convergent.

Next, let us show that G, for a fixed $(x, t) \in \Omega$, is analytic in ξ and τ, for all ξ, and $\tau < t$. To see this, let us consider for a moment that ξ and τ are complex, writing $\xi = \xi^r + i\xi^i$, $\tau = \tau^r + i\tau^i$, and $\xi_l = \xi_l^r + \xi_l^i$, $l = 1, \cdots, n$. Define $A(\xi_0, \eta, \tilde{\eta}, t_0)$ as the open subset of $\mathbb{C}^{n+1}$, complex $(n+1)$-dimensional space, given by

$$A(\xi_0, \eta, \tilde{\eta}, t_0) = \{(\xi, \tau) \| \xi_l^r - (\xi_0)_l| < \eta, |\xi_l^i| < \tilde{\eta}, \tau^r < t_0, |\tau^i| < \tilde{\eta}, l = 1, \cdots, n\},$$

where η, ξ_0 and t_0 are as before, and $\tilde{\eta} = \min\{\min_l(\sqrt{t - t_0}/2)L_l, \min_l \frac{1}{2}L_l\}$. Looking upon $\Gamma(x, t; \zeta(M, K), \tau)$, now, as a function of complex rather than real ξ and τ, we see that $|\Gamma(x, t; \zeta(M, K), \tau)| \leqq \beta/K'(M)^2$, for fixed $(x, t) \in \Omega$, $(\xi, \tau) \in A(\xi_0, \eta, \tilde{\eta}, t_0)$. In the preceding, β is a positive constant depending on η, $\tilde{\eta}$ and t_0, and $K'(M)$ and $K'(M)^2$ are the same as before, only with ξ_l there replaced by ξ_l^r. Thus, in each of the open sets $A(\xi_0, \eta, \tilde{\eta}, t_0)$, the series for G is uniformly convergent. Further, each of the $\Gamma(x, t; \zeta(M, K), \tau)$ is complex analytic. As a uniformly convergent sum of analytic functions, G is a complex analytic function in each of the $A(\xi_0, \eta, \tilde{\eta}, t_0)$. Taking $\xi^i = 0$, $\tau^i = 0$, we see that for ξ and τ now considered real, G is analytic in $\{(\xi, \tau) \| \xi_l - (\xi_0)_l| < \eta,\ l = 1, \cdots, n,\ \tau < t_0\}$ for all ξ_0, and hence G is analytic for all $\xi \in R^n$, $\tau < t$. This further implies that all derivatives $(\partial/\partial\xi)^K(\partial/\partial\tau)^k G$, all K, $k = 0, 1, \cdots$, are also analytic for such ξ and τ.

Lastly, let us take $C(\eta) = \bigcup_{j=1}^n (C_j(\eta) \cup C_j'(\eta))$, where

$$\begin{aligned} C_j(\eta) = (-\eta, L_1 + \eta) \times \cdots \times (-\eta, L_{j-1} + \eta) \times (-\eta, \eta) \\ \times (-\eta, L_{j+1} + \eta) \times \cdots \times (-\eta, L_n + \eta), \end{aligned}$$

and

$$\begin{aligned} C_j'(\eta) = (-\eta, L_1 + \eta) \times \cdots \times (-\eta, L_{j-1} + \eta) \\ \times (L_j - \eta, L_j + \eta) \times (-\eta, L_{j+1} + \eta) \times \cdots \times (-\eta, L_n - \eta). \end{aligned}$$

For fixed $(x, t) \in \Omega$ and any η such that $0 < \eta < \max_l \{\max(|x_l|, |L_l - x_l|)\}$, we have that $(\partial/\partial\xi)^K(\partial/\partial\tau)^k G \to 0$, uniformly in $C(\eta)$, for all K, $k = 0, 1, \cdots$, as $\tau \uparrow t$. This is easily seen by looking at each $(\partial/\partial\xi)^K(\partial/\partial\tau)^k G$ and bounding the individual terms in the sums that represent these functions. Using (4.4) for appropriate choices of p, for any pair of (K, k), there will be some positive exponent $m(K, k)$, depending on K and k, such that $(\partial/\partial\xi)^K(\partial/\partial\tau)^k G = O((t - \tau)^{m(K,k)})$, uniformly in $C(\eta) \times \{\tau < t\}$, and this goes to zero as $\tau \uparrow t$. Hence, the function given by G in $C(\eta) \times \{\tau < t\}$, and by zero in $C(\eta) \times \{\tau \geqq t\}$, and all derivatives of this function, are solutions of $H^*g = 0$ in $C(\eta) \times R_\tau$.

If we define the function $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}'_1, \cdots, \mathscr{G}_n, \mathscr{G}'_n)$ by specifying

$$\mathscr{G}_0(x, t; \xi) = G(x, t; \xi, 0), \tag{4.5}$$

$$\mathscr{G}_j(x, t; \xi_{,j}, \tau) = \begin{cases} \dfrac{\partial G(x, t; \xi_{,j}, 0, \tau)}{\partial \xi_j}, & 0 \leqq \tau < t, \\ 0, & t \leqq \tau \leqq T, \end{cases} \tag{4.6}$$

$$\mathscr{G}'_j(x, t; \xi_{,j}, \tau) = \begin{cases} \dfrac{\partial G(x, t; \xi_{,j}, L_j, \tau)}{\partial \xi_j}, & 0 \leqq \tau < t, \\ 0, & t \leqq \tau \leqq T, \end{cases} \tag{4.7}$$

$j = 1, \cdots, n$, then (4.1) takes the form

$$\begin{aligned} u(x, t) = \int_{\bar{C}} \mathscr{G}_0 u_0(\xi)\, d\xi &+ \sum_{j=1}^{n} \int_0^T \int_{\bar{C}_j} \mathscr{G}_j u_j(\xi_{,j}, \tau)\, \widehat{d\xi}(j)\, d\tau \\ &- \sum_{j=1}^{n} \int_0^T \int_{\bar{C}_j} \mathscr{G}'_j u'_j(\xi_{,j}, \tau)\, \widehat{d\xi}(j)\, d\tau. \end{aligned} \tag{4.8}$$

By the remarks of the past several pages for G and its derivatives, employing Lemma 3.1, and making use of properties of solutions of $H^*g = 0$ in the pertinent regions, it is not hard to verify that we have the following lemma.

LEMMA 4.1. *For each fixed* $(x, t) \in \Omega$, $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}'_1, \cdots, \mathscr{G}_n, \mathscr{G}'_n)$ *is a member of* $\mathscr{B}$.

5. The representation theorem. Particularized to our situation, Malgrange [12, pp. 292–294], has proved the following theorem.

THEOREM 5.1. *The space of polynomial solutions of* $Hu = 0$ *in* Ω *is dense in* $\{u \in \mathscr{E}(\Omega) | Hu = 0\}$ *provided with the* $\mathscr{E}(\Omega)$ *topology.*

In the above $\mathscr{E}(\Omega)$ is the space of complex-valued infinitely differentiable functions in Ω with the usual Schwartz topology.

Let us define

$$\mathscr{H}_0 = \{u | u \in \mathscr{D}'(\Omega), Hu = 0\}. \tag{5.1}$$

Since all such distributional solutions of $Hu = 0$ in Ω are C^∞ in this region, elements of $\mathscr{H}_0$ can be considered as C^∞ functions in Ω. Further, the polynomial solutions in the theorem can be considered as $\mathscr{D}(\bar{\Omega})$ solutions of the equation. We thus have the following lemma.

LEMMA 5.1. $\mathscr{H}_0 \cap \mathscr{D}(\bar{\Omega})$ *is dense in* $\mathscr{H}_0$ *provided with the* $\mathscr{E}(\Omega)$ *topology.*

By (4.8), any $u(x, t)$ in $\mathscr{H}_0 \cap \mathscr{D}(\bar{\Omega})$, $(x, t) \in \Omega$, can be written

$$\begin{aligned} u(x, t) = \int_{\bar{C}} \mathscr{G}_0 u(\xi)\, d\xi &+ \sum_{j=1}^{n} \int_0^T \int_{\bar{C}_j} \mathscr{G}_j u(\xi_{,j}, 0, \tau)\, \widehat{d\xi}(j)\, d\tau \\ &+ \sum_{j=1}^{n} \int_0^T \int_{\bar{C}_j} \mathscr{G}'_j u(\xi_{,j}, L_j, \tau)\, \widehat{d\xi}(j)\, d\tau, \end{aligned} \tag{5.2}$$

or alternatively,

$$u(x, t) = \langle \mathrm{tr}\,(u), \mathscr{G}(x, t; \cdot, \cdot) \rangle, \tag{5.3}$$

with $\operatorname{tr}(u) = (u(\xi, 0), u(\xi_{,1}, 0, \tau), u(\xi_{,1}, L_1, \tau), \cdots, u(\xi_{,n}, L_n, \tau))$ considered a continuous linear functional on $\mathscr{B}$, and the brackets representing the duality relation between $\mathscr{B}'$ and $\mathscr{B}$. Let u be any member of $\mathscr{H}_0$ that has trace $\operatorname{tr}(u) \in \mathscr{B}'$ of form (2.21). By Lemma 5.1, corresponding to each such u, there is a sequence $\{u_k\} \subset \mathscr{H}_0 \cap \mathscr{D}(\bar{\Omega})$, such that $u_k(x, t) \to u(x, t)$ in $\mathscr{E}(\Omega)$, and

$$u_k(x, t) = \langle \operatorname{tr}(u_k), \mathscr{G}(x, t; \cdot, \cdot) \rangle. \tag{5.4}$$

Further, the fact that $u_k \to u$ in $\mathscr{E}(\Omega)$, implies that $u_k \to u$ in the $\sigma(\mathscr{H}, \mathscr{H}')$ topology. Thus, by Theorem 3.1, the right-hand side of (5.4) converges to $\langle \operatorname{tr}(u), \mathscr{G}(x, t; \cdot, \cdot) \rangle$, where $\operatorname{tr}(u)$ now has the form (2.21), and does not necessarily belong to $\mathscr{D}(\bar{C}) \times \bigtimes_{j=1}^{n} (\mathscr{D}(\bar{C}_j \times [0, T]) \times \mathscr{D}(\bar{C}_j \times [0, T]))$, as does u_k. Since the left-hand side of this same relation goes to $u(x, t)$, we then have

$$u(x, t) = \langle \operatorname{tr}(u), \mathscr{G}(x, t; \cdot, \cdot) \rangle$$

for $(x, t) \in \Omega$. By a result analogous to that given at the top of [11, p. 333], for any $\tilde{u} \in \mathscr{B}'$ of form (2.21), we have that there is a unique u in $\mathscr{H}_0$ for which $\operatorname{tr}(u) = \tilde{u}$. We have proved the following theorem.

THEOREM 5.2. *Let $u \in \mathscr{D}'(\Omega)$ be the unique solution of $Hu = 0$, with the initial-boundary condition prescribed by requiring that* $\operatorname{tr}(u)$ *be a particular functional in $\mathscr{B}'$ of form* (2.21) *for some choice of $\{u_k^0\}, \cdots, \{M_k'^n\}$. Then u can be represented as*

$$\begin{aligned} u(x, t) = {} & \sum_{k=0}^{\infty} \int_{\bar{C}} \Delta^k \mathscr{G}_0(x, t; \xi)\, d\mu_k^0(\xi) \\ & + \sum_{j=1}^{n} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \int_{\dot{C}_j \times [0,T]} e^{l\tau} \left(\frac{2}{\partial \tau} + \Delta_j \right)^k \mathscr{G}_j(x, t; \xi_{,j}, \tau)\, d\mu_{kl}^j(\xi_{,j}, \tau) \\ & + \sum_{j=1}^{n} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \int_{\dot{C}_j \times [0,T]} e^{l\tau} \left(\frac{\partial}{\partial \tau} + \Delta_j \right)^k \mathscr{G}_j'(x, t; \xi_{,j}, \tau)\, d\mu_{kl}'^j(\xi_{,j}, \tau) \\ & + \sum_{j=1}^{n} \sum_{|K(j)|=0}^{\infty} \int_{\bar{C}_j} \left[\sum_{k=0}^{\infty} \frac{M_k^j \mathscr{G}_j^{(K(j),k)}(x, t; \xi_{,j}, 0)}{k!} \right] d\mu_{K(j)}(\xi_{,j}) \\ & + \sum_{j=1}^{n} \sum_{|K(j)|=0}^{\infty} \int_{\bar{C}_j} \left[\sum_{k=0}^{\infty} \frac{M_k'^j \mathscr{G}_j'^{(K(j),k)}(x, t; \xi_{,j}, 0)}{k!} \right] d\mu_{K(j)}'(\xi_{,j}), \end{aligned} \tag{5.5}$$

where the notation is that defined previously, and the operations Δ, Δ_j, etc., are performed with respect to ξ, $\xi_{,j}$ and τ.

The above is the boundary-integral-boundary-sum representation we set out to derive.

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THE REDUCED WAVE EQUATION WITH A DIFFERENT RADIATION CONDITION*

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Abstract. An existence and uniqueness theorem is presented for an exterior Dirichlet problem for the reduced wave equation with a radiation condition different from the usual one. The techniques involve an inequality of Morawetz-Ludwig used in appropriate function spaces along with spherical harmonics.

1. Introduction. Let E be the domain in R^3 exterior to a finite smooth boundary ∂E which is star shaped with respect to the origin, i.e., for some $\beta > 0$, $x \cdot n \geqq \beta$ for any point x on ∂E and the normal n from ∂E to E at x. As usual, (r, θ, ϕ) denotes the spherical coordinate of a point x. We define

$$Du = \frac{\partial u}{\partial r} - iu + \frac{u}{r}.$$

The following lemma follows from Lemma 5 of [4].

LEMMA A. *Suppose u has continuous derivatives in $\bar{E} = E \cup \partial E$, vanishes on ∂E, and satisfies*

$$\lim_{R \to \infty} \int_{|x|=R} R|Du|^2 \, ds = 0. \tag{1}$$

Here ds is the surface element on the sphere $|x| = R$. Then

$$\frac{1}{2}\beta \int_{\partial E} \left|\frac{\partial u}{\partial n}\right|^2 ds + \int_E \left(|\nabla u|^2 - |u_r|^2 + \frac{1}{2}|Du|^2\right) dx \leqq 2\int_E r^2 |\Delta u + u|^2 \, dx.$$

We remark that with a slight modification of the proof of Lemma A in [4], we can replace (1) by the weaker condition,

$$\lim_{R_m \to \infty} \int_{|x|=R_m} R_m |Du|^2 \, ds = 0 \tag{2}$$

for some sequence R_m tending to infinity. In view of Lemma A, we introduce the function spaces A and R. A is the completion of the space of all infinitely differentiable functions u with compact support in E under the norm

$$\|u\|_A^2 = \int_E (|\nabla u|^2 - |u_r|^2 + |Du|^2) \, dx. \tag{3}$$

R is the space of all functions f in E which satisfy

$$\|f\|_R^2 = \int_E r^2 |f|^2 \, dx < \infty.$$

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It is clear from the definition of A that the functions u in A satisfy the radiation condition (2). In this paper we use Lemma A, the projection theorem and the spherical harmonics to obtain the following theorem.

THEOREM. *For each f in R, there exists a unique function u in A such that*

$$Lu = \Delta u + u = f.$$

There is a vast literature about solutions of the reduced wave equation with some other radiation conditions. For example, existence and uniqueness of solutions satisfying

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iu\right) = 0$$

can be found in [5], [6] and [7]. Recently, Levine [3] extended Rellich's uniqueness theorem to functions satisfying

$$\lim_{R\to\infty} \int_{|x|=R} \left|\frac{\partial u}{\partial r} - iu\right|^2 ds = 0 \tag{4}$$

in a region exterior to a regular closed surface; however, the sufficiency of the integral form (4) was made clear in Rellich's proof. Previously Wilcox explicitly used this integral form in [8] with stronger regularity conditions. Using Levine's result and the Fredholm's theorem, P. Wolfe [9] proved the existence of the solution with the radiation condition (4) in the two-dimensional case. The result of the paper is stated in three-dimensional space, but the proofs can be easily modified to work in R^n for $n \geqq 2$.

2. Characterizations of wave functions in R^3 and outgoing wave functions. In this note, we call u a wave function in a region G if u is infinitely differentiable and satisfies $Lu = \Delta u + u = 0$ in G. Let w be a continuously differentiable function in a neighborhood of infinity. w is defined to be outgoing if w satisfies (1).

LEMMA 1. *If u is outgoing and continuously differentiable in $|x| \geqq r_0$, then u satisfies*

$$\int_{r=R} |u|^2\, ds = o(R \log R).$$

Proof. Letting x and x_0 lie on the same ray from the origin, $r = |x|$ and $r_0 = |x_0|$, we have

$$ru(x)\, e^{-ir} - r_0 u(x_0)\, e^{-ir_0} = \int_{r_0}^{r} (tu\, e^{-it})_t\, dt.$$

Application of the Schwarz inequality and the triangle inequality yields

$$r^2|u(x)|^2 \leqq \frac{2}{3} r^3 \int_{r_0}^{r} |Du|^2\, dt + 2r_0^2|u(x_0)|^2$$

so that for R tending to infinity,

$$\int_{r=R} |u|^2\, ds \leqq R \int_{r_0 \leqq |x_0| \leqq R} |Du|^2\, dx + 8\pi r_0^2|u(x)|^2$$
$$= o(R \log R)$$

since

$$\int_{r_0 \leqq |x| \leqq R} |Du|^2 \, dx = o(\log R)$$

because u is outgoing.

LEMMA 2. *Every wave function u defined in $|x| \geqq a$ for some $a > 0$ can be decomposed uniquely into a sum $u = u_1 + u_2$ such that u_1 is a wave function in R^3 and u_2 is an outgoing wave function.*

Proof. Let $k(x, y) = e^{i|x-y|}/4\pi|x - y|$. We can obtain from Green's identity that, for any given x such that $|x| > a$, and b chosen so that $|x| < b$,

$$\begin{aligned} u(x) &= \int_{|y|=b} \left(u\frac{\partial k}{\partial n} - k\frac{\partial u}{\partial n} \right) ds - \int_{|y|=a} \left(u\frac{\partial k}{\partial n} - k\frac{\partial u}{\partial n} \right) ds \\ &= u_1 + u_2 . \end{aligned}$$

It is easy to see that u_1 is a wave function in the whole space since u_1 is independent of b and u_2 is outgoing. To prove the uniqueness of the decomposition, we let u be an outgoing wave function in R^3. Then we have, for any $p > 0$,

$$\begin{aligned} u &= \int_{|y|=p} \left(u\frac{\partial k}{\partial n} - k\frac{\partial u}{\partial n} \right) ds \\ &= \int_{|y|=p} (uDk - kDu) \, ds . \end{aligned} \tag{5}$$

By virtue of Lemma 1, we conclude that u vanishes identically by letting p tend to infinity in (5).

The following lemma can be regarded as a version of the Appendix in [2].

LEMMA 3. *Let u be a wave function defined in $|x| \geqq a \geqq 0$. Then*:

(i) *u is outgoing if and only if*

$$\sqrt{r}\, u = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} a_{m,j} H^{(1)}_{m+1/2}(r) Y_{m,j}(\theta, \phi), \tag{6}$$

where $H_k(r)$ is the Hankel function of the k-th order of first kind, $\{Y_{m,j}(\theta, \phi)\}$ are the spherical harmonics.

(ii) *u can be defined as a wave function in R^3 if and only if*

$$\sqrt{r}\, u = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} b_{m,j} J_{m+1/2}(r) Y_{m,j}(\theta, \phi), \tag{7}$$

where $J_k(r)$ is the Bessel function of order k.

Proof. We begin by proving the "if" part of (i) and (ii). As u is smooth in $|x| \geqq a$, we can expand u in terms of spherical harmonics there:

$$u = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} c_{m,j}(r) Y_{m,j}(\theta, \phi).$$

It is easy to verify that $\sqrt{r}\, c_{m,j}(r)$ satisfies Bessel's equation of order $m + 1/2$. Hence we have

$$\sqrt{r}\, c_{m,j}(r) = a_{m,j} H^{(1)}_{m+1/2}(r) + b_{m,j} J_{m+1/2}(r), \qquad r \geqq a, \tag{8}$$

for some constants $a_{m,j}$ and $b_{m,j}$. If u is outgoing, we have from the orthogonality of $Y_{m,j}$ that

$$\lim_{r\to\infty} r^3|Dc_{m,j}(r)|^2 \leqq \lim_{r\to\infty} \sum_{m,j} r^3|Dc_{m,j}(r)|^2$$
$$= \lim_{r\to\infty} \int_{|x|=r} r|Du|^2\, ds = 0.$$

As $H^{(1)}_{m+1/2}$ is outgoing and $J_{m+1/2}(r)$ behaves like $[\sin(m+1/2)r]/r$ near infinity, we can conclude from (8) that $b_{m,j}(r)$ vanishes. On the other hand, if u is smooth everywhere, we can similarly conclude that $a_{m,j} = 0$ since $H^{(1)}_{m+1/2}(r)$ is singular at the origin. To prove the "only if" part, we use Lemma 2 to decompose u into a sum of u_1 and u_2 such that u_1 is a wave function in R^3 and u_2 is an outgoing wave function. From the "if" part of this lemma, we have

$$\sqrt{r}\, u_1 = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} \beta_{m,j} J_{m+1/2}(r) Y_{m,j}(\theta,\phi)$$

and

$$\sqrt{r}\, u_2 = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} \alpha_{m,j} H^{(1)}_{m+1/2}(r) Y_{m,j}(\theta,\phi).$$

Now if (6) holds, we can conclude for $r \geqq a$ that

$$a_{m,j} H^{(1)}_{m+1/2}(r) = \alpha_{m,j} H^{(1)}_{m+1/2}(r) + \beta_{m,j} J_{m+1/2}(r).$$

By comparing the behavior of the Bessel function $J_{m+1/2}$ and the Hankel function $H^{(1)}_{m+1/2}$ near infinity, we obtain $\beta_{m,j} = 0$ and hence $u = u_2$ is outgoing. Using a similar argument, we can prove (ii). The proof is complete.

3. Proof of the theorem. Let u belong to A and satisfy $Lu = 0$. As L is elliptic, u belongs to $c^\infty(\bar{E})$ (cf. [1, Thm. 8.2]). In view of the definition of A we can find a sequence R_n tending to infinity such that

$$\lim_{R_n\to\infty} \int_{|x|=R_n} R_n|Du|^2\, ds = 0.$$

Hence it follows from Lemma A that u vanishes identically. This completes the proof of the "uniqueness" part.

To prove the existence of the solution, we let $A_1 = \{u | u \in c^\infty(\bar{E}) \cap A\}$ and $L(A_1) = \{Lu | u \in A_1\}$. We assert that $L(A_1) \cap R$ is dense in R. Otherwise there exists $g \neq 0$ in R such that

$$\int_E r^2 g(Lu) = 0 \tag{9}$$

for all $u \in c^\infty(\bar{E}) \cap A$ with $\|\Delta u + u\|_R$ finite. In particular, (9) holds for all $u \in c^\infty(\bar{E})$ such that u equals zero on ∂E and vanishes for large $|x|$. It follows again from the theory of elliptic equations (cf. [1, Thm. 8.2]) that $r^2 g \in c^\infty(\bar{E})$ and

$$L(r^2 g) = \Delta(r^2 g) + r^2 g = 0. \tag{10}$$

Thus we can expand $r^2 g$ in terms of spherical harmonics in $r \geqq a$ for some a:

$$r^2 g = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} a_{m,j}(r) Y_{m,j}(\theta, \phi). \tag{11}$$

Let $h_m(r)$ equal $H^{(1)}_{m+1/2}(r)/\sqrt{r}$ and $U_{m,j}(r, \theta, \phi)$ equal $h_m(r) Y_{m,j}(\theta, \phi)$. Since $L(r^2 g) = L(U_{m,j}) = 0$, we can conclude by using Green's identity that, for $a \leqq R_1 \leqq R_2$,

$$\begin{aligned} 0 &= \int_{R_1 \leqq |x| \leqq R_2} [U_{m,j} L(r^2 g) - r^2 g L(U_{m,j})] \, dx \\ &= \int_{|x| = R_2} \left[U_{m,j} \frac{\partial r^2 g}{\partial r} - r^2 g \frac{\partial U_{m,j}}{\partial r} \right] ds \\ &\quad - \int_{|x| = R_2} \left[U_{m,j} \frac{\partial r^2 g}{\partial r} - r^2 g \frac{\partial U_{m,j}}{\partial r} \right] ds \\ &= [a_{m,j}(R_2) h'_m(R_2) - h_m(R_2) a'_{m,j}(R_2)] \\ &\quad - [a_{m,j}(R_1) h'_m(R_1) - h_m(R_2) a'_{m,j}(R_1)]. \end{aligned} \tag{12}$$

Next let $w_{m,j}(r, \theta, \phi) = h(r) U_{m,j}$, with $h(r)$ in c^∞ and vanishing in a neighborhood of ∂E and equal to one in a neighborhood of infinity. From (9) and (10) we can obtain, by using Green's identity,

$$\begin{aligned} 0 &= \int_E [r^2 g(L w_{m,j}) - w_{m,j} L(r^2 g)] \, dx \\ &= \lim_{R \to \infty} [a_{m,j}(R) h'_m(R) - h_m(R) a'_{m,j}(R)]. \end{aligned} \tag{13}$$

Combining (12) and (13), we have

$$a_{m,j}(R) = b_{m,j} h_m(R) \quad \text{for } R \geqq a.$$

By virtue of Lemma 3, we can conclude that $r^2 g$ is an outgoing wave function. Finally, letting u be any smooth function in $\bar{E}$ such that u is equal to zero on ∂E and vanishes near infinity, we have from (9) and (10) that

$$\begin{aligned} 0 &= \int_E [r^2 g(Lu) - u L(r^2 g)] \, dx \\ &= \int_{\partial E} r^2 g \frac{\partial u}{\partial n} \, ds \end{aligned}$$

which implies that $r^2 g = 0$ on ∂E. By virtue of Lemma A, $r^2 g$ must be identically zero. This contradicts our choice of g. So $L(A_1) \cap R$ is dense in R and for any f in R we can find a sequence u_n in A such that Lu_n tends to f in R. We can then apply Lemma A to conclude that u_n converges to some u in D such that $Lu = f$. The proof is complete.

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AN EXISTENCE THEOREM FOR ABEL INTEGRAL EQUATIONS*

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Abstract. An existence and smoothness theorem is given for the Abel integral equation $\int_0^s K(s,t)f(t)(s^p - t^p)^{-\alpha}\,dt = g(s)$, $0 < s \leqq T$, with given $p > 0$ and $0 < \alpha < 1$. Particular attention is given to the behavior of $g(s)$ and $f(s)$ about $s = 0$.

1. Introduction. Consider the Abel integral equation

$$\int_0^s \frac{K(s,t)f(t)\,dt}{(s^p - t^p)^\alpha} = g(s), \qquad 0 < s \leqq T, \tag{1.1}$$

with given $p > 0$ and $0 < \alpha < 1$. To avoid degeneracy, we shall assume $K(s,s) \neq 0$ for $0 \leqq s \leqq T$. This is a classical equation, and it is obtained from a variety of mathematical and physical problems; see the bibliography of Noble [7].

In the past this equation has been examined case by case (for example, see Schmeidler [8] and the references in [7]). The methods of analysis were usually constructive or explicit, and the numerical analysis of (1.1) was usually based on these methods. Within the last few years, direct numerical methods for (1.1) have been proposed and studied (see [1]–[6], [10], [11]). These are general numerical methods which depend only on the smoothness of $K(s,t)$ and $f(t)$. As a complementary study to the numerical analysis of (1.1), we give a result on the existence and smoothness of solutions.

We shall need some special function spaces. For $\gamma > -1$, let us define

$$\mathscr{X}_\gamma = \{s^\gamma f(s) \mid f \in C[0,T]\},$$
$$\mathscr{X} = \bigcup_{\gamma > -1} \mathscr{X}_\gamma.$$

It can easily be seen that if $\gamma < \delta$, then $\mathscr{X}_\gamma \supset \mathscr{X}_\delta$. The space is much, but not all, of $L(0,T) \cap C(0,T]$.

THEOREM. *Let $g(s)$ have the form*

$$g(s) = s^\beta \tilde{g}(s), \quad 0 < s \leqq T, \quad \tilde{g} \in C^{n+1}[0,T], \tag{1.2}$$

for some integer $n \geqq 0$. Let β satisfy

$$p\alpha + \beta > 0. \tag{1.3}$$

Assume $K(s,t)$ is $n + 2$ times continuously differentiable for $0 \leqq t \leqq s \leqq T$, and furthermore,

$$K(s,s) \neq 0, \qquad 0 \leqq s \leqq T. \tag{1.4}$$

Then there is a unique solution $f \in \mathscr{X}$ of (1.1), and its form is

$$f(s) = s^{p\alpha+\beta-1}[a + sl(s)] \equiv s^{p\alpha+\beta-1}\tilde{f}(s), \qquad s > 0, \tag{1.5}$$

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with $l \in C^n[0, T]$. *The constant* $a = 0$ *if and only if* $\tilde{g}(0) = 0$. (Note that the special form of $\tilde{f}(s)$ implies the existence of $\tilde{f}^{(n+1)}(0)$.) *Finally, there is a constant* $d_n > 0$, *independent of* $\tilde{g} \in C^n[0, T]$, *for which*

$$\max\{\|\tilde{f}\|, \cdots, \|\tilde{f}^{(n)}\|\} \leqq d_n \max\{\|\tilde{g}\|, \cdots, \|\tilde{g}^{(n+1)}\|\}. \tag{1.6}$$

The norm is the max norm on $[0, T]$.

In § 2 we give some standard results for $K(s, t) \equiv 1$. In § 3, we introduce a decomposition of (1.1) and prove some preliminary results about it. The proof of the theorem is given in § 4.

The theorem is true for systems as well. Let $K(s, t)$ be an $m \times m$ matrix, and let f and g be m-component column vectors. Condition (1.4) is replaced by

$$\det K(s, s) \neq 0, \qquad 0 \leqq s \leqq T;$$

all smoothness statements generalize immediately. The proof given in § 3 and § 4 generalizes by merely replacing absolute values by appropriate vector and matrix norms.

2. The Abel transform. Define the Abel transform by

$$\mathscr{A}n(s) = \int_0^s \frac{h(t)\,dt}{(s^p - t^p)^\alpha}, \qquad 0 < s \leqq T, \quad h \in L^1(0, T) \cap C(0, T].$$

(See Sneddon [9] for some properties and uses of the transform.) We give the needed properties of $\mathscr{A}$ in the following lemma.

LEMMA. *Consider the equation*

$$\int_0^s \frac{f(t)\,dt}{(s^p - t^p)^\alpha} = s^\beta \tilde{g}(s), \qquad 0 < s \leqq T, \tag{2.1}$$

with $\tilde{g} \in C^{n+1}[0, T]$, *for some* $n \geqq 0$ *and* $p\alpha + \beta > 0$. *Then there is a unique solution* $f \in L^1(0, T) \cap C(0, T]$ *and its form is*

$$f(s) = s^{p\alpha+\beta-1}[a + sk(s)] \equiv s^{p\alpha+\beta-1}\tilde{f}(s), \tag{2.2}$$

with $k \in C^n[0, T]$ *and* $a \equiv$ const. *Moreover, for some constant* d,

$$\|\tilde{f}^{(n)}\| \leqq d \max\{\|\tilde{g}\|, \cdots, \|\tilde{g}^{(n+1)}\|\}. \tag{2.3}$$

Proof. The inverse of $\mathscr{A}$ is given by

$$\mathscr{A}^{-1}h(s) = \frac{p \sin(\alpha\pi)}{\pi} \frac{d}{ds} \int_0^s \frac{r^{p-1}h(r)\,dr}{(s^p - r^p)^{1-\alpha}}, \qquad s > 0. \tag{2.4}$$

Using this and a change of the variable of integration, we obtain (2.2) with

$$\begin{aligned} a &= \frac{p \sin(\alpha\pi)}{\pi}(p\alpha + \beta)\tilde{g}(0) \int_0^1 \frac{u^{p+\beta-1}\,du}{(1 - u^p)^{1-\alpha}}, \\ k(s) &= \frac{p \sin(\alpha\pi)}{\pi} \int_0^1 \frac{u^{p+\beta}}{(1 - u^p)^{1-\alpha}}\left[\tilde{g}'(us) + (p\alpha + \beta)\frac{\tilde{g}(us) - \tilde{g}(0)}{us}\right] du. \end{aligned} \tag{2.5}$$

The remaining results follow from the formula for $k(s)$.

3. Preliminary definitions and results. Define

$$\mathscr{K}h(s) = \int_0^s \frac{K(s,t)f(t)\,dt}{(s^p - t^p)^\alpha}, \qquad 0 < s \leqq T, \quad h \in L^1(0, T) \cap C(0, T].$$

To simplify some formulas, we assume, without loss of generality, that

$$K(s, s) = 1, \qquad 0 \leqq s \leqq T.$$

Assuming $K_2(s, s) = \partial K(s, t)/\partial t|_{t=s}$ exists, let us define

$$H(s,t) = \begin{cases} \dfrac{K(s,s) - K(s,t)}{s - t}, & s > t, \\[2ex] \dfrac{\partial K(s,t)}{\partial t}, & s = t; \end{cases}$$

$$\mathscr{H}h(s) = \int_0^s \frac{H(s,t)(s-t)^{1-\alpha}h(t)\,dt}{[(s^p - t^p)/(s -)]^\alpha}, \qquad 0 < s \leqq T.$$

Then

(3.1) $$\mathscr{K} = \mathscr{A} - \mathscr{H}.$$

To solve $\mathscr{K}f = g$, equation (1.1), we solve the problem

(3.2) $$\mathscr{A}z = g, \quad f - \mathscr{A}^{-1}\mathscr{H}f = z.$$

To examine the existence and smoothness of f, we shall need a formula for $\mathscr{A}^{-1}\mathscr{H}$. An especially useful one is

(3.3) $$\begin{aligned} \mathscr{A}^{-1}\mathscr{H}h(s) = \frac{sp\sin(\alpha\pi)}{\pi}\Bigg\{&\int_0^1 \frac{u^{p-1}}{(1-u^p)^{1-\alpha}}\int_0^u \frac{h(ws)(u-w)^{1-\alpha}}{[(u^p - w^p)/(u-w)]^\alpha}[p\alpha H(us, ws) \\ &\qquad + usH_1(us, ws)]\,dw\,du \\ &+ \int_0^1 \frac{u^p}{(1-u^p)^{1-\alpha}}\int_0^u \frac{H(us, ws)h(ws)}{(u^p - w^p)^\alpha} \\ &\qquad \cdot \left[1 - \frac{\alpha p w^{p-1}(u-w)}{u^p - w^p}\right] dw\,du\Bigg\}, \end{aligned}$$

which is valid for all $h \in \mathscr{X}$. To obtain it, we take a specific form for h, say $h(s) = s^\gamma \tilde{h}(s)$, for some $\gamma > -1$, $\tilde{h} \in C[0, T]$. Substituting this into $\mathscr{H}h(s)$, we make a change of variable, and note the behavior of $\mathscr{H}h(s)$ about $s = 0$. We substitute this into (2.4), and then perform much algebraic manipulation to obtain (3.3). Note that we need the existence of the partial derivative $H_1(s, t)$, which follows from the fact that $K(s, t)$ is twice continuously differentiable.

We also need a number of special inequalities. From the identity

$$\frac{1 - s^p}{1 - s} = p\int_0^1 [1 - (1 - s)r]^{p-1}\,dr, \qquad 0 \leqq s < 1, \quad p > 0,$$

we obtain

$$\min\{1,p\} \leqq \frac{1-s^p}{1-s} \leqq \max\{1,p\}, \qquad 0 \leqq s < 1, \quad p > 0. \tag{3.4}$$

From the estimate

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}\, e^{-x+\theta/12x}, \qquad x > 0, \quad 0 < \theta < 1,$$

we obtain

$$\frac{\Gamma(x)}{\Gamma(x+\lambda)} \leqq \frac{\gamma(x)}{(x+\lambda)^{\lambda}}, \qquad x > 0, \quad 0 \leqq \lambda \leqq 1, \tag{3.5}$$

with

$$\gamma(x) = \left(1 + \frac{1}{x}\right) e^{1+1/12x},$$

a monotone decreasing function of x on $(0, \infty)$.

Define

$$\begin{aligned} A(l) &= \int_0^1 \frac{u^{p-1}}{(1-u^p)^{1-\alpha}} \int_0^u w^l(u-w)^{1-\alpha}\left[\frac{u-w}{u^p-w^p}\right]^{\alpha} dw\, du, \\ B(l) &= \int_0^1 \frac{u^p}{(1-u^p)^{1-\alpha}} \int_0^u \frac{w^l\, dw}{(u^p-w^p)^{\alpha}} du, \qquad l > -1. \end{aligned} \tag{3.6}$$

We use the change of variable $w = uv$, $0 \leqq v \leqq 1$, the bounds (3.4), and some manipulation to reduce (3.6) to new formulas involving beta functions. We evaluate these and then bound them, using (3.5), to obtain eventually

$$A(l) \leqq \frac{C_A(l)}{(l+2-\alpha)^2}, \quad B(l) \leqq \frac{C_B(l)}{(l+2-\alpha)}, \qquad l > -1, \tag{3.7}$$

with $C_A(l)$ and $C_B(l)$ monotone decreasing functions on $(-1, \infty)$.

4. Proof of theorem. The proof is divided into several parts.

(i) *Existence and uniqueness of solution $f \in \mathscr{X}$.* Recall the statement of the theorem. It is easily seen that if either $\mathscr{K}f = g$ or formulation (3.2) has a unique solution for a g of form (1.2), then so does the other. We shall use (3.2).

Let $\mathscr{A}z = g$. By the lemma,

$$z(s) = s^{p\alpha+\beta-1}[a + sk(s)] \equiv s^{p\alpha+\beta-1}\tilde{z}(s), \qquad k, \tilde{z} \in C^n[0, T]. \tag{4.1}$$

To show the unique solvability in $\mathscr{X}$ of $(I - \mathscr{A}^{-1}\mathscr{K})f = z$, we shall show that

$$I - \mathscr{A}^{-1}\mathscr{K} : \mathscr{X} \xrightarrow[\text{onto}]{1\text{-}1} \mathscr{X}.$$

This will be shown by proving that

$$I - \mathscr{A}^{-1}\mathscr{K} : \mathscr{X}_{\gamma} \xrightarrow[\text{onto}]{1\text{-}1} \mathscr{X}_{\gamma} \quad \text{for all } \gamma > -1. \tag{4.2}$$

From (4.2) and (4.1), we shall also have

$$f(s) = s^{p\alpha+\beta-1}\tilde{f}(s), \qquad \tilde{f} \in C[0, T]. \tag{4.3}$$

To prove that (4.2) holds, we begin by looking at (3.3) with $h(s) = s^{\gamma}\tilde{h}(s)$. Then

$$\begin{aligned}\mathscr{A}^{-1}\mathscr{H}h(s) = \frac{ps^{\gamma+1}\sin(\alpha\pi)}{\pi}\Bigg\{&\int_0^1 \frac{u^{p-1}}{(1-u^p)^{1-\alpha}}\int_0^u \frac{w^{\gamma}\tilde{h}(ws)(u-w)^{1-\alpha}}{[(u^p-w^p)/(u-w)]^{\alpha}}\\ &\cdot[p\alpha H(us, ws) + usH_1(us, ws)]\,dw\,du\\ &+\int_0^1 \frac{u^p}{(1-u^p)^{1-\alpha}}\int_0^u \frac{H(us, ws)w^{\gamma}\tilde{h}(ws)}{(u^p-w^p)^{\alpha}}\left[1 - \frac{\alpha p w^{p-1}(u-w)}{u^p-w^p}\right]dw\,du\Bigg\}\\ \equiv s^{\gamma+1}\tilde{y}(s), \qquad &\tilde{y} \in C[0, T].\end{aligned} \tag{4.4}$$

Thus

$$\mathscr{A}^{-1}\mathscr{H} : \mathscr{X}_{\gamma} \to \mathscr{X}_{\gamma+1}, \qquad \gamma > -1, \tag{4.5}$$

and this proves that $I - \mathscr{A}^{-1}\mathscr{H}$ maps $\mathscr{X}_{\gamma}$ into $\mathscr{X}_{\gamma}$.

Let $z \in \mathscr{X}_{\gamma}$ for some $\gamma > -1$, $z(s) = s^{\gamma}\tilde{z}(s)$. We shall show the existence of $f \in \mathscr{X}_{\gamma}$ with $(I - \mathscr{A}^{-1}\mathscr{H})f = z$ by looking at the Neumann series for the equation. Define

$$f_j = [\mathscr{A}^{-1}\mathscr{H}]^j z, \quad \tilde{f}_j(s) = s^{-\gamma}f_j(s), \qquad s \geqq 0, \quad j = 0, 1, 2, \cdots.$$

By induction, using (4.5), we have $f_j \in \mathscr{X}_{\gamma}$ for all $j \geqq 0$, and thus $\tilde{f}_j \in C[0, T]$. We shall show that

$$\tilde{f}(s) \equiv \sum_0^{\infty} \tilde{f}_j(s) \tag{4.6}$$

converges uniformly on $[0, T]$. It will follow by standard arguments for Neumann series that $f(s) \equiv s^{\gamma}\tilde{f}(s)$ is a solution of $(I - \mathscr{A}^{-1}\mathscr{H})f = z$. We shall discuss uniqueness later.

Let M be a bound on $|H(s, t)|$ and $|H_1(s, t)|$ for $0 \leqq t \leqq s \leqq T$. As an induction hypothesis, assume that for $i \leqq j$,

$$|\tilde{f}_i(s)| \leqq D_i s^i, \qquad 0 \leqq s \leqq T. \tag{4.7}$$

This is easily seen to be true for $j = 0$ since $\tilde{f}_0 \equiv \tilde{z}$; use $D_0 = \|\tilde{z}\|$. Assuming the hypothesis for a general j, we shall use (4.4) to prove it for $j + 1$. Since $\tilde{f}_{j+1}(s) = s^{-\gamma}[\mathscr{A}^{-1}\mathscr{H}f_j](s)$, from (4.4), (4.7), (3.4), and (3.6) it follows that

$$\begin{aligned}|\tilde{f}_{j+1}(s)| \leqq s^{j+1}\frac{p\sin(\alpha\pi)}{\pi}MD_j\Bigg\{&(p\alpha + T)A(j+\gamma)\\ &+\left(1 + p\alpha\max\left\{1, \frac{1}{p}\right\}\right)B(j+\gamma)\Bigg\}.\end{aligned}$$

From (3.7), with $r = \gamma + 2 - \alpha > 0$, we obtain

(4.8) $$|\tilde{f}_{j+1}(s)| \leqq D_{j+1}s^{j+1}, \quad D_{j+1} = \frac{C_0 D_j}{j+r},$$

$$C_0 = \frac{p \sin(\alpha\pi)}{\pi} M\left\{(p\alpha + T)\frac{C_A(\gamma)}{1-\alpha} + (1 + p\alpha \max\{1, 1/p\})C_B(\gamma)\right\}.$$

The constant C_0 is independent of $j \geqq 0$. Also, the induction is completed.

Using (4.8) and $D_0 = \|\tilde{z}\|$, we obtain

(4.9) $$|\tilde{f}_j(s)| \leqq \Gamma(r)\|\tilde{z}\|\frac{C_0^j s^j}{\Gamma(r+j)}, \qquad j \geqq 0, \quad 0 \leqq s \leqq T.$$

For the series (4.6),

(4.10) $$|\tilde{f}(s)| \leqq \Gamma(r)\|\tilde{z}\| \sum_0^\infty \frac{C_0^j s^j}{\Gamma(r+j)}.$$

This converges uniformly on $[0, T]$, and thus $\tilde{f}(s)$ is continuous.

To prove the uniqueness in $\mathscr{X}_\gamma$ of the previously constructed f, let us assume that

$$y - \mathscr{A}^{-1}\mathscr{K}y = 0, \quad y(s) = s^\gamma \tilde{y}(s), \qquad \tilde{y} \in C[0, T].$$

Then

$$y = [\mathscr{A}^{-1}\mathscr{K}]^j y, \qquad j \geqq 0.$$

Applying the same kind of derivation as that used to obtain (4.8), with $\tilde{z}$ replaced by $\tilde{y}$, we obtain

$$|\tilde{y}(s)| \leqq \Gamma(r)\|\tilde{y}\|\frac{C_0^j s^j}{\Gamma(r+j)}, \qquad 0 \leqq s \leqq T, \quad j \geqq 0.$$

It follows that $\tilde{y} \equiv 0$, and thus $y \equiv 0$.

We combine (4.9) with (2.3) of the lemma to obtain the stability result (1.6) for the case $n = 0$. The proof of the remaining part of (1.5) is given later.

(ii) *Case* $n = 1$. We shall show that each $\tilde{f}_j \in C^1[0, T]$ and that

(4.11) $$\sum_0^\infty \tilde{f}_j'(s)$$

converges uniformly on $[0, T]$. It then follows by standard arguments that $\tilde{f} \in C^1[0, T]$ and that $\tilde{f}'(s)$ equals the series (4.11).

From $\tilde{f}_0 = \tilde{z}_0$ and (4.1), we have that $\tilde{f}_0 \in C^1[0, T]$. By induction on j using (4.4), it follows that $\tilde{f}_j \in C^1[0, T]$ for all j. For a second induction, assume that for $1 \leqq i \leqq j$,

$$|\tilde{f}_i'(s)| \leqq D_i^{(1)} s^{i-1}, \qquad 0 \leqq s \leqq T.$$

This is true for $j = 1$ since $\tilde{f}_1'(s)$ is continuous. Let us assume it for general j, and use (4.8) and the derivative $\tilde{f}'_{j+1}(s)$ from (4.4) to obtain

$$|\tilde{f}'_{j+1}(s)| \leqq C_1\left[\frac{C_1^j\Gamma(r)}{\Gamma(r+j)} + \frac{D_j^{(1)}}{r+j+1}\right]s^j = D_{j+1}^{(1)}s^j, \tag{4.12}$$

with $C_1 = \text{const.}$, $C_1 \geqq C_0$. The induction is completed. Also choose C_1 large enough to ensure that

$$|\tilde{f}_1(s)| \leqq C_1 \max\{\|\tilde{z}\|, \|\tilde{z}'\|\}.$$

From (4.12), it follows that

$$|\tilde{f}'_j(s)| \leqq \gamma_1(j)\frac{C_1^{j-1}s^{j-1}}{\Gamma(r+j-1)}\max\{\|\tilde{z}\|, \|\tilde{z}'\|\},$$

with $\gamma_1(j)$ a linear polynomial in j, for $j \geqq 1$, $0 \leqq s \leqq T$. From this it follows that the series of (4.11) converges uniformly, concluding the proof. The stability result (1.6) follows as with $n = 0$.

(iii) *A brief sketch of the general case.* Let us assume that the result has been proven for $n \leqq m - 1$ and let us prove it for $n = m$. As part of the induction, we assume that

$$|\tilde{f}_j^{(n)}(s)| \leqq \gamma_n(j)\frac{C_1^{j-n}s^{j-n}}{\Gamma(r+j-n)}\max\{\|\tilde{z}\|, \cdots, \|\tilde{z}^{(n)}\|\} \tag{4.13}$$

for $j \geqq n$, $0 \leqq n \leqq m-1$, $0 \leqq s \leqq T$, with $\gamma_n(j)$ a polynomial in j of degree $\leqq n$. To prove the theorem for $n = m$, let us form the mth derivative of $\tilde{f}_{j+1}(s)$ using (4.4) and Leibniz's rule. Then we proceed exactly as with the case $n = 1$. The many details are omitted.

(iv) *The special form of* (1.5). Since $\tilde{f} \in C^n[0, T]$, we use $f(s) = s^{p\alpha+\beta-1}\tilde{f}(s)$ and (4.4) to obtain

$$\mathscr{A}^{-1}\mathscr{H}f(s) = s^{p\alpha+\beta}l(s), \qquad l \in C^n[0, T].$$

Using $f = z + \mathscr{A}^{-1}\mathscr{H}f$, formula (4.1), and the preceding equality we obtain

$$f(s) = s^{p\alpha+\beta-1}[a + s(k(s) + l(s))],$$

the desired form. From (2.5), it is seen that the constant $a = 0$ if and only if $\tilde{g}(0) = 0$.

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ANALYTIC REPRESENTATION OF THE DISTRIBUTIONAL FINITE FOURIER TRANSFORM*

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Abstract. We define n-dimensional finite Fourier transforms for functions and distributions which are mappings from $\mathscr{K}(A)$ to $\mathscr{Z}(\overline{2\pi})$ and $\mathscr{K}'(A)$ to $\mathscr{Z}'(\overline{2\pi})$, respectively, where A is an arbitrary n-tuple of positive real numbers. Representation theorems are obtained for the distributional finite Fourier transform in which we relate this transform to entire analytic functions in $\mathbb{C}^n$. A finite convolution is defined, and we construct from it the finite regularization of an element of $\mathscr{K}'(A)$ which is used to give another representation of the distributional finite Fourier transform. We show that the Fourier transform mapping $\mathscr{K}'$ to $\mathscr{Z}'$ can be obtained as the limit of a sequence of distributional finite Fourier transforms. Further, we give necessary and sufficient conditions for the distributional finite Fourier transform to be represented as the boundary value of a function which is analytic in the tubular radial domain $T^C = \mathbb{R}^n + iC$, C being an open connected cone; and we use these results to obtain the analytic decomposition of the distributional finite Fourier transform.

1. Introduction. In this paper we shall define an n-dimensional distributional finite Fourier transform, n being an arbitrary positive integer, which has as a special case a similar transform of Warmbrod [10]; and we shall study properties and representations of it. In particular we are interested in representing this transform as the boundary value of a function which is analytic in a subset of $\mathbb{C}^n$. More precisely we shall obtain necessary and sufficient conditions on a function which is analytic in a tubular radial domain of $\mathbb{C}^n$ such that the analytic function has the distributional finite Fourier transform as a boundary value in the distributional sense on the distinguished boundary of the tube domain. Using the necessary condition we then obtain the analytic decomposition of the distributional finite Fourier transform.

In § 2 we shall introduce the notation and definitions to be used in this paper. We shall define the n-dimensional distributional finite Fourier transform in § 3 and discuss its properties and some representations of it. We define the finite convolution of distributions in § 4 and use it to introduce the notion of finite regularization, which we in turn use to obtain a representation theorem of the distributional finite Fourier transform. In § 5 we shall show that the distributional Fourier transform on $\mathscr{K}'$ can be represented as a limit of the distributional finite Fourier transform, while § 6 will be devoted to the analytic representation results.

2. Notation and definitions. The notation to be used in this paper will be similar to that used in Carmichael [2], [3]. x, y, t, u will be points of n-dimensional Euclidean space $\mathbb{R}^n$, while z will be a point of n-dimensional complex space $\mathbb{C}^n$. We define $\langle x, t\rangle = x_1t_1 + \cdots + x_nt_n$ with a similar definition for $\langle t, z\rangle$, $t \in \mathbb{R}^n$, $z \in \mathbb{C}^n$. Let α denote an n-tuple of nonnegative integers. We define $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\alpha! = \alpha_1! \cdots \alpha_n!$. The differential operator D^α is defined by $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_j = (1/2\pi i)\partial/\partial t_j$ or $D_j = (1/2\pi i)\partial/\partial z_j$, $j = 1, \cdots, n$. We put D_t^α or D_z^α to distinguish between differentiating with respect to $t \in \mathbb{R}^n$ or $z \in \mathbb{C}^n$ whenever there

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is a possibility of confusion. We similarly define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $x \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$. We adopt the convention that for $b \in \mathbb{R}^1$, $\bar{b}$ will denote the n-tuple $(b, \cdots, b)$. If $A = (a_1, \cdots, a_n)$ is an n-tuple of positive real numbers, S_A will denote the set $S_A = \{t \in \mathbb{R}^n : |t_j| \leqq a_j, j = 1, \cdots, n\}$ throughout the paper.

The Fourier transform for L^1 functions $\phi(t)$ is

$$\hat{\phi}(x) = \mathscr{F}[\phi(t); x] = \int_{\mathbb{R}^n} \phi(t) \exp[2\pi i\langle x, t\rangle]\, dt.$$

The inverse Fourier transform of $\phi \in L^1$, denoted $\mathscr{F}^{-1}[\phi(t); x]$, is similarly defined with $-2\pi i\langle x, t\rangle$ replacing $2\pi i\langle x, t\rangle$ as the exponent of the exponential function.

We assume familiarity on the part of the reader with the test spaces of functions $\mathscr{E}$, $\mathscr{K}$ (called $\mathscr{D}$ by L. Schwartz [7]), $\mathscr{S}$ and $\mathscr{Z}$ and their dual spaces of generalized functions $\mathscr{E}'$, $\mathscr{K}'$ (i.e., $\mathscr{D}'$), $\mathscr{S}'$ and $\mathscr{Z}'$. (See Gel'fand and Shilov [4] and Schwartz [7].) In particular we call the readers attention to the notion of convergence in $\mathscr{K}$, which can be found in Schwartz [7, p. 25] or Gel'fand and Shilov [4, p. 2], and in $\mathscr{Z}$, which can be found in Gel'fand and Shilov [4, p. 158]. Further we recall that the Fourier transform is a continuous one-to-one mapping of $\mathscr{K}$ onto $\mathscr{Z}$, and this fact permits us to define the Fourier transform of $U \in \mathscr{K}'$ as that element $V \in \mathscr{Z}'$ for which

$$\langle V, \psi\rangle = \langle U, \phi\rangle, \quad \phi \in \mathscr{K}, \quad \psi = \hat{\phi} \in \mathscr{Z}.$$

(See Gel'fand and Shilov [4, Chap. 2].) We then write the Fourier transform of $U \in \mathscr{K}'$ as $V = \mathscr{F}[U]$. The Fourier and inverse Fourier transforms are both continuous linear one-to-one mappings of $\mathscr{S}$ onto $\mathscr{S}$ with the same being true of $\mathscr{S}'$ under the definitions given by Schwartz [7, Chap. 7]. For all terminology concerning generalized functions, such as support, and all definitions of operations on generalized functions, such as differentiation and convolution, we refer to Schwartz [7]. Throughout this paper the support of a function $\phi(t)$ or of a generalized function U will be denoted by $\operatorname{supp}(\phi)$ or $\operatorname{supp}(U)$, respectively.

In § 6 of this paper we shall need the following information concerning cones and tube domains. A set $C \subset \mathbb{R}^n$ is a cone with vertex at the origin if $y \in C$ implies $\lambda y \in C$ for all positive scalars λ. The intersection of a cone C with the unit sphere $|y| = 1$ is called the projection of C and is denoted $\operatorname{pr}(C)$. A cone C' for which $\operatorname{pr}(\overline{C'}) \subset \operatorname{pr}(C)$ will be called a compact subcone of C. The function

$$u_C(t) = \sup_{y \in \operatorname{pr}(C)} (-\langle t, y\rangle)$$

is the indicatrix of the cone C. $O(C)$ will denote the convex envelope of C. $T^C = \mathbb{R}^n + iC$, where C is an open connected cone, will be called a tubular radial domain.

Let C be an open connected cone. Let $f(z)$ be a function of $z \in T^C$, and let U be a generalized function. By $f(z) \to U$ in the topology (i.e., weak topology) of the generalized function space as $y = \operatorname{Im}(z) \to 0$ (i.e., $y_j \to 0$, $j = 1, \cdots, n$), $y \in C$, we mean $\langle f(z), \phi(x)\rangle \to \langle U, \phi(x)\rangle$ as $y \to 0$, $y \in C$, where ϕ is an element of the appropriate function space. U is then called the generalized function boundary value of $f(z)$. We note that this boundary value is attained on the distinguished boundary of T^C, $\{z = x + iy : x \in \mathbb{R}^n, y = (0, \cdots, 0)\}$, which is not necessarily the topological boundary of T^C.

The exponential type of an entire analytic function $f(z)$, $z \in \mathbb{C}^n$, is defined in Gel'fand and Shilov [5, p. 80] or Schwartz [7, p. 271]. We recall that the characteristic function $\lambda(t)$ of a set $S \subset \mathbb{R}^n$ is defined to be $\lambda(t) = 1, t \in S$, and $\lambda(t) = 0, t \notin S$.

3. Distributional finite Fourier transform. Let $A = (a_1, \cdots, a_n)$ be a fixed n-tuple of real numbers such that $a_j > 0, j = 1, \cdots, n$. $\mathscr{K}(A)$ is the space of all infinitely differentiable complex-valued functions which have support in

$$S_A = \{t \in \mathbb{R}^n : |t_j| \leqq a_j, j = 1, \cdots, n\}.$$

We define the finite Fourier transform of an element $\phi \in \mathscr{K}(A)$, denoted $\mathscr{F}_A[\phi(t); x]$, by

$$\mathscr{F}_A[\phi(t); x] = \frac{1}{a_1 \cdots a_n} \int_{S_A} \phi(t) \exp\left[2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dt. \tag{1}$$

Because of the compact support of ϕ, we see immediately that $\psi(x) = \mathscr{F}_A[\phi(t); x]$ is an infinitely differentiable complex-valued function which can be extended to $\mathbb{C}^n$ to be the entire analytic function

$$\psi(z) = \frac{1}{a_1 \cdots a_n} \int_{S_A} \phi(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt, \quad z \in \mathbb{C}^n. \tag{2}$$

Let α be an arbitrary n-tuple of nonnegative integers. Integrating by parts in (2) we obtain

$$\psi(z) = A^{\alpha - \bar{1}}(2\pi i)^{-|\alpha|} z^{-\alpha} \int_{S_A} D^\alpha \phi(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt. \tag{3}$$

From (3) we obtain the existence of a constant K_α such that

$$|z^\alpha \psi(z)| \leqq K_\alpha \exp[2\pi(|y_1| + \cdots + |y_n|)], \qquad y = \operatorname{Im}(z).$$

Now let $B = (b_1, \cdots, b_n)$. $\mathscr{Z}(B)$ is the space of infinitely differentiable complex-valued functions $\psi(x)$, $x \in \mathbb{R}^n$, which can be extended to be entire analytic functions $\psi(z)$, $z \in \mathbb{C}^n$, such that for every n-tuple α of nonnegative integers there exists a constant K_α for which

$$|z^\alpha \psi(z)| \leqq K_\alpha \exp[b_1|y_1| + b_2|y_2| + \cdots + b_n|y_n|], \qquad y = \operatorname{Im}(z), \quad z \in \mathbb{C}^n. \tag{4}$$

We thus have shown above that the finite Fourier transform defined in (1) maps $\mathscr{K}(A)$ into $\mathscr{Z}(\overline{2\pi})$, where, as we noted in § 2, $\overline{2\pi}$ denotes the n-tuple $(2\pi, \cdots, 2\pi)$. Let us now show that this mapping is onto. Let $\psi \in \mathscr{Z}(\overline{2\pi})$, and consider

$$\phi(t) = \int_{\mathbb{R}^n} \psi(x) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dx. \tag{5}$$

With z restricted to $\mathbb{R}^n$ we have from (4) that the function $\phi(t)$ defined in (5) exists and is an infinitely differentiable function of $t \in \mathbb{R}^n$. Changing the variable of integration in (5) we obtain

$$(6)\qquad \frac{\phi(t)}{a_1\cdots a_n} = (2\pi)^{-n}\int_{\mathbb{R}^n}\psi\left(\frac{a_1x_1}{2\pi},\cdots,\frac{a_nx_n}{2\pi}\right)e^{-i\langle t,x\rangle}\,dx.$$

Now $\psi(x)\in\mathscr{Z}(\overline{2\pi})$. Thus $\psi(a_1x_1/(2\pi),\cdots,a_nx_n/(2\pi))\in\mathscr{Z}(A)$, and by a result of Gel'fand and Shilov [4, pp. 153–158], we have from (6) that $\operatorname{supp}(\phi)\subseteq S_A$. Thus $\phi(t)$ defined by (5) is an element of $\mathscr{K}(A)$. Since $\phi\in\mathscr{K}(A)\subseteq L^1\cap L^2$ and $\psi\in\mathscr{Z}(\overline{2\pi})\subseteq L^1\cap L^2$, then from (5) or (6) and the Plancherel theory of Fourier transforms we further have that $\psi(x)=\mathscr{F}_A[\phi(t);x]$. We thus have shown that the finite Fourier transform maps $\mathscr{K}(A)$ onto $\mathscr{Z}(\overline{2\pi})$, and it is easily seen that this mapping is also one-to-one and linear. We note that if $\psi(x)=\mathscr{F}_A[\phi(t);x]$, $\phi\in\mathscr{K}(A)$, then ϕ has the representation (5) in terms of ψ, and (5) can be thought of as the inverse finite Fourier transform mapping $\mathscr{Z}(\overline{2\pi})$ onto $\mathscr{K}(A)$.

We now introduce notions of convergence on the spaces $\mathscr{K}(A)$ and $\mathscr{Z}(B)$. A topology may be defined on $\mathscr{K}(A)$ by the countable set of norms

$$\|\phi\|_m = \sup_{\substack{x\\ |\alpha|\leqq m}}|D^\alpha\phi(x)|,\qquad m=0,1,2,3,\cdots,$$

under which $\mathscr{K}(A)$ becomes a locally convex topological vector space. With this topology, we say that a sequence $\{\phi_\nu\}$, $\phi_\nu\in\mathscr{K}(A)$, converges to zero in $\mathscr{K}(A)$ as $\nu\to\nu_0$ if $\{D^\alpha\phi_\nu\}$ converges to zero uniformly on S_A as $\nu\to\nu_0$ for any n-tuple α of nonnegative integers. (See [5, pp. 77–78].)

We define a topology on the space $\mathscr{Z}(B)$ by using the countable set of norms

$$\|\psi\|_m = \sup_{\substack{z\\ |\alpha|\leqq m}}|z^\alpha\psi(x)|\exp[-b_1|y_1|-b_2|y_2|-\cdots-b_n|y_n|],\qquad y=\operatorname{Im}(z),$$

$m=0,1,2,3,\cdots$, under which $\mathscr{Z}(B)$ becomes a locally convex topological vector space. With this topology we say a sequence $\{\psi_\nu\}$, $\psi_\nu\in\mathscr{Z}(B)$, converges to zero in $\mathscr{Z}(B)$ as $\nu\to\nu_0$ if for any n-tuple α of nonnegative integers

$$|z^\alpha\psi_\nu(z)|\leqq K_\alpha\exp[b_1|y_1|+\cdots+b_n|y_n|]$$

for all $z\in\mathbb{C}^n$ and all ν (i.e., $B=(b_1,\cdots,b_n)$ and K_α are independent of ν) and if the sequence $\{\psi_\nu(x)\}$ converges uniformly to zero as $\nu\to\nu_0$ on every bounded set in $\mathbb{R}^n$. (See [5, p. 81].)

Under these topologies, $\mathscr{K}(A)$ and $\mathscr{Z}(B)$ are complete. Further, we now easily see that the finite Fourier transform defined in (1) is a topological vector space isomorphism mapping $\mathscr{K}(A)$ to $\mathscr{Z}(\overline{2\pi})$.

We now denote the spaces of continuous linear functionals defined on $\mathscr{K}(A)$ and $\mathscr{Z}(B)$ by $\mathscr{K}'(A)$ and $\mathscr{Z}'(B)$, respectively. We define the distributional finite Fourier transform of $U\in\mathscr{K}'(A)$, denoted $\mathscr{F}_A[U]$, to be that element $V\in\mathscr{Z}'(\overline{2\pi})$ such that

$$(7)\qquad \langle V,\psi\rangle = \frac{1}{a_1\cdots a_n}\langle U,\phi\rangle,\qquad \phi\in\mathscr{K}(A),\quad \psi=\mathscr{F}_A[\phi(t);x]\in\mathscr{Z}(\overline{2\pi}),$$

for all $\psi\in\mathscr{Z}(\overline{2\pi})$. We see that (7) is motivated by a Parseval relation using (1) and (5) in exactly the same way that Gel'fand and Shilov motivated their definition of Fourier transform on $\mathscr{K}'$ (see [4, p. 166].) Further, it follows from the properties of the finite Fourier transform mapping $\mathscr{K}(A)$ to $\mathscr{Z}(\overline{2\pi})$ (i.e., it is an isomorphism) and (7) that the distributional finite Fourier transform is a continuous linear one-to-one mapping of $\mathscr{K}'(A)$ onto $\mathscr{Z}'(\overline{2\pi})$.

Let us note that there is a slight difference in the distributional finite Fourier transform defined by Warmbrod [10] and the restriction of (7) above to one dimension in that Warmbrod's transform is defined on a slightly larger space of generalized functions. This difference follows from the fact that the test space on which Warmbrod defines the function finite Fourier transform [10, p. 931] is the space of infinitely differentiable functions which have support in the open interval $(-a, a)$; whereas in the restriction of the space $\mathscr{K}(A)$ of the present paper to one dimension, the elements have support in the closed interval $[-a, a]$. The techniques we have used in this section (§ 3) to obtain our distributional finite Fourier transform could be applied in exactly the same way to obtain an exact generalization to n dimensions of the transform defined by Warmbrod. However, for the results we wish to obtain in this paper, the difference is inconsequential; the results we obtain and the techniques involved in proving them are essentially the same for either. Taking $\mathscr{K}(A)$ as the test space of the distributions whose finite Fourier transform we have defined in (7) in no way restricts the results we obtain, and we shall study the distributional finite Fourier transform as we have defined it here.

We shall now obtain some representations of the distributional finite Fourier transform. If $U \in \mathscr{K}'(A)$, a result of Schwartz [8, p. 82] (see also Gel'fand and Shilov [5, pp. 111–114]) states that $U = D^\alpha f(t)$, where $f(t)$ is a continuous function. Using this fact we can prove the following representation theorem.

THEOREM 1. *Let $U \in \mathscr{K}'(A)$. Then $\mathscr{F}_A[U] = g(x)$, where $g(x)$ is a continuous function which increases as some power of $|x|$ and which can be extended to be an entire analytic function of exponential type $\leqq 2\pi$.*

Proof. If $U \in \mathscr{K}'(A)$, then from (7) there exists an element $V \in \mathscr{Z}'(\overline{2\pi})$ such that $V = \mathscr{F}_A[U]$. Let $\phi \in \mathscr{K}(A)$ and $\psi(x) = \mathscr{F}_A[\phi(t); x]$. Using the representation of $U \in \mathscr{Z}'(A)$ as noted in the paragraph preceding this theorem, (5), and a change of order of integration we have

$$\begin{aligned}
\langle V, \psi\rangle &= \frac{1}{a_1 \cdots a_n}\langle U, \phi\rangle = \frac{(-1)^{|\alpha|}}{a_1 \cdots a_n}\int_{S_A} f(t) D^\alpha \phi(t)\, dt \\
&= \frac{(-1)^{|\alpha|}}{a_1 \cdots a_n}\int_{S_A} f(t) \int_{\mathbb{R}^n} \psi(x) \\
&\quad \cdot \left[\prod_{j=1}^{n}\left(\frac{-x_j}{a_j}\right)^{\alpha_j}\right] \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dx\, dt \\
&= A^{-\alpha-\bar{1}} \int_{\mathbb{R}^n} \psi(x) x^\alpha \int_{S_A} f(t) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dt\, dx.
\end{aligned} \tag{8}$$

Now it is easily seen that the function

$$F(x) = \int_{S_A} f(t) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dt$$

can be extended to $\mathbb{C}^n$ to be the entire analytic function

$$F(z) = \int_{S_A} f(t) \exp\left[-2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt$$

which is of exponential type $\leqq 2\pi$, and $F(x)$ is a bounded function over $\mathbb{R}^n$. Thus

$$g(x) = A^{-\alpha-\bar{1}} x^\alpha F(x) \tag{9}$$

can also be extended to $\mathbb{C}^n$ as an entire analytic function having the same exponential type as the extension, $F(z)$, of $F(x)$; and for $x \in \mathbb{R}^n$, there exists a constant K_α such that

$$|g(x)| \leqq K_\alpha |x^\alpha|.$$

Thus from (8) and (9) we see that $V = \mathscr{F}_A[U] = g(x)$, where $g(x)$ has the desired properties.

COROLLARY 1. *Let $U \in \mathscr{K}'(A)$. Then*

$$\mathscr{F}_A[U] = \frac{1}{a_1 \cdots a_n}\left\langle D_t^\alpha[\lambda(t) f(t)], \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right]\right\rangle,$$

where $U = D^\alpha f(t)$ is the representation of U and $\lambda(t)$ is the characteristic function of S_A.

Proof. We obtain from (9) that

$$\begin{aligned} g(x) &= A^{-\alpha-\bar{1}} x^\alpha \int_{\mathbb{R}^n} \lambda(t) f(t) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dt \\ &= \frac{1}{a_1 \cdots a_n}\left\langle D^\alpha[\lambda(t) f(t)], \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right]\right\rangle, \end{aligned}$$

and $D_t^\alpha[\lambda(t) f(t)] \in \mathscr{E}'$ with compact support S_A. Applying Theorem 1 we immediately have the desired result.

We note that Theorem 1 and Corollary 1 generalize results of Warmbrod [10, Thm. 1, p. 937, Thm. 3, p. 941; and Cor. 1, p. 939], respectively.

Theorem 1 and Corollary 1 suggest the following result which is a Paley–Wiener–Schwartz theorem for the distributional finite Fourier transform. In the sufficiency of this result we shall need the notion of change of variable of a distribution. Let $T \in \mathscr{E}'$ and $\phi \in \mathscr{E}$. We define the change of variable $t \to (t_1/a_1, \cdots, t_n/a_n)$ in $\langle T_t, \phi(t)\rangle$ by

$$\langle T_t, \phi(t)\rangle = \frac{1}{a_1 \cdots a_n}\left\langle T_{(t_1/a_1, \cdots, t_n/a_n)}, \phi\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle.$$

Under this definition it is evident that $T_{(t_1/a_1, \cdots, t_n/a_n)} \in \mathscr{E}'$ since $T_t \in \mathscr{E}'$. This definition is obviously motivated by the usual change of variables in integrals.

THEOREM 2. *$V \in \mathscr{Z}'(\overline{2\pi})$ is the distributional finite Fourier transform of an element $U \in \mathscr{E}'$ with* supp$(U) \subseteq S_A$ *if and only if V is a continuous function $f(x)$ which grows as some power of $|x|$, and which can be extended to be an entire analytic function of exponential type $\leqq 2\pi$.*

Proof. Let $U \in \mathscr{E}'$ with $\operatorname{supp}(U) \subseteq S_A$. By Schwartz [7, Thm. XXVI, p. 91], $U = \sum_{|\alpha| \leqq m} D^\alpha f_\alpha(t)$, where the $f_\alpha(t)$ are continuous functions and m is a fixed finite positive real number; and for all α, $\operatorname{supp}(f_\alpha)$ is contained in an arbitrary neighborhood of S_A. But S_A is a regular set [7, pp. 98–99], so that in fact $\operatorname{supp}(f_\alpha) \subseteq S_A$ for all α, $|\alpha| \leqq m$. We note that $\mathscr{E}' \subset \mathscr{K}'(A)$. Using the stated representation of U, we have by exactly the same type of computation as in (8) and the proof of Corollary 1 that

$$\mathscr{F}_A[U] = \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right]\right\rangle,$$

where we recall that each $f_\alpha(t)$ in the representation of U has its support in S_A. Put

$$f(x) = \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right]\right\rangle.$$

Then by hypothesis $V = \mathscr{F}_A[U] = f(x)$, and using the representation of $U \in \mathscr{E}'$ it is easily seen that $f(x)$ is a continuous function which can be extended to be an entire analytic function

$$f(z) = \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[-2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle, \qquad z \in \mathbb{C}^n,$$

and

$$\begin{aligned} f(z) &= \frac{1}{a_1 \cdots a_n}\sum_{|\alpha| \leqq m} (-1)^{|\alpha|}\left\langle f_\alpha(t), D_t^\alpha \exp\left[-2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle \\ &= \sum_{|\alpha| \leqq m} A^{-\alpha-\bar{1}} z^\alpha \int_{S_A} f_\alpha(t) \exp\left[-2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt \end{aligned}$$

since $\operatorname{supp}(f_\alpha) \subseteq S_A$, $|\alpha| \leqq m$. Thus

$$|f(z)| \leqq \exp[2\pi(|y_1| + \cdots + |y_n|)] \sum_{|\alpha| \leqq m} |z^\alpha| \int_{S_A} |f_\alpha(t)|\, dt,$$

where $y = \operatorname{Im}(z)$, and m is a finite positive real number. From this bound it follows that $f(z)$ is of exponential type $\leqq 2\pi$, and its restriction to $\mathbb{R}^n$ grows as some power of $|x|$. This proves the necessity.

Now let V be a continuous function $f(x)$ which grows as some power of $|x|$ and which can be extended to be an entire analytic function of exponential type $\leqq 2\pi$. Then $f(x) \in \mathscr{S}'$ and applying the Paley–Wiener–Schwartz theorem [7, Thm. XVI, p. 272], we obtain the existence of an element $T \in \mathscr{E}'$ with $\operatorname{supp}(T) \subseteq S_{\bar{1}}$ such that $f(x) = \mathscr{F}^{-1}[T]$, the inverse Fourier transform in $\mathscr{S}'$. It is well known that for such a distribution T,

$$f(x) = \mathscr{F}^{-1}[T] = \langle T, \exp[-2\pi i\langle x, t\rangle]\rangle. \tag{10}$$

(This follows from the inverse Fourier transform of functions as defined in this paper and a proof similar to that in Hörmander [6, Thm. 1.7.5, pp. 20–21].) Now let $A = (a_1, \cdots, a_n)$ be an n-tuple such that $a_j > 0, j = 1, \cdots, n$. By a change of variable we have (recall the discussion in the paragraph preceding the statement of Theorem 2)

$$\begin{aligned}(11)\qquad &\langle T, \exp[-2\pi i\langle x, t\rangle]\rangle \\ &= \frac{1}{a_1 \cdots a_n}\left\langle T_{(t_1/a_1,\cdots,t_n/a_n)}, \exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]\right\rangle.\end{aligned}$$

Putting $U_t = T_{(t_1/a_1,\cdots,t_n/a_n)}$, we have from (11) that $U \in \mathscr{E}'$ and $\operatorname{supp}(U) \subseteq S_A$. Further by exactly the same type of computation as in (8) and Corollary 1 we have

$$\mathscr{F}_A[U] = \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]\right\rangle.$$

(Recall this same equality in the proof of the sufficiency of this theorem.) Combining this with (10) and (11), we have $V = f(x) = \mathscr{F}_A[U]$ as desired. This completes the proof.

We let $\bar{\eta}$, $\bar{v}$ and $\bar{\delta}$ denote the n-tuples $(\eta, \cdots, \eta), (v, \cdots, v)$ and $(\delta, \cdots, \delta)$, respectively, where η, v and δ are fixed positive real numbers. For each fixed $\eta > 0$ such that $\eta < \delta$, let $\xi_\eta(t) \in \mathscr{E}$ such that $\xi_\eta(t) = 1$, $t \in S_{A+\bar{\delta}-\bar{\eta}}$, $\xi_\eta(t) = 0$, $t \notin S_{A+\bar{\delta}+\bar{\eta}}$ and $0 \leqq \xi_\eta(t) \leqq 1$.

Now let δ and v be fixed positive real numbers and let η be chosen such that $\delta > \eta > 0$ and $v > \eta > 0$. Let $U \in \mathscr{K}'(A + \bar{\delta} + \bar{v})$, and put

$$(12)\qquad f_\eta(x) = \frac{1}{a_1 \cdots a_n}\left\langle U, \xi_\eta(t)\exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]\right\rangle.$$

From the definition of $\xi_\eta(t)$ we have that $(\xi_\eta(t)\exp[-2\pi i(x_1t_1/a_1 + \cdots + x_nt_n/a_n)]) \in \mathscr{K}(A + \bar{\delta} + \bar{\eta}) \subset \mathscr{K}(A + \bar{\delta} + \bar{v})$. Thus $f_\eta(x)$ defined in (12) exists for each fixed $\eta > 0$ restricted as above; and using the representation of $U \in \mathscr{K}'(A + \bar{\delta} + \bar{v})$ and an argument similar to that in the sufficiency of Theorem 2, we see that $f_\eta(x)$ is a continuous function which grows as some power of $|x|$ and can be extended to be an entire analytic function. Since $\mathscr{K}'(A + \bar{\delta} + \bar{v}) \subseteq \mathscr{K}'(A)$, then the distributional finite Fourier transform, $\mathscr{F}_A[U]$, of $U \in \mathscr{K}'(A + \bar{\delta} + \bar{v})$ exists. In the following result we show that it can be represented as the limit of the sequence $\{f_\eta(x)\}$ as $\eta \to 0$.

THEOREM 3. *Let δ and v be fixed positive real numbers. Let η be a real number such that $\delta > \eta > 0$ and $v > \eta > 0$. Let $U \in \mathscr{K}'(A + \bar{\delta} + \bar{v})$. Then* $\lim_{\eta\to 0} f_\eta(x) = \mathscr{F}_A[U]$ *in $\mathscr{Z}'(\overline{2\pi})$, where $f_\eta(x)$ is defined as in* (12) *for any fixed $\eta > 0$ restricted as above.*

Proof. Let $\eta > 0$ be fixed and restricted as above. As we have noted in the paragraph preceding this theorem, $f_\eta(x)$ is continuous and grows as some power of $|x|$. It is known that such a function is an element of $\mathscr{Z}'$, and $\mathscr{Z}' \subseteq \mathscr{Z}'(\overline{2\pi})$. Thus $f_\eta(x) \in \mathscr{Z}'(\overline{2\pi})$. Let $\psi \in \mathscr{Z}(\overline{2\pi})$ and $\phi \in \mathscr{K}(A)$ such that $\psi(x) = \mathscr{F}_A[\phi(t); x]$. Since $U \in \mathscr{K}'(A + \bar{\delta} + \bar{v})$, then $U = D^\alpha f(t)$, where $f(t)$ is continuous. Using this representation of U, a change of order of integration, and (5) we have

$$\begin{aligned}
&\langle f_\eta(x), \psi(x)\rangle \\
&= \frac{(-1)^{|\alpha|}}{a_1 \cdots a_n} \int_{\mathbb{R}_n} \psi(x) \int_{\mathbb{R}^n} f(t) D_t^\alpha\left(\xi_\eta(t) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right]\right) dt\, dx \\
(13)\quad &= \frac{(-1)^{|\alpha|}}{a_1 \cdots a_n} \int_{\mathbb{R}^n} f(t) \int_{\mathbb{R}^n} \psi(x) D_t^\alpha\left(\xi_\eta(t) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right]\right) dx\, dt \\
&= \frac{1}{a_1 \cdots a_n} \langle U, \xi_\eta(t)\phi(t)\rangle .
\end{aligned}$$

Now $\phi \in \mathscr{K}(A)$; thus we have that $(\xi_\eta(t)\phi(t)) \to \phi(t)$ in $\mathscr{K}(A + \bar{\delta} + \bar{v})$ (and in fact in $\mathscr{K}(A)$) as $\eta \to 0$. Since $U \in \mathscr{K}'(A + \bar{\delta} + \bar{v}) \subset \mathscr{K}'(A)$, then

$$\lim_{\eta \to 0} \langle U, \xi_\eta(t)\phi(t)\rangle = \langle U, \phi(t)\rangle . \tag{14}$$

Combining (13) and (14) and recalling (7), we thus have

$$\lim_{\eta \to 0} \langle f_\eta(x), \psi(x)\rangle = \frac{1}{a_1 \cdots a_n} \langle U, \phi\rangle = \langle V, \psi\rangle ,$$

where $V = \mathscr{F}_A[U]$, and the proof is complete.

We note that Theorem 3 generalizes a result of Warmbrod [10, Thm. 5, p. 943], for if we restrict U in Theorem 3 to be an element of $\mathscr{K}'$, then Theorem 3 concludes that $f_\eta(x) \to \mathscr{F}_A[U]$ in $\mathscr{Z}'(\overline{2\pi})$ as $\eta \to 0$ since $\mathscr{K}' \subset \mathscr{K}'(A + \bar{\delta} + \bar{v}) \subset \mathscr{K}'(A)$. We note further that if $U \in \mathscr{K}'$, we could let the functions $\xi_\eta(t) \in \mathscr{E}$ be $\xi_\eta(t) = 1$, $t \in S_{A+\bar{\eta}}$, $\xi_\eta(t) = 0$ outside some neighborhood of $S_{A+\bar{\eta}}$, and $0 \leqq \xi_\eta(t) \leqq 1$. With $U \in \mathscr{K}'$, we could then define $f_\eta(x)$ as in (12). A similarly constructed proof as in Theorem 3 shows that the sequence $\{f_\eta(x)\}$ has as limit $\mathscr{F}_A[U]$ in the topology of $\mathscr{Z}'(\overline{2\pi})$ as $\eta \to 0$.

4. Finite convolution. Let U and V be elements of $\mathscr{K}'(A)$; then by the representation of $\mathscr{K}'(A)$ we have $U = D^\alpha f(t)$ and $V = D^\beta g(t)$, where f and g are continuous functions and α and β are n-tuples of nonnegative integers. Throughout this section $\lambda(t)$ will denote the characteristic function of $S_A = \{t : |t_j| \leqq a_j,\ j = 1, \cdots, n\}$.

We define the finite convolution of U and V as

$$U \,\Delta\, V = D^\alpha(\lambda(t) f(t)) * D^\beta(\lambda(t) g(t)),$$

where $*$ denotes the ordinary distributional convolution. Since $D^\alpha(\lambda(t)f(t))$ and $D^\beta(\lambda(t)g(t))$ both have compact support in S_A as distributions, then they are both elements of $\mathscr{E}'$. Hence $(U \,\Delta\, V)$ exists and is an element of $\mathscr{K}' \subset \mathscr{K}'(A)$. (In fact $(U \,\Delta\, V) \in \mathscr{E}' \subset \mathscr{K}' \subset \mathscr{K}'(A)$. This follows from the fact that $(U \,\Delta\, V) \in \mathscr{K}'$; and by Schwartz [7, Thm. II, p. 156], $\operatorname{supp}(U \,\Delta\, V) = \operatorname{supp}(D^\alpha(\lambda(t)f(t)) * D^\beta(\lambda(t)g(t))) \subseteq S_{2A}$. Thus $(U \,\Delta\, V)$ has compact support in S_{2A}, and $(U \,\Delta\, V) \in \mathscr{E}'$.) We note also that by a property of distributional convolution, this definition of $(U \,\Delta\, V)$ is equivalent to

$$U \,\Delta\, V = D^{\alpha+\beta}((\lambda(t) f(t)) * (\lambda(t) g(t))).$$

Further, we immediately see that if $U_1, \cdots, U_m$ is a finite set of elements of $\mathscr{K}'(A)$, then $(U_1 \Delta U_2 \Delta \cdots \Delta U_m)$ exists and is an element of $\mathscr{K}' \subset \mathscr{K}'(A)$. Using properties of the distributional convolution and the definition of finite convolution, the following theorem is easily proved. The proof is left to the interested reader.

THEOREM 4. *Let U, V and W be elements of $\mathscr{K}'(A)$. Then $(U \Delta V) = (V \Delta U)$, $((U \Delta V) \Delta W) = (U \Delta (V \Delta W))$, and $D^\gamma(U \Delta V) = (D^\gamma U \Delta V) = (U \Delta D^\gamma V)$.*

The following theorem shows that the distributional finite Fourier transform maps the finite convolution into the product of the respective transforms; hence this result shows that this transform operates on finite convolution as the Fourier transform operates on ordinary distributional convolution. (See Schwartz [7, pp. 268–270].) We note also that the following result generalizes Warmbrod [10, Thm. 9, p. 948].

THEOREM 5. *Let U and V be elements of $\mathscr{K}'(A)$. Then*

$$\mathscr{F}_A[U \Delta V] = (a_1 \cdots a_n)\mathscr{F}_A[U]\mathscr{F}_A[V]$$

in $\mathscr{L}'(\overline{2\pi})$.

Proof. Using Corollary 1 and the definition of the distributional convolution we have

$$\begin{aligned}
\mathscr{F}_A[U \Delta V] &= \frac{1}{a_1 \cdots a_n}\left\langle D^\alpha(\lambda(t)f(t)) * D^\beta(\lambda(t)g(t)), \exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]\right\rangle \\
&= \frac{1}{a_1 \cdots a_n}\left\langle D^\alpha(\lambda(t)f(t)),\right. \\
&\qquad \left.\left\langle D^\beta(\lambda(u)g(u)), \exp\left[-2\pi i\left(\frac{x_1(t_1+u_1)}{a_1} + \cdots + \frac{x_n(t_n+u_n)}{a_n}\right)\right]\right\rangle\right\rangle \qquad (15) \\
&= \frac{1}{a_1 \cdots a_n}\left\langle D^\alpha(\lambda(t)f(t)), \exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]\right\rangle \\
&\qquad \left\langle D^\beta(\lambda(u)g(u)), \exp\left[-2\pi i\left(\frac{x_1u_1}{a_1} + \cdots + \frac{x_nu_n}{a_n}\right)\right]\right\rangle,
\end{aligned}$$

where $U = D^\alpha f(t)$ and $V = D^\beta g(t)$. Using Corollary 1 again, we have from (15) that $\mathscr{F}_A[U \Delta V] = (a_1 \cdots a_n)\mathscr{F}_A[U]\mathscr{F}_A[V]$ as desired.

From (15) and the proof of Corollary 1, we see that Theorem 5 also states $\mathscr{F}_A[U \Delta V] = (a_1 \cdots a_n)(h(x)q(x))$, where from Theorem 1, $h(x) = \mathscr{F}_A[U]$ and $q(x) = \mathscr{F}_A[V]$.

Regularization of a distribution (Hörmander [6, p. 15] is an important tool in distribution theory and can be used, for example, in proving the Paley–Wiener–Schwartz theorem [6, pp. 21–22]. In the following we introduce finite regularization and show that it can be used to represent the distributional finite Fourier transform of elements in $\mathscr{K}'(A)$.

It is known that there exists a function $\chi \in \mathscr{K}$ such that $\chi(t) \geqq 0$, $\operatorname{supp}(\chi) = S_{\bar{1}} = \{t : |t_j| \leqq 1, j = 1, \cdots, n\}$, and

$$\int_{\mathbb{R}^n} \chi(t)\, dt = 1.$$

(A function having these properties can be constructed similarly to the function constructed in Hörmander [6, pp. 2–3] or Schwartz [7, p. 21].) With the above function χ, we now put $\chi_\eta(t) = (\eta)^{-n}\chi(t_1/\eta, \cdots, t_n/\eta)$, $\eta > 0$. Then $\chi_\eta(t) \in \mathscr{K}$, $\chi_\eta(t) \geqq 0$, and $\operatorname{supp}(\chi_\eta) = S_{\bar{\eta}}$. Let $U \in \mathscr{K}'(A)$. Then $(U \Delta \chi_\eta(t))$ exists for each fixed $\eta > 0$ and as we have seen before is an element of $\mathscr{E}' \subset \mathscr{K}' \subset \mathscr{K}'(A)$. We call $(U \Delta \chi_\eta(t))$ the finite regularization of $U \in \mathscr{K}'(A)$. In the following two theorems $\chi_\eta(t)$ is the function defined immediately above.

THEOREM 6. *Let $U \in \mathscr{K}'(A)$. There exists a sequence $\{U_\eta\}$ of elements in $\mathscr{E}'$ with* $\operatorname{supp}(U_\eta) \subseteq S_{A+\bar{\eta}}$ *for each fixed $\eta > 0$ such that* $\lim_{\eta\to 0} U_\eta = D^\alpha(\lambda(t)f(t))$ *in $\mathscr{K}'(A)$, where $U = D^\alpha f(t)$ is the representation of U in $\mathscr{K}'(A)$, and $\lambda(t)$ is the characteristic function of S_A.*

Proof. We put $U_\eta = (U \Delta \chi_\eta(t))$, $\eta > 0$, which exists for each fixed $\eta > 0$. Since $\operatorname{supp}(D^\alpha(\lambda(t)f(t))) \subseteq S_A$ and $\operatorname{supp}(\chi_\eta) = S_{\bar{\eta}}$, the result of Schwartz [7, Thm. II, p. 156] yields $\operatorname{supp}(U \Delta \chi_\eta) = \operatorname{supp}(D^\alpha(\lambda(t)f(t)) * (\lambda(t)\chi_\eta(t))) \subseteq S_{A+\bar{\eta}}$. Thus $(U \Delta \chi_\eta) \in \mathscr{K}'$ and has compact support in $S_{A+\bar{\eta}}$ for each fixed $\eta > 0$; hence $(U \Delta \chi_\eta) \in \mathscr{E}'$ for each fixed $\eta > 0$. Now let $\phi \in \mathscr{K}(A)$. From the definition of distributional convolution we have

$$\langle (U_\eta), \phi(t)\rangle = (-1)^{|\alpha|}\langle \lambda(t)f(t), \langle \lambda(u)\chi_\eta(u), D_t^\alpha\phi(t+u)\rangle\rangle. \tag{16}$$

Now $\operatorname{supp}(\lambda) = S_A$. Using this, the definition of χ_η, and a change of variable, we have

$$\begin{aligned} \langle \lambda(u)\chi_\eta(u), D_t^\alpha\phi(t+u)\rangle &= \int_{S_A} \chi_\eta(u)D_t^\alpha\phi(t+u)\,du \\ &= \int_{S_{(1/\eta)A}} \chi(u)D_t^\alpha\phi(t+\eta u)\,du, \end{aligned} \tag{17}$$

where as usual $S_{(1/\eta)A} = \{t : |t_j| \leqq (1/\eta)a_j, j = 1, \cdots, n\}$. We now show that

$$\lim_{\eta\to 0} \int_{S_{(1/\eta)A}} \chi(u)D_t^\alpha\phi(t+\eta u)\,du = D_t^\alpha\phi(t) \tag{18}$$

uniformly in t on all compact sets of $\mathbb{R}^n$. First let us restrict $\eta > 0$ such that $\eta \leqq \min\{a_1, \cdots, a_n\}$. For this choice of η, $S_{\bar{1}} \subseteq S_{(1/\eta)A}$. Recalling that $\operatorname{supp}(\chi) = S_{\bar{1}}$, we have for any η of this choice that

$$\int_{S_{(1/\eta)A}} \chi(u)D_t^\alpha\phi(t+\eta u)\,du = \int_{S_1} \chi(u)D_t^\alpha\phi(t+\eta u)\,du.$$

Thus for any $\eta \leqq \min\{a_1, \cdots, a_n\}$, we have by the properties of χ that

$$\int_{S_{(1/\eta)A}} \chi(u)D_t^\alpha\phi(t+\eta u)\,du - D_t^\alpha\phi(t) = \int_{S_1} \chi(u)(D_t^\alpha\phi(t+\eta u) - D_t^\alpha\phi(t))\,du. \tag{19}$$

Now recall that $\phi \in \mathscr{K}(A)$. Thus ϕ and any derivative $D_t^\alpha\phi(t)$ are continuous functions, and if t is restricted to an arbitrary compact set in $\mathbb{R}^n$, then ϕ and $D_t^\alpha\phi(t)$ are uniformly continuous there. Using this fact and the Lebesgue dominated convergence theorem in (19), we see that if we now let $\eta \to 0$, the desired convergence in (18) is obtained. Since $\operatorname{supp}(\lambda(t)f(t)) = S_A$, the convergence in (18) and another use of the Lebesgue dominated convergence theorem give

$$\lim_{\eta\to 0}\left\langle \lambda(t)f(t), \int_{S_{(1/\eta)A}} \chi(u)D_t^\alpha\phi(t+\eta u)\,du\right\rangle = \langle \lambda(t)f(t), D^\alpha\phi(t)\rangle. \tag{20}$$

Combining (16), (17) and (20) we thus have

$$\lim_{\eta\to 0}\langle (U_\eta), \phi(t)\rangle = (-1)^{|\alpha|}\langle \lambda(t)f(t), D^\alpha\phi(t)\rangle = \langle D^\alpha(\lambda(t)f(t)), \phi\rangle,$$

where ϕ is an arbitrary element of $\mathscr{K}(A)$. This proves the desired result.

THEOREM 7. *Let $U \in \mathscr{K}'(A)$. There exists a sequence $\{U_\eta\}$ of elements in $\mathscr{E}'$ with $\operatorname{supp}(U_\eta) \subseteq S_{A+\bar{\eta}}$ for each fixed $\eta > 0$ such that $\lim_{\eta\to 0}\mathscr{F}_A[U_\eta] = \mathscr{F}_A[U]$ in $\mathscr{Z}'(\overline{2\pi})$.*

Proof. Putting $U_\eta = (U \,\Delta\, \chi_\eta(t))$, $\eta > 0$, we have as in Theorem 6 that $U_\eta \in \mathscr{E}'$ with $\operatorname{supp}(U_\eta) \subseteq S_{A+\bar{\eta}}$ for each $\eta > 0$. Now $\mathscr{F}_A[U_\eta] \in \mathscr{Z}'(\overline{2\pi})$ for any fixed $\eta > 0$ since $U_\eta \in \mathscr{E}' \subset \mathscr{K}' \subset \mathscr{K}'(A)$. By the proofs of Theorem 5 (i.e., see (15)) and Corollary 1, we have

$$\mathscr{F}_A[U_\eta] = g(x)\left\langle \lambda(t)\chi_\eta(t), \exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]\right\rangle \tag{21}$$

in $\mathscr{Z}'(\overline{2\pi})$, where $g(x)$ is the function from Theorem 1 such that $g(x) = \mathscr{F}_A[U]$. Letting $\psi \in \mathscr{Z}(\overline{2\pi})$ and recalling that $\lambda(t)$ is the characteristic function of S_A, we have from (21) that

$$\langle \mathscr{F}_A[U_\eta], \psi(x)\rangle = \left\langle g(x)\int_{S_A}\chi_\eta(t)\exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]dt, \psi(x)\right\rangle. \tag{22}$$

Recalling the definition of $\chi_\eta(t)$ and using a change of variable, we obtain

$$\begin{aligned}\int_{S_A}&\chi_\eta(t)\exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]dt\\ &= \int_{S_{(1/\eta)A}}\chi(t)\exp\left[-2\pi i\left(\frac{\eta x_1t_1}{a_1} + \cdots + \frac{\eta x_nt_n}{a_n}\right)\right]dt.\end{aligned} \tag{23}$$

Let us now restrict η so that $\eta \leqq \min\{a_1, \cdots, a_n\}$. Then $S_{\bar{1}} \subseteq S_{(1/\eta)A}$. Since $\operatorname{supp}(\chi) = S_{\bar{1}}$, then for this choice of η,

$$\begin{aligned}\int_{S_{(1/\eta)A}}&\chi(t)\exp\left[-2\pi i\left(\frac{\eta x_1t_1}{a_1} + \cdots + \frac{\eta x_nt_n}{a_n}\right)\right]dt\\ &= \int_{S_{\bar{1}}}\chi(t)\exp\left[-2\pi i\left(\frac{\eta x_1t_1}{a_1} + \cdots + \frac{\eta x_nt_n}{a_n}\right)\right]dt.\end{aligned} \tag{24}$$

Letting $\eta \to 0$ now in (24), we have by the Lebesgue dominated convergence theorem that

$$\lim_{\eta\to 0}\int_{S_{(1/\eta)A}}\chi(t)\exp\left[-2\pi i\left(\frac{\eta x_1t_1}{a_1} + \cdots + \frac{\eta x_nt_n}{a_n}\right)\right]dt = \int_{S_{\bar{1}}}\chi(t)\,dt = 1. \tag{25}$$

Recalling that the function $g(x) = \mathscr{F}_A[U]$ from Theorem 1 is a continuous function which increases as some finite power of $|x|$, and using (23) and (25), we have by another application of the Lebesgue dominated convergence theorem that

$$\lim_{\eta \to 0} \left\langle g(x) \int_{S_A} \chi_n(t) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dt, \psi(x) \right\rangle \tag{26}$$
$$= \langle g(x), \psi(x) \rangle = \langle \mathscr{F}_A[U], \psi(x) \rangle.$$

Combining (22) and (26), we obtain $\lim_{\eta \to 0} \mathscr{F}_A[U_\eta] = \mathscr{F}_A[U]$ in $\mathscr{Z}'(\overline{2\pi})$ as desired. This completes the proof.

5. Limit of the finite Fourier transform. In this section we shall show that the Fourier transform of an element in $\mathscr{K}'$ can be represented as the limit of the finite Fourier transform. Let $\phi \in \mathscr{K}(A)$, and let $U \in \mathscr{K}' \subset \mathscr{K}'(A)$; then the Fourier transform of U restricted to $\mathscr{K}(A)$ is (recall the definition of Fourier transform on $\mathscr{K}'$ given in § 2)

$$\langle \mathscr{F}[U], \mathscr{F}[\phi(t); x] \rangle = \langle U, \phi \rangle, \qquad \phi \in \mathscr{K}(A).$$

From (7) we have

$$(a_1 \cdots a_n) \langle \mathscr{F}_A[U], \mathscr{F}_A[\phi(t); x] \rangle = \langle U, \phi \rangle, \qquad \phi \in \mathscr{K}(A).$$

Thus if $U \in \mathscr{K}'$, then its Fourier transform and finite Fourier transform are connected by

$$\langle \mathscr{F}[U], \mathscr{F}[\phi(t); x] \rangle = (a_1 \cdots a_n) \langle \mathscr{F}_A[U], \mathscr{F}_A[\phi(t); x] \rangle, \qquad \phi \in \mathscr{K}(A). \tag{27}$$

In the following we shall make use of the concept of change of variable in a distribution. Let $\psi(x) \in \mathscr{Z}(2\pi A)$ where A is an arbitrary n-tuple of positive real numbers, and let $V \in \mathscr{Z}'(\overline{2\pi})$. We define the generalized function $V_{(a_1 x_1, \cdots, a_n x_n)} \in \mathscr{Z}'(2\pi A)$ by

$$\langle V_{(a_1 x_1, \cdots, a_n x_n)}, \psi(x) \rangle = \frac{1}{a_1 \cdots a_n} \left\langle V_x, \psi\left(\frac{x_1}{a_1}, \cdots, \frac{x_n}{a_n}\right) \right\rangle. \tag{28}$$

Here $V_{(a_1 x_1, \cdots, a_n x_n)}$ denotes as usual that the generalized function V is acting on the variable $(a_1 x_1, \cdots, a_n x_n)$ where $A = (a_1, \cdots, a_n)$ is a fixed n-tuple of positive real numbers. The definition in (28) is meaningful, for if $\psi(x) \in \mathscr{Z}(2\pi A)$, then $\psi(x_1/a_1, \cdots, x_n/a_n) \in \mathscr{Z}(\overline{2\pi})$. It is obvious that the definition of distributional change of variable in (28) is suggested by the usual change of variable in integrals.

We are now ready to represent the Fourier transform of an element $U \in \mathscr{K}'$ as the limit of a sequence of finite Fourier transforms. By $A = (a_1, \cdots, a_n) \to \infty$ in the following two theorems, we mean $a_j \to \infty, j = 1, \cdots, n$.

THEOREM 8. *Let* $U \in \mathscr{K}'$. *Then*

$$\lim_{A \to \infty} (a_1 \cdots a_n) \mathscr{F}_A[U]_{(a_1 x_1, \cdots, a_n x_n)} = \mathscr{F}[U] \tag{29}$$

in $\mathscr{Z}'$.

Proof. Let $\psi \in \mathscr{Z}$. Then there exists an n-tuple B of positive real numbers such that $\psi \in \mathscr{Z}(2\pi B)$. Hence for all n-tuples A of positive real numbers such that $A \geqq B$, $\psi \in \mathscr{Z}(2\pi A)$. For the present let A be arbitrary but fixed such that $A \geqq B$. By Gel'fand and Shilov [4, pp. 154–155 for 1 dimension, p. 158 for n dimensions] there exists an element $\phi \in \mathscr{K}(A)$ such that $\psi(x) = \mathscr{F}[\phi(t); x] \in \mathscr{Z}(2\pi A)$. By (27) we have

(30) $$\langle \mathscr{F}[U], \psi(x)\rangle = (a_1 \cdots a_n)\langle \mathscr{F}_A[U], \mathscr{F}_A[\phi(t); x]\rangle.$$

It is obvious that

$$\mathscr{F}_A[\phi(t); x] = \frac{1}{a_1 \cdots a_n}\mathscr{F}\left[\phi(t); \left(\frac{x_1}{a_1}, \cdots, \frac{x_n}{a_n}\right)\right]$$

for $\phi \in \mathscr{K}(A)$. Using this fact and (28), we have

$$(a_1 \cdots a_n)\langle \mathscr{F}_A[U], \mathscr{F}_A[\phi(t); x]\rangle = \left\langle \mathscr{F}_A[U], \mathscr{F}\left[\phi(t); \left(\frac{x_1}{a_1}, \cdots, \frac{x_n}{a_n}\right)\right]\right\rangle$$

(31) $$= \left\langle \mathscr{F}_A[U], \psi\left(\frac{x_1}{a_1}, \cdots, \frac{x_n}{a_n}\right)\right\rangle$$

$$= (a_1 \cdots a_n)\langle \mathscr{F}_A[U]_{(a_1x_1, \cdots, a_nx_n)}, \psi(x)\rangle.$$

Combining (30) and (31) we have

(32) $$\langle \mathscr{F}[U], \psi(x)\rangle = (a_1 \cdots a_n)\langle \mathscr{F}_A[U]_{(a_1x_1, \cdots, a_nx_n)}, \psi(x)\rangle,$$

and (32) holds for A being any n-tuple of positive real numbers such that $A \geqq B$. Hence (29) follows from (32), and the proof is complete.

Let U_A denote an element of $\mathscr{K}'(A)$ for any fixed A, and let $\mathscr{F}_A[U_A]$ denote the finite Fourier transform of U_A for this fixed A. The following theorem relates the limit of $\{U_A\}$ in $\mathscr{K}'$ with the limit of $\{(a_1 \cdots a_n)\mathscr{F}_A[U_A]_{(a_1x_1, \cdots, a_nx_n)}\}$ in $\mathscr{Z}'$ as $A \to \infty$.

THEOREM 9. *$\lim_{A\to\infty}(a_1 \cdots a_n)\mathscr{F}_A[U_A]_{(a_1x_1, \cdots, a_nx_n)} = V$ in the $\mathscr{Z}'$ topology, where $V \in \mathscr{Z}'$, if and only if $\lim_{A\to\infty} U_A = U$ in the $\mathscr{K}'$ topology, where $U = \mathscr{F}^{-1}[V] \in \mathscr{K}'$.*

Proof. We establish the sufficiency here. Similar arguments will establish the necessity, and the details will be left to the interested reader. To prove the sufficiency we let $\phi \in \mathscr{K}$ and assume that there exists an element $U \in \mathscr{K}'$ such that $\lim_{A\to\infty}\langle U_A, \phi\rangle = \langle U, \phi\rangle$. Since $\phi \in \mathscr{K}$, then there exists an n-tuple B of positive real numbers such that $\phi \in \mathscr{K}(B)$. Let A be an arbitrary n-tuple of positive real numbers such that $A \geqq B$. Then $\phi \in \mathscr{K}(A)$, and $\psi(x) = \mathscr{F}[\phi(t); x] \in \mathscr{Z}(2\pi A) \subset \mathscr{Z}$. Using the definition of finite Fourier transform and a calculation similar to that in (31), we have

$$\langle U_A, \phi\rangle = (a_1 \cdots a_n)\langle \mathscr{F}_A[U_A], \mathscr{F}_A[\phi(t); x]\rangle$$
$$= (a_1 \cdots a_n)\langle \mathscr{F}_A[U_A]_{(a_1x_1, \cdots, a_nx_n)}, \psi(x)\rangle,$$

where $\psi(x) = \mathscr{F}[\phi(t); x]$. This equality holds for all $A \geqq B$. Thus

(33) $$\langle U, \phi\rangle = \lim_{A\to\infty}\langle U_A, \phi\rangle = \lim_{A\to\infty}\langle (a_1 \cdots a_n)\mathscr{F}_A[U_A]_{(a_1x_1, \cdots, a_nx_n)}, \psi\rangle.$$

Since $U \in \mathscr{K}'$, there exists an element $V \in \mathscr{Z}'$ such that $V = \mathscr{F}[U]$ and $U = \mathscr{F}^{-1}[V]$. From (33) and the definition of the distributional Fourier transform on $\mathscr{K}'$, we thus obtain

$$\langle V, \psi\rangle = \langle U, \phi\rangle = \lim_{A\to\infty}\langle (a_1 \cdots a_n)\mathscr{F}_A[U_A]_{(a_1x_1, \cdots, a_nx_n)}, \psi\rangle,$$

and the proof of the sufficiency is complete.

6. Representation of the finite Fourier transform as the boundary value of analytic functions. In this section we shall obtain necessary and sufficient conditions on a function which is analytic in a tubular radial domain such that the function attains the distributional finite Fourier transform as boundary value on the distinguished boundary of the tube domain. We then use these results to obtain the analytic decomposition of the finite Fourier transform and a representation theorem for the analytic functions which have this transform as the boundary value.

Throughout this section, C will denote an open connected cone, C' will denote an arbitrary compact subcone of C, and $T^C = \mathbb{R}^n + iC$ is a tubular radial domain. We consider functions $f(z)$, $z \in T^C$, which satisfy

$$(34) \quad |f(z)| \leqq P(C')(1 + |z|)^N \exp[2\pi(b + \sigma)|y|], \qquad z = x + iy \in T^{C'}, \quad C' \subset C,$$

for all real numbers $\sigma > 0$, where b is a nonnegative real number, N is a real number, and $P(C')$ is a constant depending on C'.

THEOREM 10. *Let $A = (a_1, \cdots, a_n)$ be a fixed n-tuple of positive real numbers. Let $f(z)$ be analytic in T^C and satisfy* (34), *where C is an open connected cone. Then there exists an element $U \in \mathscr{K}'(A)$ having support in $\{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq b\}$ such that $f(z) \to \mathscr{F}_A[U]$ in $\mathscr{Z}'(\overline{2\pi})$ as $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset C$.*

Proof. Since $f(z)$ satisfies (34), we may choose an n-tuple $K = (k_1, \cdots, k_n)$ of nonnegative integers such that

$$(35) \quad |z^{-K}f(z)| \leqq P(C')(1 + |z|)^{-n-\varepsilon} \exp[2\pi(b + \sigma)|y|], \qquad z \in T^{C'}, \quad C' \subset C,$$

where n is the dimension and ε is any fixed positive real number. Put

$$(36) \quad g_y(t) = \int_{\mathbb{R}^n} z^{-K}f(z) \exp\left[-2\pi i\left(\frac{z_1t_1}{a_1} + \cdots + \frac{z_nt_n}{a_n}\right)\right] dx, \quad z \in T^{C'}, \quad C' \subset C.$$

Because of (35), $g_y(t)$ exists and is a continuous function of $t \in \mathbb{R}^n$ for any fixed $y \in C' \subset C$. By a similar argument as in the proof of [2, Thm. 1, p. 846, first paragraph of proof], we have that in fact the function in (36) is independent of $y \in C' \subset C$. Hence we now denote the function in (36) as $g(t)$. Using the same method of proof as in [2, Thm. 1, pp. 846–847, second paragraph of proof], we have that $\operatorname{supp}(g) \subseteq \{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq b\}$. From (36) we have

$$\begin{aligned}
&\exp\left[-2\pi\left(\frac{y_1t_1}{a_1} + \cdots + \frac{y_nt_n}{a_n}\right)\right]g(t) \\
&\quad = \int_{\mathbb{R}^n} z^{-K}f(z) \exp\left[-2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right] dx \\
&\quad = \mathscr{F}^{-1}\left[z^{-K}f(z); \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right],
\end{aligned}$$

$z = x + iy \in T^{C'}$, $C' \subset C$. By (35), $(z^{-K}f(z)) \in L^1 \cap L^2$ as a function of $x \in \mathbb{R}^n$ for $y \in C' \subset C$, $z = x + iy$. We thus have by Plancherel's theory that

$$\left(\exp\left[-2\pi\left(\frac{y_1t_1}{a_1} + \cdots + \frac{y_nt_n}{a_n}\right)\right]g(t)\right) \in L^2$$

and

$$(37)\qquad z^{-K}f(z) = \mathscr{F}\left[\exp\left[-2\pi\left(\frac{y_1t_1}{a_1} + \cdots + \frac{y_nt_n}{a_n}\right)\right]g(t); \left(\frac{x_1}{a_1}, \cdots, \frac{x_n}{a_n}\right)\right],$$

$z = x + iy \in T^{C'}$, $C' \subset C$, where this Fourier transform is in the L^2 sense.

Now let $\phi \in \mathscr{K}(A)$ and $\psi(x) = \mathscr{F}_A[\phi(t); x]$. By a change of variable,

$$(38)\qquad \psi(x) = \mathscr{F}_A[\phi(t); x] = \mathscr{F}[\phi(a_1t_1, \cdots, a_nt_n); x].$$

Since $(z^{-K}f(z)) \in L^1 \cap L^2$ as a function of x for $y \in C' \subset C$, then it is an element of $\mathscr{Z}'$ as a function of $x \in \mathbb{R}^n$. Hence from (37) and a change of variable,

$$(39)\qquad \begin{aligned} &\langle z^{-K}f(z), \psi(x)\rangle \\ &= \int_{\mathbb{R}^n} \psi(x) \int_{\mathbb{R}^n} \exp\left[-2\pi\left(\frac{y_1t_1}{a_1} + \cdots + \frac{y_nt_n}{a_n}\right)\right]g(t) \\ &\qquad \exp\left[2\pi i\left(\frac{x_1t_1}{a_1} + \cdots + \frac{x_nt_n}{a_n}\right)\right]d\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right) dx \\ &= \langle \mathscr{F}[\exp[-2\pi\langle y, t\rangle]g(a_1t_1, \cdots, a_nt_n); x], \psi(x)\rangle, \end{aligned}$$

and $\mathscr{F}[\exp[-2\pi\langle y, t\rangle]g(a_1t_1, \cdots, a_nt_n); x]$ is well-defined as an element of $\mathscr{Z}'$ since $(\exp[-2\pi\langle y, t\rangle]g(a_1t_1, \cdots, a_nt_n)) \in L^2 \subset \mathscr{K}'$. We now have from (38), (39), and the distributional Fourier transform on $\mathscr{K}'$ that

$$(40)\qquad \langle z^{-K}f(z), \psi(x)\rangle = \langle \exp[-2\pi\langle y, t\rangle]g(a_1t_1, \cdots, a_nt_n), \phi(a_1t_1, \cdots, a_nt_n)\rangle,$$

$z \in T^{C'}$, $C' \subset C$. It is straightforward to show that $(\exp[-2\pi\langle y,t\rangle]\phi(a_1t_1, \cdots, a_nt_n)) \to \phi(a_1t_1, \cdots . a_nt_n)$ in $\mathscr{K}$ as $y \to 0$, $y \in C' \subset C$. Since $g(a_1t_1, \cdots, a_nt_n)$ is a continuous function, it is an element of $\mathscr{K}' \subset \mathscr{K}'(A)$. Thus

$$(41)\qquad \begin{aligned} &\langle \exp[-2\pi\langle y, t\rangle]g(a_1t_1, \cdots, a_nt_n), \phi(a_1t_1, \cdots, a_nt_n)\rangle \\ &= \langle g(a_1t_1, \cdots, a_nt_n), \exp[-2\pi\langle y, t\rangle]\phi(a_1t_1, \cdots, a_nt_n)\rangle \\ &\to \langle g(a_1t_1, \cdots, a_nt_n), \phi(a_1t_1, \cdots, a_nt_n)\rangle \end{aligned}$$

as $y \to 0$, $y \in C' \subset C$. By a change of variable and (7) we have

$$(42)\qquad \langle g(a_1t_1, \cdots, a_nt_n), \phi(a_1t_1, \cdots, a_nt_n)\rangle = \frac{1}{a_1 \cdots a_n}\langle g(t), \phi(t)\rangle = \langle \mathscr{F}_A[g], \psi\rangle.$$

Combining (40), (41) and (42) we have shown that

$$\lim_{\substack{y \to 0 \\ y \in C' \subset C}} \langle z^{-K}f(z), \psi(x)\rangle = \langle \mathscr{F}_A[g], \psi(x)\rangle, \qquad \psi \in \mathscr{Z}(\overline{2\pi}).$$

It now follows immediately that

$$(43)\qquad \langle f(z), \psi(x)\rangle = \langle z^{-K}f(z), z^K\psi(x)\rangle \to \langle \mathscr{F}_A[g], x^K\psi(x)\rangle = \langle x^K\mathscr{F}_A[g], \psi(x)\rangle$$

as $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset C$. We now define the differential operator Ω as follows:

$$\Omega = A^K(2\pi i)^{-|K|}\frac{\partial^{|K|}}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}},$$

where $K = (k_1, \cdots, k_n)$ is the n-tuple of nonnegative integers chosen at the beginning of the proof. We now put $U = \Omega g(t)$. Then $U \in \mathscr{K}'(A)$ since $g(t)$ is a continuous function, and $\operatorname{supp}(U) = \operatorname{supp}(g) \subseteq \{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq b\}$. Further, from (7) and (5) we obtain

$$\begin{aligned} \langle \mathscr{F}_A[U], \psi\rangle &= \frac{1}{a_1 \cdots a_n}\langle \Omega g, \phi\rangle \\ &= \frac{1}{a_1 \cdots a_n}\left\langle \Omega g, \int_{\mathbb{R}_n} \psi(x) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dx\right\rangle \\ &= \frac{1}{a_1 \cdots a_n}\left\langle g, \int_{\mathbb{R}^n} x^K \psi(x) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dx\right\rangle. \end{aligned} \tag{44}$$

Since $\psi \in \mathscr{Z}(\overline{2\pi})$, $(x^K\psi(x)) \in \mathscr{Z}(\overline{2\pi})$. By our work at the beginning of § 3 (see (5)), we have that

$$\omega(t) = \int_{\mathbb{R}^n} x^K \psi(x) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dx$$

is an element of $\mathscr{K}(A)$, and $\mathscr{F}_A[\omega(t); x] = (x^K\psi(x))$. Thus from (7) we have

$$\begin{aligned} &\langle \mathscr{F}_A[g], x^K\psi(x)\rangle \\ &= \frac{1}{a_1 \cdots a_n}\langle g(t), \omega(t)\rangle \\ &= \frac{1}{a_1 \cdots a_n}\left\langle g(t), \int_{\mathbb{R}^n} x^K \psi(x) \exp\left[-2\pi i\left(\frac{x_1 t_1}{a_1} + \cdots + \frac{x_n t_n}{a_n}\right)\right] dx\right\rangle. \end{aligned} \tag{45}$$

Combining (44) and (45) gives

$$\langle \mathscr{F}_A[U], \psi(x)\rangle = \langle \mathscr{F}_A[g], x^K\psi(x)\rangle = \langle x^K \mathscr{F}_A[g], \psi(x)\rangle. \tag{46}$$

By (43) and (46) we obtain the desired result

$$\lim_{\substack{y \to 0 \\ y \in C' \subset C}} \langle f(z), \psi(x)\rangle = \langle \mathscr{F}_A[U], \psi(x)\rangle,$$

and the proof is complete.

We note that by our proof, the element $U \in \mathscr{K}'(A)$, whose existence is claimed in the statement of Theorem 10, is a distributional derivative of the continuous function $g(t)$ defined in (36) which has support in $\{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq b\}$. From (35) we see further that $g(t)$ satisfies the growth condition

$$|g(t)| \leqq M(C') \exp\left[2\pi((b + \sigma)|y| + \langle y, (t_1/a_1 + \cdots, t_n/a_n)\rangle)\right] \tag{47}$$

for all $\sigma > 0$, where $M(C')$ is a constant depending on C'. It is important to note that the bound in (47) holds for all $y \in C' \subset C$.

Also we note the one-dimensional corollary to Theorem 10. Let $a > 0$ be given, and let $C = \{y \in \mathbb{R}^1 : y > 0\}$ in Theorem 10. Consider a function $f(z)$ which is analytic in the upper half-plane $T^C = \mathbb{R}^1 + iC$ and which satisfies

$$|f(z)| \leqq P_\delta(1 + |z|)^N \exp[2\pi b|y|], \qquad y = \operatorname{Im}(z) \geqq \delta > 0.$$

(For the one-dimensional case we do not need the σ in this boundedness condition as we did in (34).) The proof of Theorem 10 concludes the existence of an element $U \in \mathscr{K}'(a)$ having support in $\{t \in \mathbb{R}^1 : t \geqq -ab\}$ such that $f(z) \to \mathscr{F}_a[U]$ in $\mathscr{Z}'(2\pi)$ as $y = \operatorname{Im}(z) \to 0$. A similar result holds for the lower half-plane in which case the cone $C = \{y \in \mathbb{R}^1 : y < 0\}$.

We now obtain a converse result to Theorem 10. Letting C be an open connected cone, we recall that $O(C)$ denotes the convex envelope of C. If $U \in \mathscr{K}'(A)$, we define $\check{U}$ by $\langle \check{U}, \phi \rangle = \langle U, \check{\phi} \rangle$, $\phi \in \mathscr{K}(A)$, where $\check{\phi}(t) = \phi(-t)$, and it is evident that $\check{U} \in \mathscr{K}'(A)$ under this definition. We let $N(0, R)$ denote a neighborhood of the origin in $\mathbb{R}^n$ with radius R, and we put $Q_C = \{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq 0\}$. $C' \subset O(C)$ will denote that C' is an arbitrary compact subcone of $O(C)$.

THEOREM 11. *Let $A = (a_1, \cdots, a_n)$ be a fixed n-tuple of positive real numbers. Let $U = \sum_{|\alpha| \leqq m} D^\alpha g_\alpha(t), m < \infty$, where each $g_\alpha(t)$ is a continuous function on $\mathbb{R}^n$ and has the growth condition*

$$|g_\alpha(t)| \leqq M_\alpha \exp\left[2\pi\left\langle \Lambda, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right] \tag{48}$$

for all $\Lambda \in O(C)$, where M_α is a constant depending on α. Let $\operatorname{supp}(g_\alpha) \subseteq Q_C$ for each α, $|\alpha| \leqq m$. Then there exists a function $f(z)$ which is analytic in $T^{O(C)}$ such that

$$|f(z)| \leqq P(m; C')(1 + |z|)^N, \quad z \in T^{C' \setminus (C' \cap N(O,R))}, \quad C' \subset O(C), \tag{49}$$

where $P(m; C')$ is a constant depending on m and C' and N is a positive real number, and $f(z) \to \mathscr{F}_A[\check{U}]$ in $\mathscr{Z}'(\overline{2\pi})$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$.

Proof. We consider the function

$$\begin{aligned} f(z) &= \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle \\ &= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} A^{-\alpha} z^\alpha \int_{Q_C} g_\alpha(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt \end{aligned} \tag{50}$$

for $z \in T^{O(C)}$. To prove the desired existence and analyticity of $f(z)$ it suffices to consider

$$h_\alpha(z) = \int_{Q_C} g_\alpha(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt, \qquad z \in T^{O(C)}, \tag{51}$$

for any fixed α, $|\alpha| \leqq m$. Let z_0 be an arbitrary but fixed point of $T^{O(C)}$, and let $N(z_0, r) \subset T^{O(C)}$ be an arbitrary neighborhood of z_0 with radius r whose closure is in $T^{O(C)}$. Let $z \in N(z_0, r)$, and let γ be an arbitrary n-tuple of nonnegative integers. From Vladimirov [9, Lem. 2, p. 223] we obtain the existence of a real number $d > 0$ such that

$$\left\langle \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right), y \right\rangle \geqq d\left|\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right|, \qquad z = x + iy \in N(z_0, r), \tag{52}$$

for all $t \in \mathbb{R}^n$ such that $u_C(t_1/a_1, \cdots, t_n/a_n) \leqq 0$. Since (48) holds for all $\Lambda \in O(C)$, we now choose $\Lambda = \frac{1}{2}y$, $z = x + iy \in N(z_0, r)$. (Obviously $\Lambda = \frac{1}{2}y \in O(C)$, $z = x + iy \in N(z_0, r) \subset T^{O(C)}$, since $O(C)$ is a cone; hence (48) holds by assumption for this choice of Λ.) With this choice of Λ in (48) and using (52), we obtain a constant M_α depending on α such that

$$\begin{aligned} &\left| \int_{Q_C} g_\alpha(t) t^\gamma \exp\left[2\pi i \left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt \right| \\ &\quad \leqq M_\alpha \int_{Q_C} |t^\gamma| \exp\left[-\pi\left\langle y, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right] dt \\ (53) \qquad &\quad \leqq M_\alpha \int_{Q_C} |t^\gamma| \exp\left[-d\pi \left|\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right|\right] dt \\ &\quad \leqq M_\alpha A^{\gamma + \bar{1}}(S_n) \int_0^\infty r^{|\gamma| + n - 1} \exp[-d\pi r]\, dr \\ &\quad < \infty, \end{aligned}$$

where $z = x + iy \in N(z_0, r)$ and S_n is the area of the unit sphere in $\mathbb{R}^n$ (See [8, Thm. 32, p. 39].) From (53) we conclude that the integral defining $h_\alpha(z)$ in (51) and any derivative $D^\gamma h_\alpha(z)$ of it converges uniformly for $z \in N(z_0, r)$. Since z_0 is an arbitrary point in $T^{O(C)}$, we thus have that $h_\alpha(z)$ exists and is analytic for $z \in T^{O(C)}$ for any α, $|\alpha| \leqq m$. From (50) it thus follows that $f(z)$ is analytic in $T^{O(C)}$.

Now let C' be an arbitrary compact subcone of $O(C)$. Since (48) holds for all $\Lambda \in O(C)$, we choose $\Lambda = \frac{1}{2}y$, $y \in C' \subset O(C)$. (Obviously $\Lambda = \frac{1}{2}y \in C' \subset O(C)$ if $y \in C' \subset O(C)$ since C' is a cone.) Again applying Vladimirov [9, Lem. 2, p. 223], we obtain a real number $d > 0$ such that

$$\left\langle \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right), y \right\rangle \geqq d|y| \left|\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right|, \qquad y \in C' \subset O(C),$$

for all $t \in Q_C$. Using this inequality and the above noted choice of $\Lambda = \frac{1}{2}y$, $y \in C' \subset O(C)$, in (48), we have by an estimate as in (53) that

$$\begin{aligned} &\left| \int_{Q_C} g_\alpha(t) \exp\left[2\pi i \left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt \right| \\ &\quad \leqq M_\alpha \int_{Q_C} \exp\left[-\pi d|y| \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right] dt \\ &\quad \leqq M_\alpha A^{\bar{1}}(S_n) \int_0^\infty r^{n-1} \exp[-\pi d|y| r]\, dr, \end{aligned}$$

$y \in C' \subset O(C)$, where again S_n is the area of the unit sphere in $\mathbb{R}^n$ and M_α is a constant depending on α. From this estimate and (50) we obtain

$$\begin{aligned} (54) \qquad |f(z)| &\leqq (S_n) \sum_{|\alpha| \leqq m} M_\alpha A^{-\alpha} |z^\alpha| \int_0^\infty r^{n-1} \exp[-\pi d|y| r]\, dr \\ &\leqq (S_n) \sum_{|\alpha| \leqq m} M_\alpha A^{-\alpha} (n-1)! (\pi d|y|)^{-n} |z^\alpha|, \end{aligned}$$

$z \in T^{C'}$, $C' \subset O(C)$, where we have integrated by parts $n - 1$ times on the integral in (54). The boundedness condition (49) follows immediately from (54).

It remains to prove that $f(z) \to \mathscr{F}_A[U]$ in $\mathscr{Z}'(\overline{2\pi})$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$. Let $\psi \in \mathscr{Z}(\overline{2\pi})$, and for the present let y be a fixed point of $O(C)$. From (54) we see that $f(z) \in \mathscr{Z}'(\overline{2\pi})$, $z = x + iy$, as a function of $x \in \mathbb{R}^n$. By (50) and a change of order of integration we obtain

$$\langle f(z), \psi(x)\rangle$$

$$\begin{aligned}
&= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} A^{-\alpha} \int_{\mathbb{R}^n} z^\alpha \psi(x) \int_{Qc} g_\alpha(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt\,dx \\
(55)\quad &= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} A^{-\alpha} \int_{Qc} g_\alpha(t) \int_{\mathbb{R}^n} z^\alpha \psi(x) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dx\,dt \\
&= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^\alpha \int_{Qc} g_\alpha(t) \left(D_t^\alpha \int_{\mathbb{R}^n} \psi(x) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dx\right) dt.
\end{aligned}$$

Since $\psi \in \mathscr{Z}(\overline{2\pi})$, then $\phi(t)$ defined in (5) is an element of $\mathscr{K}(A)$ such that $\psi(x) = \mathscr{F}_A[\phi(t); x]$. Recalling that $\check{\phi}(t) = \phi(-t)$, we thus have from (5) that

$$\left(\exp\left[-2\pi\left\langle y, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right]\check{\phi}(t)\right) = \int_{\mathbb{R}^n} \psi(x) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dx.$$

Putting this in (55) we obtain

$$\langle f(z), \psi(x)\rangle$$

$$\begin{aligned}
(56)\quad &= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} \int_{Qc} g_\alpha(t) \left(D_t^\alpha\left(\exp\left[-2\pi\left\langle y, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right]\check{\phi}(t)\right)\right) dt \\
&= \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[-2\pi\left\langle y, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right]\check{\phi}(t)\right\rangle,
\end{aligned}$$

where $z = x + iy \in T^{O(C)}$. Until now y is an arbitrary but fixed point in $O(C)$. Now $(\exp[-2\pi\langle y, (t_1/a_1, \cdots, t_n/a_n)\rangle]\check{\phi}(t)) \to \check{\phi}(t)$ in $\mathscr{K}(A)$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$, and $U \in \mathscr{K}'(A)$. Thus from (56) and (7),

$$\langle f(z), \psi(x)\rangle \to \frac{1}{a_1 \cdots a_n}\langle U, \check{\phi}(t)\rangle = \frac{1}{a_1 \cdots a_n}\langle \check{U}, \phi\rangle = \langle \mathscr{F}_A[\check{U}], \psi\rangle$$

as $y \to 0$, $y \in O(C)$, and the proof is complete.

We note the following more general setting for Theorem 11. Let

$$U = \sum_{|\alpha| \leqq m} D^\alpha g_\alpha(t), \qquad m < \infty,$$

where each $g_\alpha(t)$ is continuous on $\mathbb{R}^n$ and bounded as

$$|g_\alpha(t)| \leqq M_\alpha \exp\left[2\pi\left((b + \sigma)|\Lambda| + \left\langle \Lambda, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right)\right]$$

for all $\sigma > 0$ and all $\Lambda \in O(C)$ (recall (47);) and $\operatorname{supp}(g_\alpha) \subseteq \{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq b\}$, $|\alpha| \leqq m$, $b \geqq 0$. If the cone C is such that $\{t : 0 < u_C(t_1/a_1, \cdots, t_n/a_n) \leqq b\}$

is a bounded set in $\mathbb{R}^n$, then using essentially the same proof as in Theorem 11, we have that

$$f(z) = \frac{1}{a_1 \cdots a_n}\left\langle U, \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle$$

is an analytic function in $T^{O(C)}$; $f(z)$ satisfies

$$|f(z)| \leqq P(m; C')(1 + |z|)^N \exp[\pi(b + \sigma)|y|], \qquad z = x + iy \in T^{C' \setminus (C' \cap N(O,R))},$$

$C' \subset O(C)$, and $f(z) \to \mathscr{F}_A[U]$ in $\mathscr{L}'(\overline{2\pi})$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$. The method of proof for this more general setting is essentially the same as that in Theorem 11; further, we are interested in applying this type of result in the form given in Theorem 11 to obtain the analytic decomposition of the distributional finite Fourier transform. Thus we leave the details of the above more general setting for Theorem 11 to the interested reader, for we will not use this in the remainder of this paper.

We now show that the distributional finite Fourier transform can be decomposed into a finite sum of elements each of which is the distributional boundary value of a function which is analytic in a tubular radial domain. From Theorems 10 and 11 we know that any element in $\mathscr{K}'(A)$ whose finite Fourier transform can be so represented must be the distributional derivative of a continuous function which has growth as we have studied.

Let $A = (a_1, \cdots, a_n)$ be a fixed n-tuple of positive real numbers. Let C be an open cone such that $C = \bigcup_{j=1}^r C_j$, where $C_j, j = 1, \cdots, r$, are open connected cones such that

$$\mathbb{R}^n \Big\backslash \bigcup_{j=1}^r \left\{ t : u_{C_j}\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right) \leqq 0 \right\} \tag{57}$$

and

$$\left\{ t : u_{C_j}\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right) \leqq 0 \right\} \cap \left\{ t : u_{C_k}\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right) \leqq 0 \right\}, \tag{58}$$

$$j \neq k, \quad j = 1, \cdots, r, \quad k = 1, \cdots, r,$$

are sets of Lebesgue measure zero. Let the C'_j denote arbitrary compact subcones of $O(C_j), j = 1, \cdots, r$.

THEOREM 12. *Let $C = \bigcup_{j=1}^r C_j$ be an open cone such that the properties in the preceding paragraph are satisfied for the open connected cones C_j, $j = 1, \cdots, r$. Let $U = \sum_{|\alpha| \leq m} D^\alpha g_\alpha(t)$, $m < \infty$, where the $g_\alpha(t)$ are continuous functions on $\mathbb{R}^n$ which satisfy (48) for all $\Lambda \in O(C)$. Then $\mathscr{F}_A[\check{U}] = \sum_{j=1}^r W_j$ where each $W_j \in \mathscr{L}'(\overline{2\pi})$ is the $\mathscr{L}'(\overline{2\pi})$ boundary value as $y = \operatorname{Im}(z) \to 0$, $y \in O(C_j)$, of a function $f_j(z)$ which is analytic in $T^{O(C_j)}$ and satisfies (49) for $z \in T^{C'_j \setminus (C'_j \cap N(O,R))}$, $C'_j \subset O(C_j)$. Also $W_j = \mathscr{F}_A[\check{U}_j]$, where each $U_j \in \mathscr{K}'(A)$ and $\operatorname{supp}(U_j) \subseteq \{t : u_{C_j}(t_1/a_1, \cdots, t_n/a_n) \leqq 0\}$, $j = 1, \cdots, r$.*

Proof. Put $U_j = \sum_{|\alpha| \leq m} D^\alpha(\lambda_j(t) g_\alpha(t))$, where $\lambda_j(t)$ is the characteristic function of the set

$$Q_{C_j} = \left\{ t : u_{C_j}\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right) \leqq 0 \right\}, \qquad j = 1, \cdots, r.$$

We have that $U_j \in \mathscr{K}'(A)$ and $\operatorname{supp}(U_j) \subseteq Q_{C_j}, j = 1, \cdots, r$. Further, we have by hypothesis that

$$|g_\alpha(t)| \leqq M_\alpha \exp\left[2\pi\left\langle \Lambda, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right]$$

for all $\Lambda \in O(C)$ and hence for all $\Lambda \in O(C_j), j = 1, \cdots, r$. We now put

$$f_j(z) = \frac{1}{a_1 \cdots a_n}\left\langle U_j, \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle, \quad z \in T^{O(C_j)}, \quad j = 1, \cdots, r.$$

Then

$$\begin{aligned} f_j(z) &= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} A^{-\alpha} z^\alpha \int_{\mathbb{R}^n} \lambda_j(t) g_\alpha(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt \\ &= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} A^{-\alpha} z^\alpha \int_{Q_{C_j}} g_\alpha(t) \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right] dt. \end{aligned}$$

From the proof of Theorem 11 we obtain that $f_j(z)$ is analytic in $T^{O(C_j)}$, satisfies (49) for $z \in T^{C_j' \setminus (C_j' \cap N(O,R))}$, $C_j' \subset O(C_j)$, and $f_j(z) \to \mathscr{F}_A[\check{U}_j] = W_j \in \mathscr{Z}'(2\pi)$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C_j)$, $j = 1, \cdots, r$. Now let $\psi \in \mathscr{Z}(2\pi)$ and $\phi \in \mathscr{K}(A)$ such that $\psi(x) = \mathscr{F}_A[\phi(t); x]$. Using (7) and the assumptions (57) and (58) we have

$$\begin{aligned} \left\langle \sum_{j=1}^r W_j, \psi \right\rangle &= \sum_{j=1}^r \left\langle \mathscr{F}_A[\check{U}_j], \psi \right\rangle = \sum_{j=1}^r \frac{1}{a_1 \cdots a_n} \langle \check{U}_j, \phi \rangle \\ &= \frac{1}{a_1 \cdots a_n} \sum_{j=1}^r \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} \langle \lambda_j(t) g_\alpha(t), D^\alpha \check{\phi}(t) \rangle \\ &= \frac{1}{a_1 \cdots a_n} \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} \langle g_\alpha(t), D^\alpha \check{\phi}(t) \rangle \\ &= \frac{1}{a_1 \cdots a_n} \langle \check{U}, \phi \rangle \\ &= \langle \mathscr{F}_A[\check{U}], \psi \rangle. \end{aligned}$$

Thus $\mathscr{F}_A[\check{U}] = \sum_{j=1}^r W_j$, and the proof is complete.

It is interesting to note the restriction of Theorem 12 to one dimension. Let $C_1 = \{y : y > 0\}$ and $C_2 = \{y : y < 0\}$ be the open connected cones. Then $C = C_1 \cup C_2 = \mathbb{R}^1 \setminus \{0\}$, and $O(C) = \mathbb{R}^1$. Since $\{t : u_{C_1}(t/a) \leqq 0\} = \{t : t \geqq 0\}$ and $\{t : u_{C_2}(t/a) \leqq 0\} = \{t : t \leqq 0\}$, the hypotheses (57) and (58) in Theorem 12 hold for these cones C_1 and C_2, where a is an arbitrary positive real number. Now let $U = \sum_{k=1}^m D^k g_k(t)$, where the $g_k(t)$ are continuous functions on $\mathbb{R}^1$. The assumption (48) for all $\Lambda \in O(C)$ in Theorem 12 becomes

$$|g_k(t)| \leqq M_k \exp\left[2\pi\Lambda\left(\frac{t}{a}\right)\right]$$

for all $\Lambda \in O(C) = \mathbb{R}^1$. Theorem 12 concludes the existence of elements $W_1 = \mathscr{F}_a[\check{U}_1]$

and $W_2 = \mathscr{F}_a[\check{U}_2]$ such that $\mathscr{F}_a[\check{U}] = W_1 + W_2$. Here W_1 is the $\mathscr{Z}'(2\pi)$ boundary value of a function $f_1(z)$ which is analytic in the upper half-plane and satisfies

$$|f_1(z)| \leqq P(m;\delta)(1 + |z|)^N, \qquad \operatorname{Im}(z) \geqq \delta > 0,$$

and $U_1 \in \mathscr{K}'(a)$ such that $\operatorname{supp}(U_1) \subseteq \{t : t \geqq 0\}$. Further W_2 is the $\mathscr{Z}'(2\pi)$ boundary value of a function $f_2(z)$ which is analytic in the lower half-plane and satisfies the above boundedness condition in $\operatorname{Im}(z) \leqq \delta < 0$, and $U_2 \in \mathscr{K}'(a)$ such that $\operatorname{supp}(U_2) \subseteq \{t : t \leqq 0\}$.

Let us note that the preceding paragraph describes a Hilbert decomposition problem for the distributional finite Fourier transform in one dimension; that is this transform is represented as the sum of two boundary values of functions analytic in the upper and lower half-planes, respectively. Theorem 12 may thus be considered to be a generalization of the Hilbert decomposition problem to functions analytic in tubular radial domains. In Theorem 12 we have decomposed the distributional finite Fourier transform into the sum of a finite number of generalized functions each of which is the boundary value of an analytic function in a tubular radial domain.

Using Theorem 10 and part of the proof of Theorem 11, we now prove a representation theorem for the analytic functions which attain the distributional finite Fourier transform as boundary value. This result shows that the choice of the function $f(z)$ in the proof of Theorem 11 is the necessary choice.

THEOREM 13. *Let C be an open connected cone, and let C' be an arbitrary compact subcone of C. Let $f(z)$ be analytic in T^C and satisfy* (34) *for $b = 0$. Then there exists an element $V \in \mathscr{K}'(A)$ with* $\operatorname{supp}(V) \subseteq Q_C = \{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq 0\}$ *such that*

$$f(z) = \frac{1}{a_1 \cdots a_n}\left\langle V, \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle, \quad z \in T^{C'}, \quad C' \subset C.$$

Proof. From Theorem 10 we obtain the existence of an element $U \in \mathscr{K}'(A)$ such that $f(z) \to \mathscr{F}_A[U]$ in $\mathscr{Z}'(\overline{2\pi})$ as $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset C$, and $\operatorname{supp}(U) \subseteq Q_C = \{t : u_C(t_1/a_1, \cdots, t_n/a_n) \leqq 0\}$. From the proof of Theorem 10, we know that this element $U = \Omega g(t)$, where $g(t)$ is the continuous function defined in (36) which has support in Q_C, and Ω is the differential operator defined in the proof of Theorem 10. Further, from (47) we have that

$$|g(t)| \leqq M(C') \exp\left[2\pi\left(\sigma|\Lambda| + \left\langle \Lambda, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle\right)\right] \tag{59}$$

for all $\sigma > 0$ and all $\Lambda \in C' \subset C$. Now let y be an arbitrary but fixed point of $C' \subset C$. We have again by Vladimirov [9, Lemma 2, p. 223] that there exists a real number $d > 0$ such that

$$\left\langle y, \left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right\rangle \geqq d|y|\left|\left(\frac{t_1}{a_1}, \cdots, \frac{t_n}{a_n}\right)\right|, \qquad y \in C' \subset C,$$

for all $t \in Q_C$. Using this inequality and choosing $\Lambda = \frac{1}{2}y$, $y \in C' \subset C$, in (59) (this is a suitable choice of Λ since C' is a cone and $\frac{1}{2}y \in C' \subset C$ if $y \in C' \subset C$, and

(59) holds for all choices of $\Lambda \in C' \subset C$) we have

$$\left|\exp\left[-2\pi\left\langle y,\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right\rangle\right]g(t)\right| \le M(C')\exp[\pi\sigma|y|]\exp\left[-\pi d|y|\left|\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right|\right], \tag{60}$$

$y \in C' \subset C$, for all $t \in Q_C$. Now let $1 \leqq p < \infty$. Since $\operatorname{supp}(g) \subset Q_C$, then the estimate in (60) and integration by parts as in (54) show that

$$\begin{aligned}
\int_{\mathbb{R}^n}&\left|\exp\left[-2\pi\left\langle y,\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right\rangle\right]g(t)\right|^p dt \\
&= \int_{Q_C}\left|\exp\left[-2\pi\left\langle y,\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right\rangle\right]g(t)\right|^p dt \\
&\leqq (M(C'))^p\exp[\pi\sigma p|y|]\int_{Q_C}\exp\left[-\pi\, dp|y|\left|\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right|\right]dt \\
&\leqq (M(C'))^p\exp[\pi\sigma p|y|](S_n)\int_0^\infty r^{n-1}\exp[-\pi\, dp|y|r]\,dr \\
&\leqq (M(C'))^p(S_n)(A^{\bar{1}})\exp[\pi\sigma p|y|](n-1)!(\pi\, dp|y|)^{-n},
\end{aligned} \tag{61}$$

for all $\sigma > 0$, where (S_n) is the area of the unit sphere in $\mathbb{R}^n$. Thus for y arbitrary but fixed in $C' \subset C$, (61) shows in particular that

$$\left(\exp\left[-2\pi\left\langle y,\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right\rangle\right]g(t)\right) \in L^1 \cap L^2.$$

Recalling the definition of the differential operator Ω in the proof of Theorem 10 and using a straightforward calculation, we have

$$\begin{aligned}
\frac{(-1)^{|K|}}{a_1\cdots a_n}&\left\langle \Omega g(t),\exp\left[2\pi i\left(\frac{z_1t_1}{a_1}+\cdots+\frac{z_nt_n}{a_n}\right)\right]\right\rangle \\
&= z^K\mathscr{F}\left[\exp\left[-2\pi\left\langle y,\left(\frac{t_1}{a_1},\cdots,\frac{t_n}{a_n}\right)\right\rangle\right]g(t);\left(\frac{x_1}{a_1},\cdots,\frac{x_n}{a_n}\right)\right],
\end{aligned} \tag{62}$$

$z \in T^{C'}, C' \subset C$. The Fourier transform here can be interpreted in both the L^1 and L^2 sense. Recalling that $U = \Omega g(t)$ and combining (62) and (37), we thus have

$$f(z) = \frac{1}{a_1\cdots a_n}\left\langle(-1)^{|K|}U,\exp\left[2\pi i\left(\frac{z_1t_1}{a_1}+\cdots+\frac{z_nt_n}{a_n}\right)\right]\right\rangle,\quad z \in T^{C'},\quad C' \subset C.$$

We now put $V = (-1)^{|K|}U$. Then $V \in \mathscr{K}'(A)$ and $\operatorname{supp}(V) \subseteq Q_C$ since U has these properties, and the proof is complete.

When $b = 0$ in Theorem 10, the proof of Theorem 13 shows that the Fourier transform in (37) can in fact be interpreted in both the L^1 and L^2 sense. Also the equality

$$f(z) = \frac{1}{a_1 \cdots a_n}\left\langle V, \exp\left[2\pi i\left(\frac{z_1 t_1}{a_1} + \cdots + \frac{z_n t_n}{a_n}\right)\right]\right\rangle$$

in Theorem 13 is immediately seen to hold for $z \in T^{O(C')}$, $C' \subset C$. This is true because by the proof of Theorem 11, $\langle V, \exp[2\pi i(z_1 t_1/a_1 + \cdots + z_n t_n/a_n)]\rangle$ is analytic in $T^{C'}$, $C' \subset C$; hence by Bochner's analytic extension theorem [1, Chap. V], it is analytic in $T^{O(C')}$, $C' \subset C$. By this same theorem of Bochner, $f(z)$ is analytic in $T^{O(C)}$ since by assumption in Theorem 13 it is analytic in T^C. Hence $f(z)$ is analytic in $T^{O(C')}$, $C' \subset C$, since $O(C') \subset O(C)$. Thus the identity theorem for analytic functions gives the equality in Theorem 13 in $T^{O(C')}$, $C' \subset C$.

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EXTENDED CHEBYSHEV SYSTEMS ON $(-\infty, \infty)$*

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Abstract. Let $0 \leqq t_0 < t_1 < \cdots < t_m$ be a sequence of integers. Necessary and sufficient conditions are obtained for $\{x^{t_0}, x^{t_1}, \cdots, x^{t_m}\}$ to form an extended Chebyshev system of order $n + 1$ on $(-\infty, \infty)$.

A system of functions $\mu_i \in C[a, b]$, $i = 0, 1, \cdots, m$, is said to be a Chebyshev system (TS) on $[a, b]$, if $\sum_{i=0}^{m} a_i\mu_i(x)$ has at most m zeros in $[a, b]$, for any nontrivial choice of real $\{a_i\}$.

A sequence of integers $0 \leqq t_0 < t_1 < \cdots < t_m$ is said to have the alternating parity property (APP), if, for all i, t_{2i} is even and t_{2i+1} is odd.

In [3] the following theorem was proved.

THEOREM 1. *$\{x^{t_i}\}$, $i = 0, 1, \cdots, m$, is a TS on $(-\infty, \infty)$ if and only if $t_0 = 0$ and $\{t_i\}$, $i = 0, 1, \cdots, m$, has APP.*

It is well known [2, p. 24] that the following are equivalent:

(i) $\{\mu_i\}$, $i = 0, 1, \cdots, m$, is a TS on $[a, b]$;

(ii) for any $a \leqq x_0 < x_1 < \cdots < x_m \leqq b$; $y_0, y_1, \cdots, y_m$, there exists a unique $p(x) = \sum_{i=0}^{m} c_i\mu_i(x)$ satisfying $p(x_j) = y_j$, $j = 0, 1, \cdots, m$.

If the functions $\{\mu_i\}$ are differentiable, then we can attempt to solve interpolation problems of a more general nature than in (ii). In particular, suppose $\mu_i \in C^n[a, b]$, $i = 0, 1, \cdots, m$. We consider the problem of Hermite interpolation i.e., let $a \leqq x_0 < x_1 < \cdots < x_k \leqq b$; $y_j^{(0)}, y_j^{(1)}, \cdots, y_j^{(a_j-1)}$, $j = 0, 1, \cdots, k$, be arbitrary, with $\sum_{j=0}^{k} a_j = m + 1$ and $\max_j (a_j - 1) \leqq n$. We ask the following: Does there exist a unique

$$p(x) = \sum_{i=0}^{m} c_i\mu_i(x)$$

such that

$$p^{(r)}(x_j) = y_j^{(r)}, \qquad j = 0, 1, \cdots, k, \quad r = 0, 1, \cdots, a_j - 1?$$

If a unique solution exists for every choice of the parameters (with the given restrictions) then $\{\mu_i\}$ is said to be an extended Chebyshev system (ETS) of order $n + 1$ (cf. [1, p. 6] for an equivalent formulation). Note that if $\{\mu_i\}$ is an ETS of order $n + 1$ then it is an ETS of any lower order, and that a TS is an ETS of order 1. If $n = m$ then $\{\mu_i\}$ will be referred to simply as an extended Chebyshev system, with no mention of the order.

Since we are only concerned with a unique solution it follows that $\{\mu_i\}$ is an ETS of order $n + 1$ if and only if the following holds: Let

$$p(x) = \sum_{i=0}^{m} c_i\mu_i(x)$$

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be a solution to

$$p^{(r)}(x_j) = 0, \qquad j = 0, 1, \cdots, k, \quad r = 0, 1, \cdots, a_j - 1,$$

where $\sum_{j=0}^{k} a_j = m + 1$ and $\max_j (a_j - 1) \leqq n$. Then $p(x) \equiv 0$.

In this paper we let $0 \leqq t_0 < t_1 < \cdots < t_m$ be a sequence of integers, set $\mu_i(x) = x^{t_i}$, $i = 0, 1, \cdots, m$, and seek necessary and sufficient conditions on $\{t_i\}$ for $\{x^{t_i}\}$ to be an ETS of order $n + 1$ on $(-\infty, \infty)$.

THEOREM 2. *Let $0 \leqq t_0 < t_1 < \cdots < t_m$ be a sequence of integers. Then $\{x^{t_i}\}$, $i = 0, 1, \cdots, m$, is an ETS of order $n + 1$ on $(-\infty, \infty)$ if and only if $t_i = i$, $i = 0, 1, \cdots, n$, and $\{t_i\}$ has APP.*

Proof. If $\{t_i\}$ does not have APP, then $\{x^{t_i}\}$ is not a TS, hence not an ETS of any order. Now suppose that there exists i, $0 \leqq i \leqq n$, such that $t_i > i$. Let j be the smallest such index. We shall construct a Hermite interpolation problem which has no solution. Let $x_0 = 0$; $y_0^{(j)} \neq 0$. The remaining parameters may be chosen arbitrarily. Let

$$p(x) = \sum_{i=0}^{m} c_i x^{t_i}.$$

Then

$$p^{(j)}(x) = \sum_{i=j}^{m} t_i(t_i - 1) \cdots (t_i - j + 1) c_i x^{t_i - j},$$

so that $p^{(j)}(0) = 0$. Since p was arbitrary, interpolation is impossible. Thus the conditions are necessary.

Now suppose that $\{t_i\}$ satisfies the conditions of the theorem and $p(x) = \sum_{i=0}^{m} c_i x^{t_i}$ satisfies $p^{(r)}(x_j) = 0, j = 0, 1, \cdots, k$; $r = 0, 1, \cdots, a_j - 1$. Without loss of generality we can assume that $\max_j (a_j - 1) = n$. The theorem is true for $n = 0$. Assume the result is true for $n - 1$. Applying Rolle's theorem to p, we obtain points z_j, $x_{j-1} < z_j < x_j$, $j = 1, 2, \cdots, k$, such that $p'(z_j) = 0$. Applying the induction hypothesis to

$$p'(x) = \sum_{i=1}^{m} t_i c_i x^{t_i - 1} = \sum_{i=0}^{m-1} b_i x^{s_i},$$

where $s_i = i$, $i = 0, 1, \cdots, n - 1$, and $\{s_i\}$ has APP, we obtain $p'(x) \equiv 0$. Therefore, $p(x) \equiv 0$, and the theorem is proved.

COROLLARY. *Let $0 \leqq t_0 < t_1 < \cdots < t_m$ be a sequence of integers. Then $\{x^{t_i}\}$, $i = 0, 1, \cdots, m$, is an extended Chebyshev system on $(-\infty, \infty)$ if and only if $t_i = i, i = 0, 1, \cdots, m$.*

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INVERSION OF GENERAL INTEGRAL TRANSFORMS*

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Abstract. Following the terminology of Titchmarsh, we call an integral transform with a kernel depending on the product of its arguments

$$\int_D k(xy)\phi(y)\,dy = f(x), \qquad x \in D,$$

a general integral transform. In this paper, the transform is inverted with $D = (-\infty, \infty)$, by means of the ordinary and generalized Mellin transforms.

1. Introduction. A common characteristic of many integral transforms, such as the Fourier (sine, cosine and exponential) transforms, the Laplace transform, and the Hankel transform is that their kernels depend upon the product of their arguments. Because of this, it is of interest to study an integral transform of the general form

$$\int_{-\infty}^{\infty} k(xy)\phi(y)\,dy = f(x), \qquad x \in R. \tag{1.1}$$

Indeed the transforms mentioned above are all of the form (1.1), with $k(xy)$ replaced by the appropriate kernel. Of course, this has been known for some time, and Titchmarsh [11] considered the inversion of integral transforms

$$\int_{0}^{\infty} k(xy)\phi(y)\,dy = f(x), \qquad x \in (0, \infty). \tag{1.2}$$

Considered as an integral equation of the first kind with given free term $f(x)$, (1.2) arises in certain inverse problems in the theory of light scattering [6], [10] and has been investigated in this connection by Perelman and others [7], [8], [10]. More generally, Fox [4] has shown how the Laplace transform can be used to solve a variety of integral equations of the form (1.2).

While work has been done on the inversion of (1.2) and equations of the form

$$\int_D k(xy)\phi(y)\,dy = f(x), \qquad x \in D,$$

where D is a finite interval [9], we have seen no work on the inversion of (1.1). Hence, we consider (1.1) in this paper. First, the ordinary Mellin transform is used to invert (1.1) and, second, a generalized Mellin transform [5] is used so that the inversion can be accomplished for a larger class of kernels and free terms than is possible with the ordinary Mellin transform.

The Mellin transform, $F(s)$, of a complex-valued function $f(x)$ defined for almost all $x > 0$, is defined by

$$F(s) \equiv \int_0^{\infty} f(x)x^{s-1}\,dx,$$

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where s is in C, the complex plane, whenever the integral exists. When the integral exists, it exists on a vertical line, in a vertical strip, or in a right or left half-plane in C. We shall always denote the Mellin transform of a function $f(x)$ by its upper-case $F(s)$. The inverse Mellin transform $f(x)$, $x > 0$, of $F(s)$ a complex-valued function of the complex variable s, is defined by

$$f(x) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}\,ds,$$

where c is a real number, whenever the integral exists. Theorems concerning the Mellin transform may be found in Titchmarsh [11]. The theorems to be used here are listed in § 6. Let R^+ and R^- denote $(0, \infty)$ and $(-\infty, 0)$, respectively. Let $L^p(S)$, where $S \subseteq R$ or S is a vertical line in C, be the Banach space of complex-valued, measurable functions $f(x)$ defined on S such that the norm of f,

$$\|f\| \equiv \left(\int_S |f(x)|^p\,dx\right)^{1/p}, \qquad 1 \leqq p < +\infty,$$

is finite. The letter s will always represent a complex variable, $s \equiv \sigma + i\tau$, and we denote the real and imaginary parts of s by $\mathrm{Re}\,(s)$ and $\mathrm{Im}\,(s)$, respectively. Throughout, E will denote the vertical line $\{s \in C \mid \mathrm{Re}\,(s) = 1/2\}$.

2. L^2 case. The first problem under consideration is to find, for a given kernel $k(u)$ in $L^2(R)$ and a given free term $f(x)$ in $L^2(R)$, a function $\phi(y)$ in $L^2(R)$ which satisfies for almost all x in R, the integral equation

$$\int_{-\infty}^{\infty} k(xy)\phi(y)\,dy = f(x). \tag{2.1}$$

More precisely, we seek a function $\phi(y)$ in $L^2(R)$, such that for almost all x in R, the limit

$$\lim_{\substack{a\to+\infty\\ b\to+\infty}} \int_{-a}^{b} k(xy)\phi(y)\,dy$$

exists and defines a function equal to $f(x)$ for almost all x in R.

Define functions $f_1(x)$ and $f_2(x)$ as

$$f_1(x) \equiv \begin{cases} f(x), & x > 0,\\ 0, & x < 0,\end{cases} \qquad f^-(x) \equiv \begin{cases} 0, & x > 0,\\ f(x), & x < 0,\end{cases} \tag{2.2}$$
$$f_2(x) = f^-(-x),$$

and in a similar way, define $\phi_1(x)$, $k_1(x)$, $\phi_2(x)$, $k_2(x)$, $k^-(x)$ and $\phi^-(x)$.

With these definitions and conditions it is not difficult to show that *the problem of solving* (2.1) *is equivalent to the problem of solving the system of integral equations*

$$\begin{aligned} &\int_0^\infty k_2(xy)\phi_2(y)\,dy + \int_0^\infty k_1(xy)\phi_1(y)\,dy = f_1(x), \qquad x > 0,\\ &\int_0^\infty k_1(xy)\phi_2(y)\,dy + \int_0^\infty k_2(xy)\phi_1(y)\,dy = f_2(x), \qquad x > 0. \end{aligned} \tag{2.3}$$

If (2.3) is solved for $\phi_1(x)$ and $\phi_2(x)$ in $L^2(R^+)$ then $\phi(x) = \phi_1(x) + \phi_2(-x)$ is a solution for (2.1) in $L^2(R)$.

The use of the Mellin transform to solve system (2.3) is contained in the proof of the following theorem.

THEOREM 2.1. *If $k(x)$ and $f(x)$ are in $L^2(R)$, if $K_1(s)$ and $K_2(s)$ are multipliers of $L^2(1/2 - i\infty, 1/2 + i\infty)$ and if the quotients*

$$G_j(s) \equiv \frac{F_i(1-s)K_2(1-s) - F_j(1-s)K_1(1-s)}{K_2^2(1-s) - K_1^2(1-s)}, \quad (i,j) = (1,2),(2,1), \tag{2.4}$$

are in $L^2(1/2 - i\infty, 1/2 + i\infty)$ then the equation

$$\int_{-\infty}^{\infty} k(xy)\phi(y)\,dy = f(x), \qquad x \in R,$$

has a unique solution in $L^2(R)$, obtainable by application of the Mellin transform to the system (2.3).

Proof. Since $G_1(s)$ and $G_2(s)$ are in $L^2(E)$, we may define the inverse Mellin transforms

$$\begin{aligned}
\phi_1(x) &\equiv \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_E G_1(s)x^{-s}\,ds, & x > 0, \\ 0, & x < 0, \end{cases} \\
\phi_2(x) &\equiv \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_E G_2(s)x^{-s}\,ds, & x > 0, \\ 0, & x < 0. \end{cases}
\end{aligned} \tag{2.5}$$

Both of these functions belong to $L^2(R^+)$ and have Mellin transforms $G_1(s)$ and $G_2(s)$, respectively, for all s with $\mathrm{Re}(s) = 1/2$. We claim that the functions defined in (2.5) solve the system (2.3). To this end, we replace s by $1 - s$ in $G_1(s)$ and $G_2(s)$ as in (2.4). Algebraic manipulation shows that

$$\begin{aligned}
K_2(s)G_2(1-s) + K_1(s)G_1(1-s) &= F_1(s), \\
K_1(s)G_2(1-s) + K_2(s)G_1(1-s) &= F_2(s),
\end{aligned} \tag{2.6}$$

where $\mathrm{Re}(s) = 1/2$. Since $K_1(s)$ and $K_2(s)$ are multipliers of $L^2(E)$, all the terms on the left-hand side of (2.6) are in $L^2(E)$. Plancherel's theorem for Mellin transforms (Theorem 6.1, § 6) implies that $F_1(s)$ and $F_2(s)$ are also in $L^2(E)$, so that every term in (2.6) has an inverse Mellin transform.

Multiplying the equations in (2.6) by $(1/2\pi i)x^{-s}$ and integrating over E, we obtain for almost all $x > 0$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_E K_2(s)G_2(1-s)x^{-s}\,ds + \frac{1}{2\pi i}\int_E K_1(s)G_1(1-s)x^{-s}\,ds &= f_1(x), \\
\frac{1}{2\pi i}\int_E K_1(s)G_2(1-s)x^{-s}\,ds + \frac{1}{2\pi i}\int_E K_2(s)G_1(1-s)x^{-s}\,ds &= f_2(x).
\end{aligned} \tag{2.7}$$

An application of Corollary 6.1 shows that

$$\frac{1}{2\pi i}\int_E K_i(s)G_j(1-s)x^{-s}\,ds = \int_0^\infty k_i(xy)\phi_j(y)\,dy$$

for all $x > 0$, and $i, j = 1, 2$. Therefore, we conclude that for almost all $x > 0$, $\phi_1(x)$ and $\phi_2(x)$ as defined in (2.5) satisfy (2.3). Furthermore, $\phi(x) = \phi_1(x) + \phi_2(-x)$ solves equation (2.1). The solution is unique in $L^2(R)$ by virtue of the uniqueness theorem for the Mellin transform.

Remark. It may be noted that the quotients $G_j(s)$ in (2.4) are obtainable from (2.6) by a formal application of Cramer's rule. That is, the system of integral equations in (2.3) is transformed via the Mellin transform to the algebraic equations in (2.6), then Cramer's rule is applied to the algebraic equations to obtain the quotients in (2.4). Finally, the inverse Mellin transform transforms the G_j's back to the solution pair of the system of integral equations.

3. Example. Consider the kernel

$$k(x) = \begin{cases} (x+1)\log|x|, & |x| \leqq 1, \\ 0, & |x| > 1, \end{cases}$$

and the right-hand side

$$f(x) = \begin{cases} (\log|x|)^2, & |x| \leqq 1, \\ 0, & |x| > 1. \end{cases}$$

Then the respective Mellin transforms are [1, with (14), p. 315, corrected]

$$K_1(s) = \frac{-1}{(s+1)^2} - \frac{1}{s^2}, \qquad \mathrm{Re}(s) > 0,$$

$$K_2(s) = \frac{1}{(s+1)^2} - \frac{1}{s^2}, \qquad \mathrm{Re}(s) > 0,$$

$$F_1(s) = F_2(s) = \frac{+2}{s^3}, \qquad \mathrm{Re}(s) > 0.$$

Therefore, the quotients $G_1(s)$ and $G_2(s)$ are both equal to

$$\frac{1}{s-1}, \qquad \mathrm{Re}(s) < 1,$$

and so

$$\phi_1(x) = \phi_2(x) = \begin{cases} -1/x, & x > 1, \\ 0, & \text{otherwise}. \end{cases}$$

Hence the solution to (2.1) is

$$\phi(x) = \begin{cases} -1/|x|, & |x| > 1, \\ 0, & |x| \leqq 1. \end{cases}$$

This can be verified by direct integration.

4. Application of generalized Mellin transforms. Although the exponential Fourier transform is of the form (1.1), Theorem 2.1 cannot be used for its inversion. In order to obtain an inversion for this important case as well as for other transforms which do not yield to the result in § 2, we turn to a generalized Mellin transform [5]. The reader is now referred to § 6.1, where the definitions that we need from [5] are listed.

Let P^- be defined the same as P^+, with R^+ replaced by R^-. Extending the functions in P^+ and P^- to all of R by defining the functions to be zero on R^- and R^+, respectively, we define P as the space of all functions

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

where $\phi^+ \in P^+$ and $\phi^- \in P^-$. A sequence $\{\phi_n(x)\}$ converges to zero in P if and only if $\{\phi_n^+(x)\}$ and $\{\phi_n^-(x)\}$ converge to zero in P^+ and P^-, respectively.

The relationship among P, P^+ and P^- is easy to see. The expected connection among P', $(P^+)'$, and $(P^-)'$ is proved in the following lemma.

LEMMA 4.1. *The space of continuous linear functionals on* P, *denoted by* P', *is identical to*

$$(P^+)' + (P^-)'.$$

Proof. For $f \in P'$ denote by f^+ and f^- the restrictions of f to P^+ and P^-, respectively. Then for every ϕ in P we have

$$\langle f, \phi\rangle = \langle f, \phi^+ + \phi^-\rangle = \langle f, \phi^+\rangle + \langle f, \phi^-\rangle = \langle f^+, \phi^+\rangle + \langle f^-, \phi^-\rangle.$$

On the other hand, f^+ given in $(P^+)'$ and g^- given in $(P^-)'$ may be extended to continuous linear functionals on all of P by setting

$$\langle f^+, \phi\rangle = \langle f^+, \phi^+\rangle, \qquad \langle g^-, \phi\rangle = \langle g^-, \phi^-\rangle,$$

$\phi \in P$. These equalities, combined with the inclusions

$$\begin{aligned} &P^+ \subset P \subset P' \subset (P^+)', \\ &P^- \subset P \subset P' \subset (P^-)', \\ &P^+ + P^- = P \subset P' = (P^+ + P)', \end{aligned}$$

give the desired conclusion.

Now, knowing the structure of P and P', we turn to defining the generalized analogue of the transform

$$\int_{-\infty}^{\infty} k(xy)\phi(y)\, dy.$$

To this end we state the following definition.

DEFINITION 4.1. The *reflection*, ϕ_*^-, *of* ϕ^- *in* P^- is defined by

$$\phi_*^-(x) = \phi^-(-x), \qquad x > 0.$$

Similarly, the *reflection*, ϕ_*^-, *of* ϕ^+ *in* P^+ is defined by

$$\phi_*^+(x) = \phi^+(-x), \qquad x < 0.$$

The *reflections*, $f_*^- \in (P^+)'$, $f_*^+ \in (P^-)'$, *of* $f^- \in (P^-)'$ *and* $f^+ \in (P^+)'$, respectively, are defined by

$$\begin{aligned} \langle f_*^+, \phi^- \rangle &= \langle f^+, \phi_*^- \rangle, \qquad \phi^- \in P^-, \\ \langle f_*^-, \phi \rangle &= \langle f^-, \phi_*^+ \rangle, \qquad \phi^+ \in P^+. \end{aligned} \tag{4.1}$$

The generalized analogue of the standard convolution

$$\int_0^\infty k(xy)\phi(y)\,dy$$

is contained in Definition 6.6. Let k^+ and g^+ be elements of $(P^+)'$ such that $k^+ \wedge g^+$ is defined and in $(P^+)'$. Then $(k^+ \wedge g^+)_*$ is in $(P^-)'$ and furthermore

$$(k^+ \wedge g^+)_* = k^+ \wedge g_*^+ = k_*^+ \wedge g^+. \tag{4.2}$$

Indeed, from the definitions of reflection and convolution we obtain

$$\begin{aligned} \langle (k^+ \wedge g^+)_*, \phi_*^+ \rangle &\equiv \langle k^+ \wedge g^+, \phi^+ \rangle \\ &= \langle g^+(x), x^{-1} \langle k^+(y), \phi^+(x^{-1}y) \rangle \rangle \\ &= \langle g^+(-x), -x^{-1} \langle k^+(y), \phi_*^+(-x^{-1}y) \rangle \rangle \\ &= \langle k^+ \wedge g_*^+, \phi_*^+ \rangle, \end{aligned}$$

for the first equality, and the second equality follows similarly. We define the generalized analogue for the convolution

$$\int_{-\infty}^0 k(xy)\phi(y)\,dy$$

in the obvious way, and note that if $k^- \wedge g^-$ is well-defined, then

$$(k^- \wedge g^-)_* = k^- \wedge g_*^- = k_*^- \wedge g^-, \qquad k^-, g^- \in (P^-)', \tag{4.3}$$

is a P^+ distribution. Equalities (4.2) and (4.3) show that if the convolutions of k^+ with all elements of $(P^+)'$ and k^- with all elements in $(P^-)'$ are defined then so are their convolutions with all elements of $(P^-)'$ and $(P^+)'$ respectively.

With this information about the reflection of convolution on $(P^+)'$ and $(P^-)'$, we formulate the next definition.

DEFINITION 4.2. Let k and g in P' be such that $k = k^+ + k^-$ and $g = g^+ + g^-$. Then $k \wedge g$ in P' is defined by

$$k \wedge g = (k^+ \wedge g^+) + (k^+ \wedge g^-) + (k^- \wedge g^+) + (k^- \wedge g^-) \tag{4.4}$$

provided all the convolutions on the right-hand side of (4.4) exist.

Having defined this convolution on P' we turn to the solution of (1.1) under the following conditions upon $k(x)$ and $f(x)$:

(4.5) $K_1(s)$ and $K_2(s)$, the ordinary Mellin transforms of k_1 and k_2, exist and are analytic in a vertical strip containing the line $\mathrm{Re}(s) = 1/2$.

(4.6) $f_i(x)$ is an ordinary function such that $x^{1/2}f_i(x)$ is in $(P^+S)'$, $i = 1, 2$. Note that $f_i \in L^2(R^+)$ is sufficient for $x^{1/2}f_i(x)$ to be in $(P^+S)'$ (see [5, Thm. 3]).

(4.7) $G_j(s)$ is an ordinary function which is locally integrable and of finite order on $(1/2 - i\infty, 1/2 + i\infty)$, $j = 1, 2$.

Our result concerning (1.1) is stated as the following theorem.

THEOREM 4.1. *If conditions* (4.5) *through* (4.7) *hold, then* (1.1) *has a solution* $\phi \in P'$ *which is unique among all* $\phi = \phi_1 + \phi_{2*}$ *such that* $x^{1/2}\phi_1$ *and* $x^{1/2}\phi_2$ *are in* $(P^+S)'$.

Proof. The argument is patterned after the proof of the previous theorem. We shall define ϕ_1 and ϕ_2 in $(P^+)'$ as the generalized inverse Mellin transforms of $G_1(s)$ and $G_2(s)$, and then show that ϕ_1 and ϕ_2 satisfy a system of equations analogous to (2.3). Then $\phi \equiv \phi_1 + \phi_{2*}$ will be a solution to (1.1) in the sense that

$$\langle k \wedge \phi, \psi \rangle = \langle f, \psi \rangle$$

for all ψ in P.

Since $G_1(s)$ and $G_2(s)$ are only required to be of finite order as $|\tau| \to \infty$ on E, the ordinary inverse Mellin transforms of G_1 and G_2 need not exist. However, the finite order and local integrability of G_1 and G_2 imply that the generalized inverse Mellin transforms of G_1 and G_2 (considered as elements of Q') exist, so we define two P^+ distributions

$$\phi_i(x) \equiv (M^{-1}(G_i)), \qquad i = 1, 2,$$

where the inversion is taken along E. Furthermore, $x^{1/2}\phi_1(x)$ and $x^{1/2}\phi_2(x)$ are in $(P^+S)'$.

Considering (2.6) again, we see, in view of condition (4.5), that all the terms on the left-hand side of (2.6) are locally integrable and of finite order on E. Hence, all these terms can be considered as elements of Q' and we have for all $\Psi, Z \in Q$,

$$\langle K_2(s)G_2(1-s), \Psi(s)\rangle + \langle K_1(s)G_1(1-s), \Psi(s)\rangle = \langle F_1(s), \Psi(s)\rangle,$$
$$\langle K_1(s)G_2(1-s), Z(s)\rangle + \langle K_2(s)G_1(1-s), Z(s)\rangle = \langle F_2(s), Z(s)\rangle,$$

$s \in E$. By (4.6), $F_1(s)$ and $F_2(s)$ are Q distributions.

Taking generalized inverse Mellin transforms in these equations, we get [5, Thm. 4] with $M^{-1}\Psi = \psi^+$, $M^{-1}Z = \zeta^+$,

$$\begin{aligned} &\langle k_2 \wedge \phi_2, \psi^+\rangle + \langle k_1 \wedge \phi_1, \psi^+\rangle = \langle f_1, \psi^+\rangle, \\ &\langle k_1 \wedge \phi_2, \zeta^+\rangle + \langle k_2 \wedge \phi_1, \zeta^+\rangle = \langle f_2, \zeta^+\rangle \end{aligned} \tag{4.8}$$

for all ψ^+, $\zeta^+ \in P^+$, which means that the generalized analogue of (2.3) holds. Now a straightforward application of the equalities (4.2) and (4.3), followed by the addition of the corresponding sides of (4.8), gives for all $\psi^+ \in P^+$, $\psi^- \equiv \zeta_*^+ \in P^-$,

$$\begin{aligned} \langle k_{2*} \wedge \phi_{2*}, \psi^+\rangle + \langle k_1 \wedge \phi_1, \psi^+\rangle + \langle k_1 \wedge \phi_{2*}, \psi^-\rangle + \langle k_{2*} \wedge \phi_1, \psi^-\rangle \\ = \langle f_1, \psi^+\rangle + \langle f_{2*}, \psi^-\rangle = \langle f_1 + f_{2*}, \psi^+ + \psi^-\rangle, \end{aligned}$$

which means that

$$\langle k \wedge \phi, \psi \rangle = \langle (k_1 + k_{2*}) \wedge (\phi_1 + \phi_{2*}), \psi \rangle = \langle f, \psi \rangle$$

for all $\psi \in P$. That is, $\phi = \phi_1 + \phi_{2*}$ solves (1.1) in the P distribution sense. The solution is unique among all $\phi = \phi_1 + \phi_{2*}$ with $x^{1/2}\phi_1$ and $x^{1/2}\phi_2$ in $(P^+S)'$, by virtue of the uniqueness theorem for the generalized Mellin transform.

5. Examples. Consider the equation

$$\int_{-\infty}^{\infty} e^{-ixy}\phi(y)\,dy = e^{-ix}, \qquad x \in R, \tag{5.1}$$

so that $k_1(x) = e^{-ix}$, $k_2(x) = e^{ix}$, $f_1(x) = e^{-ix}$, $f_2(x) = e^{ix}$. Hence [1],

$$K_1(s) = F_1(s) = \Gamma(s)\exp\left(-i\frac{\pi}{2}s\right), \qquad -1 < \operatorname{Re}(s) < 1,$$

$$K_2(s) = F_2(s) = \Gamma(s)\exp\left(i\frac{\pi}{2}s\right), \qquad 0 < \operatorname{Re}(s) < 1,$$

and we have

$$G_1(s) = 1, \qquad G_2(s) = 0, \qquad 0 < \operatorname{Re}(s) < 1,$$

so that the solution to (5.1) is

$$M^{-1}(1) + (M^{-1}(0))_* = \delta(y - 1).$$

For another example consider the equation

$$\int_{-\infty}^{\infty} e^{-ixy}\phi(y)\,dy = e^{-|x|}, \qquad x \in R. \tag{5.2}$$

In this case we have $K_1(s)$ and $K_2(s)$ as before and

$$F_1(s) = F_2(s) = \Gamma(s), \qquad \operatorname{Re}(s) > 0.$$

Therefore,

$$G_1(s) = G_2(s) = \frac{1}{2\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right), \qquad 0 < \operatorname{Re}(s) < 1,$$

and since G_1 and G_2 are analytic in a strip of positive width an application of the convolution theorem gives

$$\phi_1(x) = \phi_2(x) = \frac{1}{\pi}\int_0^{\infty} e^{-x^2 y}\, e^{-y}\,dy = \frac{1}{\pi}\,\frac{1}{1 + x^2}.$$

Since $G_1(s)$ and $G_2(s)$ are of exponential decrease as $|\tau| \to \infty$, the ordinary convolution theorem may be used and

$$\phi(x) = \frac{1}{\pi}\,\frac{1}{1 + x^2}$$

is a solution in the ordinary sense also.

6. Theorems about Mellin transforms and definitions from [5].

6.1. Mellin transforms.

THEOREM 6.1. [11, p. 94, $k = 1/2$]. *Let $f(x)$ belong to $L^2(R^+)$. Then*

$$\int_{1/a}^{a} f(x)x^{s-1}\,dx, \qquad (\operatorname{Re}(s) = 1/2)$$

converges in mean square over $(1/2 - i\infty, 1/2 + i\infty)$ to a function $F(s)$;

$$\frac{1}{2\pi i}\int_{1/2-ia}^{1/2+ia} F(s)x^{-s}\,ds$$

converges in mean to $f(x)$; and

$$\int_0^\infty |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(1/2 + it)|^2\,dt.$$

THEOREM 6.2. [11, p. 95, $k = 1/2$]. *Let $f(x)$ and $g(x)$ belong to $L^2(R^+)$. Then*

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(s)G(1 - s)\,ds.$$

COROLLARY 6.1. *Let $f(x)$ and $g(x)$ belong to $L^2(R^+)$. Then for $x > 0$,*

$$\int_0^\infty f(xy)g(y)\,dy = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(s)G(1 - s)x^{-s}\,ds.$$

6.2. Generalized Mellin transforms.

DEFINITION 6.1. P^+ is the vector space of all complex-valued, infinitely differentiable functions defined on R^+ with compact support in R^+. P^+ is equipped with the usual Schwartz topology.

DEFINITION 6.2. $(P^+)'$ is the vector space of all continuous linear functionals defined on P^+. A sequence $\{f_j^+\}$ *converges to zero in* $(P^+)'$ if $\langle f_j^+, \phi^+\rangle \to 0$ for all ϕ^+ in P^+. The symbol $\langle f, \phi\rangle$ denotes the value of f at ϕ.

DEFINITION 6.3. P^+S is the vector space of all infinitely differentiable functions $\phi(x)$ such that for all polynomials $p(x)$,

$$p(\log x)x(D_x x)^k\phi(x)$$

is bounded for all integers $k \geqq 0$. A sequence $\{\phi_j\}$ *converges to zero in* P^+S if

$$p(\log x)x(D_x x)^k\phi(x) \to 0$$

uniformly on R^+ for any polynomial p and any integer $k \geqq 0$.

DEFINITION 6.4. $(P^+S)'$ is the vector space of all elements of $(P^+)'$ that can be extended to continuous linear functionals on P^+S.

DEFINITION 6.5. Q is the vector space of all complex-valued entire functions $\Psi(s)$, $s \in C$, such that

$$|\Psi(\sigma + i\tau)| \leqq A\, e^{B|\sigma|},$$

where A and B are fixed positive constants, and such that for every fixed $\sigma \in R$, $\Psi(\sigma + i\tau)$ is a rapidly decreasing function of τ. Q is equipped with the usual Schwartz topology.

DEFINITION 6.6. The *convolution*, $f^+ \wedge g^+$, of f^+ and g^+ in $(P^+)'$ is an element of $(P^+)'$ defined by

$$\langle f^+ \wedge g^+, \phi^+ \rangle = \langle g^+(x), \langle f^+(y), x^{-1}\phi^+(x^{-1}y)\rangle\rangle, \quad \phi^+ \in P^+,$$

provided $\langle f^+(y), x^{-1}\phi^+(x^{-1}y)\rangle$ defines a function in P^+.

DEFINITION 6.7. $F:C \to C$ *is of finite order on the line* $\operatorname{Re}(s) = k$ if for some integer n, $F(s) = O(|\tau|^n)$ as $|\tau| \to \infty$ on this line. F is said to be of *finite order in the closed strip* $\{s|c \leqq \operatorname{Re}(s) \leqq d; c, d \in R\}$ if the order relation holds uniformly in this strip. F is said to be of *finite order in the open strip* $\{s|c < \operatorname{Re}(s) < d; c, d \in R\}$ if F is of finite order in every closed substrip of the open strip.

DEFINITION 6.8. The *generalized Mellin transform*, Mf^+, *of* f^+ in $(P^+)'$ is an element of Q' defined by the equality

$$\langle Mf^+, \Psi\rangle = \langle f^+, \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Psi(s)x^{-s}\, ds\rangle,$$

Ψ in Q, $c \in R$. The *generalized inverse Mellin transform*, $M^{-1}F$, *of* F in Q' is an element of $(P^+)'$ defined by the equality

$$\langle M^{-1}F, \phi\rangle = \langle F, \int_0^\infty x^{s-1}\phi(x)\, dx\rangle.$$

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GENERATING FUNCTIONS FOR SPHERICAL HARMONICS. PART I: THREE-DIMENSIONAL HARMONICS*

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Abstract. A new approach to the theory of spherical harmonics and Legendre functions of integer degree and order is presented, based on a generating function

$$\zeta(\mathbf{r}) = \exp(z - iy) = \sum r^l \mathrm{P}_l^m(\cos\theta)\frac{e^{im\phi}}{(l+m)!}$$

and modifications thereof. The analytic properties of the harmonics, their recurrence relations and addition theorems under translation are derived in a simple and consistent way; the mathematical apparatus required is minimal. Integrals of products of surface harmonics over the unit sphere can be studied by integrating products of generating functions; with two factors this yields the orthogonality relations and normalization constants. The case of three factors is best discussed in terms of coefficients, closely related to Wigner's $3j$-symbols, but which are integers of moderate size and satisfy a 3-term recurrence relation analogous to that of the binomial coefficients. The Regge symmetries are obtained for modified coefficients which are rational, but not integer.

1. Introduction. The theory of Legendre functions and spherical harmonics has been established for many generations, and it may appear pointless to try and base this theory on yet another new approach. However, during work on a series of papers [24]–[27] dealing with the expansion of scalar functions of 3-dimensional vector sums in spherical harmonics, the writer has become aware of a major difficulty connected with the existing methods. The elementary theory is most easily presented on an analytic basis, whereas the more refined properties of the harmonics, such as their transformations under rotations of the coordinate system or the re-expansion of their products, have been almost exclusively treated by algebraic or group-theoretical methods. Yet these latter properties are extensively used in theoretical physics, and in consequence even such comprehensive works based on an analytic approach as Hobson's [10] are inadequate for present-day needs as they only cover part of the required theorems. For a complete understanding of the properties of spherical harmonics one first has to relearn even their elementary theory in an algebraic description. One indication of the drawback of this dichotomy is that the almost trivial addition theorems of the solid spherical harmonics appear to have been found only within the last two decades [21], [25], [6], [7], [29], [30].

In [24]–[27] the radial coefficients multiplying the spherical harmonics in the spatial expansions were derived and their properties studied from the differential equations which they satisfy and from the analytic theory of hypergeometric functions. The writer found himself in the paradoxical situation of having to apply several algebraically and group-theoretically derived properties of the spherical harmonics in an otherwise completely analytic context. He therefore felt the need for a simple new approach which would lead to the establishment of the usual analytic theory of Legendre functions (at least for integral orders and degrees), but could easily be extended to cover also those properties which have hitherto been dealt with essentially on an algebraic or group-theoretical basis.

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Such an approach has presented itself through an exponential generating function (G.F.) in which the associated Legendre functions $\mathrm{P}_l^m(u)$ are defined as the expansion coefficients of the harmonic function

$$\zeta(\mathbf{r}) = \exp(z - iy) = \exp[r(u - i\sigma \sin\phi)], \tag{1.1}$$

$$\mathbf{r} = (x, y, z) = (r, \theta, \phi), \quad u = \cos\theta, \quad \sigma = \sin\theta. \tag{1.2}$$

If we define the $\mathrm{P}_l^m(u)$ by the expansion

$$\zeta(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r^l e^{im\phi} \mathrm{P}_l^m(u)}{(l+m)!}, \tag{1.3}$$

all their properties can be derived from those of $\zeta(\mathbf{r})$ and certain generalizations thereof. The only mathematical knowledge required are Taylor's series, series solutions of linear ordinary differential equations and their singular points, the form of the Laplacean operator in Cartesian and polar coordinates.

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right], \end{aligned} \tag{1.4}$$

the solution of the two-dimensional Laplace equation

$$\nabla^2(F(z - iy) + G(z + iy)) = 0 \tag{1.5}$$

with its generalization to three dimensions

$$\nabla^2 F(z - iy\cos\psi - ix\sin\psi) = 0 \tag{1.6}$$

and the elementary theory of Fourier series. For the discussion of the irregular solid spherical harmonics some elementary knowledge of the exponential integral and asymptotic series is also required.

The standard theory of spherical harmonics usually starts with the G.F. for the Legendre polynomials

$$(1 - 2ru + r^2)^{-1/2} = \sum_l P_l(u) r^l \tag{1.7}$$

from which their differential equation is deduced, and Rodrigues' formula appears as the solution of this equation. The associated Legendre functions $\mathrm{P}_l^m(u)$ are introduced as angular factors in the solution of Laplace's equation by separation of variables; at each stage new concepts have to be introduced. An alternative approach is based on a definition of the form (cf. [3,(3.7.25)])

$$(z - iy)^l = r^l l! \sum_m \frac{e^{im\phi} \mathrm{P}_l^m(u)}{(l+m)!} \tag{1.8}$$

or a similar expansion of $(z - iy)^{-l-1}$. Here all the Legendre functions with $m =$ or $\neq 0$ occur at the same level. It is clear that (1.1)–(1.3) is obtained from (1.8) merely by summation over l, and it is surprising that this further systematization has not been employed hitherto. Yet an extensive search of the literature [1], [10], [11], [15], [19], [33], [12], [14], [17], [32] failed to reveal any G.F. resembling (1.1)–(1.3); the last four works especially deal extensively with generating functions. The Bateman Manuscript Project [3] gives a G.F., apparently due to Herglotz [3, (11.5.21)],

$$[1 - stu - \tfrac{1}{2}s\sigma(1 - t^2)]^{-1} = \sum_l \sum_k l!\mathrm{P}_l^{k-l}(u)s^l \frac{t^k}{k!}, \tag{1.9}$$

where the summation is taken over l and k simultaneously. On substituting $r = st$, $t = e^{i\phi}$, (1.9) is seen to equal $(1 - z + iy)^{-1}$. The main difference between the two expansions is the presence of $l!$ in the numerator of (1.9), leading to a geometric expansion in contrast to the exponential (1.1). For the purpose of applying the G.F., the exponential form appears to be more fruitful as the product of two exponentials is again an exponential.

Since the first draft of this paper was submitted, a referee has pointed out that a G.F. equivalent to (1.1)–(1.3) has been given implicitly by W. Miller in equations (4.12), (4.13) of [13] as a special case of a G.F. for spherical Bessel functions. But, whereas Miller uses the more general function to study the properties of spherical Bessel functions, the special G.F. is not used to derive any properties of spherical harmonics.

As with all analytic approaches, the definition (1.1)–(1.3) leads naturally to Legendre functions and spherical harmonics in their unnormalized forms;[1] thus we write

$$Y_l^m(\theta, \phi) = \mathrm{P}_l^m(\cos\theta)\, e^{im\phi} \tag{1.10}$$

for the surface harmonics; (in [25]–[27] the symbol Ω was used instead of Y). With an algebraic approach these functions are more easily handled in their normalized forms, denoted in this paper by the prefix n,

$$^{n}Y_{lm}(\theta, \phi) = \left[\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} Y_l^m(\theta, \phi); \tag{1.11}$$

their use leads to more symmetric formulas in most cases, but at the price of introducing square roots in many places, which are entirely avoided with the present approach. Similarly, we define the regular and irregular solid harmonics as

$$\mathcal{Y}_l^m(\mathbf{r}) = r^l Y_l^m(\theta, \phi), \qquad \Upsilon_l^m(\mathbf{r}) = Y_l^m(\theta, \phi) r^{-l-1}, \tag{1.12}$$

respectively, even though conventionally they are defined with the normalized surface harmonics as the angular factors; again a prefix n can serve to distinguish the normalized forms where required.

The elementary properties of the Legendre functions P_l^m and the harmonics Y and $\mathcal{Y}$ are rederived on the basis of the definition (1.1)–(1.3) in § 2; in § 3 the same is done for the irregular solid harmonics Υ on the basis of a related, asymptotic, G.F. To keep the paper reasonably short, most proofs and some of the well-

[1] The adjectives *associated* with Legendre functions and *spherical* with harmonics will be omitted in the sequel when no confusion can arise.

known results are omitted. Recurrence relations and addition theorems are derived in § 4; the Laplace expansion (1.7), which usually serves as the starting point for the theory, appears here as a special case of the general addition theorem for the irregular harmonics Υ.

When considering integrals of ν products of surface harmonics taken over the whole unit sphere, the condensation of the individual formulas (1.8) into the exponential form (1.3) becomes of crucial importance as it enables generating functions for the integrals to be set up from which their essential properties can easily be derived. The general approach to these integrals is indicated in § 5, together with a discussion of the case $\nu = 2$, which yields the orthogonality and normalization conditions of the surface harmonics. The cases $\nu > 2$ are treated in § 6, especially $\nu = 3$, which leads to a theory of $3j$-symbols in an unnormalized form previously introduced in [25]–[27]. In order not to make the present paper excessively long, some of the properties of the spherical harmonics, especially their rotational transformation properties and their 4-dimensional analogues, will be discussed on the basis of exponential G.F.'s in subsequent publications.

2. Elementary properties.

2.1. Rodrigues' formula and series expansions. With the definition (1.2),

$$z - iy = r(u - \tfrac{1}{2}\sigma\, e^{i\phi} + \tfrac{1}{2}\sigma\, e^{-i\phi}), \tag{2.1}$$

the exponential (1.1) can be factorized in the form

$$\zeta(\mathbf{r}) = \exp(ru)\cdot\exp(-\tfrac{1}{2}r\sigma\, e^{i\phi})\cdot\exp(\tfrac{1}{2}r\sigma\, e^{-i\phi}) \tag{2.2}$$

which shows that the only terms occurring in the expansion (1.3) satisfy the relation

$$-l \leqq m \leqq l; \tag{2.3}$$

the equality signs apply only if the power of r derives exclusively from the third or second factor in (2.2) respectively. Only the condition (2.3) justifies the insertion of the factorial $(l + m)!$ in the denominator of (1.3) as nonvanishing terms with $l + m < 0$ could not be rendered by the expansion (1.3). Alternatively, the argument in (1.1) can be written in the form

$$z - iy = r\frac{(u - e^{i\phi}\sigma)^2 - 1}{-2\, e^{i\phi}\sigma} \tag{2.4}$$

which yields for the expansion of the exponential

$$\zeta(\mathbf{r}) = \sum_l \frac{r^l}{l!}(u - i\sigma\sin\phi)^l = \sum_l \frac{r^l}{l!}\,\frac{[(u - e^{i\phi}\sigma)^2 - 1]^l}{[-2\sigma\, e^{i\phi}]^l} \tag{2.5}$$

$$= \sum_l \sum_\mu \frac{r^l}{2^l l!}\,\frac{(-\sigma\, e^{i\phi})^{\mu - l}}{\mu!}\left(\frac{d}{du}\right)^\mu (u^2 - 1)^l, \tag{2.6}$$

the summation over μ following from Taylor's theorem. Comparison with (1.3) and identification of μ with $l + m$ yields Rodrigues' formula for the Legendre functions

$$\mathrm{P}_l^m(u) = (-)^m(1-u^2)^{m/2}\left(\frac{d}{du}\right)^{l+m}\frac{(u^2-1)^l}{2^l l!} \tag{2.7}$$

with its concomitant expansion in powers of u

$$\mathrm{P}_l^m(u) = \frac{(-)^m(1-u^2)^{m/2}}{2^l}\sum_\lambda \frac{(-)^\lambda u^{l-2\lambda-m}(2l-2\lambda)!}{\lambda!(l-\lambda)!(l-m-2\lambda)!}. \tag{2.8}$$

An alternative expansion of the P_l^m is obtained by grouping together all those terms in the product of the exponential series in (2.2) which have the factor $r^l e^{im\phi}$; the order of the term in one of the factors may be varied, the powers of the other arguments being uniquely determined thereby:

$$\mathrm{P}_l^m(u) = \sum_\nu \frac{(l+m)!(-)^{\nu+m}u^{l-m-2\nu}(\frac{1}{2}\sigma)^{m+2\nu}}{\nu!(\nu+m)!(l-2\nu-m)!}. \tag{2.9}$$

The summations in (2.8) and (2.9) are to be taken over all values of the indices for which the arguments of all the factorials are nonnegative.

2.2. Linear independence of harmonics. The solid harmonics $\mathscr{Y}_l^m(\mathbf{r})$ defined in (1.12) are polynomials in x, y, z, as, in view of (2.1), they contain only products of positive powers of z, $x+iy$ and $x-iy$. They are linearly independent in view of their different dependence on r or ϕ or both. For a given value of m any $\mathrm{P}_l^m(u)$ is linearly independent of the functions $\mathrm{P}_\lambda^m(u)(\lambda < l)$ as it alone contains the power u^{l-m}; hence all the Legendre functions $\mathrm{P}_l^m(u)$ at constant m are linearly independent, and so are the surface harmonics Y_l^m for any l and m, in view of their dependence on $e^{im\phi}$.

2.3. Parity in u and m; special values. If we replace $\mathbf{r}$ by $-\mathbf{r}$ we obtain, in view of (1.1)–(1.3) and (1.12),

$$\mathscr{Y}_l^m(-\mathbf{r}) = r^l\mathrm{P}_l^m(-\cos\theta)\, e^{im(\phi+\pi)} = (-)^l\mathscr{Y}_l^m(\mathbf{r}). \tag{2.10}$$

and hence,

$$\mathrm{P}_l^m(-u) = (-)^{l+m}\mathrm{P}_l^m(u); \tag{2.11}$$

the latter could also have been inferred from the expansion (2.9). For $l+m$ even and $u=0$ only one term survives in (2.8) or (2.9) giving

$$\mathrm{P}_l^m(0) = \begin{cases} (-)^{(l+m)/2}\dfrac{(l+m)!}{[\frac{1}{2}(l+m)]![\frac{1}{2}(l-m)]!}, & l+m \text{ even},\\ 0 & l+m \text{ odd}. \end{cases} \tag{2.12}$$

For $u=\pm 1$, $\sigma = 0$ and the right-hand side of (2.2) reduces to its first factor which is independent of ϕ; comparison with (1.3) yields

$$\mathrm{P}_l^m(\pm 1) = (\pm 1)^l\delta_{m0}. \tag{2.13}$$

Other properties can be derived by means of a generalization of ζ which differs from (1.1) by putting kr for r and $\phi + \psi$ for ϕ:

$$\begin{aligned}\zeta(\mathbf{r}; k, \psi) &= \exp\left[k(z - iy\cos\psi - ix\sin\psi)\right] \\ &= \exp\{kr[u - i\sigma\sin(\phi + \psi)]\} = \sum \frac{k^l e^{im\psi}\mathcal{Y}_l^m(\mathbf{r})}{(l+m)!}.\end{aligned} \tag{2.14}$$

In particular,

$$\begin{aligned}\zeta(\mathbf{r}; 1, \pi/2) &= \exp\left[r(u - i\sigma\cos\phi)\right] \\ &= \sum \frac{r^l i^m \mathrm{P}_l^m(u)\, e^{im\phi}}{(l+m)!}\end{aligned} \tag{2.15}$$

is an even function in ϕ; hence the right-hand side remains invariant on substituting $-\phi$ for ϕ, or alternatively $-m$ for m; in consequence

$$\frac{\mathrm{P}_l^{-m}(u)}{(l-m)!} = \frac{(-)^m \mathrm{P}_l^m(u)}{(l+m)!}. \tag{2.16}$$

2.4. Differential equations. Application of the Laplacean (1.6) to (2.14) yields

$$\nabla^2\zeta(\mathbf{r}; k, \psi) = \sum \frac{k^l\, e^{im\psi}\nabla^2\mathcal{Y}_l^m(\mathbf{r})}{(l+m)!} = 0; \tag{2.17}$$

but this is an identity in k and ψ; hence each solid harmonic $\mathcal{Y}_l^m$ individually satisfies Laplace's equation

$$\nabla^2\mathcal{Y}_l^m(\mathbf{r}) = 0. \tag{2.18}$$

From (1.3), (1.4) and (2.18) there follows the ordinary differential equation for the Legendre functions

$$\left[\frac{d}{du}(1 - u^2)\frac{d}{du} + l(l+1) - \frac{m^2}{1 - u^2}\right]\mathrm{P}_l^m(u) = 0. \tag{2.19}$$

One of the solutions of this second order equation is given by (2.8), a polynomial in u multiplied by $(1 - u^2)^{|m|/2}$; the second solution becomes infinite at both singular points ($u = \pm 1$).

As stated in the Introduction, the definition (1.1)–(1.3) of the Legendre functions and the derivation of their properties as given above is entirely equivalent to the well-known definition (1.8) and the conclusions drawn therefrom. The two approaches differ only in that l occurs as a constant parameter in (1.8), but as a summation index in (2.5) and (2.6). Hence for the results obtained hitherto the approach based on (1.1)–(1.3) can, at best, be considered as a systematization of a well-known method. Other results lean even more heavily on the expansion of single powers of $z - iy$; thus the Fourier expansion of (1.8) yields directly

$$\mathrm{P}_l^m(u) = \frac{(l+m)!}{2\pi l!}\int_0^{2\pi} (u - i\sigma\sin\phi)\, e^{-im\phi}\, d\phi, \tag{2.20}$$

which is essentially Laplace's first integral [3, (3.7.25)]. The proof of the completeness of the set of the regular solid harmonics $\mathscr{Y}_l^m(\mathbf{r})$ to represent all the harmonic polynomials again follows conventional lines (Hobson [10, Chap. 4]) and will not be repeated here. The orthogonality of the surface harmonics is derived in § 5.

One further set of formulas follows straightforwardly from (1.1)–(1.3) and the generating function for the Bessel functions of integer order in the form

$$e^{i\rho\sin\phi} = \sum_{-\infty}^{\infty} e^{im\phi} J_m(\rho) \tag{2.21}$$

[3, (7.2.26)]; comparison of terms in $e^{im\phi}$ yields

$$e^{-z}J_m(\rho) = (-)^m \sum_{l=|m|}^{\infty} \frac{r^l \mathrm{P}_l^m(u)}{(l+m)!}, \qquad \rho = r\sigma. \tag{2.22}$$

Again this formula is not altogether new; Rainville [17] devotes considerable space to the case $m = 0$, and the equivalent for $m \neq 0$ is given in equation (22.9.5) of [1]. However, the only derivations the writer has seen are by tedious comparison of coefficients or by Laplace transforms, in contrast to the very concise derivation given above.

3. Irregular solid harmonics. The function

$$\mathrm{Exp}(a) = \int_0^\infty e^{-at}\,dt/(1+t) = -e^a\,\mathrm{Ei}(-a), \qquad \mathrm{Re}(a) > 0, \tag{3.1}$$

cannot be expanded in a convergent power series, but has an asymptotic expansion [3, (9.7.7)]

$$\mathrm{Exp}(a) \sim \sum_{\nu=0}^{\infty} \frac{(-)^\nu \nu!}{a^{\nu+1}}. \tag{3.2}$$

The notation Exp, which is not the usual one, has been chosen here to underline the similarity of some of its properties to those of the exponential function. In particular, from the expansion

$$\int_0^\infty \frac{e^{-ta}}{1+t} t^n\,dt \sim \sum_{\nu=n} \frac{(-)^{n+\nu}\nu!}{a^{\nu+1}} \tag{3.3}$$

we obtain the formal addition theorem

$$\mathrm{Exp}(a+b) = [\mathrm{Exp}(a)\cdot\exp(b)]^* \sim \sum_{n=0}^{\infty}\sum_{\nu=n}^{\infty} \frac{(-)^\nu \nu!}{a^{\nu+1}}\frac{b^n}{n!}, \qquad a > |b|, \tag{3.4}$$

where the asterisk on the product implies that those terms in the product of the expansions for which $n > \nu$ are to be suppressed. A corresponding expansion for three terms in the argument

$$\mathrm{Exp}(a+b+c) = [\mathrm{Exp}(a)\exp(b)\exp(c)]^* \tag{3.5}$$

requires the condition

$$|b| + |c| < |a| \tag{3.6}$$

in order to make the partial sums

$$\frac{N!}{(a+b+c)^{N+1}} = \sum \frac{(-)^{n+m}(N+n+m)!}{a^{N+n+m+1}} \frac{b^n}{n!} \frac{c^m}{m!} \tag{3.7}$$

absolutely convergent; similar considerations apply to sums of more than three terms in the argument.

Consider now the generating function

$$Z(\mathbf{r}) = \operatorname{Exp}(z - iy) \sim \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-\infty}^{\infty} (-)^{l+m} \mathrm{p}_l^m(u)\, e^{im\phi} \tag{3.8}$$

and its generalization analogous to (2.14)

$$\begin{aligned} Z(\mathbf{r}; k, \psi) &= \operatorname{Exp}[k(z - iy\cos\psi - ix\sin\psi)] \\ &\sim \sum_{l=0}^{\infty} \frac{1}{(kr)^{l+1}} \sum_{m=-\infty}^{\infty} (-)^{l+m} \mathrm{p}_l^m(u)\, e^{im(\phi+\psi)}. \end{aligned} \tag{3.9}$$

For real values of (x, y, z) comparison of (2.1) with (3.5)–(3.7) shows that each summation over m at constant l in (3.8) converges for $\sigma < |u|$ and, since u must be positive, (3.8) is meaningful for $0 \leqq \theta \leqq \pi/4$. In particular, all the $\mathrm{p}_l^m(u)$ must be finite at $u = 1 - 0$. The function $Z(\mathbf{r}; k, \psi)$ satisfies Laplace's equation for any value of k and ψ; hence, in view of (1.4), each function $\mathrm{p}_l^m(u)$ must satisfy (2.19) individually. However, it was shown, following (2.19), that for $|m| \leqq l$ those solutions of (2.19) which remain finite at $u = 1 - 0$ are given by the Legendre functions P_l^m; hence p_l^m and P_l^m can only differ by a constant factor. This proportionality constant is found by comparing the lowest powers of σ coupled with $e^{im\phi}/r^{l+1}$, i.e. ignoring the third or the second factor in (2.2), when substituted in (1.3) and (3.8). According to (2.2), (3.4) and (3.8) we have

$$\mathrm{p}_l^m(u) = \begin{cases} (-\tfrac{1}{2}\sigma)^m (l+m)!/m!(1 - \cdots), & m \geqq 0, \\ (\tfrac{1}{2}\sigma)^{|m|}(l+|m|)!/|m|!(1 - \cdots), & m < 0. \end{cases} \tag{3.10}$$

Comparison with (2.9) yields

$$\mathrm{p}_l^m(u) = (l-m)!\,\mathrm{P}_l^m(u), \qquad -l \leqq m \leqq l. \tag{3.11}$$

We can therefore write for (3.8),

$$\begin{aligned} Z(\mathbf{r}) &\sim \sum_{l=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-)^{l+m}(l-m)!}{r^{l+1}} \mathrm{P}_l^m(u)\, e^{im\phi} \\ &= \sum_l \sum_m (-)^{l+m}(l-m)!\,\Upsilon_l^m(\mathbf{r}), \end{aligned} \tag{3.12}$$

where the $\mathrm{P}_l^m(u)$ are the Legendre functions discussed in § 2 for $|m| \leqq l$, and the Υ_l^m are the irregular solid harmonics defined in (1.12). For $|m| > l$ the formula (2.7) becomes meaningless when substituted in (3.12): for $m < -l$, (2.7) involves derivatives of negative order, i.e., integrals whose values depend on where the lower limit is chosen, and for $m > l$ the $\mathrm{P}_l^m(u)$ vanish identically, but are multiplied by the gamma function of nonpositive integer argument, so that the terms contain the indefinite factor $0 \cdot \infty$. In view of the nature of the differential equation

(2.19) for the $P_l^m(u)$, which has singularities at $u = \pm 1$ only, any solution which is finite at $u = 1 - 0$ remains analytic at least in the interval $1 > u > -1$. However, the more detailed theory of Legendre functions with nonintegral l or m or not satisfying (2.3) lies outside the scope of this paper.

The expansion of the individual inverse powers of $z - iy$ occurring in (3.8) is again standard in the theory of spherical harmonics; the determination of the various Fourier components in (3.8), together with (3.11), leads immediately to

$$P_l^m(u) = \frac{(-)^m l!}{2\pi(l-m)!} \int_0^{2\pi} \frac{e^{-im\phi}\, d\phi}{(u - i\sigma \sin\phi)^{l+1}} \tag{3.13}$$

which is essentially Laplace's second integral [10, p. 103].

4. Recurrence relations and addition theorems. Differentiation of (1.3) with respect to ϕ yields

$$-\frac{i\partial\zeta}{\partial\phi} = \sum \frac{mr^l e^{im\phi} P_l^m(u)}{(l+m)!}, \tag{4.1}$$

but, in view of (1.1), this is also

$$-i\frac{\partial\zeta}{\partial\phi} = -r\sigma\cos\phi\,\zeta = -\frac{1}{2}r\sigma(e^{i\phi} + e^{-i\phi})\sum\frac{e^{im\phi}r^l P_l^m(u)}{(l+m)!}. \tag{4.2}$$

The total factors multiplying $r^l e^{im\phi}/(l+m)!$ in (4.1) and (4.2) respectively are

$$mP_l^m(u) = -\tfrac{1}{2}\sigma[P_{l-1}^{m+1} + (l+m)(l+m-1)P_{l-1}^{m-1}]. \tag{4.3a}$$

Similarly from a differentiation of (1.1)–(1.3) with regard to r we obtain by comparison

$$lP_l^m(u) = (l+m)uP_{l-1}^m + \tfrac{1}{2}\sigma[P_{l-1}^{m+1} - (l+m)(l+m-1)P_{l-1}^{m-1}], \tag{4.4a}$$

and from differentiation with respect to u,

$$\frac{dP_l^m}{du} = (l+m)P_{l-1}^m - \frac{1}{2}\frac{u}{\sigma}[P_{l-1}^{m+1} - (l+m)(l+m-1)P_{l-1}^{m-1}]. \tag{4.5a}$$

Alternatively, we could differentiate (3.8) and (3.12) with respect to the same three variables; this yields a set of formulas (4.3b)–(4.5b) which differ from (4.3a)–(4.5a) only by the substitution of $-l-1$ for l. All the numerous 3-term recurrence relations satisfied by the Legendre functions [3, (3.8.11–19)] can be deduced by suitable linear combinations of (4.3)–(4.5), though some of the formulas are obtained with less effort if both sets are used simultaneously. The detailed calculations and results will not be quoted here.

In order to derive the addition theorem for the regular solid harmonics $\mathcal{Y}_l^m(\mathbf{r})$ where

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 \tag{4.6}$$

it is best to use the generalized G.F. $\zeta(\mathbf{r}; k, \psi)$ defined in (2.14). Observing that

$$\zeta(\mathbf{r}; k, \psi) = \zeta(\mathbf{r}_1; k, \psi)\cdot\zeta(\mathbf{r}_2; k, \psi), \tag{4.7}$$

expanding each G.F. according to (2.14) and comparing the coefficients of $k^l\, e^{im\psi}$ on both sides we find

$$\frac{\mathcal{Y}_l^m(\mathbf{r})}{(l+m)!} = \sum \frac{\mathcal{Y}_\lambda^\mu(\mathbf{r}_1)}{(\lambda+\mu)!}\,\frac{\mathcal{Y}_{\lambda'}^{\mu'}(\mathbf{r}_2)}{(\lambda'+\mu')!}, \tag{4.8}$$

the sum to be taken over all values of λ, λ', μ, μ' subject to

$$\lambda + \lambda' = l, \qquad \mu + \mu' = m, \tag{4.9}$$

and for which, in addition, the equivalent formula to (2.3) applies to each factor. More compactly this can be written as

$$\mathcal{Y}_l^m(\mathbf{r}) = \sum_{\lambda\mu} \binom{l+m}{\lambda+\mu} \mathcal{Y}_\lambda^\mu(\mathbf{r}_1)\mathcal{Y}_{l-\lambda}^{m-\mu}(\mathbf{r}_2). \tag{4.10}$$

If one of the polar angles vanishes, say θ_2, this formula simplifies, in view of (2.13), to

$$\mathcal{Y}_l^m(\mathbf{r}) = \sum_\lambda \binom{l+m}{\lambda+m} r_1^\lambda r_2^{l-\lambda} Y_\lambda^m(\theta_1, \phi_1). \tag{4.11}$$

Whereas (4.11) goes back to Hobson [10], this writer has been unable to find any publication giving the more general case (4.10) earlier than Rose [21], who derived the equivalent formula for the normalized harmonics by algebraic methods; for unnormalized harmonics (4.10) has been derived in [25] starting from the definition (2.20) (see also [7]). The generalization of (4.6)–(4.10) for

$$\mathbf{r} = \sum_1^N \mathbf{r}_j \tag{4.12}$$

is straightforward, leading to

$$\frac{\mathcal{Y}_l^m(\mathbf{r})}{(l+m)!} = \sum_{\{\lambda,\mu\}} \prod_{j=1}^N \frac{\mathcal{Y}_{\lambda_j}^{\mu_j}(\mathbf{r}_j)}{(\lambda_j+\mu_j)!}, \tag{4.13}$$

the sum to be taken over all sets of λ and μ subject to

$$\sum \lambda_j = l, \qquad \sum \mu_j = m. \tag{4.14}$$

For integer l and m this formula is valid irrespective of the relative magnitudes of the r_j.

To find the addition theorem for the irregular solid harmonics $\Upsilon_l^m(\mathbf{r})$ of the sum vector (4.6) we use the formal factorization (3.4) for the G.F. (3.8):

$$Z(\mathbf{r}; k, \psi) = [Z(\mathbf{r}_1; k, \psi)\zeta(\mathbf{r}_2; k, \psi)]^* \tag{4.15}$$

valid for

$$r_1 > r_2. \tag{4.16}$$

By sorting out terms in $k^{-l-1} e^{im\psi}$ we obtain

$$\Upsilon_l^m(\mathbf{r}) = \sum_{\lambda,\mu} (-)^{\lambda+\mu} \binom{l+\lambda-m+\mu}{l-m} \Upsilon_{l+\lambda}^{m-\mu}(\mathbf{r}_1)\mathscr{Y}_\lambda^\mu(\mathbf{r}_2). \tag{4.17}$$

This addition theorem has previously been derived by Chiu [6] by means of irreducible tensor algebra (Chiu uses normalized harmonics), by the writer [25] as a special case of a more general addition theorem and by Dahl and Barnett [7] by induction. We see that, whenever the condition (2.3) applies to the indices of $\Upsilon(\mathbf{r})$, it also applies to all the factors $\Upsilon(\mathbf{r}_1)$ occurring on the right-hand side of (4.17), i.e., no harmonics with singularities at $\theta = 0$ or $\theta = \pi$ enter into the expansions. Again the summation over μ reduces to a single term if one of the polar angles vanishes, but in this case it matters whether it belongs to the longer or the shorter of the vectors. If $r_1 > r_2$ and $\theta_1 = 0$ the expansion becomes

$$\Upsilon_l^m(\mathbf{r}) = \sum_{\lambda} (-)^{\lambda+m} \binom{l+\lambda}{l-m} r_2^\lambda r_1^{-(l+\lambda+1)} Y_\lambda^m(\theta_2, \phi_2), \tag{4.18}$$

and if $r_1 > r_2$ and $\theta_2 = 0$,

$$\Upsilon_l^m(\mathbf{r}) = \sum_{\lambda} (-)^{\lambda} \binom{l+\lambda-m}{l-m} r_2^\lambda r_1^{-(l+\lambda+1)} Y_{\lambda+l}^m(\theta_1, \phi_1). \tag{4.19}$$

Of special importance is the case $l = 0$, i.e., the inverse distance $|\mathbf{r}_1 + \mathbf{r}_2|^{-1}$. If the direction of one of the vectors is reversed, this merely introduces an additional factor $(-)^\lambda$ in the expansion, so that (4.17) becomes

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{\lambda,\mu} (-)^\mu \frac{r_<^\lambda}{r_>^{\lambda+1}} Y_\lambda^\mu(\theta_1, \phi_1) Y_\lambda^{-\mu}(\theta_2, \phi_2), \tag{4.20}$$

where $r_>$ and $r_<$ denote the larger and smaller of r_1 and r_2 respectively. On the other hand, $|\mathbf{r}_1 - \mathbf{r}_2|$ depends on the polar and azimuthal angles only through the angle ω subtended by the two vectors; substituting ω for θ_2 in (4.18) or for θ_1 in (4.19) we obtain

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{(r_1^2 - 2r_1 r_2 \cos\omega + r_2^2)^{1/2}} = \sum_{\lambda} \frac{r_<^\lambda}{r_>^{\lambda+1}} \mathrm{P}_\lambda^0(\cos\omega), \tag{4.21}$$

which is essentially Laplace's generating function (1.7) for the Legendre polynomials $P_l(u) = \mathrm{P}_l^0(u)$. Furthermore, comparison of the terms at constant $\lambda = l$ in (4.20) and (4.21) gives the addition theorem for the surface zonal spherical harmonics [3,(3.11.2)]

$$\begin{aligned} P_l(\cos\omega) &= \sum_{\mu=-l}^{l} (-)^\mu Y_l^\mu(\theta_1, \phi_1) Y_l^{-\mu}(\theta_2, \phi_2) \\ &= P_l(\cos\theta_1) P_l(\cos\theta_2) \\ &\quad + 2\sum_{\mu=1}^{l} \frac{(l-\mu)!}{(l+\mu)!} \mathrm{P}_l^m(\cos\theta_1) \mathrm{P}_l^m(\cos\theta_2) \cos m(\phi_1 - \phi_2), \end{aligned} \tag{4.22}$$

the last equation following from (2.16). In particular, $l = 1$ yields the standard expression

(4.23) $$\cos \omega = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2).$$

There is a striking similarity between the expansions (4.10) and (4.17), and the binomial series for positive and negative powers of $(a + b)$ respectively if $\mathcal{Y}_l^m$ is compared with a^l and Υ_l^m with a^{-l-1}. This is an immediate consequence of the (actual or formal) factorizability of their G.F.'s $\zeta(\mathbf{r})$ and $Z(\mathbf{r})$. A further analogous consequence is that the straightforward generalization of (4.17) for the addition of more than two vectors as in (4.12),

(4.24) $$\Upsilon_l^m(\mathbf{r}) = \sum_{\{\lambda,\mu\}} \frac{(\lambda_1 - \mu_1)!}{(l - m)!} \Upsilon_{\lambda_1}^{\mu_1}(\mathbf{r}_1) \prod_{j=2}^{N} \left[\frac{(-)^{\lambda_j + \mu_j} \mathcal{Y}_{\lambda_j}^{\mu_j}(\mathbf{r}_j)}{(\lambda_j + \mu_j)!} \right]$$

subject to

(4.25) $$\lambda_1 = l + \sum_{j=2}^{N} \lambda_j, \qquad \sum_{j=1}^{N} \mu_j = m,$$

is absolutely convergent only in the case

(4.26) $$r_1 > \sum_{j=2}^{N} r_j.$$

When no vector is larger than the algebraic sum of all the other vectors, the multiple series (4.24), (4.25) are at most conditionally convergent; it depends on the mutual orientation of the $\mathbf{r}_j$ and on the order of summing over λ_j and μ_j whether they converge. Thus if (4.26) is not valid, but

(4.27) $$r_1 > \left| \sum_{2}^{N} \mathbf{r}_j \right|,$$

convergence can be achieved provided the summations over λ_j, μ_j $(j > 1)$ are first carried out for fixed values of λ_1 and μ_1 and the final sum taken over λ_1 and μ_1. Since the limits of validity of (4.27) depend on the angular variables as well as the radii, some of the advantages of the addition theorems, especially the orthogonality of the surface harmonics, are lost. Formulas equivalent to (4.24)–(4.25) have been derived by Chiu [6] for normalized harmonics and by Steinborn [29], [30] in a new standardization; both authors quote conditions more general than (4.26), without pointing out their pitfalls. Thus for the Coulomb interaction of two interpenetrating spheres, each having a uniform surface charge q and radius a with their centers a distance a apart, only the term $l_1 = l_2 = l_3 = 0$ would be expected to contribute; Chiu's formula [6] yields explicitly q^2/a for this term, and an uncritical application of Steinborn's [29] formula likewise, whereas the correct result [5], [26] is $3q^2/4a$.

For $N = 3$ series expansions for Υ_0^0 have been given, valid in the overlap region $|r_1 - r_2| < r_3 < (r_1 + r_2)$ by Buehler and Hirschfelder [5] for a special choice of axes and by the writer [26] for arbitrary orientations (other possible expansions involve Hankel transforms [23], [28]). The resulting expressions are more complex than (4.24) by an order of magnitude; the underlying cause for this is that the Υ_l^m do not satisfy Laplace's equation at the origin, hence any result based directly or indirectly on their harmonicity must break down if the magnitudes of the vectors are such that with appropriate mutual orientations their geometric sum can be made to vanish.

In a recent paper [30] Steinborn and Ruedenberg have rederived the addition theorems for normalized harmonics equivalent to (4.10) and (4.17) on the basis of the transformation properties of harmonics under rotations. They temporarily rotate the coordinate system so as to make the polar axis coincide with the direction of one of the vectors, and the terms obtained by applying a translation along the z-axis according to (4.11), (4.18) and (4.19) are transformed back to the old directions. This approach, though it follows logically on their prior treatment of rotations, is very involved, and in the writer's opinion requires considerably greater effort than any of the previous derivations [21], [25], [6], [7]. They also define a new set of regular and irregular solid harmonics which differ from the standard ones by the absence of the factorials in (1.3) when applied to (1.10) and in (3.12); the resulting addition theorems differ from (4.10) and (4.17) in that the binomial coefficients are missing, their generalizations to more than two constituent vectors follow easily and again differ from (4.13) and (4.24) only in that the factorials are absent.

A different type of addition theorem applicable to solid or surface harmonics of a *single* vector is more easily deduced from the single power G.F.'s (1.8) or (3.8), (3.12) summed over m only, l being kept constant. Identification of terms in $e^{im\phi}$ on both sides of the equation $(z - iy)^l = (z - iy)^\lambda(z - iy)^{l-\lambda}$ yields

$$\mathscr{Y}_l^m(\mathbf{r}) = \frac{\lambda!(l-\lambda)!}{l!}\sum_\mu \binom{l+m}{\lambda+\mu}\mathscr{Y}_\lambda^\mu(\mathbf{r})\mathscr{Y}_{l-\lambda}^{m-\mu}(\mathbf{r}) \tag{4.28}$$

and for negative powers

$$\Upsilon_l^m(\mathbf{r}) = \frac{l!\lambda!}{(l+\lambda)!}\sum_\mu (-)^\mu \binom{l+\lambda-m+\mu}{l-m}\mathscr{Y}_\lambda^\mu(\mathbf{r})\Upsilon_{l+\lambda}^{m-\mu}(\mathbf{r}). \tag{4.29}$$

Two further sets of formulas can be derived, in one of which the degree of regular harmonic on the r.h.s. exceeds that of the Υ factor, and in the other both factors are irregular; but these expansions necessarily involve harmonics outside the range (2.3). In view of the fixed powers of r involved, the solid harmonics in (4.28), (4.29) may be replaced by the appropriate surface harmonics Y. For $\lambda = 1$ the equations become equivalent to (4.4).

5. Angle averages of products and orthogonality. In § 4 the product of two or more generalized G.F.'s ζ in (2.14) with common parameters k and ψ, but different vector arguments $\mathbf{r}_j$, served to deduce the addition theorems for the solid harmonics $\mathscr{Y}_l^m(\mathbf{r})$. Conversely, the integrals of products of ν surface harmonics over the unit sphere or, on dividing by 4π, the angle average of these products

$$I_\nu\begin{pmatrix} l_1, \cdots, l_\nu \\ m_1, \cdots, m_\nu \end{pmatrix} = I_\nu(\mathbf{l}, \mathbf{m}) = \frac{1}{4\pi}\int d^2S_1 \prod_1^\nu Y_{l_j}^{m_j}(\theta, \phi), \tag{5.1}$$

where d^2S_1 means integration over the unit sphere, are most easily studied by means of a G.F. Ξ:

$$\Xi_\nu\begin{pmatrix} k_1, \cdots, k_\nu \\ \psi_1, \cdots, \psi_\nu \end{pmatrix} = \Xi_\nu(\mathbf{k}, \boldsymbol{\psi}) = \frac{1}{4\pi}\int d^2S_1 \prod_1^\nu \zeta(\mathbf{r}; k_j, \psi_j). \tag{5.2}$$

Here the various factors ζ depend on the same vector argument $\mathbf{r}$, but different parameters k_j, ψ_j. In view of (2.14), (5.1) and (5.2) are related by

$$\Xi_\nu(\mathbf{k}, \boldsymbol{\psi}) = \frac{\sum_{\mathbf{lm}} I_\nu(\mathbf{l}, \mathbf{m})\mathbf{k}^{\mathbf{l}} e^{i\mathbf{m}\boldsymbol{\psi}}}{\prod(l_j + m_j)!} \tag{5.3}$$

where we abbreviate

$$\mathbf{k}^{\mathbf{l}} e^{i\mathbf{m}\boldsymbol{\psi}} = \prod_1^\nu [k_j^{l_j} \exp(im_j\psi_j)]. \tag{5.4}$$

On the other hand, in view of (2.14), (5.2) can be written

$$\Xi_\nu(\mathbf{k}, \boldsymbol{\psi}) = \frac{1}{4\pi}\int d^2S_1[u\Psi_0 - \tfrac{1}{2}\sigma(e^{i\phi}\Psi_1 - e^{-i\phi}\Psi_{-1})], \tag{5.5}$$

where

$$\Psi_\alpha = \sum k_j \exp(i\alpha\psi_j), \qquad \alpha = -1, 0, 1. \tag{5.6}$$

Now for any constant vector $\mathbf{h}$ with real components and magnitude h, we have

$$\frac{1}{4\pi}\int d^2S_1\, e^{\mathbf{r}\cdot\mathbf{h}} = \frac{\sinh h}{h} = \frac{\sum h^{2L}}{(2L+1)!}. \tag{5.7}$$

This is easily proved by re-expressing $\mathbf{r}$ in a coordinate system the z-axis of which points along the direction of $\mathbf{h}$. But (5.7), considered as a function of a complex variable h, has no finite singularities, and as it is an even function in h it is also an integer function of $H = h^2$. Hence (5.7) remains valid even if the components of $\mathbf{h}$ are complex as in (5.5), (5.6) with the result

$$\Xi_\nu(\mathbf{k}, \boldsymbol{\psi}) = \frac{\sum H^L}{(2L+1)!}, \tag{5.8}$$

$$H = \Psi_0^2 - \Psi_1\Psi_{-1} = \sum_{j,t} k_j k_t[2 - 2\cos(\psi_j - \psi_t)], \tag{5.9}$$

the sum to be taken over all pairs (j, t) once only. For $\nu = 2$, this sum reduces to a single term

$$\begin{aligned} H &= 4k_1k_2 \sin^2[\tfrac{1}{2}(\psi_1 - \psi_2)] \\ &= -k_1k_2\{\exp\tfrac{1}{2}i(\psi_1 - \psi_2) - \exp\tfrac{1}{2}i(\psi_2 - \psi_1)\}^2. \end{aligned} \tag{5.10}$$

Hence the expansion (5.8) for $\Xi_2(k_1, \psi_1, k_2, \psi_2)$ contains only terms for which

$$l_1 = l_2, \qquad m_1 = -m_2, \tag{5.11}$$

so that in view of (5.3)

$$I_2\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} = 0, \quad \text{unless } l_1 = l_2; \quad m_1 = -m_2. \tag{5.12}$$

If (5.11) is satisfied, application of the binomial theorem to (5.3), (5.8) and (5.10) yields

$$I_2\begin{pmatrix} l & l \\ m & -m \end{pmatrix} = (-)^m \frac{(l+m)!(l-m)!}{(2l+1)!}\begin{pmatrix} 2l \\ l-m \end{pmatrix} = \frac{(-)^m}{2l+1}. \tag{5.13}$$

Since, in view of (2.16), $(-)^m Y_l^{-m}$ is proportional to $(Y_l^m)^*$, (5.12) expresses the orthogonality of any two different harmonics over the unit sphere. Similarly, (5.13) yields the normalization constants for the surface harmonics Y_l^m, and with (2.16), the factor 4π in (5.2) and the conservation of phase leads to the definition (1.11) for the normalized surface harmonics ${}^nY_{lm}$. The expansion of an arbitrary function of the angular variables in surface harmonics follows from (5.12) and (5.13) in the usual way:

$$F(\theta, \phi) = \sum_{l,m} c_{lm} Y_l^m(\theta, \phi), \tag{5.14}$$

$$\begin{aligned} c_{lm} &= (-)^m \frac{2l+1}{4\pi} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi F(\theta, \phi) Y_l^{-m}(\theta, \phi) \\ &= \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi F(\theta, \phi)[Y_l^m(\theta, \phi)]^*. \end{aligned} \tag{5.15}$$

6. Generalized Gaunt's coefficients. Integrals of the form (5.1) with $\nu = 3$ were first discussed by Gaunt [9] and since then have been extensively studied mainly on the basis of Wigner coefficients and $3j$-symbols (cf. [8], [16], [20], [34]). However, these algebraic methods relate the overlap integrals (5.1) to the transformation properties of the harmonics under rotation, whereas with the G.F.'s Ξ_ν of (5.2) they can be studied independently of any concept of rotations, quite apart from the fact that ν may be taken larger than 3 (generalized Gaunt's coefficients).

Some general theorems follow immediately from (5.1)–(5.3), (5.8)–(5.9). Since H in (5.9) contains the parameters k_j only in pairs and the phases only as differences $\psi_j - \psi_t$, the only nonvanishing terms in Ξ_ν satisfy

$$\sum l_j = \text{even} = 2\Lambda, \tag{6.1a}$$

$$\sum m_j = 0. \tag{6.1b}$$

Furthermore, since H does not contain any squares of k_j but only cross products, no one exponent l_j can exceed the sum of all the others; with (6.1a) this is equivalent to

$$\lambda_j = \Lambda - l_j \geqq 0, \qquad j = 1, 2, \cdots, \nu. \tag{6.1c}$$

Also, in view of (2.3),

$$|m_j| \leqq l_j, \qquad j = 1, 2, \cdots, \nu. \tag{6.1d}$$

This implies that all the Gaunt's coefficients $I_\nu(\mathbf{l}, \mathbf{m})$ vanish unless (6.1a)–(6.1d) are satisfied. To investigate the nonzero integrals I_ν it is convenient to modify Ξ_ν slightly by suppressing the denominators in (5.8). Thus

$$\Xi'_\nu(\mathbf{k}, \boldsymbol{\psi}) = \sum (-H)^\Lambda = (1 + H)^{-1} \tag{6.2a}$$

$$= \sum U'_\nu(\mathbf{l}, \mathbf{m})\mathbf{k}^{\mathbf{l}} e^{i\mathbf{m}\boldsymbol{\psi}}. \tag{6.2b}$$

Comparison of (5.3), (5.8) and (6.2) yields

$$I(\mathbf{l}, \mathbf{m}) = (-)^\Lambda U'_\nu(\mathbf{l}, \mathbf{m}) \prod \frac{(l_j + m_j)!}{(2\Lambda + 1)!}. \tag{6.3}$$

The advantage of using the symbols U'_ν rather than the averages I_ν is that, since H in (5.9) contains terms in $k_j k_t \exp[i\alpha(\psi_j - \psi_t)]$ with coefficients -1 and $+2$ only, all the coefficients $U'_\nu(\mathbf{l}, \mathbf{m})$ in the expansion (6.2b) for H^Λ are integers. This simplifies both their generation and a discussion of their properties. They are obviously symmetric functions of the ν pairs of indices (l_j, m_j), invariant on simultaneous reversal of the signs of all m_j:

$$U'(\mathbf{l}, \mathbf{m}) = U'(\mathbf{l}, -\mathbf{m}). \tag{6.4}$$

One further important property is their recurrence relation, based on the identity $H^\Lambda = H^\lambda \cdot H^{\Lambda - \lambda}$. Applying this to (6.2) one obtains, on expanding the respective powers of H,

$$U'_\nu\begin{pmatrix} l_1, \cdots, l_\nu \\ m_1, \cdots, m_\nu \end{pmatrix} = \sum U'_\nu\begin{pmatrix} l'_1, \cdots, l'_\nu \\ m'_1, \cdots, m'_\nu \end{pmatrix} U'_\nu\begin{pmatrix} l''_1, \cdots, l''_\nu \\ m''_1, \cdots, m''_\nu \end{pmatrix}, \tag{6.5}$$

the sum to be taken over all $\mathbf{l}'$, $\mathbf{m}'$, $\mathbf{l}''$, $\mathbf{m}''$ subject to

$$\mathbf{l}' + \mathbf{l}'' = \mathbf{l}, \quad \mathbf{m}' + \mathbf{m}'' = \mathbf{m}, \quad \sum l'_j = 2\Lambda' = \text{const}. \tag{6.6}$$

In the particular case $\Lambda' = 1$, this simplifies to

$$\begin{aligned} U'_\nu\begin{pmatrix} l_1, \cdots, l_\nu \\ m_1, \cdots, m_\nu \end{pmatrix} = \sum_{j,t} \Bigg[& U'_\nu\begin{pmatrix} l_1, \cdots, l_j - 1, \cdots, l_t - 1, \cdots, l_\nu \\ m_1, \cdots, m_j - 1, \cdots, m_t + 1, \cdots, m_\nu \end{pmatrix} \\ & + U'_\nu\begin{pmatrix} l_1, \cdots, l_j - 1, \cdots, l_t - 1, \cdots, l_\nu \\ m_1, \cdots, m_j + 1, \cdots, m_t - 1, \cdots, m_\nu \end{pmatrix} \\ & - 2U'_\nu\begin{pmatrix} l_1, \cdots, l_j - 1, \cdots, l_t - 1, \cdots, l_\nu \\ m_1, \cdots, m_j \cdots, m_t, \cdots, m_\nu \end{pmatrix} \Bigg], \end{aligned} \tag{6.7}$$

where the sum is to be taken over all pairs (j, t). Since there are $\frac{1}{2}\nu(\nu - 1)$ such pairs, the relations (6.7) would involve 19 terms even for ν as low as 4. In addition, with increasing Λ the U'_ν become rapidly very large; so instead of investigating their properties in detail, further modifications of the generating function are sought, which lead to smaller, but still integer, coefficients which satisfy recurrence relations less cumbersome than (6.5)–(6.7). This can be achieved for arbitrary ν at the cost of losing the symmetry in all the indices. The double sum for H in (5.9) can be split into two partial sums, one of which contains all the terms with a given k_j as a factor, say k_1, and the other the rest:

$$H = H_1 + H_{-1}, \qquad H_1 = \sum k_1 k_t[2 - 2\cos(\psi_1 - \psi_t)]. \tag{6.8}$$

A reduced G.F. can be defined as

$$\begin{aligned} \Xi''_{\nu,1}(\mathbf{k},\boldsymbol{\psi}) &= (1 + H_1)^{-1}(1 + H_{-1})^{-1} = \sum (-H_1)^{l_1}(-H_{-1})^{\lambda_1} \\ &= \sum U''_{\nu,1}(\mathbf{l},\mathbf{m})\mathbf{k}^{\mathbf{l}} e^{i\mathbf{m}\boldsymbol{\psi}}. \end{aligned} \tag{6.9}$$

In view of the definition (6.1c) and the binomial expansion of $H^\Lambda = (H_1 + H_{-1})^\Lambda$, the symmetric symbols U'_ν are related to the asymmetric symbols $U''_{\nu,1}$ by

$$U'_\nu(\mathbf{l},\mathbf{m}) = U''_{\nu,1}(\mathbf{l},\mathbf{m}) \cdot \binom{l_1 + \lambda_1}{l_1}. \tag{6.10}$$

The doubly primed coefficients are thus smaller than the U'_ν, but (6.9) shows that they are still integers. The separation of the powers of H_1 and H_{-1} implies that the $U''_{\nu,1}$ satisfy recurrence relations entirely analogous to (6.5), but subject to the condition

$$l'_1 = \text{const}. \tag{6.11}$$

in addition to (6.6). In particular, for $\Lambda' = 1$, $l'_1 = 0$ this leads to a formula corresponding to (6.7), but with the summation taken over all pairs (j, t), not including the index 1. By contrast, the case $\Lambda' = l'_1 = 1$ yields a single sum

$$\begin{aligned} U''_{\nu,1}\begin{pmatrix} l_1, l_2, \cdots, l_\nu \\ m_1, m_2, \cdots, m_\nu \end{pmatrix} = \sum_{j \neq 1} \Bigg[& U''_{\nu,1}\begin{pmatrix} l_1 - 1, \cdots, l_j - 1, \cdots, l_\nu \\ m_1 - 1, \cdots, m_j + 1, \cdots, m_\nu \end{pmatrix} \\ & + U''_{\nu,1}\begin{pmatrix} l_1 - 1, \cdots, l_j - 1, \cdots, l_\nu \\ m_1 + 1, \cdots, m_j - 1, \cdots, m_\nu \end{pmatrix} \\ & - 2U''_{\nu,1}\begin{pmatrix} l_1 - 1, \cdots, l_j - 1, \cdots, l_\nu \\ m_1, \cdots, m_j, \cdots, m_\nu \end{pmatrix} \Bigg]. \end{aligned} \tag{6.12}$$

It is relevant to ask whether the U''-symbols can be further reduced in magnitude without losing their integer properties; but for general ν this appears impossible as one cannot further partition H so that each product of powers $\mathbf{k}^{\mathbf{l}} \exp(i\mathbf{m}\boldsymbol{\psi})$ arises out of terms corresponding to a fixed set of powers of these constituents of H; this is essential for getting a definite ratio between the symbols before and after reduction, as in (6.10). The exception is the case $\nu = 3$ for which each H_{-j} consists of a single bracket in (5.9):

$$H_{-1} = k_2 k_3[2 - 2\cos(\psi_2 - \psi_3)] \tag{6.13}$$

with cyclic permutations of the indices. Hence any given set of powers (l_1, l_2, l_3) of $\mathbf{k}$ corresponds to a unique set of powers $(\lambda_1, \lambda_2, \lambda_3)$ of (H_{-1}, H_{-2}, H_{-3}), the

λ_j being defined in (6.1c). We can thus form a further reduced G.F.

$$\begin{aligned}\Xi_3'''(\mathbf{l}, \mathbf{m}) &= [(1 + H_{-1})(1 + H_{-2})(1 + H_{-3})]^{-1} \\ &= \sum_{\boldsymbol{\lambda}} (-H_{-1})^{\lambda_1}(-H_{-2})^{\lambda_2}(-H_{-3})^{\lambda_3} = \sum U(\mathbf{l}, \mathbf{m})\mathbf{k}^{\mathbf{l}} \exp(i\mathbf{m}\boldsymbol{\psi}),\end{aligned} \tag{6.14}$$

where the primes and the suffix 3 to the symbols U remain suppressed. From the multinomial theorem we have by comparison with (6.2),

$$U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = U_3'\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\frac{\lambda_1!\lambda_2!\lambda_3!}{\Lambda!}, \tag{6.15}$$

but (6.14) shows that the unprimed coefficients (henceforth referred to as unnormalized $3j$-symbols) are still integers, and the full symmetry in the 3 pairs of indices is retained. Comparison with (6.3) shows

$$I_3\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-)^\Lambda \Lambda!}{(2\Lambda + 1)!}\prod_1^3 \frac{(l_s + m_s)!}{\lambda_s!} U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{6.16}$$

The unnormalized $3j$-symbols $U(\mathbf{l}, \mathbf{m})$ were first introduced in [25], where the relation (6.16) was given in (28) and the connection with the normalized Wigner $3j$-symbols

$$U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\left[(2\Lambda + 1)!\prod \frac{(2\lambda_s)!}{(l_s - m)!(l_s + m_s)!}\right]^{1/2} \tag{6.17}$$

in (26). Since the values of the $3j$-symbols with $\mathbf{m} = \mathbf{0}$, which enter into the evaluation of integrals of products of three normalized harmonics (Edmonds [8, (4.6.3)]), are most easily derived when dealing with 4-dimensional harmonics, the proof of (6.17) will be postponed until the relevant section. Similarly, although in the present context the U-symbols are meaningful only for integer l_1, l_2, l_3 and Λ, they can be generalized to half-integral values by an ad hoc generating function. We can define

$$\eta_{23} = (-H_{-1})^{1/2} = (k_2k_3)^{1/2}\{\exp \tfrac{1}{2}i(\psi_2 - \psi_3) - \exp \tfrac{1}{2}i(\psi_3 - \psi_2)\} \tag{6.18}$$

with a strictly cyclic permutation of indices. Now in the expansion of

$$\begin{aligned}\Xi_3^{\mathrm{IV}}(\mathbf{k}, \boldsymbol{\psi}) &= [(1 - \eta_{23})(1 - \eta_{31})(1 - \eta_{12})]^{-1} \\ &= \sum_{\boldsymbol{\lambda}} \boldsymbol{\eta}^{\boldsymbol{\lambda}} = \sum U(\mathbf{l}, \mathbf{m})\mathbf{k}^{\mathbf{l}} e^{i\mathbf{m}\boldsymbol{\psi}}\end{aligned} \tag{6.19}$$

the sum is taken over all integer and half-integral values of (l_1, l_2, l_3), provided all $(l_s - m_s)$ as well as the sum $l_1 + l_2 + l_3$ are integers. The conditions (6.1b, c, d) are automatically satisfied, and those U-values in (6.19) for which (6.1a) is also satisfied with integer Λ agree with those obtained from (6.14). If the definition of $\Lambda = \frac{1}{2}(l_1 + l_2 + l_3)$ is retained from (6.1a), then (6.16) is valid only for integer Λ; for integer $\mathbf{l}$ and half-odd Λ the left-hand side of (6.16) vanishes. The symmetry relations follow from (6.18) and (6.19):

$$U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = U\begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = U\begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

$$(6.20) \qquad = (-)^{2\Lambda} U\begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

$$= (-)^{2\Lambda} U\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix};$$

these are the same relations as between the normalized $3j$-symbols [8], [12], [14], [20], [34].

The recurrence relations analogous to (6.5) considerably simplify for the $3j$-symbols since a simultaneous decomposition $\mathbf{l} = \mathbf{l}' + \mathbf{l}''$ for all three powers (l_1, l_2, l_3) is possible. The formula thus reduces to a double sum of products of which both factors satisfy (6.1b, c, d):

$$(6.21) \quad U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sum_{\mu\nu} U\begin{pmatrix} l'_1 & l'_2 & l'_3 \\ \mu & \nu & -\mu-\nu \end{pmatrix} U\begin{pmatrix} l-l'_1 & l_2-l'_2 & l_3-l'_3 \\ m_1-\mu & m_2-\nu & m_3+\mu+\nu \end{pmatrix};$$

for normalized $3j$-symbols the equivalent formula has been given on the last two pages of Chap. III of Vilenkin's book [32]. In particular, since

$$(6.22) \quad U\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1, \qquad U\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} = -U\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & +\frac{1}{2} & 0 \end{pmatrix} = 1,$$

$$U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = U\begin{pmatrix} l_1-\frac{1}{2} & l_2-\frac{1}{2} & l_3 \\ m_1-\frac{1}{2} & m_2+\frac{1}{2} & m_3 \end{pmatrix}$$

$$(6.23) \qquad - U\begin{pmatrix} l_1-\frac{1}{2} & l_2-\frac{1}{2} & l_3 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix}, \qquad l_3 < \Lambda,$$

valid also for all cyclic permutations of the indices (1, 2, 3) (cf. Vilenkin [32] and Edmonds [8, (3.7.12)]). Apart from signs, (6.23) corresponds to a 5-dimensional generalization of Pascal's triangle (cf. [25, (A5)]). All the symbols with, say, $l_3 = 0$ can be obtained by raising η_{12} in (6.18) to the relevant powers, so that

$$(6.24) \qquad U\begin{pmatrix} l & l & 0 \\ m & -m & 0 \end{pmatrix} = (-)^{l-m}\binom{2l}{l-m}$$

(cf. also (5.13)). For $l_3 = l_1 + l_2$ the sum in (6.21) can be reduced to a single term

$$U\begin{pmatrix} l_1 & l_2 & l_1+l_2 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} = U\begin{pmatrix} l_1 & 0 & l_1 \\ m_1 & 0 & -m_1 \end{pmatrix} U\begin{pmatrix} 0 & l_2 & l_2 \\ 0 & m_2 & -m_2 \end{pmatrix}$$

$$(6.25) \qquad = (-)^{l_1+l_2+m_1-m_2}\binom{2l_1}{l_1-m_1}\binom{2l_2}{l_2-m_2}.$$

In the case of general $\mathbf{l}$, the decomposition

$$(l_1, l_2, l_3) = (\lambda_3, \lambda_3, 0) + (\lambda_2, \lambda_1, l_3) \tag{6.26}$$

reduces (6.21) to a single sum over all integral or half-odd values of μ, which in view of (6.24) and (6.25) becomes

$$U\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sum_\mu (-)^{\Lambda + m_1 - m_2 + \mu} \binom{2\lambda_3}{\lambda_3 - \mu} \binom{2\lambda_2}{\lambda_2 - m_1 + \mu} \binom{2\lambda_1}{\lambda_1 - m_2 - \mu} \tag{6.27}$$

which is equivalent to Racah's [16] expansion for the Wigner coefficients and, with the use of (6.16), to Gaunt's [9] formula for the integrals I_3 as defined in (5.1). In the writer's opinion the use of unnormalized $3j$-symbols is advisable, as their values can be generated by integer arithmetic by means of the simple equation (6.23) and remain of manageable size; even for $2\Lambda = 32$ none of the U-symbols exceeds 10^9. Tabulation would also be simpler in the form of integers than in the usual form of products of prime factors in the numerator and denominator [4], [22].

By sacrificing the integer nature of the U-symbols, it is easy to obtain the full Regge symmetries [18] of the $3j$-symbols. Thus the symbols

$$U^{\mathrm{V}}(\mathbf{l}, \mathbf{m}) = \frac{U(\mathbf{l}, \mathbf{m})}{(2\lambda_1)!(2\lambda_2)!(2\lambda_3)!} \tag{6.28}$$

possess the generating function

$$\Xi_3^{\mathrm{V}}(\mathbf{k}, \boldsymbol{\psi}) = \sum_{\mathbf{l},\mathbf{m}} U^{\mathrm{V}}(\mathbf{l}, \mathbf{m}) \mathbf{k}^{\mathbf{l}} e^{i\mathbf{m}\boldsymbol{\psi}} = \exp(\eta_{23} + \eta_{31} + \eta_{12}) \tag{6.29}$$

in view of (6.19). By pairwise bracketing of the factorials in the expansion derived from (6.27)

$$U^{\mathrm{V}}(\mathbf{l}, \mathbf{m}) = \sum_\mu \frac{(-)^{\Lambda + m_1 - m_2 + \mu}}{(\lambda_3 - \mu)!(\lambda_2 + m_1 - \mu)!(\lambda_1 - m_2 - \mu)!(\lambda_3 + \mu)!(\lambda_2 - m_1 + \mu)!(\lambda_1 + m_2 + \mu)!} \tag{6.30}$$

in all possible ways such that the sum of the arguments in each pair remains constant as μ varies, one obtains Regge's result that the symbols remain invariant if the $\mathbf{l}$ and $\mathbf{m}$ are modified in such a way that the array

$$\begin{pmatrix} 2\lambda_1 & 2\lambda_2 & 2\lambda_3 \\ l_1 + m_1 & l_2 + m_2 & l_3 + m_3 \\ l_1 - m_1 & l_2 - m_2 & l_3 - m_3 \end{pmatrix} \tag{6.31}$$

suffers *either* a cyclic permutation of the rows, *or* a cyclic permutation of the columns, *or* an interchange of rows and columns. If the array is subjected to an odd permutation of rows or columns, the symbol has to be multiplied by $(-)^{2\Lambda}$. The same result can be obtained by substituting (6.18) in (6.29) and taking all possible pairwise groupings of positive and negative terms in the exponent, or more plainly by putting in (6.29)

$$(6.32)\qquad (\eta_{23} + \eta_{31} + \eta_{12}) = \begin{vmatrix} \sqrt{k_2k_3} & \sqrt{k_3k_1} & \sqrt{k_1k_2} \\ e^{i\phi_1/2} & e^{i\phi_2/2} & e^{i\phi_3/2} \\ e^{-i\phi_1/2} & e^{-i\phi_2/2} & e^{-i\phi_3/2} \end{vmatrix};$$

the G.F., and hence the coefficients U^{V}, exhibit all the symmetries of the determinant (the fact that in (6.32) the elements of the third row are the inverses of those in the second row is immaterial). For normalized $3j$-symbols a G.F. equivalent to (6.28)–(6.32) has been given by Miller [14, p. 262)].

Further properties of the U-symbols will be discussed in later publications.

7. Conclusions. In this paper, the elementary theory of spherical harmonics has been reformulated on the basis of the generating functions ζ defined in (1.1)–(1.3) and (2.14). The approach is essentially a 19th century one, and it is astonishing that it should not have been explored before. It appears to run counter to the current trend which aims at reformulating the theory of most of the special functions of analysis (not only of spherical harmonics) in terms of group theory and Lie algebras [12], [13], [14], [31], [32]. However, the writer feels that the use of scalar functions as a starting point is simpler in many ways since commutativity is retained throughout. One further drawback of group-theoretical methods is that attention is usually focused on representations of a given degree, corresponding to fixed values of l for the harmonics themselves and fixed sets of $\{l_1, \cdots, l_v\}$ for the product integrals. Thus, although W. Miller [12], [14], Akim and Levin [2] and others have established G.F.'s for Clebsch-Gordan coefficients and employed them to determine their properties and have derived recurrence relations far more numerous than given in the brief outline of § 6, these G.F.'s are mostly confined to fixed upper indices of the $3j$-symbols and are therefore less general than (6.2), (6.9), (6.14) and (6.19). Hence the writer has been unable to find any formulas in the literature corresponding to the recurrence relations (6.7) or (6.12). For the case of three factors none of the results (6.21)–(6.32) are new; the derivation of (6.21)–(6.23) and (6.28)–(6.32) have essentially been given by Vilenkin [32] and W. Miller [14] respectively. Nevertheless, I have felt it useful to present these results in a coherent way; it should be noted, for example, that Edmonds [8] in deriving his formula (3.7.12), which is the equivalent of (6.23), has to make use of $6j$-symbols.

As mentioned in the Introduction, a G.F. equivalent to (1.1)–(1.3) occurs implicitly in a paper by W. Miller [13] dealing with the special functions associated with the complex Euclidean group in 3-dimensional space. These functions are essentially spherical Bessel functions and the spherical harmonics arise as a special case thereof. However, Miller does not apply his G.F. (4.12)–(4.13) to study the properties of spherical harmonics. Miller (in [12, (5.142)]) has further given an exponential G.F. incorporating a sum over all degrees of unitary representations of the group SU_2; but, although the 3-dimensional spherical harmonics are equivalent to a subset thereof, this G.F. cannot be reduced to (1.3) since the P_l^m enter the two G.F.'s with different coefficients. It can, however, serve as a G.F. for 4-dimensional harmonics, and the second part of this series is based on its use. Miller derives the orthogonality of the rotational functions in a way analogous to (5.12).

In this paper, §§ 2–5 are essentially didactic; all the results have been derived before by analytic means; but in almost every case the new presentation is simpler than or at least as simple as by conventional methods. The exceptions are Laplace's expansion (1.7) and the orthogonality and normalization relations (5.12) and (5.13); these equations arise here as special cases of more general formulas and hence require a more powerful apparatus than is normally required. If only the special theorems are required, it may be better to introduce them by a more conventional approach. It has not been the aim of the treatment in §§ 2–5 to establish a complete set of formulas, but rather to concentrate on essentials. The derivation of the higher irregular solid harmonics Υ_l^m by differentiation of Υ_0^0 would need considerable elaboration in a more comprehensive exposition, and a large number of further formulas following from other operations on the G.F.'s have been omitted. Similarly, only the basic properties of the angle averages and $3j$-symbols have been presented in § 6, though they will be considered in greater detail in the following publications.

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EXPANSION OF ANALYTIC FUNCTIONS IN JACOBI SERIES*

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Abstract. A classical theorem on expansion of an analytic function in a series of Jacobi polynomials $P_n^{(\alpha,\beta)}$ is extended so that α and β, instead of being assumed real, may have any complex values provided $\alpha + \beta \neq -2, -3, -4, \cdots$. The function to be expanded may be analytic inside an arbitrary ellipse in the complex plane, since we use notation which does not fix the foci at -1 and 1. If the ellipse is a circle, Jacobi's series reduces to Taylor's series as a special case, not a limiting case. Cauchy's inequality for the coefficients of Taylor's series has an analogue for Jacobi's series. The coefficients are determined uniquely by the analytic function. Some special series are listed.

1. Introduction and summary. The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are defined for all complex values of α and β and are orthogonal on $[-1, 1]$ if $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$. (At least in the case of real α and β, these conditions are necessary as well as sufficient for orthogonality.) If f is holomorphic inside an ellipse in the complex plane with foci at -1 and 1, it is well known [8, p. 245] that f can be represented by a series of orthogonal Jacobi polynomials with real α and β satisfying $\alpha > -1$ and $\beta > -1$.

Several facts about the representability of f by Jacobi series are less well known, and a new one will be proved in this paper. Taken together they give a viewpoint from which the Jacobi series of an analytic function is regarded as a generalized Taylor series rather than in the customary way as a series of special orthogonal polynomials.

(i) Orthogonality of the Jacobi polynomials is unnecessary.

(ii) Reality of α and β is unnecessary. The indices may have any complex values provided $\alpha + \beta \neq -2, -3, -4, \cdots$.

(iii) The ellipse inside which f is holomorphic may have its foci at arbitrary points r and s of the complex plane. This is obvious if $r \neq s$, for it suffices to substitute $z = (2w - r - s)/(s - r)$. If $r = s$, however, the ellipse degenerates to a circle and Jacobi's series to Taylor's series. In customary notation this can be verified by a limiting process, but there is a convenient notation in which Jacobi's series with arbitrary foci appears [see (1.9)] as a two-point generalization of Taylor's series, the latter being a special rather than a limiting case.

(iv) If one focus is fixed at 0 while the other tends to $+\infty$ (along with one of the indices), the ellipse becomes a parabola and the Jacobi series becomes a Laguerre series.

(v) If the two foci recede symmetrically to $-\infty$ and $+\infty$, the ellipse becomes the boundary of a strip and the Jacobi series becomes an Hermite series.

Point (i) has been observed by Boas and Buck [1, p. 61] and developed in detail by Colton [6], who showed that the indices may have any real values provided $\alpha + \beta \neq -2, -3, -4, \cdots$ and that representation by Jacobi series is not in general possible in these exceptional cases. Point (ii) is the subject of the present paper, and (iii) has been made previously by the author [3, (3.16a)]. Except

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for (7.12) and (7.13) below, we shall not touch on (iv) and (v), which incidentally show one disadvantage of fixing the foci at -1 and 1. We shall use the notation mentioned in (iii), which we now summarize.

The set of nonnegative integers will be denoted by $\mathbb{N}$, the set of nonnegative real numbers by $\mathbb{R}_+$, the complex plane by $\mathbb{C}$, and the right half-plane by $\mathbb{C}_> = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The inner region of an ellipse will be called an open elliptic disk, assumed to be nonempty. The class of functions holomorphic on an open set $\Omega \subset \mathbb{C}$ will be denoted by $H(\Omega)$. If $(b, b') \in \mathbb{C}^2_>$ a complex measure is defined on $[0, 1]$ by

$$d\mu_{(b,b')}(u) = [B(b, b')]^{-1} u^{b-1}(1 - u)^{b'-1}\, du, \tag{1.1}$$

where B is the beta function. Note that $[0, 1]$ has unit measure. Let $\Omega \subset \mathbb{C}$ be a convex open set, and let $f \in H(\Omega)$ and $f^{(n)} = d^n f/dz^n$ with $f^{(0)} = f$. For every $n \in \mathbb{N}$, $(b, b') \in \mathbb{C}^2_>$, and $(x, y) \in \Omega^2$, we define

$$F^{(n)}(b, b'; x, y) = \int_0^1 f^{(n)}[ux + (1 - u)y]\, d\mu_{(b,b')}(u). \tag{1.2}$$

We shall sometimes write F for $F^{(0)}$. We may think of $F^{(n)}$ as an average (strictly speaking, only if b and b' are positive) of $f^{(n)}$ over the line segment with endpoints x and y, which we denote by

$$[x, y] = \{z \in \mathbb{C} : z = ux + (1 - u)y, 0 \leqq u \leqq 1\}. \tag{1.3}$$

Note that

$$F^{(n)}(b, b'; x, x) = f^{(n)}(x). \tag{1.4}$$

In the particular case $f(z) = z^t$, with $t \in \mathbb{C}$ and $0 \notin \Omega$, we define

$$R_t(b, b'; x, y) = \int_0^1 [ux + (1 - u)y]^t\, d\mu_{(b,b')}(u). \tag{1.5}$$

If $t = n \in \mathbb{N}$, binomial expansion of the integrand leads to the formula

$$R_n(b, b'; x, y) = \frac{n!}{(b + b')_n} \sum_{m=0}^{n} \frac{(b)_m (b')_{n-m}}{m!(n - m)!} x^m y^{n-m}, \tag{1.6}$$

where $(a)_0 = 1$ and $(a)_m = a(a + 1) \cdots (a + m - 1)$, $m \geqq 1$. Thus $R_n(b, b'; x, y)$ is a homogeneous polynomial in x and y and is defined for all b, b' with $b + b' \neq 0$, $-1, \cdots, -n + 1$. Comparison of (1.6) with [8, (4.3.2)] shows that

$$P_n^{(\alpha,\beta)}(z) = (2^{-n}/n!)(1 + \alpha + \beta + n)_n R_n(-\alpha - n, -\beta - n; z + 1, z - 1). \tag{1.7}$$

The coefficient of z^n in the R-polynomial is unity. If $\alpha + \beta$ has one of the values $-2, -3, -4, \cdots$, there is at least one positive integer n for which

$$(1 + \alpha + \beta + n)_n = 0.$$

Then R_n is not defined and $P_n^{(\alpha,\beta)}$, although defined, is of degree strictly less than n.

Let $(r, s) \in \mathbb{C}^2$ be fixed. It is shown in [3, Lem. 1] that, for every $n \in \mathbb{N}$, $R_{-n-1}(b, b'; w - r, w - s)$ can be continued holomorphically to all complex

b, b', w such that $-b - b' \notin \mathbb{N}$ and $w \notin [r, s]$. If r and s are in Ω, this function is the kernel of a generalized Cauchy integral formula [3, (5.4)],

$$F^{(n)}(b, b'; r, s) = n!(2\pi i)^{-1} \int_\gamma f(w) R_{-n-1}(b, b'; w - r, w - s)\, dw, \tag{1.8}$$

where the rectifiable Jordan curve γ lies in Ω and encircles $[r, s]$ in the positive direction. If $r = s$, (1.8) reduces to the ordinary Cauchy formula by (1.4). According to [3, Thm. 3] the right side of (1.8) provides the holomorphic continuation of the left side to all complex b, b' such that $-b - b' \notin \mathbb{N}$. This guarantees the existence of the coefficients in the following expansion, the proof of which is the object of the present paper.

THEOREM 1.1. *Let $\Omega \subset \mathbb{C}$ be an open elliptic disk with foci r and s, and let $f \in H(\Omega)$. Let $(\alpha, \beta) \in \mathbb{C}^2$ and assume $\alpha + \beta \neq -2, -3, -4, \cdots$. Then, for every $z \in \Omega$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(1 + \alpha + n, 1 + \beta + n; r, s) R_n(-\alpha - n, -\beta - n; z - r, z - s). \tag{1.9}$$

The series converges absolutely on Ω and uniformly on every compact subset of Ω.

The coefficients are proved unique in § 6, which contains also two noteworthy relations, (6.2) and (6.6), for the Jacobi function of the second kind. If $r = s$ the coefficient $F^{(n)}$ reduces to $f^{(n)}(r)$ by (1.4) and R_n reduces to $(z - r)^n$, so Taylor's series is a special case. An inequality for $F^{(n)}$, analogous to Cauchy's inequality for $f^{(n)}$, is given in (3.8). If $r = -1$ and $s = 1$, (1.9) becomes a standardized Jacobi series according to (1.7).

The proof of Theorem 1.1, like that of the classical theorem [8, pp. 251–252], will follow the same route as Cauchy's proof of Taylor's series. The principal task is to prove (1.9) when $f(z)$ is the Cauchy kernel $1/(w - z)$. Substitution of this special case in Cauchy's integral formula will then prove the general case. Since Cauchy's formula requires f to be holomorphic, we shall not discuss Jacobi's series for functions which are square-integrable on a line segment (for a summary with references, see [7, § 10.19]).

2. Jacobi's series with remainder for the Cauchy kernel. We shall use the abbreviations

$$\begin{aligned} p_n(z) &= R_n(-\alpha - n, -\beta - n; z - r, z - s), \\ q_n(w) &= R_{-n-1}(1 + \alpha + n, 1 + \beta + n; w - r, w - s). \end{aligned} \tag{2.1}$$

If α, β, z, w, r, s are complex numbers such that $\alpha + \beta \neq -2, -3, -4, \cdots$ and $w \notin [r, s]$, p_n and q_n are well-defined for every $n \in \mathbb{N}$. Note that $p_0 = 1$. Similarly $q_{-1} = 1$ by (1.5) and analytic continuation, and p_{-1} is well-defined if $\alpha + \beta \neq 2, 3, 4, \cdots$ and $z \notin [r, s]$.

From elementary relations between associated R-functions [2, (4.1)–(4.3)] it follows after a certain amount of algebra that

$$\begin{aligned} &R_{t+1}(b - 1, b' - 1; x, y) + \left[\frac{(2t + c)(b' - b)(x - y)}{2c(c - 2)} - \frac{x + y}{2}\right] R_t(b, b'; x, y) \\ &\quad - \frac{bb't(t + c)(x - y)^2}{c^2(c^2 - 1)} R_{t-1}(b + 1, b' + 1; x, y) = 0, \end{aligned} \tag{2.2}$$

where $c = b + b'$ and $2 - c \notin \mathbb{N}$. For every $n \in \mathbb{N}$ we define U_n and V_n by the following equations if $\alpha + \beta \neq 1, 0, -1, \cdots$ and require continuity at $\alpha + \beta = 1, 0, -1$:

$$(2.3)\qquad \begin{aligned} U_n &= \frac{(\alpha^2 - \beta^2)(r - s)}{2(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)} + \frac{r + s}{2}, \\ V_n &= \frac{n(\alpha + n)(\beta + n)(\alpha + \beta + n)(r - s)^2}{(\alpha + \beta + 2n)^2[(\alpha + \beta + 2n)^2 - 1]}. \end{aligned}$$

If $z \notin [r, s]$, $w \notin [r, s]$, $\alpha + \beta \neq \pm 2, \pm 3, \pm 4, \cdots$, and $n \in \mathbb{N}$, (2.2) yields

$$(2.4)\qquad \begin{aligned} p_{n+1}(z) + (U_n - z)p_n(z) + V_n p_{n-1}(z) &= 0, \\ q_{n-1}(w) + (U_n - w)q_n(w) + V_{n+1}q_{n+1}(w) &= 0. \end{aligned}$$

We multiply the first equation by $q_n(w)$, the second by $p_n(z)$, and subtract. Defining

$$(2.5)\qquad X_n(z, w) = p_n(z)q_{n-1}(w) - V_n p_{n-1}(z)q_n(w), \qquad n \in \mathbb{N},$$

we find

$$(2.6)\qquad X_{n+1}(z, w) = X_n(z, w) + (z - w)p_n(z)q_n(w), \qquad n \in \mathbb{N}.$$

Summing (2.6) over $n = 0, 1, \cdots, N$ yields

$$(2.7)\qquad X_{N+1}(z, w) = X_0(z, w) + (z - w) \sum_{n=0}^{N} p_n(z)q_n(w).$$

Now $X_0 = 1$ under the restrictions imposed on (2.4). All other terms in (2.7) are polynomials in z and continuous functions of α and β provided $\alpha + \beta \neq -2, -3, -4, \cdots$. Thus the restrictions on z, α, β may be relaxed by continuity, and we have proved the following Jacobi series with remainder for $1/(w - z)$ (cf. [8, (4.62.19)]).

LEMMA 2.1. *Let α, β, z, w, r, s be complex numbers with $\alpha + \beta \neq -2, -3, -4, \cdots$ and $w \notin [r, s]$. For every $n \in \mathbb{N}$ define p_n and q_n by* (2.1) *and V_{n+1} by* (2.3). *Then, for every $N \in \mathbb{N}$,*

$$(2.8)\qquad 1 = (w - z) \sum_{n=0}^{N} p_n(z)q_n(w) + p_{N+1}(z)q_N(w) - V_{N+1}p_N(z)q_{N+1}(w).$$

If $r = s$ then $V_{N+1} = 0$ and (2.8) reduces to Taylor's series with remainder for $1/(w - z)$ in powers of $z - r$. For general r and s we shall determine conditions under which the two remainder terms tend uniformly to zero as N tends to infinity. To avoid proving that asymptotic approximations to q_n hold uniformly, we shall carry the method of steepest descents only as far as is needed to obtain inequalities. For p_n we shall use inequalities proved in [5].

3. Inequalities for the Jacobi function of the second kind and the expansion coefficients. An inequality for q_n will be obtained from the following inequality for an R-function in two variables.

INEQUALITY 3.1. *Let $(\alpha, \beta) \in \mathbb{C}^2$ and define $\mathbb{N}(\alpha, \beta) = \{n \in \mathbb{N} : \mathrm{Re}\,(1 + \alpha + n) > 0, \mathrm{Re}\,(1 + \beta + n) > 0\}$. Let $D = \{(x, y) \in \mathbb{C}^2 : 0 \notin [x, y]\}$, where $[x, y]$ is defined by*

(1.3). *Then there exists a continuous function* $M : D \to \mathbb{R}_+$ *such that, for every* $(x, y) \in D$ *and every* $n \in \mathbb{N}(\alpha, \beta)$,

$$\begin{aligned}(3.1)\qquad &|B(1 + \alpha + n, 1 + \beta + n)R_{-n-1}(1 + \alpha + n, 1 + \beta + n; x, y)| \\ &\leqq M(x, y)|x^{1/2} + y^{1/2}|^{-2n-2},\end{aligned}$$

where the square roots are chosen so that $|x^{1/2} + y^{1/2}| > |x^{1/2} - y^{1/2}|$. *The function* M *is homogeneous of degree zero in* x *and* y *and depends on* α, β *but not on* n.

Proof. For every $n \in \mathbb{N}(\alpha, \beta)$ define

$$A_n = B(1 + \alpha + n, 1 + \beta + n)R_{-n-1}(1 + \alpha + n, 1 + \beta + n; x, y),$$
$$\psi_n(u) = [ux + (1 - u)y]^{-n-1}u^{\alpha+n}(1 - u)^{\beta+n}.$$

Let m be the least element of $\mathbb{N}(\alpha, \beta)$. By (1.1) and (1.5),

$$A_n = \int_0^1 \psi_n(u)\, du = \int_0^1 [\varphi(u)]^{n-m}\psi_m(u)\, du,$$

where

$$(3.2)\qquad \varphi(u) = \frac{u(1 - u)}{ux + (1 - u)y}.$$

Note that ψ_m is absolutely integrable on the unit interval. The function φ has zeros at 0 and 1, a simple pole at $u_0 = y/(y - x)$, and saddle points at

$$u_1 = y^{1/2}/(y^{1/2} + x^{1/2}) \quad \text{and} \quad u_2 = y^{1/2}/(y^{1/2} - x^{1/2}),$$

provided $x \neq y$. The pole is halfway between the saddle points.

Since $(x, y) \in D$ we may suppose $|\arg x - \arg y| < \pi$. If $\arg x = \arg y$ then $0 < u_1 < 1$, while u_0 and u_2 are either greater than unity or negative. The saddle point u_1 is on the path of integration, and $|\varphi(u)| \leqq |\varphi(u_1)|$ for $0 \leqq u \leqq 1$. (The same inequality holds in the trivial case $x = y$.)

If $-\pi < \arg x - \arg y < 0$, we choose the square roots so that $-\pi/2 < \arg x^{1/2} - \arg y^{1/2} < 0$. Then $u_1 = [1 + x^{1/2}/y^{1/2}]^{-1}$ lies in the first quadrant of the open unit disk, while u_0 and u_2 are in the lower half-plane. By Cauchy's theorem we may deform the path of integration into a contour C leading from 0 to 1 over the saddle point at u_1, such that $|\varphi(u)| \leqq |\varphi(u_1)|$ for all points u on the contour. The same remark holds for the case $0 < \arg x - \arg y < \pi$, in which u_1 lies in the fourth quadrant of the open unit disk while u_0 and u_2 are in the upper half-plane. Therefore, in every case,

$$(3.3)\qquad |A_n| \leqq |\varphi(u_1)|^{n-m} \int_C |\psi_m(u)\, du|.$$

Since $|\varphi(u_1)| = |x^{1/2} + y^{1/2}|^{-2}$, we have proved (3.1) with $M(x, y)$ replaced by

$$(3.4)\qquad \tilde{M}(x, y) = |x^{1/2} + y^{1/2}|^{2m+2} \int_C |ux + (1 - u)y|^{-m-1}|u^{\alpha+m}(1 - u)^{\beta+m}\, du|.$$

We may choose C to depend continuously on $(x, y) \in D$, for example, by taking the path of steepest descents from u_1. Then $\tilde{M}$ is continuous on D. Since R_{-n-1}

is homogeneous of degree $-n-1$ in x and y, (3.1) still holds if $\tilde{M}(x, y)$ is replaced by $M(x, y) = \tilde{M}(x/y, 1)$.

LEMMA 3.2. *Let $(r, s) \in \mathbb{C}^2$, $\rho \in \mathbb{R}_+$, and $\rho > |r - s|$. Then the set of all points $z \in \mathbb{C}$ satisfying*

$$|(z - r)^{1/2} + (z - s)^{1/2}| = \rho^{1/2} \tag{3.5}$$

is an ellipse with foci r and s. The lengths of the principal axes have sum ρ and difference $|r - s|^2/\rho$.

Proof. Equation (3.5) implies

$$|(z - r)^{1/2} - (z - s)^{1/2}| = |r - s|/\rho^{1/2} \tag{3.6}$$

and hence, by adding the squares of (3.5) and (3.6),

$$|z - r| + |z - s| = \tfrac{1}{2}\rho + \tfrac{1}{2}|r - s|^2/\rho.$$

This is the equation of an ellipse with foci r and s. The right side is the length of the major axis, and the length of the minor axis is therefore $\frac{1}{2}\rho - \frac{1}{2}|r - s|^2/\rho$.

INEQUALITY 3.3. *Let $\Omega \subset \mathbb{C}$ be an open elliptic disk with foci r and s. Denote the sum of the lengths of the principal axes by ρ. Let $(\alpha, \beta) \in \mathbb{C}^2$ and define $\mathbb{N}(\alpha, \beta)$ as in Inequality 3.1. If q_n is defined by (2.1), there exists $A \in \mathbb{R}_+$ such that, for every $n \in \mathbb{N}(\alpha, \beta)$ and every $w \in \mathbb{C} - \Omega$,*

$$|B(1 + \alpha + n, 1 + \beta + n)q_n(w)| \leqq A\rho^{-n-1}. \tag{3.7}$$

Proof. Define D as in Inequality 3.1. By (2.1), (3.1) and Lemma 3.2, there exists a continuous homogeneous function $M : D \to \mathbb{R}_+$ such that, for every $n \in \mathbb{N}(\alpha, \beta)$ and every $w \in \mathbb{C} - \Omega$,

$$|B(1 + \alpha + n, 1 + \beta + n)q_n(w)| \leqq M(w - r, w - s)\rho^{-n-1} = M\left(\frac{w - r}{w - s}, 1\right)\rho^{-n-1}.$$

Let $K = \{((w - r)/(w - s), 1) : w \in \mathbb{C} - \Omega\}$. Since K is a compact subset of D and M is continuous on D, M is bounded on K. Let A be the maximum of M on K.

From Inequality 3.3 we can obtain for $F^{(n)}$ an inequality analogous to Cauchy's inequality for $f^{(n)}$ but less precise.

INEQUALITY 3.4. *Define Ω, ρ and $\mathbb{N}(\alpha, \beta)$ as in Inequality 3.3. Let $f \in H(\Omega)$ and assume $|f|$ is bounded on Ω. Define $F^{(n)}$ by (1.2). Then there exists $L \in \mathbb{R}_+$ such that, for every $n \in \mathbb{N}(\alpha, \beta)$,*

$$|B(1 + \alpha + n, 1 + \beta + n)F^{(n)}(1 + \alpha + n, 1 + \beta + n; r, s)| \leqq Ln!\rho^{-n}. \tag{3.8}$$

Proof. Let γ be a positively oriented ellipse lying in Ω with foci r and s, and let $\rho' < \rho$ be the sum of the lengths of its principal axes. By (1.8) and (2.1),

$$F^{(n)}(1 + \alpha + n, 1 + \beta + n; r, s) = n!(2\pi i)^{-1} \int_\gamma f(w)q_n(w)\, dw.$$

Let λ be the length of γ, and note that $\lambda < 2\pi\rho'$. Let μ be the supremum of $|f|$ on Ω. By Inequality 3.3 (with ρ and A replaced by ρ' and A', respectively) we find, for every $n \in \mathbb{N}(\alpha, \beta)$,

$$\begin{aligned}|B(1 + \alpha + n, 1 + \beta + n)F^{(n)}(1 + \alpha + n, 1 + \beta + n; r, s)| &\leqq n!(2\pi)^{-1}\lambda\mu A'/(\rho')^{n+1} \\ &\leqq n!\mu A'/(\rho')^n.\end{aligned}$$

Since γ can be chosen to make ρ' arbitrarily close to ρ, the inequality still holds if $A'/(\rho')^n$ is replaced by A/ρ^n. Putting $L = \mu A$ we have proved (3.8).

4. Jacobi's infinite series for the Cauchy kernel. Retaining the notation and conditions of Inequality 3.3, we obtain from [5, Thm. 5.6] the following inequality for the Jacobi polynomial $p_n(z)$ defined by (2.1). For every $n \in \mathbb{N}$ there exists a positive number $g_n(\alpha, \beta)$ such that

$$|(1 + \alpha + \beta + n)_n p_n(z)| \leqq n! g_n(\alpha, \beta)\rho^n, \qquad z \in \Omega, \tag{4.1a}$$

$$\lim_{n \to \infty} [g_n(\alpha, \beta)]^{1/n} = 1. \tag{4.1b}$$

The inequalities for q_n and p_n give enough information about the behavior of the remainder terms in (2.8) to prove the following theorem.[1]

THEOREM 4.1. *Let $\Omega \subset \mathbb{C}$ be an open elliptic disk with foci r and s. Let $(\alpha, \beta) \in \mathbb{C}^2$ and assume $\alpha + \beta \neq -2, -3, -4, \cdots$. Then, for every $z \in \Omega$ and every $w \in \mathbb{C} - \Omega$,*

$$\begin{aligned} \frac{1}{w - z} &= \sum_{n=0}^{\infty} q_n(w) p_n(z) \\ &= \sum_{n=0}^{\infty} R_{-n-1}(1 + \alpha + n, 1 + \beta + n; w - r, w - s) \\ &\quad \cdot R_n(-\alpha - n, -\beta - n; z - r, z - s). \end{aligned} \tag{4.2}$$

The series converges absolutely, uniformly in w on $\mathbb{C} - \Omega$, and uniformly in z on every compact subset of Ω.

Proof. Let ρ denote the sum of the lengths of the principal axes of Ω. Let $\rho' < \rho$ be the corresponding sum for a confocal elliptic disk $\Omega' \subset \Omega$. Assume $z \in \Omega'$ and $w \in \mathbb{C} - \Omega$. By (3.7) and (4.1a) the first remainder term in (2.8) satisfies, for every $n \in \mathbb{N}(\alpha, \beta)$,

$$|p_{N+1}(z) q_N(w)| \leqq \frac{A(N + 1)! g_{N+1}(\alpha, \beta)(\rho'/\rho)^{N+1}}{|(1 + \alpha + \beta + N + 1)_{N+1} B(1 + \alpha + N, 1 + \beta + N)|}.$$

According to (4.1b) and [7, (1.18(4))] the Nth root of the right side tends to $\rho'/\rho < 1$ as $N \to \infty$, and hence the left side tends to zero uniformly on $\Omega' \times (\mathbb{C} - \Omega)$. So also does the second remainder term, $|V_{N+1} p_N(z) q_{N+1}(w)|$, by a similar calculation. Given any compact set $K \subset \Omega$ we can choose Ω' so that $K \subset \Omega' \subset \Omega$, and hence the series in (4.2) converges uniformly to $1/(w - z)$ on $K \times (\mathbb{C} - \Omega)$. The series converges absolutely by the root test, as one verifies by applying (3.7) and (4.1) to $|p_n(z) q_n(w)|$.

5. Proof of Theorem 1.1. Let γ be a positively oriented ellipse lying in Ω with foci r and s and interior Ω'. For every $z \in \Omega'$ Cauchy's integral formula gives

$$f(z) = (2\pi i)^{-1} \int_\gamma f(w)(w - z)^{-1}\, dw. \tag{5.1}$$

[1] The classical theorem for real $\alpha, \beta > -1$ is proved in [8, pp. 251–252] by using the asymptotic formulas of Darboux. Since Darboux's formula for the Jacobi polynomial [8, p. 196] does not hold uniformly on any neighborhood of the foci, the proof of uniform convergence is incomplete. It can be completed, following a suggestion of Professor R. Askey, by using the maximum-modulus theorem.

Since the Jacobi expansion (4.2) of $(w - z)^{-1}$ converges uniformly in w on $\mathbb{C} - \Omega'$ and therefore on γ, we may integrate term by term to obtain

$$f(z) = \sum_{n=0}^{\infty} p_n(z)(2\pi i)^{-1} \int_{\gamma} f(w) q_n(w)\, dw. \tag{5.2}$$

By (2.1) and (1.8) this is the same as (1.9). If K is any compact subset of Ω, we can choose γ so that $K \subset \Omega'$. Since (4.2) converges uniformly in z on K, so do (5.2) and (1.9).

To prove that (1.9) converges absolutely for every $z \in \Omega$, choose elliptic disks Ω' and Ω'' confocal with Ω such that $z \in \Omega' \subset \Omega'' \subset \Omega$. Let the sums of lengths of principal axes satisfy $\rho' < \rho'' < \rho$. Since $f \in H(\Omega)$, $|f|$ is bounded on Ω''. By (3.8) and (4.1) the nth term of (1.9) is bounded in modulus for every $n \in \mathbb{N}(\alpha, \beta)$ by

$$\frac{L n! g_n(\alpha, \beta)(\rho'/\rho'')^n}{|B(1 + \alpha + n, 1 + \beta + n)(1 + \alpha + \beta + n)_n|}.$$

Since the nth root of this quantity tends to $\rho'/\rho'' < 1$ as $n \to \infty$, (1.9) converges absolutely by the root test.

6. Uniqueness of the coefficients. If (1.9) were not the only representation of f by a series of Jacobi polynomials, subtraction of two different representations would give a nontrivial series of the form

$$0 = \sum_{m=0}^{\infty} a_m p_m(z), \qquad z \in \Omega. \tag{6.1}$$

There is a simple example [9, (14)], [10, (33)] of a series which converges to zero on the interfocal line segment provided $\alpha < -\frac{1}{2}$ and $\beta \neq -1, -2, -3, \cdots$ (the polynomials may even be orthogonal), but it diverges everywhere else for all values of α. We shall show in this section that convergence to zero cannot occur at every point of an open set in the plane. The method, although not the shortest possible, is interesting because of (6.2) and (6.6), which are not widely known. The latter can be found in different notation in [6, (16)].

If the complex plane is cut along the line segment $[r, s]$, $q_n(w)$ is holomorphic in the cut plane but discontinuous across the cut. The following lemma states that the discontinuity in q_n is the Jacobi polynomial p_n multiplied by the associated weight function. This is a generalization of a familiar property of Legendre functions ($\alpha = \beta = 0$) [7, (10.10(36))].

LEMMA 6.1. *Let α, β, r, s be complex numbers with $\alpha + \beta \neq -2, -3, -4, \cdots$ and $r \neq s$. Let $z \in [r, s] - \{r\} - \{s\}$, and let $\varepsilon \in \mathbb{C}$ satisfy* $\arg \varepsilon = \arg(s - r) + \pi/2$. *Define p_n and q_n by* (2.1). *Then, for every $n \in \mathbb{N}$,*

$$\lim_{\varepsilon \to 0} [q_n(z - \varepsilon) - q_n(z + \varepsilon)] = \frac{2\pi i}{I_n}(z - r)^{\beta}(s - z)^{\alpha} p_n(z), \tag{6.2}$$

where $\arg(z - r) = \arg(s - r) = \arg(s - z)$ *and*

$$I_n = \frac{n! B(1 + \alpha + n, 1 + \beta + n)}{(1 + \alpha + \beta + n)_n}(s - r)^{1 + \alpha + \beta + 2n}. \tag{6.3}$$

Remark. The Jacobi function of the second kind is usually defined as $q_n(w)$ multiplied by $(w - r)^{-\beta}(w - s)^{-\alpha}$ and a constant, and the cut extends through r (usually -1) to infinity. Since the elementary factor also is discontinuous across the cut, the discontinuity of the product does not have the simplicity of (6.2) unless α is an integer [7, (10.8(21))], [8, (4.62.8)]. We shall see in the proof of Theorem 6.2 that (6.2) is intimately connected with the orthogonality and normalization of the Jacobi polynomials.

Proof. Given $n \in \mathbb{N}$ we first prove (6.2) for values of α and β such that $n \in \mathbb{N}(\alpha, \beta)$ (see Inequality 3.1). By (2.1) and (1.5) we find, assuming $w \notin [r, s]$ and putting $t = ur + (1 - u)s$,

$$(6.4)\qquad \begin{aligned} &B(1 + \alpha + n, 1 + \beta + n)(s - r)^{1+\alpha+\beta+2n}q_n(w) \\ &= \int_r^s (w - t)^{-n-1}(s - t)^{\alpha+n}(t - r)^{\beta+n}\,dt. \end{aligned}$$

If $w = z + \varepsilon$ we may deform the path of integration into a path γ_+ which follows the cut except for a small semicircular detour to the right of the cut (as one proceeds from r to s) with radius δ and center z. Since $|z + \varepsilon - t| \geqq \delta$ for every t on γ_+, it follows from Lebesgue's theorem of dominated convergence that the integral is continuous at $\varepsilon = 0$. Hence its limit as $w = z + \varepsilon \to z$ is

$$\int_{\gamma+} (z - t)^{-n-1}(s - t)^{\alpha+n}(t - r)^{\beta+n}\,dt.$$

The same result holds if $w = z - \varepsilon \to z$ and γ_+ is replaced by γ_-, for which the semicircular detour around z lies to the left of the cut. When we take the difference in (6.2), the straight portions of γ_+ and γ_- cancel, leaving the two semicircles. These combine to form a complete circle with center z and radius δ, described clockwise. Evaluation by the Cauchy integral formula yields $(-1)^{n+1}(-2\pi i)/n!$ times

$$(6.5)\qquad \frac{d^n}{dz^n}[(s - z)^{\alpha+n}(z - r)^{\beta+n}] = (-1)^n(1 + \alpha + \beta + n)_n(z - r)^{\beta}(s - z)^{\alpha}p_n(z),$$

this being Rodrigues' formula for p_n. Having proved (6.2), we now observe that both sides are holomorphic in α and β for $\alpha + \beta \neq -2, -3, -4, \cdots$. By analytic continuation, (6.2) is valid subject to this condition.

THEOREM 6.2. *Let α, β, r, s be complex numbers, assume $\alpha + \beta \neq -2, -3, -4, \cdots$, and let γ be a rectifiable Jordan curve encircling the segment $[r, s]$ in the positive direction. Let $(m, n) \in \mathbb{N}^2$ and let $\delta_{mn} = 1$ if $m = n$ and 0 otherwise. Define p_m and q_n by* (2.1). *Then*

$$(6.6)\qquad \frac{1}{2\pi i}\int_{\gamma} p_m(w)q_n(w)\,dw = \delta_{mn}.$$

Proof. Since $p_m(w)$ is a polynomial of degree m with unit coefficient of w^m, (6.6) follows at once from (1.8) if $m \leqq n$. Also, the case $r = s$ is elementary, for the integrand reduces by (1.4) to $(w - r)^{m-n-1}$. If $r \neq s$ we deform γ into two small circles around r and s, joined by the two edges of the cut. We now assume $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$. It follows easily from (6.4) that $(w - s)q_n(w) \to 0$ as

$w \to s$ [for example, write $w - s = (w - t) - (s - t)$], and similarly with s replaced by r. Hence the integrals around the two small circles tend to zero with the radius. By Lemma 6.1 the integrals along the edges of the cut combine to give

$$(6.7)\qquad (2\pi i)^{-1} \int_\gamma p_m(w) q_n(w)\, dw = I_n^{-1} \int_r^s p_m(z) p_n(z)(z - r)^\beta (s - z)^\alpha\, dz.$$

We have already seen that this vanishes if $m < n$. Since the integral on the right is symmetric in m and n (and $I_n^{-1} \neq 0$), it must vanish also if $m > n$. In addition to proving (6.6), we obtain as a by-product the orthogonality integral

$$(6.8)\qquad \int_r^s p_m(z) p_n(z)(z - r)^\beta (s - z)^\alpha\, dz = \delta_{mn} I_n, \qquad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1.$$

Provided $\alpha + \beta \neq -2, -3, -4, \cdots$ the integrand of (6.6) is bounded on γ for every fixed α, β and holomorphic in α and β for every fixed w on γ. Therefore the integral is holomorphic in α and β, and the validity of (6.6) can be extended by the principle of analytic continuation.

An alternative proof of (6.6) which uses neither (6.2) nor analytic continuation will be sketched briefly. As before, we need to consider only the case $m > n$. In the definition (2.1) of p_m, expand the right side by the binomial theorem [3, (3.4)] for R-polynomials. Using (1.8) we find that the left side of (6.6) equals

$$(6.9)\qquad \frac{m!}{n!} \sum_{k=0}^{m-n} \frac{R_k(-\alpha - m, -\beta - m; -r, -s) R_{m-n-k}(1 + \alpha + n, 1 + \beta + n; r, s)}{k!(m - n - k)!}.$$

By [3, (4.5), (2.18), (4.15)] the sum is the term proportional to $(r - s)^{m-n}$ in

$$(6.10)\qquad {}_1F_1(-\alpha - m; -\alpha - \beta - 2m; s - r)\, {}_1F_1(1 + \alpha + n; 2 + \alpha + \beta + 2n; r - s).$$

Multiplication of the two series[2] shows that this term is proportional to

$$(6.11)\qquad \begin{aligned} &{}_2F_1(-m + n, 1 + \alpha + \beta + m + n; 2 + \alpha + \beta + 2n; 1) \\ &\quad = \frac{(1 - m + n)_{m-n}}{(2 + \alpha + \beta + 2n)_{m-n}} = 0. \end{aligned}$$

This completes the alternative proof.

We now suppose $\alpha + \beta \neq -2, -3, -4, \cdots$ and $\sum a_m p_m = 0$ on an open set in the plane. The essential fact is that convergence of $\sum a_m p_m(z)$ at a point $z \notin [r, s]$ implies convergence on an open elliptic disk Ω (with foci r and s and with z on the boundary) and uniform convergence on compact subsets of Ω. This statement is readily proved by Darboux's asymptotic formula [8, (8.21.9), (8.23.1)], which we have not used previously in this paper, and the maximum-modulus theorem. (Darboux's formula is valid for all complex α and β, although a printed statement to this effect is hard to find.) It follows that the sum of the series is holomorphic on Ω and must vanish on Ω if it vanishes on an open subset. We multiply by q_n and integrate term by term (since the convergence is uniform) around a contour γ which lies in Ω and encircles $[r, s]$. By Theorem 6.2 we find

[2] On the right side of [7, (4.3(13))] the first parameter of the ${}_3F_2$-series should be a' instead of $-a'$.

$a_n = 0$ for every $n \in \mathbb{N}$. Thus there is no nontrivial representation of zero on an open set in the plane, and the coefficients of (1.9) are unique.

7. Some special cases. We note here some special cases of Theorem 1.1. Let α, β, r, s be complex numbers with $\alpha + \beta \neq -2, -3, -4, \cdots$, and define p_n by (2.1). The S-function [3, (2.16)] is defined by an equation similar to (1.5),

$$S(b, b'; x, y) = \int_0^1 \exp\,[ux + (1-u)y]\, d\mu_{(b,b')}(u), \tag{7.1}$$

and continued analytically by (1.8) with $f(w) = e^w$. For all complex κ and z,

$$e^{\kappa z} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} S(1 + \alpha + n, 1 + \beta + n; \kappa r, \kappa s) p_n(z). \tag{7.2}$$

The Bessel function J_ν and the Gegenbauer polynomial C_n^ν (including the Legendre polynomial $P_n = C_n^{1/2}$) are given by

$$\Gamma(1 + \nu) J_\nu(z) = (z/2)^\nu S(\tfrac{1}{2} + \nu, \tfrac{1}{2} + \nu; iz, -iz), \tag{7.3}$$

$$n! C_n^\nu(z) = 2^n (\nu)_n R_n(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; z + 1, z - 1). \tag{7.4}$$

Thus (7.2) contains (for $\alpha = \beta$) Sonine's formula [7, (7.10(5))] and in particular (for $\alpha = \beta = 0$) the expansion of a plane wave in spherical Bessel functions and Legendre polynomials.

The Chebyshev polynomial with unit coefficient for the highest power of z is

$$2^{1-n}(1 + \delta_{n0})^{-1} \cos n(\arccos z) = R_n(\tfrac{1}{2} - n, \tfrac{1}{2} - n; z + 1, z - 1). \tag{7.5}$$

From this we deduce the following Fourier cosine series. Let $(A, B) \in \mathbb{C}^2$, let $\Omega \subset \mathbb{C}$ be an open elliptic disk with foci $A \pm B$, and let $f \in H(\Omega)$. Then, for any complex θ such that $A + B\cos\theta \in \Omega$,

$$\begin{aligned} f(A + B\cos\theta) &= F(\tfrac{1}{2}, \tfrac{1}{2}; A + B, A - B) \\ &\quad + 2 \sum_{n=1}^{\infty} \frac{B^n}{n! 2^n} F^{(n)}(\tfrac{1}{2} + n, \tfrac{1}{2} + n; A + B, A - B) \cos n\theta. \end{aligned} \tag{7.6}$$

A particular case is the Fourier cosine series of a plane wave, in which the expansion coefficients are Bessel coefficients. A useful corollary of (7.6) is the relation

$$\int_0^\pi f(A + B\cos\theta) \cos n\theta \, d\theta = \frac{\pi B^n}{n! 2^n} F^{(n)}(\tfrac{1}{2} + n, \tfrac{1}{2} + n; A + B, A - B), \quad n \in \mathbb{N}. \tag{7.7}$$

Let $(A, B) \in \mathbb{C}^2$ and let Ω be an open elliptic disk with foci r and s such that $A/B \notin \Omega$. If $\lambda \in \mathbb{C}$ and $z \in \Omega$, then

$$(A - Bz)^{-\lambda} = \sum_{n=0}^{\infty} (\lambda)_n \frac{B^n}{n!} R_{-\lambda-n}(1 + \alpha + n, 1 + \beta + n; A - Br, A - Bs) p_n(z). \tag{7.8}$$

Special cases and an application of this series are given in [4]. One special case is an expansion of a Coulomb potential in Gegenbauer polynomials.

With the same conditions as for (7.8), we find

$$\log(A - Bz) = L(1 + \alpha, 1 + \beta; A - Br, A - Bs) \tag{7.9}$$
$$- \sum_{n=1}^{\infty} \frac{B^n}{n} R_{-n}(1 + \alpha + n, 1 + \beta + n; A - Br, A - Bs)p_n(z),$$

where the L-function [3, (4.6)] is defined by

$$L(b, b'; x, y) = \int_0^1 \log[ux + (1 - u)y]\, d\mu_{(b,b')}(u) \tag{7.10}$$

and continued analytically by (1.8) with $f(w) = \log w$.

If $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$ and $r \neq s$, the coefficients in (1.9) can be written in the form

$$F^{(n)}(1 + \alpha + n, 1 + \beta + n; r, s) = \frac{n!}{I_n} \int_r^s f(z)p_n(z)(z - r)^\beta (s - z)^\alpha\, dz, \tag{7.11}$$

where $\arg(z - r) = \arg(s - r) = \arg(s - z)$ and I_n is defined by (6.3). This representation follows from (1.9) and (6.8), or alternatively from (1.8) and (6.2) by squeezing the contour γ down onto the cut.

Finally we mention that the Laguerre and Hermite polynomials, defined as in [7, §§ 10.12, 10.13], satisfy

$$L_n^\beta(z) = \frac{(-1)^n}{n!} \lim_{s \to \infty} R_n(-s - n, -\beta - n; z, z - s), \tag{7.12}$$

$$H_n(z) = 2^n \lim_{s \to \infty} R_n(-s^2 - n, -s^2 - n; z + s, z - s). \tag{7.13}$$

Rodrigues' formula (6.5) provides a convenient method of proof.

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PARSEVAL RELATION AND TAUBERIAN THEOREMS FOR THE HANKEL TRANSFORM*

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Abstract. Let $F_\nu(x)$, the Hankel transform of f, be defined by $F_\nu(x) = \int_0^\infty f(t)\sqrt{xt}\,J_\nu(xt)\,dt$. It is proved that the Parseval relation $\int_{\to 0}^{\to\infty} F_\nu(x)G_\nu(x)\,dx = \int_0^\infty f(x)g(x)\,dx$ holds, if (i) $x^{\nu+1/2}f(x) \in L(0, R)$ for each finite $R > 0$, f is of bounded variation in $[a, \infty)$ for some $a > 0$, and $f(x) \to 0$ as $x \to \infty$; (ii) $g \in L(0, \infty)$, g is of bounded variation in a neighborhood of every point where f is not and $G_\nu(x) = O(x^{-\lambda-\nu})$ as $x \to \infty$ for some $\lambda > 3/2$. As an application, results of Tauberian character for the Hankel transform are obtained.

1. Introduction. Let $F(x)$, the k-transform of f, be defined by

$$F(x) = \int_0^\infty f(u)k(xu)\,du. \tag{1.1}$$

The Hankel transform, with $k(u) = u^{1/2}J_\nu(u)$, arises in the study of multiple Fourier integrals [1, p. 69], the summability of multiple Fourier series [3] and the eigenvalue problems for the partial differential operators [2]. It is well known [1, Chap. 5] that, if k is a Fourier kernel, (1.1) defines a unitary transformation on $L_2(0, \infty)$. Moreover, if $f \in L_2(0, \infty)$ and $g \in L_2(0, \infty)$, we have

$$\int_0^\infty F(x)G(x)\,dx = \int_0^\infty f(t)g(t)\,dt. \tag{1.2}$$

The formula (1.2), however, may be true even when f and g are not square integrable. Edmonds [4], [5] has proved a number of results concerning the validity of (1.2) when k is the Fourier sine or the cosine kernel and the integrals defining F or G converge only in the Cauchy sense. Macaulay-Owen [10] considered the Hankel transform and obtained (1.2) under conditions comparable to a result of Titchmarsh [14, Thm. 38] for the trigonometric transforms. More recently, Soni and Soni [13] have considered the general Fourier kernels with some additional restrictions on the kernel. However, only monotone functions are considered in [13].

We generalize the result proved by Macauley-Owen [10] in a manner analogous to Edmonds [4] for the sine and the cosine transforms. It is well known [6, (10), p. 29] that

$$\int_0^\infty t^{\nu+1/2}\,e^{-st^2}(xt)^{1/2}J_\nu(xt)\,dt = \frac{x^{\nu+1/2}}{(2s)^{\nu+1}}\exp\left(\frac{-x^2}{4s}\right), \qquad s > 0. \tag{1.3}$$

Our generalization, together with (1.3), allows us to prove Tauberian theorems for the Hankel transform. If the kernel belongs to $L(0, \infty)$, Wiener's Tauberian theorem [7, Thm. 232] may be directly applicable. Bureau [2] and Cheng [3] proved Tauberian theorems for a modified form of the Hankel transform, with $k(u) = u^{-\nu}J_\nu(u)$, $\nu > 0$. Both authors impose order conditions on the function and do not allow slowly varying functions in the asymptotic behavior of the

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transform. Pitman [11] does allow slowly varying functions; however, his results are only for the sine and cosine transforms. Our results are for the Hankel transform which, for the particular cases $\nu = \pm 1/2$, reduces to the sine and the cosine transforms respectively.

Karamata [7, Thm. 110] proved the following Tauberian theorem for the Laplace transform.

THEOREM A. *Assume that* $\sigma > 0$, $\alpha(u)$ *is nondecreasing,* $I(y) = \int_0^\infty e^{-yu}\, d\alpha(u)$ *converges for* $y > 0$, *and* $g(u)$ *is of bounded variation in* $[0, 1]$. *Further let*

$$\chi(s) = \int_0^\infty e^{-su} g(e^{-su})\, d\alpha(u). \tag{1.4}$$

If $L(x)$ is slowly varying, then

$$I(y) \sim y^{-\sigma} L(1/y), \qquad y \to 0+ \tag{1.5}$$

implies

$$\chi(s) \sim \frac{1}{\Gamma(\sigma)} s^{-\sigma} L(1/s) \int_0^\infty e^{-u} g(e^{-u}) u^{\sigma - 1}\, du \tag{1.6}$$

as $s \to 0+$, except for an exceptional set.

With the help of the Parseval relation, we are able to give analogous Tauberian theorems for the Hankel transform. Except for the results given here, to the best of our knownledge, there are no results similar to Theorem A for any other integral transform. By specializing g, we then obtain asymptotic behavior of the function (or an integral related to the function) when the Hankel transform behaves like $x^\sigma L(x)$.

2. Notation and basic assumptions. All functions are assumed to be real and measurable. The parameter ν satisfies the condition $\nu \geqq -1/2$. $F_\nu(x)$, the Hankel transform of f, is defined by

$$F_\nu(x) = \int_0^\infty f(u)(xu)^{1/2} J_\nu(xu)\, du. \tag{2.1}$$

($G_\nu(x)$ is similarly related to g.) The convergence of an integral in the Cauchy sense is indicated by an arrow [14, p. 9]. The function $\alpha(t; f)$ is defined by

$$\alpha(t; f) = \int_0^t u^{\nu/2 - 1/4} f(u^{1/2})\, du. \tag{2.2}$$

The function $L(x)$ is slowly varying in the sense of Karamata [8]. $\mathrm{BV}[a, b]$ is the class of functions of bounded variation in $a \leqq x \leqq b$.

The class $\mathscr{P}_\alpha$: $f(x) \in \mathscr{P}_\alpha$ if $x^\alpha f(x) \in L(0, R)$ for each finite $R > 0$, $f(x) \in \mathrm{BV}[a, \infty)$ for some $a > 0$ and $f(x) \to 0$ as $x \to \infty$.

The constants ρ and K are defined by

$$\rho = (\nu - \gamma + \tfrac{1}{2})/2, \tag{2.3}$$

$$K = 2^{\gamma + 1/2} \frac{\Gamma(\nu/2 + \gamma/2 + 3/4)}{\Gamma(\rho + 1)}. \tag{2.4}$$

3. Main results. The first theorem gives a generalization of a result proved by Macaulay-Owen [10].

THEOREM 1. *Let* $\nu \geqq -\frac{1}{2}$. *If* $f \in \mathscr{P}_{\nu+1/2}$ *and* g *satisfies the conditions*

(i) $g \in L(0, \infty)$ *and is of bounded variation in a neighborhood of every point where* f *is not*;

(ii) $G_\nu(x) = O(x^{-\nu-\lambda})$ *as* $x \to \infty$ *for some* $\lambda > \frac{3}{2}$;

then the Parseval relation

$$\int_{\to 0}^{\to \infty} F_\nu(x) G_\nu(x)\, dx = \int_0^\infty f(t) g(t)\, dt \tag{3.1}$$

holds.

Theorems 2–5 give results analogous to Theorem A for the Hankel transform.

THEOREM 2. *Let* $-1 < \gamma < \nu + 1/2$, $f \geqq 0$. *If* $f \in L(0, \infty)$ *and* $h \in \mathrm{BV}[0, 1]$, *then*

$$\chi(s) = \int_0^\infty e^{-su} h(e^{-su})\, d\alpha(u; f) \tag{3.2}$$

exists for $s > 0$. *Furthermore,*

$$F_\nu(x) \sim x^\gamma L(1/x), \qquad x \to 0+ \tag{3.3}$$

implies

$$\chi(s) \sim \rho K L(s^{-1/2}) s^{-\rho} \int_0^\infty e^{-u} h(e^{-u}) u^{\rho-1}\, du, \tag{3.4}$$

as $s \to 0+$.

THEOREM 3. *Let* $\nu > -1/2$, $-1 < \gamma < \nu + 1/2$, $f \geqq 0$. *If* $f \in \mathscr{P}_{\nu+1/2}$ *and* $h \in \mathrm{BV}[0, 1]$, *then* (3.3) *implies* (3.4).

THEOREM 4. *Let* $\nu \geqq -1/2$ *and* h *be continuous in* $[0, 1]$. *If* f *satisfies the conditions of Theorem* 2, *then*

$$F_\nu(x) \sim x^{\nu+1/2} L(1/x), \qquad x \to 0+, \tag{3.5}$$

implies

$$\chi(s) \sim 2^{\nu+1} \Gamma(\nu + 1) L(s^{-1/2}) h(1), \qquad s \to 0+. \tag{3.6}$$

THEOREM 5. *Let* $\nu > -\frac{1}{2}$ *and* h *be continuous in* $[0, 1]$. *If* f *satisfies the conditions of Theorem* 3, *then* (3.5) *implies* (3.6).

Theorems 6 and 7 give the behavior of $\alpha(u; f)$ and $f(u)$ in terms of the behavior of $F_\nu(x)$.

THEOREM 6. *Let* $\nu \geqq -\frac{1}{2}$, $-1 < \gamma \leqq \nu + \frac{1}{2}$. *If* f *satisfies the conditions of Theorem* 2 *or* 3, *according as* $\nu \geqq -\frac{1}{2}$ *or* $\nu > -\frac{1}{2}$, *then*

$$F_\nu(x) \sim x^\gamma L(1/x), \qquad x \to 0+, \tag{3.7}$$

implies

$$\alpha(u; f) \sim K L(u^{1/2}) u^\rho, \qquad u \to \infty. \tag{3.8}$$

THEOREM 7. *Let* $\nu \geqq -\frac{1}{2}$, $-1 < \gamma < \nu + \frac{1}{2}$. *Assume that* f *satisfies conditions of Theorems* 6 *and one of the following*:

(i) $u^{\nu - 1/2}f(u)$ *is monotone decreasing for* t *sufficiently large.*

(ii) $u^{\nu - 1/2}f(u)$ *is monotone increasing for* t *sufficiently large. Then,*

$$F_\nu(x) \sim x^\gamma L(1/x), \qquad x \to 0+, \tag{3.9}$$

implies

$$f(u) \sim \rho K u^{-\gamma-1}L(u), \qquad u \to \infty. \tag{3.10}$$

Results similar to those contained in Theorems 2–7, when the behavior of $f(u)$ and $\alpha(u; f)$ as $u \to 0$ is obtained in terms of the behavior of $F_\nu(x)$ as $x \to \infty$, can be given by using the dual results of Karamata [9, Haupsatz 1]. We also note that the restriction $\gamma > -1$ is necessary in view of Lemma 1.

4. Preliminary results. In this section we give properties of the Hankel transform needed to prove the main results. We note that $F_\nu(x)$, the Hankel transform of $f \in L(0, \infty)$, exists and is a bounded, continuous function for $x \geqq 0$.

LEMMA 1. *If* $f \in \mathscr{P}_{\nu+1/2}$, *then* $F_\nu(x)$ *exists for* $x > 0$. *Moreover,*

$$F_\nu(x) = o(x^{-1}), \qquad x \to 0+, \tag{4.1}$$

$$F_\nu(x) = o(x^{\nu+1/2}), \qquad x \to \infty. \tag{4.2}$$

The next lemma gives a property of the Hankel transform analogous to a known result [14, p. 13] for the Fourier transform.

LEMMA 2. *If* $0 \leqq \alpha < a < b < \beta \leqq \infty$, $f \in L(0, \infty)$ *and* f *vanishes in* (α, β), *then*

$$\int_0^{\to\infty} \sqrt{xu}\, J_\nu(xu)\, du \int_0^\infty f(y)\sqrt{uy}\, J_\nu(uy)\, dy = 0 \tag{4.3}$$

uniformly for $x \in [a, b]$.

Although a direct proof of these lemmas can be given, the following known result [12, Lemma, § 2] considerably shortens the proofs.

LEMMA 3. *Let* $f(t) \in L(0, \infty)$. *If* $k(t)$ *is essentially bounded and* $k_1(t) = o(t)$, $t \to \infty$, *where*

$$k_1(t) = \int_0^t k(u)\, du,$$

then $F(x) = o(1)$, $x \to \infty$. *If* $k_1(t) \sim ct$, $t \to \infty$, *then*

$$F(x) \sim c \int_0^\infty f(t)\, dt, \qquad x \to \infty.$$

Proof of Lemma 1. It is known that $\sqrt{x}\, J_\nu(x)$ and $\int_0^x \sqrt{u}\, J_\nu(u)\, du$ are uniformly bounded so that the existence of $F_\nu(x)$ is obvious.

$$\begin{aligned} F_\nu(x) &= \int_0^N + \int_N^\infty \sqrt{xt}\, J_\nu(xt) f(t)\, dt \\ &= I_1 + I_2. \end{aligned}$$

Since $f(t)$ is of bounded variation in $[N, \infty)$ for N sufficiently large, and tends to 0 as $t \to \infty$, we may write $f(t) = f_1(t) - f_2(t)$ where f_1 and f_2 are nonincreasing and tend to 0 as $t \to \infty$. Therefore,

$$I_2 = O\left(\frac{f_1(N) + f_2(N)}{x}\right). \tag{4.4}$$

Also

$$I_1 = x^{\nu+1/2} \int_0^N (xt)^{-\nu} J_\nu(xt) t^{\nu+1/2} f(t)\, dt.$$

Again, $u^{-\nu}J_\nu(u)$ and $\int_0^x u^{-\nu}J_\nu(u)\,du$ are uniformly bounded. Therefore, by the dominated convergence theorem,

$$x^{-\nu-1/2} I_1 = O(1), \qquad x \to 0+. \tag{4.5}$$

This proves (4.1).

By Lemma 3,

$$x^{-\nu-1/2} I_1 = o(1), \qquad x \to \infty.$$

This, together with (4.4), proves (4.2).

Proof of Lemma 2. By absolute convergence,

$$\int_0^\lambda \sqrt{xu}\, J_\nu(xu) F_\nu(x)\, dx = \sqrt{x} \int_0^\infty f(y) y^{1/2}\, dy \int_0^\lambda u J_\nu(xu) J_\nu(xy)\, du.$$

By [15, (8), p. 134], the right-hand side becomes

$$\begin{aligned}
&\lambda\sqrt{x} \int_0^\infty \sqrt{y} f(y) \frac{x J_{\nu+1}(\lambda x) J_\nu(\lambda y) - y J_{\nu+1}(\lambda y) J_\nu(\lambda x)}{x^2 - y^2}\, dy \\
&= \lambda x^{3/2} J_{\nu+1}(\lambda x) \int_0^\alpha y^{1/2} f(y) \frac{J_\nu(\lambda y)}{x^2 - y^2}\, dy - \lambda x^{1/2} J_\nu(\lambda x) \\
&\quad \cdot \int_0^\alpha y^{3/2} f(y) \frac{J_{\nu+1}(\lambda y)}{x^2 - y^2}\, dy + \lambda x^{3/2} J_{\nu+1}(\lambda x) \int_\beta^\infty y^{1/2} f(y) \frac{J_\nu(\lambda y)}{x^2 - y^2}\, dy \\
&\qquad - \lambda x^{1/2} J_\nu(\lambda x) \int_\beta^\infty y^{3/2} f(y) \frac{J_{\nu+1}(\lambda y)}{x^2 - y^2}\, dy \\
&= I_3 - I_4 + I_5 - I_6.
\end{aligned}$$

Let

$$E(t, \lambda) = \int_0^t f(y) (\lambda y)^{1/2} J_\nu(\lambda y)\, dy.$$

Integrating by parts we obtain

$$I_3 = O(1)\left\{\left|\frac{E(\alpha, \lambda)}{x^2 - \alpha^2} - \int_0^\alpha E(y, \lambda) \frac{2y}{(x^2 - y^2)^2}\, dy\right|\right\}.$$

By Lemma 3, $E(\alpha, \lambda)/(x^2 - \alpha^2) = o(1)$, $\lambda \to \infty$. Moreover, Lemma 3, together with the dominated convergence theorem, implies

$$\int_0^\alpha E(y, \lambda)\frac{2y}{(x^2 - y^2)^2}\,dy = o(1), \qquad \lambda \to \infty.$$

Hence,

$$I_3 = o(1), \qquad \lambda \to \infty.$$

Similarly,

$$I_4 = o(1), \qquad \lambda \to \infty.$$

$$I_6 = (\lambda x)^{1/2} J_\nu(\lambda x) \int_\beta^\infty \frac{y}{x^2 - y^2} f(y)(\lambda y)^{1/2} J_{\nu+1}(\lambda y)\,dy.$$

Let

$$F(t, \lambda) = \int_\beta^t f(y)(\lambda y)^{1/2} J_{\nu+1}(\lambda y)\,dy.$$

Integrating by parts we obtain

$$I_6 = O(1) \int_\beta^\infty F(y, \lambda)\left\{\frac{1}{y^2 - x^2} - \frac{2y^2}{(y^2 - x^2)^2}\right\} dy.$$

$F(t, \lambda)$ is uniformly bounded and, by Lemma 3, tends to 0 as $\lambda \to \infty$. $I_6 = o(1)$ as $\lambda \to \infty$ now follows by the dominated convergence theorem. Similarly,

$$I_5 = o(1), \qquad \lambda \to \infty.$$

Clearly, the estimates are uniform for $x \in [a, b]$.

Remark. If $\alpha = 0$ in Lemma 2, the condition $\alpha < a$ is not required. In other words, if f vanishes in $[0, \beta]$ and $0 < b < \beta$, then the convergence to zero in Lemma 2 is uniform for $0 \leqq x \leqq b$. A similar statement is valid for the case $\beta = \infty$.

The next lemma is due to Macaulay-Owen [10].

LEMMA 4. *If $f \in L[0, \infty)$, $g \in \mathrm{BV}[0, \infty)$ and $g(u) \to 0$ as $u \to \infty$, then*

$$\int_{\to 0}^{\to \infty} F_\nu(x) G_\nu(x)\,dx = \int_0^\infty f(t) g(t)\,dt.$$

5. Proof of Theorems 1–7.

Proof of Theorem 1. By Lemma 4, it is sufficient to prove the theorem for the special case when $f \equiv 0$ in $[a, \infty)$. Now

$$|F_\nu(x)| \leqq M x^{\nu + 1/2} \int_0^a t^{\nu + 1/2} |f(t)|\,dt$$

so that $F_\nu(x)$ is bounded in $0 \leqq x \leqq 1$. Also $G_\nu(x)$ is a bounded, continuous function in $0 \leqq x \leqq 1$. Thus $F_\nu(x) G_\nu(x) \in L(0, 1)$. Let us first assume that there is exactly one point b, $b > 0$, such that f is of bounded variation in $(0, b - \delta)$ and $(b + \delta, a)$, while g is of bounded variation in $(b - 2\delta, b + 2\delta)$ for some $\delta > 0$.

Define the functions $f^A(x), f^B(x)$ and $f^C(x)$ by

$$f^A(x) = \begin{cases} f(x), & x \in (0, b-\delta), \\ 0, & \text{otherwise}, \end{cases}$$

$$f^B(x) = \begin{cases} f(x), & x \in (b-\delta, b+\delta), \\ 0, & \text{otherwise}, \end{cases}$$

$$f^C(x) = \begin{cases} f(x), & x \in (b+\delta, a), \\ 0, & \text{otherwise}. \end{cases}$$

By Lemma 4,

$$\int_0^\infty G_\nu(x) F_\nu^A(x)\, dx = \int_0^\infty g(x) f^A(x)\, dx$$

and

$$\int_0^\infty G_\nu(x) F_\nu^C(x)\, dx = \int_0^\infty g(x) f^C(x)\, dx.$$

Hence we only need to prove the result for f^B and g.

Define

$$g^A(x) = \begin{cases} g(x), & x \in (0, b-2\delta), \\ 0, & \text{otherwise}, \end{cases}$$

$$g^B(x) = \begin{cases} g(x), & x \in (b-2\delta, b+2\delta), \\ 0, & \text{otherwise}, \end{cases}$$

$$g^C(x) = \begin{cases} g(x), & x \in (b+2\delta, \infty), \\ 0, & \text{otherwise}. \end{cases}$$

Since $f^B \in L(0, \infty)$, $g^B \in \mathrm{BV}[0, \infty)$ and $g^B(x) \to 0$ as $x \to \infty$, by Lemma 4

$$\int_0^\infty G_\nu^B(x) F_\nu^B(x)\, dx = \int_0^\infty g^B(x) f^B(x)\, dx.$$

Also $g^A \in L(0, \infty)$, so that

$$\begin{aligned} \int_0^X G_\nu^A(x) F_\nu^B(x)\, dx &= \int_0^X F_\nu^B(x)\, dx \int_0^\infty g^A(x)(xt)^{1/2} J_\nu(xt)\, dt \\ &= \int_0^{b-2\delta} g^A(t)\, dt \int_0^X F_\nu^B(x)(xt)^{1/2} J_\nu(xt)\, dx. \end{aligned}$$

By the remark after the proof of Lemma 2,

$$\lim_{X\to\infty} \int_0^X F_\nu^B(x)(xt)^{1/2} J_\nu(xt)\, dx = 0$$

uniformly for $t \in [0, b-2\delta]$. Hence,

$$\int_0^{\to\infty} G_\nu^A(x) F_\nu^B(x)\, dx = 0 = \int_0^\infty g^A(x) f^B(x)\, dx.$$

Similarly

$$\int_0^{\to\infty} G_\nu^C(x)F_\nu^B(x)\,dx = 0 = \int_0^\infty g^C(x)f^B(x)\,dx.$$

If $b = 0$, the proof is slightly different since f may not be integrable into the origin. By assumption, $g \in \mathrm{BV}[0, 2\delta]$ and $f \in \mathrm{BV}[\delta, a]$ for some $\delta > 0$. Define functions f^{A^*} and f^{B^*} as follows:

$$f^{A^*}(x) = \begin{cases} f(x), & x \in (0, \delta), \\ 0, & \text{otherwise}, \end{cases}$$

$$f^{B^*}(x) = \begin{cases} f(x), & x \in [\delta, a], \\ 0, & \text{otherwise}. \end{cases}$$

Then, by Lemma 4,

$$\int_0^{\to\infty} F_\nu^{B^*}(x)G_\nu(x)\,dx = \int_0^\infty f^{B^*}(x)g(x)\,dx.$$

Also

$$\int_0^X F_\nu^{A^*}(x)G_\nu(x)\,dx = \int_0^\delta f^{A^*}(t)\,dt \int_0^X G_\nu(x)(xt)^{1/2}J_\nu(xt)\,dx$$

$$= \int_0^\delta t^{\nu+1/2}f^{A^*}(t)\,dt \int_0^X G_\nu(x)x^{\nu+1/2}\frac{J_\nu(xt)}{(xt)^\nu}\,dx.$$

The inner integral is uniformly bounded and $g(x)$ is of bounded variation in $(0, \delta)$ so that, by the dominated convergence theorem,

$$\int_0^{\to\infty} F_\nu^{A^*}(x)G_\nu(x)\,dx = \int_0^\infty f^{A^*}(x)g(x)\,dx.$$

This completes the proof for the case of the single point.

Next let E be the set of points x for which f is not of bounded variation in any neighborhood of x. Then E is closed and bounded. For each $x \in E$, there is an interval $(x - 2\delta_x, x + 2\delta_x)$ in which g is of bounded variation. The collection of intervals $(x - \delta_x, x + \delta_x)$, $x \in E$, is an open cover for E so that there is a finite subcover $(x_1 - \delta_{x_1}, x_1 + \delta_{x_1}), \cdots, (x_n - \delta_{x_n}, x_n + \delta_{x_n})$, say. We may assume that $x_1 < x_2 < \cdots < x_n$ and that no interval is completely contained in another one. Let I_k denote the interval $(x_k - \delta_{x_k}, x_k + \delta_{x_k})$, $k = 1, 2, \cdots, n$. Define the functions $f^{A_0}(x), f^{A_1}(x), \cdots, f^{A_n}(x)$ by

$$f^{A_0}(x) = \begin{cases} f(x), & x \notin \bigcup_{k=1}^n I_k, \\ 0, & \text{otherwise}, \end{cases}$$

$$f^{A_1}(x) = \begin{cases} f(x), & x \in I_1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f^{A_k}(x) = \begin{cases} f(x), & x \in (\max\{x_{k-1} + \delta_{x_{k-1}}, x_k - \delta_{x_k}\}, x_k + \delta_{x_k}), \\ 0, & \text{otherwise}, \end{cases}$$

for $k = 2, 3, \cdots, n$. Then

$$f(x) = \sum_{k=0}^{n} f^{A_k}(x)$$

except possibly at a finite number of points. By Lemma 4,

$$\int_0^{\to\infty} G_\nu(x) F_\nu^{A_0}(x)\, dx = \int_0^\infty g(x) f^{A_0}(x)\, dx.$$

Furthermore, by the same reasoning as for the single point, we have

$$\int_0^{\to\infty} G_\nu(x) F_\nu^{A_k}(x)\, dx = \int_0^\infty g(x) f^{A_k}(x)\, dx$$

for $k = 1, 2, \cdots, n$. Thus

$$\begin{aligned} \int_0^\infty f(x) g(x)\, dx &= \int_0^\infty g(x) \left(\sum_{k=0}^{n} f^{A_k}(x) \right) dx \\ &= \sum_{k=0}^{n} \int_0^{\to\infty} G_\nu(x) F_\nu^{A_k}(x)\, dx \\ &= \int_0^{\to\infty} G_\nu(x) F_\nu(x)\, dx. \end{aligned}$$

This completes the proof.

We note that the condition $G_\nu(x) = O(x^{-\nu-\lambda})$ is needed only when f is not integrable into the origin. For the case $f \in L(0, 1)$, we obtain an improved form of Lemma 4. The cases $\nu = \pm 1/2$ give Edmonds' results for the sine and the cosine transforms [4].

Proof of Theorem 2. Let $g(x) = x^{\nu+1/2} e^{-sx^2}$. Then, by [6, (10), p. 29],

$$G_\nu(x) = x^{\nu+1/2} (2s)^{-\nu-1} \exp\left(-\frac{x^2}{4s} \right), \qquad s > 0.$$

By Theorem 1,

$$H(s) = \int_0^{\to\infty} \frac{x^{\nu+1/2}}{(2s)^{\nu+1}} e^{-x^2/4s} F_\nu(x)\, dx = \int_0^\infty x^{\nu+1/2} e^{-sx^2} f(x)\, dx. \tag{5.1}$$

$$\begin{aligned} H(1/s) &= \left(\frac{s}{2} \right)^{\nu+1} \int_0^{\to\infty} x^{\nu+1/2} e^{-sx^2/4} F_\nu(x)\, dx \\ &= 2^{-1/2} s^{\nu+1} \int_0^{\to\infty} e^{-su} p(u)\, du, \end{aligned}$$

where $p(u) = u^{\nu/2-1/4}F_\nu(2u^{1/2})$. Assume (3.3) holds; then

$$p(u) \sim 2^\gamma u^{(\gamma/2+\nu/2+3/4)-1}L(u^{-1/2}), \qquad u \to 0+.$$

By a well-known Abelian argument, which involves a change of variable followed by an application of the dominated convergence theorem, we obtain

$$\int_0^\infty e^{-su}p(u)\,du \sim 2^\gamma \Gamma\left(\frac{\nu}{2}+\frac{\gamma}{2}+\frac{3}{4}\right)L(s^{1/2})s^{-\nu/2-\gamma/2-3/4}, \qquad s \to \infty.$$

Hence

$$H(s) \sim 2^{\gamma-1/2}\Gamma\left(\frac{\nu}{2}+\frac{\gamma}{2}+\frac{3}{4}\right)L(s^{-1/2})s^{-\nu/2+\gamma/2-1/4}, \qquad s \to 0+. \tag{5.2}$$

Also

$$\begin{aligned} H(s) &= \int_0^\infty x^{\nu+1/2}\,e^{-sx^2}f(x)\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-su}\,d\alpha(u;f). \end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3), we get

$$\int_0^\infty e^{-su}\,d\alpha(u;f) \sim 2^{\gamma+1/2}\Gamma\left(\frac{\nu}{2}+\frac{\gamma}{2}+\frac{3}{4}\right)L(s^{-1/2})s^{-\rho}, \qquad s \to 0+,$$

where ρ is defined by (2.3). $\alpha(u;f)$ is nondecreasing and continuous and $L(s^{-1/2})$ is slowly varying. Hence, by [7, Thm. 110], for $\rho > 0$,

$$\chi(s) \sim \rho K L(s^{-1/2})s^{-\rho}\int_0^\infty e^{-u}h(e^{-u})u^{\rho-1}\,du, \qquad s \to 0+. \tag{5.4}$$

This completes the proof.

The proof of Theorem 3 is similar.

Proof of Theorem 4. By an argument similar to the one employed in the proof of Theorem 2, (3.5) implies

$$\int_0^\infty e^{-su}p(u)\,du \sim 2^{\nu+1/2}\Gamma(\nu+1)L(s^{1/2})s^{-\nu-1}, \qquad s \to \infty,$$

so that

$$\int_0^\infty e^{-su}\,d\alpha(u;f) \sim 2^{\nu+1}\Gamma(\nu+1)L(s^{-1/2}), \qquad s \to 0+.$$

Hence, by [7, Thm. 111], the integral

$$\chi(s) = \int_0^\infty e^{-su}h(e^{-su})\,d\alpha(u;f)$$

exists for $s > 0$ and

$$\chi(s) \sim 2^{\nu+1}\Gamma(\nu + 1)L(s^{-1/2})h(1), \qquad s \to 0+.$$

The proof of Theorem 5 is similar.

Proof of Theorem 6. *Case* (i). $-1 < \gamma < \nu + 1/2$. In Theorem 2 or 3 (according to $\nu \geqq -1/2$ or $\nu > -1/2$), let $h(u)$ be the function defined by

$$h(u) = \begin{cases} u^{-1}, & e^{-1} \leqq u \leqq 1, \\ 0, & 0 \leqq u < e^{-1}. \end{cases}$$

Then,

$$\begin{aligned} \chi(s) &= \int_0^\infty e^{-su}h(e^{-su})\, d\alpha(u; f) \\ &= \int_0^{1/s} d\alpha(u; f) = \alpha(1/s; f). \end{aligned}$$

Also

$$\int_0^\infty e^{-u}h(e^{-u})u^{\rho-1}\, du = 1/\rho. \tag{5.5}$$

Combining (5.4), (5.5) and (3.4), we get

$$\alpha(u; f) \sim KL(u^{1/2})u^\rho, \qquad u \to \infty.$$

Case (ii). $\gamma = \nu + 1/2$. Let

$$h(u) = \begin{cases} u^{-1}(1 + \log u), & e^{-1} \leqq u \leqq 1, \\ 0, & 0 \leqq u \leqq e^{-1}. \end{cases}$$

Then

$$\begin{aligned} \chi(s) &= \int_0^\infty e^{-su}h(e^{-su})\, d\alpha(u; f) \\ &= s\int_0^{1/s} \alpha(u; f)\, du. \end{aligned} \tag{5.6}$$

By Theorem 4 or 5,

$$\chi(s) \sim 2^{\nu+1}\Gamma(\nu + 1)L(s^{-1/2}), \qquad s \to 0+. \tag{5.7}$$

Combining (5.6), (5.7) and (3.6), we obtain

$$\int_0^u \alpha(t; f)\, dt \sim 2^{\nu+1}\Gamma(\nu + 1)L(u^{1/2})u, \qquad u \to \infty. \tag{5.8}$$

By a well-known argument [7, p. 170], (5.8) implies

$$\alpha(u; f) \sim 2^{\nu+1}\Gamma(\nu + 1)L(u^{1/2}), \qquad u \to \infty.$$

Proof of Theorem 7. (i) $p_1(t) = t^{\nu/2-1/4}f(t^{1/2})$ is nonincreasing for $t \gg 1$. By Theorem 6,

$$\int_0^u p_1(t)\,dt \sim KL(u^{1/2})u^\rho, \qquad u \to \infty. \tag{5.9}$$

Let δ be any constant such that $0 < \delta < 1$. Then

$$\int_0^{u-\delta u} p_1(t)\,dt \sim KL((u-\delta u)^{1/2})(u-\delta u)^\rho, \qquad u \to \infty, \tag{5.10}$$

so that

$$\frac{\int_{u-\delta u}^u p_1(t)\,dt}{L(u^{1/2})u^\rho} \to K[1-(1-\delta)^\rho], \qquad u \to \infty. \tag{5.11}$$

Since $p_1(t)$ is nonincreasing, (5.11) implies

$$\limsup_{u\to\infty} \frac{p_1(u)}{L(u^{1/2})u^{\rho-1}} \leqq \frac{K}{\delta}[1-(1-\delta)^\rho]. \tag{5.12}$$

Similarly, by considering

$$\int_0^{u+\delta u} p_1(t)\,dt \sim KL((u+\delta u)^{1/2})(u+\delta u)^\rho, \qquad u \to \infty,$$

we obtain

$$\liminf_{u\to\infty} \frac{p_1(u)}{L(u^{1/2})u^{\rho-1}} \geqq \frac{K}{\delta}[(1+\delta)^\rho - 1]. \tag{5.13}$$

By choosing δ sufficiently small, the bounds in (5.12) and (5.13) can be made arbitrarily close to $K\rho$. Hence,

$$p_1(u) \sim \rho KL(u^{1/2})u^{\rho-1}, \qquad u \to \infty. \tag{5.14}$$

However, (5.14) is equivalent to (3.10). The proof of the other case is similar.

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LIE THEORY AND SEPARATION OF VARIABLES. II: PARABOLIC COORDINATES*

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Abstract. Winternitz and coworkers have characterized those solutions of the equation $(\Delta_3 + \omega^2)f(x) = 0$ which are expressible as products of functions of the paraboloid of revolution, as simultaneous eigenfunctions of the commuting quadratic operators

$$E = J_1P_2 + P_2J_1 - P_1J_2 - J_2P_1, J_3^2$$

in the enveloping algebra of the Lie algebra of the Euclidean group in three-space $E(3)$. Here we study the representation theory of the real and complex Euclidean groups in an $E - J_3^2$ basis and use the results to derive some addition and expansion theorems for parabolic functions which simplify and in some cases extend identities due to Buchholz and Hochstadt. We also give the decomposition of the quasi-regular representation of $E(3)$ in an $E - J_3^2$ basis.

Introduction. This paper is the second in a series analyzing the relationship between group theory and the method of separation of variables in the principal partial differential equations of mathematical physics [1]. Here we study a relationship between the Euclidean group in three-space $E(3)$ and the separation of the reduced wave equation $(\Delta_3 + \omega^2)f(\mathbf{x}) = 0$ in parabolic coordinates.

The Lie algebra $\mathscr{E}(3)$ of $E(3)$ is six-dimensional with basis $\{P_k, J_k : k = 1, 2, 3\}$ and commutation relations

$$[J_j, J_k] = \varepsilon_{jkl}J_l, \quad [J_j, P_k] = \varepsilon_{jkl}P_l, \quad [P_j, P_k] = 0, \quad j, k, l = 1, 2, 3,$$

where ε_{jkl} is the completely skew-symmetric tensor such that $\varepsilon_{123} = +1$. A three-variable model of $\mathscr{E}(3)$ is

$$P_k = \partial_{x_k}, \quad J_1 = x_3\partial_{x_2} - x_2\partial_{x_3}, \quad J_2 = x_1\partial_{x_3} - x_3\partial_{x_1},$$
$$J_3 = x_2\partial_{x_1} - x_1\partial_{x_2}$$

and the reduced wave equation is

$$(*) \qquad (P_1^2 + P_2^2 + P_3^2)f(\mathbf{x}) = -\omega^2 f(\mathbf{x}),$$

where ω is a nonzero constant. In [2] it is shown that the elementary solutions f of (*) in parabolic coordinates, i.e., products of functions of the paraboloid of revolution, are characterized by the requirement that they are simultaneous eigenvectors of the quadratic operators $E = J_1P_2 + P_2J_1 - P_1J_2 - J_2P_1$ and J_3^2 in the enveloping algebra of $\mathscr{E}(3)$:

$$Ef = \lambda f, \qquad J_3^2 f = -m^2 f.$$

Here, we exploit this observation by studying the spectral resolution of E corresponding to class 1 irreducible representations of $E(3)$, and of the complexified group $CE(3)$. In particular we determine the relationship between an $E = J_3^2$

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basis for the representation space and the more natural $J \cdot J - J_3^2$ basis. As is well known [3], [4] studies of the representation theory of $E(3)$ using the latter basis lead to addition theorems and expansion formulas for spherical Bessel functions. Similarly we here use the former basis to derive addition theorems and expansion formulas for products of functions of the paraboloid of revolution. We also show explicitly how to directly use the representation theory of $E(3)$ to obtain solutions of (*) in parabolic coordinates. (For a classical discussion of the separation of (*) in parabolic coordinates see [5, Chap. 8].)

The spectral resolution of the self-adjoint operator E and the explicit group theoretic methods used to study parabolic coordinates in this paper are new. However, the addition theorems (2.13) are simplifications and extensions of results derived by Hochstadt [6]. Relation (4.7) expressing spherical waves in terms of parabolic functions is due to Buchholtz [7]. Neither of these authors made use of representation theory in their proofs. To the author's knowledge the present paper is the first to explicitly relate parabolic coordinates to the representation theory of $E(3)$.

Applying the methods of this paper to the group $PE(3)$, the Poincaré group in three-space, and its covering groups one can derive expansion formulas much more general than those given here.

1. The complex Euclidean group $CE(3)$. We denote by $\mathscr{CE}(3)$ the Lie algebra of the complex Euclidean group in three-space. This algebra has a basis with commutation relations

$$\begin{aligned}&[\mathscr{J}^3,\mathscr{J}^\pm] = \mathscr{J}^\pm, \quad [\mathscr{J}^+,\mathscr{J}^-] = 2\mathscr{J}^3, \quad [\mathscr{P}^3,\mathscr{P}^\pm] = [\mathscr{P}^+,\mathscr{P}^-] = 0,\\ &[\mathscr{J}^3,\mathscr{P}^\pm] = [\mathscr{P}^3,\mathscr{J}^\pm] = \pm\mathscr{P}^\pm, \quad [\mathscr{J}^+,\mathscr{P}^-] = [\mathscr{P}^+,\mathscr{J}^-] = 2\mathscr{P}^3,\\ &[\mathscr{J}^+,\mathscr{P}^+] = [\mathscr{J}^-,\mathscr{P}^-] = [\mathscr{J}^3,\mathscr{P}^3] = 0.\end{aligned} \tag{1.1}$$

The complex Euclidean group $CE(3)$ is the 6-parameter Lie group consisting of all ordered pairs $\{\mathbf{w}, A\}$,

$$\begin{aligned}&\mathbf{w} = (u, v, w) \in \mathbb{C}^3, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}),\\ &ad - bc = 1,\end{aligned} \tag{1.2}$$

with group multiplication

$$\{\mathbf{w}, A\}\{\mathbf{w}', A'\} = \{\mathbf{w} + A\mathbf{w}', AA'\}, \tag{1.3}$$

where

$$\begin{aligned}A\mathbf{w} = (&a^2u - b^2v + abw, -c^2u + d^2v - c\,dw,\\ &2acu - 2b\,dv + (bc + ad)w) \in \mathbb{C}^3;\end{aligned} \tag{1.4}$$

see [4], [8]. It is easy to verify that $\mathscr{CE}(3)$ is the Lie algebra of $CE(3)$. Indeed the generators of $\mathscr{C}\mathscr{E}(3)$ can be chosen so

$$\{\mathbf{w}, A\} = \exp(u\mathscr{P}^+ + v\mathscr{P}^- + w\mathscr{P}^3)\exp\left(\frac{-b}{d}\mathscr{J}^+\right) \cdot \exp(-c\,d\mathscr{J}^-)\exp(-2\ln d\mathscr{J}^3) \tag{1.5}$$

in a neighborhood of the identity element. Here "exp" is the exponential mapping.

A realization of $\mathscr{C}\mathscr{E}(3)$ is given by the type E operators [4],

$$J^3 = t\partial_t, \qquad J^{\pm} = \mp t^{\pm 1}\left(\sqrt{1-z^2}\,\partial_z \pm \frac{zt}{\sqrt{1-z^2}}\partial_t\right), \tag{1.6}$$
$$P^3 = \omega z, \quad P^{\pm} = \omega\sqrt{1-z^2}\,t^{\pm 1}, \quad \omega \in \mathbb{C}, \quad \omega \neq 0,$$

acting on analytic functions $f(z, t)$. Note that

$$P \cdot P = P_1^2 + P_2^2 + P_3^2 = -P^+P^- - P^3P^3 = -\omega^2, \tag{1.7}$$
$$P \cdot J = P_1J_1 + P_2J_2 + P_3J_3 = 0,$$

where

$$P^{\pm} = \mp P_2 + iP_1, \quad P^3 = iP_3, \quad i = \sqrt{-1}, \tag{1.8}$$
$$J^{\pm} = \mp J_2 + iJ_1, \qquad J^3 = iJ_3.$$

Using this realization we study the eigenvalue problem

$$Eh = \lambda h, \qquad J^3h = mh, \tag{1.9}$$
$$E = J_1P_2 + P_2J_1 - P_1J_2 - J_2P_1 = i(P^+J^- - P^-J^+ - 2P^3)$$

for analytic $h(z, t)$. Since

$$E = -2i\omega(z - (1-z^2)\partial_z), \qquad J^3 = t\partial_t, \tag{1.10}$$

in this model we find that, to within a constant multiple, the eigenfunctions are given by

$$h_m^\alpha(z, t) = (1-z^2)^{-1/2}\left(\frac{1-z}{1+z}\right)^{\alpha/2} t^m, \quad \alpha = \frac{i\lambda}{2\omega}, \quad \lambda, m \in \mathbb{C}. \tag{1.11}$$

Introducing new variables

$$\xi^2 = \frac{1-z}{1+z}, \qquad \tau = t, \tag{1.12}$$

and the multiplier $\rho(\xi) = (1+\xi^2)/(2\xi)$ we obtain transformed eigenfunctions and operators,

$$f_m^\alpha(\xi, \tau) = \rho(\xi)^{-1}h_m^\alpha(z, t) = \xi^\alpha\tau^m, \tag{1.13}$$
$$J_\rho^3 = \tau\partial_\tau, \quad P_\rho^{\pm} = \frac{2\omega\xi\tau^{\pm 1}}{1+\xi^2}, \quad P_\rho^3 = \omega\left(\frac{1-\xi^2}{1+\xi^2}\right),$$
$$J_\rho^{\pm} = \frac{\tau^{\pm 1}}{2}((\xi - \xi^{-1})\tau\partial_\tau \pm (1+\xi^2)\partial_\xi \pm (\xi - \xi^{-1})),$$

where

$$J_\rho = \rho^{-1}J\rho, \qquad P_\rho = \rho^{-1}P\rho.$$

Dropping the subscript ρ in (1.13) we see that

$$J^3 f_m^\alpha = m f_m^\alpha, \qquad J^+ f_m^\alpha = \frac{(m+\alpha+1)}{2} f_{m+1}^{\alpha+1} + \frac{(\alpha-m-1)}{2} f_{m+1}^{\alpha-1},$$

$$J^- f_m^\alpha = \frac{(-\alpha+m-1)}{2} f_{m-1}^{\alpha+1} + \frac{(-\alpha-m+1)}{2} f_{m-1}^{\alpha-1},$$

$$(1.14) \qquad \begin{aligned} P^+ f_m^\alpha &= \sum_{k=0}^{\infty} 2\omega(-1)^k f_{m+1}^{\alpha+2k+1}, \\ P^- f_m^\alpha &= \sum_{k=0}^{\infty} 2\omega(-1)^k f_{m-1}^{\alpha+2k+1}, \end{aligned}$$

$$P^3 f_m^\alpha = \omega f_m^\alpha + 2\omega \sum_{k=1}^{\infty} (-1)^k f_m^{\alpha+2k},$$

$$E f_m^\alpha = \lambda f_m^\alpha, \qquad \lambda = -2i\omega\alpha.$$

If we choose arbitrary complex numbers α_0, m_0 and let $\alpha = \alpha_0 + n$, $m = m_0 + l$, n, l integers, we see that at least formally, expressions (1.14) define a representation $T^\omega_{\alpha_0,m_0}$ of $\mathscr{C}\mathscr{E}(3)$ which is not equivalent to any of the representations classified in [4].

In the following we set $\alpha_0 = m_0 = 0$ so that α and m range over the integers, though most of our formulas are true for arbitrary α, m.

The Lie algebra representation (1.14) induces a local group representation by operators

$$(1.15) \qquad \begin{aligned} \mathbf{T}(\mathbf{w}, A) = {} & \exp(uP^+ + vP^- + wP^3)\exp\left(-\frac{b}{d}J^+\right) \\ & \cdot \exp(-cdJ^-)\exp(-2\ln dJ^3). \end{aligned}$$

We define the matrix elements $\{\mathbf{w}, A\}_{\beta,n}^{\alpha,m}$ of these operators by

$$(1.16) \qquad \mathbf{T}(\mathbf{w}, A) f_m^\alpha = \sum_{\beta,n=-\infty}^{\infty} \{\mathbf{w}, A\}_{\beta,n}^{\alpha,m} f_n^\beta.$$

It follows that

$$(1.17) \qquad \sum_{\alpha,m=-\infty}^{\infty} \{\mathbf{w}, A\}_{\beta,n}^{\alpha,m} \{\mathbf{w}', A'\}_{\alpha,m}^{\gamma,k} = \{\mathbf{w} + A\mathbf{w}', AA'\}_{\beta,n}^{\gamma,k}$$

for $\{\mathbf{w}, A\}$, $\{\mathbf{w}', A'\}$ sufficiently close to the identity element. Using the differential operators (1.13) and standard results from Lie theory [4], we obtain

$$\mathbf{T}(u, v, w)f(\xi, \tau) = \mathbf{T}(\mathbf{w})f(\xi, \tau)$$

$$= \exp\left[\frac{\omega}{1+\xi^2}(2u\xi\tau + 2v\xi\tau^{-1} + w(1-\xi^2))\right]f(\xi, \tau),$$

$$\text{(1.18)} \quad \mathbf{T}(A)f(\xi, \tau) = \left[\frac{(1 - b\tau/d\xi)(1 + b\tau\xi/d)(1 - c\xi/a\tau)}{1 + c/a\xi\tau}\right]^{1/2}$$

$$\cdot f\left(\xi\left[\frac{(1 - \tau b/d\xi)(1 + c/a\tau\xi)}{(1 + b\tau\xi/d)(1 - c\xi/a\tau)}\right]^{1/2}, \frac{a\tau}{d}\left[\frac{(1 + c/a\tau\xi)(1 - c\xi/a\tau)}{(1 - b\tau/d\xi)(1 + b\tau\xi/d)}\right]^{1/2}\right),$$

$$|b/d| < |\xi/\tau| < |a/c|, \quad |b/d| < |1/\xi\tau| < |a/c|,$$

where

$$\mathbf{T}(u, v, w) = \mathbf{T}(\mathbf{w}, I), \qquad \mathbf{T}(A) = \mathbf{T}(\mathbf{0}, A),$$

$$\mathbf{w} = (u, v, w), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{0} = (0, 0, 0).$$

Substituting these results into (1.16) and using well-known generating functions for Laguerre and Jacobi polynomials [4], [5], we find

$$\text{(1.19)} \quad \begin{aligned} &\{0, 0, w\}^{\alpha,m}_{\alpha+2l,m} = e^{w\omega}(-1)^l L_l^{(-1)}(-2w\omega), \qquad l = 0, 1, 2, \cdots, \\ &\{0, 0, w\}^{\alpha,m}_{\beta,n} = 0 \quad \text{otherwise}. \end{aligned}$$

$$\{u, v, 0\}^{\alpha,m}_{\alpha+l+2p,m+l} = (2u\omega)^l \frac{\Gamma(p+l)}{p!\,l!\,\Gamma(l)}\, {}_2F_3\left(\begin{matrix} p+l, -p \\ l+1, \frac{l+1}{2}, \frac{l}{2} \end{matrix}\middle| -uv\omega^2\right)$$

$$\text{if } p = 1, 2, \cdots, \quad l = 0, 1, 2, \cdots,$$

$$\text{(1.20)} \quad \{u, v, 0\}^{\alpha,m}_{\alpha+l,m+l} = \frac{(2u\omega)^l}{l!}, \qquad l = 0, 1, 2, \cdots,$$

$$\{u, v, 0\}^{\alpha,m}_{\alpha-l+2p,m-l} = \left(\frac{v}{u}\right)^l \{u, v, 0\}^{\alpha,m}_{\alpha+l+2p,m+l}$$

$$\text{if } p = 0, 1, 2, \cdots, \quad -l = 1, 2, \cdots$$

$$\{u, v, 0\}^{\alpha,m}_{\beta,n} = 0 \quad \text{otherwise}.$$

$$\{A\}^{\alpha,m}_{\alpha+l-k,m-l-k} = (-1)^l a^m d^{-m+l+k} b^{-l-k}$$

$$\cdot \frac{\Gamma\left(\dfrac{3+\alpha-m}{2}\right)\Gamma\left(\dfrac{3-\alpha-m}{2}\right)}{\Gamma\left(\dfrac{3+\alpha-m+2l}{2}\right)\Gamma\left(\dfrac{3-\alpha-m+2k}{2}\right)} \frac{1}{\Gamma(-l+1)\Gamma(-k+1)}$$

$$(1.21)\qquad \cdot {}_2F_1\left(\begin{matrix}\dfrac{m-\alpha-2l-1}{2}, \dfrac{-m-\alpha-1}{2} \\ -l+1\end{matrix}\middle| \frac{bc}{ad}\right)$$

$$\cdot {}_2F_1\left(\begin{matrix}\dfrac{m+\alpha-2k-1}{2}, \dfrac{-m-\alpha+1}{2} \\ -k+1\end{matrix}\middle| \frac{bc}{ad}\right)$$

$$\text{if } l, k = 0, \pm 1, \pm 2, \cdots,$$

$\{A\}^{\alpha,m}_{\beta,n} = 0$ otherwise.

Here $L_n^{(\alpha)}(z)$ is a generalized Laguerre polynomial and ${}_pF_q\left(\begin{matrix}\alpha_1 \cdots \alpha_p \\ \beta_1 \cdots \beta_q\end{matrix}\middle| z\right)$ is a generalized hypergeometric function [9]. In (1.21) we use the fact that ${}_2F_1\left(\begin{matrix}a, b \\ c\end{matrix}\middle| z\right)\Big/\Gamma(c)$ is an entire function of c.

Alternatively we can employ the Hille–Hardy formula [4, p. 193] to derive

$$\{u, v, w\}^{\alpha,m}_{\alpha+l+2k,m+l} = (-1)^k e^{w\omega}(2u\omega)^l$$

$$\cdot \left[\frac{k!}{\Gamma(k+l+1)} L_k^{(l)}(\rho) L_k^{(l)}(\eta) - \frac{(k-1)!}{\Gamma(k+l)} L_{k-1}^{(l)}(\rho) L_{k-1}^{(l)}(\eta)\right],$$

$$l = 0, 1, 2, \cdots, \quad k = 1, 2, \cdots.$$

$$\{u, v, w\}^{\alpha,m}_{\alpha+l,m+l} = \frac{e^{w\omega}(2u\omega)^l}{l!}, \qquad l = 0, 1, 2, \cdots.$$

(1.22)

$$\{u, v, w\}^{\alpha,m}_{\alpha-l+2k,m-l} = \left(\frac{v}{u}\right)^l \{u, v, w\}^{\alpha,m}_{\alpha+l+2k,m+l}, \quad l = 1, 2, \cdots, \quad k = 0, 1, 2, \cdots,$$

$\{u, v, w\}^{\alpha,m}_{\beta,n} = 0$ otherwise.

Here,

$$(1.23)\qquad \frac{\rho+\eta}{2} = -w\omega, \qquad \frac{\sqrt{\rho\eta}}{2} = \sqrt{uv}\,\omega.$$

The general matrix element $\{\mathbf{w}, A\}^{\alpha,m}_{\beta,n}$ can be computed from (1.19)–(1.22) and the addition theorem (1.17).

2. A three-variable model for $CE(3)$. We next construct a model of relations (1.14) in which the J and P operators are differential operators in three complex variables x, y, z. Namely we set

$$\mathbf{P}_1 = \partial_x, \quad \mathbf{P}_2 = \partial_y, \quad \mathbf{P}_3 = \partial_z, \tag{2.1}$$
$$\mathbf{J}_1 = -y\partial_z + z\partial_y, \quad \mathbf{J}_2 = -z\partial_x + x\partial_z, \quad \mathbf{J}_3 = -x\partial_y + y\partial_x.$$

It follows easily that the operators

$$\mathbf{P}^\pm = \mp\mathbf{P}_2 + i\mathbf{P}_1, \quad \mathbf{P}^3 = i\mathbf{P}_3, \quad \mathbf{J}^\pm = \mp\mathbf{J}_2 + i\mathbf{J}_1, \quad \mathbf{J}^3 = i\mathbf{J}_3 \tag{2.2}$$

satisfy the commutation relations (1.1). Furthermore, $\mathbf{P}\cdot\mathbf{J} \equiv 0$.

For this model the eigenvalue problem (1.7), (1.9) becomes

$$(\mathbf{P}_1^2 + \mathbf{P}_2^3 + \mathbf{P}_3^2)f = -\omega^2 f, \quad i(\mathbf{P}^+\mathbf{J}^- - \mathbf{P}^-\mathbf{J}^+ - 2\mathbf{P}^3)f = \lambda f, \tag{2.3}$$
$$\mathbf{J}^3 f = mf,$$

where $f = f(x, y, z)$. Note that the equation $\mathbf{P}\cdot\mathbf{P}f = -\omega^2 f$ becomes

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 + \omega^2)f = 0.$$

The equations (2.3) have solutions which are products of functions of the paraboloid of revolution. Instead of verifying this we shall directly construct basis functions $f_m^\alpha(x, y, z)$ satisfying (1.14). The plane wave functions $h(x, y, z) = \exp[i\omega(a_1x + a_2y + a_3z)]$, $a_1^2 + a_2^2 + a_3^2 = 1$ satisfy

$$-\mathbf{P}\cdot\mathbf{P}h = \omega^2 h, \quad \mathbf{P}_j h = i\omega a_j h, \quad j = 1, 2, 3.$$

We shall construct our basis functions as integrals over plane waves:

$$f(x, y, z) = \int_S\int d\beta\frac{dt}{t}F(\beta, t)\exp\left\{\frac{-i\omega}{2}[x\sqrt{1-\beta^2}(t + t^{-1}) + iy\sqrt{1-\beta^2}(t^{-1} - t) + 2\beta z]\right\} = I(F).$$

We assume that the surface S and the analytic function F are such that $I(F)$ converges absolutely and arbitrary differentiation in x, y and z is permitted under the integral sign. Integrating by parts we find

$$i(-x\partial_y + y\partial_x)f = I(t\partial_t F) \tag{2.4}$$

provided that S and F are chosen such that the boundary terms vanish. Similarly, if the boundary terms vanish we find

$$\mathbf{J}^\pm f = I(J^\pm F), \quad \mathbf{P}^\pm f = I(P^\pm F), \quad \mathbf{P}^3 f = I(P^3 F), \tag{2.5}$$

where the $\mathbf{J}$, $\mathbf{P}$ operators are given by (2.1), (2.2) and the J, P operators are given by (1.6) (with $\beta = z$). Thus, $\mathbf{P}\cdot\mathbf{P}f = -\omega^2 f$ and the action of $\mathscr{CE}(3)$ on f corresponds exactly to the action of the operators (1.6) on F.

In particular, the functions

$$
\begin{aligned}
f_m^{\alpha}(x, y, z) &= \int_{C_1} d\beta \int_{C_2} \frac{dt}{t}(1 - \beta^2)^{-1/2}\left(\frac{1 - \beta}{1 + \beta}\right)^{\alpha/2} t^m \\
&\quad \cdot \exp\left\{\frac{-i\omega}{2}[x\sqrt{1 - \beta^2}\,(t + t^{-1}) + iy\sqrt{1 - \beta^2}\,(t^{-1} - t) + 2\beta z]\right\} \qquad (2.6)\\
&= I(h_m^{\alpha}(\beta, t)), \qquad \alpha, m = 0, \pm 1, \pm 2, \cdots,
\end{aligned}
$$

satisfy relations (1.14) where C_1 is the contour shown in Fig. 1

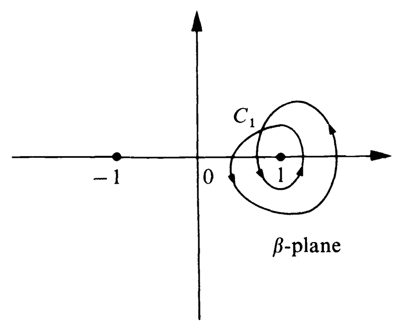

FIG. 1

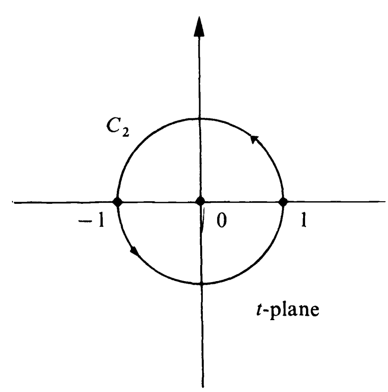

FIG. 2

and C_2 is the contour shown in Fig. 2 in the β- and t-planes respectively. In terms of the variables ξ, τ where

$$z = \frac{1 - \xi^2}{1 + \xi^2}, \quad t = \tau,$$

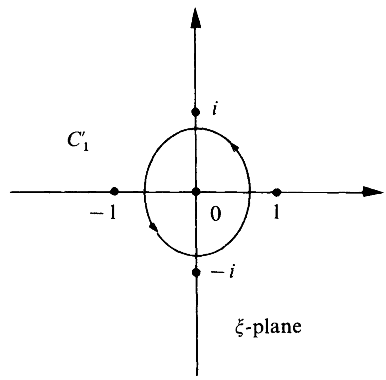

FIG. 3

the integral becomes

$$f_m^\alpha(x,y,z) = -2\int_{C_1'} d\xi \int_{C_2} d\tau\,\tau^{m-1}\frac{\xi^\alpha}{1+\xi^2}\exp\left\{-i\omega\left[\frac{\tau\xi}{1+\xi^2}(x-iy) + \frac{\tau^{-1}\xi}{1+\xi^2}(x+iy) + z\frac{(1-\xi^2)}{1+\xi^2}\right]\right\},$$

where C_1' is the contour in the ξ-plane shown in Fig. 3. Using the standard exponential generating function for Bessel functions and the Hille–Hardy formula we obtain

$$\begin{aligned}f_m^\alpha(x,y,z) &= \frac{8\pi^2(i)^{|m|}(-1)^k k!}{(|m|+k)!}(2i\omega\rho)^{|m|/2}(-2i\omega\eta)^{|m|/2}\\ &\quad\cdot e^{i\omega(\eta-\rho)}L_k^{(|m|)}(2i\omega\rho)L_k^{(|m|)}(-2i\omega\eta)\,e^{im\beta}\\ &\text{if } \alpha = -|m|-2k-1,\quad k=0,1,2,\cdots,\quad m=0,\pm1,\pm2,\cdots,\\ f_m^\alpha(x,y,z) &= 0 \quad \text{otherwise}.\end{aligned} \tag{2.7}$$

Here ρ, η, β are parabolic coordinates:

$$x = 2\sqrt{\rho\eta}\cos\beta,\quad y = 2\sqrt{\rho\eta}\sin\beta,\quad z = \rho-\eta. \tag{2.8}$$

We conclude that the functions (2.7) and the operators (2.1), (2.2) satisfy relations (1.14). This is true even though some of the functions $f_m^\alpha(x,y,z)$ in our model are identically zero. This Lie algebra representation of $\mathscr{CE}(3)$ can be extended to a local Lie group representation of $CE(3)$ in the usual way. In particular the Lie derivatives (2.1), (2.2) induce group operators

$$\begin{aligned}[\mathbf{T}(u,v,w)f](\mathbf{x}) &= \exp(u\mathbf{P}^+ + v\mathbf{P}^- + w\mathbf{P}^3)f(x)\\ &= f(x+i(u+v),\, y-u+v,\, z+iw),\qquad \mathbf{x}=(x,y,z),\\ [\mathbf{T}(A)f](\mathbf{x}) &= \exp\left(-\frac{b}{d}\mathbf{J}^+\right)\exp(-cd\mathbf{J}^-)\exp(-2\ln d\mathbf{J}^3)f(x)\\ &= f(R(A^{-1})\mathbf{x}),\end{aligned} \tag{2.9}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$$

and $R(A)$ is the complex orthogonal matrix

$$R(A) = \begin{pmatrix} \frac{1}{2}(a^2 - b^2 + d^2 - c^2) & \frac{i}{2}(d^2 + c^2 - a^2 - b^2), & ab - cd \\ \frac{i}{2}(a^2 - b^2 - d^2 + c^2) & \frac{1}{2}(d^2 + c^2 + a^2 + b^2), & i(ab + cd) \\ ac - bd & -i(ac + bd) & ad + bc \end{pmatrix}. \tag{2.10}$$

Finally we obtain the relations

$$\begin{aligned} [\mathbf{T}(u, v, w)f_m^\alpha](x, y, z) &= \sum_{\beta,n=-\infty}^{\infty} \{u, v, w\}_{\beta,n}^{\alpha,m} f_n^\beta(x, y, z), \\ [\mathbf{T}(A)f_m^\alpha](x, y, z) &= \sum_{\beta,n=-\infty}^{\infty} \{A\}_{\beta,n}^{\alpha,m} f_n^\beta(x, y, z), \qquad |b/d| < 1 < |a/c|, \end{aligned} \tag{2.11}$$

for the functions (2.7) where the matrix elements are given by (1.19)–(1.23). Expressions (2.11) are addition theorems for the functions of the paraboloid of revolution (2.7) and they contain most of the formulas of Hochstadt [6] as special cases.

By altering the integration contours in (2.6) it is not difficult to construct functions $f_m^\alpha(x, y, z)$ of the general form (2.7) but with m an arbitrary noninteger complex number, and such that the relations (1.14) corresponding to $P^\pm$ and P^3 hold. Thus one can derive addition theorems corresponding to the operators $\mathbf{T}(u, v, w)$. For some of these results see [6]. However, it does not appear possible to satisfy relations (1.14) corresponding to $J^\pm$ and J^3 for arbitrary m.

3. The real Euclidean group $E(3)$. Let $\mathscr{E}(3)$ be the Lie algebra of the real Euclidean group in three-space, i.e., the 6-dimensional real Lie algebra with basis $\{\mathscr{J}_k, \mathscr{P}_k : k = 1, 2, 3\}$ and commutation relations

$$[\mathscr{J}_j, \mathscr{J}_k] = \varepsilon_{jkl}\mathscr{J}_l, \quad [\mathscr{J}_j, \mathscr{P}_k] = \varepsilon_{jkl}\mathscr{P}_l, \quad [\mathscr{P}_j, \mathscr{P}_k] = 0, \quad j, k, l = 1, 2, 3, \tag{3.1}$$

where ε_{jkl} is the completely skew-symmetric tensor such that $\varepsilon_{123} = +1$. The real Euclidean group $E(3)$ consists of all ordered pairs $(\mathbf{r}, A)$, where $\mathbf{r} = (r_1, r_2, r_3) \in R_3$ and

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2), \qquad |a|^2 + |b|^2 = 1. \tag{3.2}$$

The group multiplication law is

$$(\mathbf{r}, A)(\mathbf{r}', A') = (\mathbf{r} + R(A)\mathbf{r}', AA'), \tag{3.3}$$

where $R(A)$ is the real 3×3 orthogonal matrix given by (2.10) with $c = -\bar{b}$, $d = \bar{a}$. Note that $E(3)$ is the simply connected covering group of the Euclidean group of all rotations and translations in three-space [4, p. 256]. We relate the Lie algebra generators to the finite group elements via the expression

$$(\mathbf{r}, A) = \exp(r_1 P_1 + r_2 P_2 + r_3 P_3) \exp \varphi_1 \mathscr{J}_3 \exp \theta \mathscr{J}_1 \exp \varphi_2 \mathscr{J}_3. \tag{3.4}$$

Here $(\varphi_1, \theta, \varphi_2)$ are the Euler coordinates of A (see [4, p. 217]). Using relations (1.8) we can easily verify that $\mathscr{E}(3)$ is a real form of $\mathscr{C}\mathscr{E}(3)$.

The class one faithful irreducible representations of $E(3)$ are defined by operators

$$\mathbf{T}(\mathbf{r}, A) f(\hat{\mathbf{k}}) = \exp(-i\omega \mathbf{r} \cdot \hat{\mathbf{k}}) f(R(A^{-1})\hat{\mathbf{k}}) \tag{3.5}$$

acting on the Hilbert space $L_2(S_2)$ of Lebesgue square integrable functions $f(\hat{\mathbf{k}})$ on the unit 2-sphere, with inner product

$$\langle f, g \rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta f(\hat{\mathbf{k}}) \overline{g(\hat{\mathbf{k}})} \sin \theta,$$

$$\hat{\mathbf{k}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta);$$

see [3], [4]. The representations are indexed by the constant $\omega > 0$. The induced Lie algebra representation is defined by operators

$$\begin{aligned} &P_1 = -i\omega \sin \theta \cos \varphi, \qquad P_2 = -i\omega \sin \theta \sin \varphi, \\ &P_3 = -i\omega \cos \theta, \qquad J_1 = \sin \varphi \partial_\theta + \cos \varphi \cot \theta \partial_\varphi, \\ &J_2 = -\cos \varphi \partial_\theta + \sin \varphi \cot \theta \partial_\varphi, \qquad J_3 = -\partial_\varphi. \end{aligned} \tag{3.6}$$

The operator E on $L_2(S_2)$ is given formally as

$$\begin{aligned} E &= J_1 P_2 + P_2 J_1 - P_1 J_2 - J_2 P_1 = 2(P_3 + P_2 P_1 - P_1 J_2) \\ &= -2i\omega(\cos \theta + \sin \theta \partial_\theta). \end{aligned} \tag{3.7}$$

To be definite we initially define E by (3.7) with domain the space of all C^∞ functions on S_2 which vanish in neighborhoods of $\theta = 0$ and $\theta = \pi$. It is easy to show that E is symmetric on this domain and essentially self-adjoint.

For any function $f(\hat{\mathbf{k}}) \in L_2(S_2)$ let

$$\mathscr{F}_m(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} f(\hat{\mathbf{k}})\, e^{-im\varphi} \left(\tan \frac{\theta}{2}\right)^{-i\lambda/2\omega} \frac{d\theta}{2\omega}, \qquad \lambda \in R, \quad m = 0, \pm 1, \pm 2, \cdots. \tag{3.8}$$

Then

$$f(\hat{\mathbf{k}}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{\infty} \mathscr{F}_m(\lambda) \frac{(\tan(\theta/2))^{i\lambda/2\omega}}{\sin \theta} d\lambda \tag{3.9}$$

in the sense of L_2-convergence. Furthermore,

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta |f(\hat{\mathbf{k}})|^2 \sin \theta = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathscr{F}_m(\lambda)|^2 d\lambda, \tag{3.10}$$

and for f in the domains of E and $J^3 = iJ_3$,

$$
\begin{aligned}
Ef(\hat{\mathbf{k}}) &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im\varphi}\int_{-\infty}^{\infty} \lambda\mathscr{F}_m(\lambda)\frac{(\tan(\theta/2))^{i\lambda/2\omega}}{\sin\theta}\,d\lambda, \\
J^3 f(\hat{\mathbf{k}}) &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im\varphi}\int_{-\infty}^{\infty} m\mathscr{F}_m(\lambda)\frac{(\tan(\theta/2))^{i\lambda/2\omega}}{\sin\theta}\,d\lambda.
\end{aligned} \tag{3.11}
$$

Thus E and J^3 can be extended to unique self-adjoint operators on $L_2(S_2)$.

In the usual treatments of the representation theory of $E(3)$ (see [3], [4]) one chooses an orthonormal basis for $L_2(S_2)$ which consists of simultaneous eigenvectors $j^l_m(\hat{\mathbf{k}})$ of the operators $J^3 = iJ_3$ and $J \cdot J = J_1^2 + J_2^2 + J_3^2$. Indeed, the functions

$$j^l_m(\hat{\mathbf{k}}) = (-1)^l Y^m_l(\theta, \varphi), \qquad l = 0, 1, 2, \cdots, \quad m = l, l-1, \cdots, -l, \tag{3.12}$$

form an orthonormal basis and

$$J^3 j^l_m = m j^l_m, \qquad J \cdot J j^l_m = -l(l+1) j^l_m. \tag{3.13}$$

Here $Y^m_l(\theta, \varphi)$ is a spherical harmonic [3]. To determine the relationship between the $J \cdot J - J^3$ spectral representation and the $E - J^3$ representation we compute the transforms $\mathscr{F}^{l,m}_n(\lambda)$ corresponding to the function $f(\hat{\mathbf{k}}) = j^l_m(\hat{\mathbf{k}})$. The result is

$$
\begin{aligned}
\mathscr{F}^{l,m}_m(\lambda) &= (-1)^{l+m}\frac{2^{m-2}}{\pi\omega}\left[\frac{(2l+1)(l-m)!}{(l+m)!}\right]^{1/2}\Gamma\left(m+\frac{1}{2}\right) \\
&\quad\cdot\int_{-1}^{1}(1-z)^{-i\lambda/4\omega+(m-1)/2}(1+z)^{i\lambda/4\omega+(m-1)/2}C^{m+1/2}_{l-m}(z)\,dz \\
&= \frac{(-1)^{l+m}}{(m!)^2}\left[\frac{(2l+1)(l+m)!}{4\pi(l-m)!}\right]^{1/2}\Gamma\left(\frac{-i\lambda}{4\omega}+\frac{m+1}{2}\right)\Gamma\left(\frac{i\lambda}{4\omega}+\frac{m+1}{2}\right) \\
&\quad\cdot{}_3F_2\left(\begin{matrix}-l+m, l+m+1, -i\lambda/4\omega+(m+1)/2 \\ m+1, m+1\end{matrix}\,\middle|\,1\right), \quad m = 0, 1, \cdots, l, \\
\mathscr{F}^{l,m}_m(\lambda) &= \frac{(-1)^l}{(k!)^2}\left[\frac{(2l+1)(l+k)!}{4\pi(l-k)!}\right]^{1/2}\Gamma\left(\frac{-i\lambda}{4\omega}+\frac{k+1}{2}\right)\Gamma\left(\frac{i\lambda}{4\omega}+\frac{k+1}{2}\right) \\
&\quad\cdot{}_3F_2\left(\begin{matrix}-l+k, l+k+1, -i\lambda/4\omega+(k+1)/2 \\ k+1, k+1\end{matrix}\,\middle|\,1\right), \\
&\qquad\qquad m = -k, \quad k = 1, 2, 3, \cdots, l, \\
\mathscr{F}^{l,m}_n(\lambda) &= 0, \qquad \text{if } n \neq m.
\end{aligned} \tag{3.14}
$$

Now that we have determined the unitary transformation relating the two bases we can use the known matrix elements, Clebsch–Gordan coefficients, etc. in the $J \cdot J - J^3$ basis and express them in terms of the $E - J^3$ basis [3], [4], [10]. The routine computations are omitted.

4. A three-variable model for $E(3)$. We now show how to construct a three-variable model for the class one faithful irreducible representations of $E(3)$. Consider the functions

$$h_{\mathbf{x}}(\hat{\mathbf{k}}) = \exp[i\omega\mathbf{x}\cdot\hat{\mathbf{k}}] \in L_2(S_2),$$
$$\mathbf{x} = (x, y, z) \in R_3, \qquad \hat{\mathbf{k}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{4.1}$$

Computing the expansion coefficients of $h_{\mathbf{x}}$ with respect to the basis $\{j^l_m\}$, (3.12), we find

$$H^l_m(\mathbf{x}) = \langle h_{\mathbf{x}}, j^l_m\rangle = 4\pi(i)^l j_l(\omega r)\overline{Y^m_l}(\theta', \varphi'),$$
$$m = -l, -l+1, \cdots, l, \quad l = 0, 1, 2, \cdots, \tag{4.2}$$

where $\mathbf{x} = (r\sin\theta'\cos\varphi', r\sin\theta'\sin\varphi', r\cos\theta')$ and

$$j_l(\omega r) = \left(\frac{\pi}{2\omega r}\right)^{1/2} J_{l+1/2}(\omega r)$$

is a spherical Bessel function [5], [4, p. 263].

On the other hand, in the $E - J^3$ basis we find, from (3.8),

$$\begin{aligned}\mathscr{H}^{\mathbf{x}}_m(\lambda) &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \exp[i\omega\mathbf{x}\cdot\hat{\mathbf{k}}]\, e^{-im\varphi}\left(\tan\frac{\theta}{2}\right)^{-i\lambda/2\omega} \frac{d\theta}{2\omega}\\
&= \frac{e^{im\beta}}{2\omega} i^m \int_0^{\pi} \exp[i\omega(\rho - \eta)\cos\theta]\, J_m(2\omega\sqrt{\rho\eta}\sin\theta)\left(\tan\frac{\theta}{2}\right)^{-i\lambda/2\omega} d\theta\\
&= \frac{i^m}{2^{3/2}\omega^2}\Gamma\left(\frac{1+m}{2} + \frac{i\lambda}{4\omega}\right)\Gamma\left(\frac{1+m}{2} - \frac{i\lambda}{4\omega}\right)(\eta\rho)^{-1/2}\\
&\quad\cdot \mathscr{M}_{i\lambda/4\omega, m/2}(e^{-i\pi/2}\omega\rho\sqrt{2})\mathscr{M}_{i\lambda/4\omega, m/2}(e^{i\pi/2}\omega\eta\sqrt{2})\, e^{im\beta};\end{aligned} \tag{4.3}$$

see [7, p. 83]. Here

$$\mathscr{M}_{\alpha,\mu/2}(z) = \frac{z^{(1+\mu)/2}e^{-z/2}}{\Gamma(1+\mu)}\, {}_1F_1\left(\frac{1+\mu}{2} - \alpha; 1+\mu; -z\right)$$

and ${}_1F_1(a; c; z)$ is a confluent hypergeometric function. The parabolic coordinates ρ, η, β are defined by (2.8).

We can better understand the significance of these results by noting the action of the operators $\mathbf{T}(\mathbf{r}, A)$, (3.5), on the plane wave functions $h_{\mathbf{x}}(\hat{\mathbf{k}})$:

$$\mathbf{T}(\mathbf{r}, A)h_{\mathbf{x}}(\hat{\mathbf{k}}) = \exp(-i\omega\mathbf{r}\cdot\hat{\mathbf{k}})h_{\mathbf{x}}(R(A^{-1})\hat{\mathbf{k}}) = h_{R(A)\mathbf{x}-\mathbf{r}}(\hat{\mathbf{k}}).$$

Thus, the induced group action on the functions $H^l_m(\mathbf{x})$ is given by

$$\begin{aligned}\mathbf{T}(\mathbf{0}, A)H^l_m(\mathbf{x}) &\equiv \langle h_{\mathbf{x}}, \mathbf{T}(\mathbf{0}, A)j^l_m\rangle\\
&= \langle \mathbf{T}(0, A^{-1})h_{\mathbf{x}}, j^l_m\rangle = H^l_m(R(A^{-1})\mathbf{x}),\\
\mathbf{T}(\mathbf{r}, E)H^l_m(\mathbf{x}) &\equiv \langle h_{\mathbf{x}}, \mathbf{T}(\mathbf{r}, E)j^l_m\rangle\\
&= \langle \mathbf{T}(-\mathbf{r}, E)h_{\mathbf{x}}, j^l_m\rangle = H^l_m(\mathbf{x} + \mathbf{r}),\end{aligned} \tag{4.4}$$

where E is the 2×2 identity matrix. It follows that the $H_m^l(\mathbf{x})$ transform under the group action induced by the Lie derivatives (2.1) where $\mathbf{x} = (x, y, z)$. Furthermore,

$$\text{(4.5)} \quad \mathbf{P}\cdot\mathbf{P}H_m^l = -\omega^2 H_m^l, \qquad \mathbf{J}\cdot\mathbf{J}H_m^l = -l(l+1)H_m^l, \qquad \mathbf{J}^3 H_m^l = mH_m^l,$$

so the $\{H_m^l\}$ are solutions of the equation $(\nabla^2 + \omega^2)H_m^l = 0$ which transform as a $\mathbf{J}\cdot\mathbf{J} - \mathbf{J}^3$ (spherical wave) basis under the action of $E(3)$. A similar computation shows that the functions $\{\mathscr{H}_m^{\mathbf{x}}(\lambda)\}$ also transform under the operators (2.1) and satisfy the relations

$$\text{(4.6)} \quad \begin{aligned} &\mathbf{P}\cdot\mathbf{P}\mathscr{H}_m^{\mathbf{x}}(\lambda) = -\omega^2\mathscr{H}_m^{\mathbf{x}}(\lambda), \qquad \mathbf{E}\mathscr{H}_m^{\mathbf{x}}(\lambda) = \lambda\mathscr{H}_m^{\mathbf{x}}(\lambda),\\ &\mathbf{J}^3\mathscr{H}_m^{\mathbf{x}}(\lambda) = m\mathscr{H}_m^{\mathbf{x}}(\lambda). \end{aligned}$$

Thus the $\{\mathscr{H}_m^{\mathbf{x}}(\lambda)\}$ are solutions of $(\nabla^2 + \omega^2)H_m^l = 0$ which transform as a $\mathbf{E} - \mathbf{J}^3$ basis under the action of $E(3)$. This suggests that the $\mathscr{H}_m^{\mathbf{x}}(\lambda)$ should be simply expressible in terms of functions of the paraboloid of revolution.

Using the coefficients (3.14) we can relate the functions $H_m^l(\mathbf{x})$ and $\mathscr{H}_m^{\mathbf{x}}(\lambda)$:

$$\text{(4.7)} \qquad H_m^l(\mathbf{x}) = \langle h_{\mathbf{x}}, j_m^l\rangle = \int_{-\infty}^{\infty} \mathscr{H}_m^{\mathbf{x}}(\lambda)\overline{\mathscr{F}_m^{l,m}}(\lambda)\, d\lambda.$$

This formula, which follows easily from (3.10), constitutes the expansion of a spherical wave in parabolic functions [7, p. 170].

Direct computation yields

$$\text{(4.8)} \quad \langle h_{\mathbf{x}}, h_{\mathbf{x}'}\rangle = 4\pi\frac{\sin \omega R}{\omega R}, \qquad R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

On the other hand,

$$\text{(4.9)} \quad \begin{aligned} \langle h_{\mathbf{x}}, h_{\mathbf{x}'}\rangle &= \sum_{l,m} \langle h_{\mathbf{x}}, j_m^l\rangle\langle j_m^l, h_{\mathbf{x}'}\rangle\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} H_m^l(\mathbf{x})\overline{H_m^l}(\mathbf{x}'), \end{aligned}$$

$$\text{(4.10)} \qquad \langle h_{\mathbf{x}}, h_{\mathbf{x}'}\rangle = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{H}_m^{\mathbf{x}}(\lambda)\overline{\mathscr{H}_m^{\mathbf{x}'}}(\lambda)\, d\lambda.$$

Thus, the expression (4.8) can be regarded as a bilinear generating function for the $H_m^l(\mathbf{x})$ and $\mathscr{H}_m^{\mathbf{x}}(\lambda)$.

The decomposition of the quasi-regular representation of $E(3)$ into a direct integral of irreducible representations is well known, e.g., [11]. The results are usually expressed in terms of the $\mathbf{J}\cdot\mathbf{J} - \mathbf{J}^3$ basis. Here we briefly describe the decomposition in terms of the $\mathbf{E} - \mathbf{J}^3$ basis. As we have shown the bases are related by (4.7).

Let $L_2(R_3)$ be the Hilbert space of Lebesgue square integrable functions $f(\mathbf{x}) = f(x, y, z)$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\mathbf{x})|^2\, d\mathbf{x} < \infty.$$

Then

$$f(\mathbf{x}) = \frac{1}{\pi}\int_0^\infty k^3\,dk \sum_{m=-\infty}^{\infty} \int_0^\infty \mathscr{H}^{\mathbf{x},k}_{-m}(-\lambda)\mathscr{F}^k_m(\lambda)\,d\lambda, \tag{4.11}$$

where

$$\mathscr{F}^k_m(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \mathscr{H}^{-\mathbf{x},k}_m(\lambda) f(\mathbf{x})\,d\mathbf{x} \tag{4.12}$$

and $\mathscr{H}^{\mathbf{x},k}_m(\lambda)$ is given by (4.3) with $\omega = k$.

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ON BOUNDED SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACE*

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Abstract. The main result can be stated roughly as follows: Let u be a Hilbert-space-valued solution of $u'(t) = Au(t)$, $-\infty < t < \infty$, where A differs from a symmetric linear operator by a suitable nonlinear perturbation. Then either u is constant or else $\sup_t \|u(t)\| = \infty$. More generally, A can depend on t, and the equation $u''(t) = A(t)u(t)$ is also considered.

1. Introduction. Let A be a symmetric linear operator on a Hilbert space $\mathscr{H}$. Let $u:\mathbb{R} = (-\infty, \infty) \to \mathscr{H}$ be a strongly continuously differentiable solution of $u'(t) = Au(t)$, $t \in \mathbb{R}$, where $'$ means differentiation with respect to t. Then either u is identically constant, or else u is unbounded. This result is due to H. Levine [3], who generalized earlier work of S. Zaidman [5]. Levine also allowed $A = A(t)$ to depend on t. The present note generalizes Levine's result to allow $A(t)$ to be nonlinear. Our nonlinear theorems ought to be useful in the study of almost-periodic solutions of nonlinear parabolic equations (cf. e.g., Foias–Zaidman [1]). We also consider the equation $u''(t) = A(t)u(t)$ and give some examples.

2. First order equations. Let $\mathscr{H}$ be a real or complex Hilbert space. The norm and inner product in $\mathscr{H}$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. For each $t \in \mathbb{R}$ let $A(t)$ be an operator with domain $\mathscr{D}(A(t))$ and range contained in $\mathscr{H}$. Let $\mathscr{D}$ be a subset of $\mathscr{H}$ such that $\mathscr{D} \subset \bigcap_{t\in\mathbb{R}} \mathscr{D}(A(t))$.

By a *strong $\mathscr{D}$-solution* of

$$u'(t) = A(t)u(t) \tag{1}$$

we mean a strongly absolutely continuous function $u:\mathbb{R} \to \mathscr{D}$ such that (1) holds a.e. For nonlinear equations, this notion is more appropriate than the notion of strongly continuously differentiable solution (cf. e.g., Pazy [4]).

THEOREM 1. *Let u be a strong $\mathscr{D}$-solution of* (1), *and suppose u' is $\mathscr{D}$-weakly absolutely continuous, i.e., $\langle u'(\cdot), x\rangle$ is absolutely continuous for all $x \in \mathscr{D}$. Suppose there exists a Lebesgue measurable function $\alpha:\mathbb{R} \to \mathbb{R}$ such that*

$$\operatorname{Re}\left(\frac{d}{ds}\langle u'(s), u(t)\rangle|_{s=t}\right) \geqq -\alpha(t)\|u'(t)\|^2 \tag{2}$$

and $\alpha(t) < 1$ for a.e. $t \in \mathbb{R}$. Then either $u \equiv$ const. or else

$$\sup\left\{\int_t^{t+1} \|u(s)\|^2\, ds | t \in \mathbb{R}\right\} = \infty.$$

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Proof. The idea of the proof is as in [3] and is quite simple. Let

$$F(t) = \int_t^{t+1} \|u(s)\|^2 \, ds.$$

Then for a.e. $t \in \mathbb{R}$,

$$\begin{aligned} F'(t) &= \|u(t+1)\|^2 - \|u(t)\|^2 = \int_t^{t+1} \frac{d}{ds}\|u(s)\|^2 \, ds \\ &= 2\int_t^{t+1} \operatorname{Re}\langle u'(s), u(s)\rangle \, ds, \\ F''(t) &= 2\operatorname{Re}\langle u'(s), u(s)\rangle|_t^{t+1} \\ &= 2\int_t^{t+1} \frac{d}{ds}\operatorname{Re}\langle u'(s), u(s)\rangle \, ds \\ &= 2\int_t^{t+1} (G(s) + H(s))\, ds, \end{aligned}$$

where $G(s) = \operatorname{Re}\langle u'(s), u'(s)\rangle = \|u'(s)\|^2$, and $H(s) = (d/dr)\operatorname{Re}\langle u'(r), u(s)\rangle|_{r=s} \geqq -\alpha(s)\|u'(s)\|^2$. Consequently,

$$F''(t) \geqq 2\int_t^{t+1} (1 - \alpha(s))\|u'(s)\|^2 \, ds \geqq 0. \tag{3}$$

Hence, F is a convex function on $\mathbb{R}$. If u is a nonconstant solution of (1), then $u' \not\equiv 0$ and (3) implies $F''(t_0) > 0$ for some $t_0 \in \mathbb{R}$ since $(1 - \alpha(s)) > 0$ a.e. by hypothesis. Thus, either $\lim_{t\to-\infty} F(t) = \infty$ or $\lim_{t\to\infty} F(t) = \infty$, and the theorem follows.

Remark 1. Suppose $A(t)$ is linear and symmetric for each t, and for each $x \in \mathscr{D}$, $A(\cdot)x$ is strongly continuously differentiable on $\mathbb{R}$ and $\langle A'(t)x, x\rangle \geqq -\beta(t)\|A(t)x\|^2$ for each $t \in \mathbb{R}$, $x \in \mathscr{D}$, where $\beta : \mathbb{R} \to \mathbb{R}$ is measurable, independent of x, and $\beta(t) < 2$ for each t. Then the hypotheses of Theorem 1 hold with $\alpha(t) = \beta(t) - 1$. Thus, Levine's result [3, Thm. 1] is included as a special case. It is worth noting that $u' = Au$ is actually strongly absolutely continuous in this case because of the smoothness assumption on A. Thus our assumption that u' is $\mathscr{D}$-weakly absolutely continuous can occur as a consequence of regularity assumptions on A.

Example 1. Let $A(t)x = g(x)S(t)x$, where $S(t)$ is a linear symmetric operator for each $t \in \mathbb{R}$, $\mathscr{D} \subset \bigcap_{t\in\mathbb{R}} \mathscr{D}(S(t))$, and $g : \mathscr{D} \to \mathbb{R}$ is differentiable. Suppose that there is a measurable $\beta : \mathbb{R} \to \mathbb{R}$ such that $S(\cdot)x$ is strongly absolutely continuous for each $x \in \mathscr{D}$ and $\langle S'(t)x, x\rangle \geqq -\beta(t)\|S(t)x\|^2$ for each $x \in \mathscr{D}$ and a.e., $t \in \mathbb{R}$. Let $\gamma > 0$ be such that $\gamma \leqq g(x)$ for each $x \in \mathscr{D}$, and suppose $\beta(t) < 2\gamma$ for each $t \in \mathbb{R}$. Let u be a strong $\mathscr{D}$-solution of (1) and suppose that either

(i) $S(t) \geqq 0$ for each $t \in \mathbb{R}$ and $g'(u(t))u'(t) \geqq 0$ a.e., or

(ii) $S(t) \leqq 0$ for each $t \in \mathbb{R}$ and $g'(u(t))u'(t) \leqq 0$ a.e.

((ii) becomes equivalent to (i) if t is replaced by $-t$ in (1).) Then the hypotheses of Theorem 1 hold, so $\sup_t \|u(t)\| = \infty$ if u is nonconstant. We now verify this.

For a.e. t we have

$$\frac{d}{dt}[g(u(t))S(t)u(t)] = g(u(t))S(t)u'(t) + g(u(t))S'(t)u(t) + [g'(u(t))u'(t)]S(t)u(t),$$

so $\operatorname{Re}((d/ds)\langle u'(s), u(t)\rangle|_{s=t}) = J_1 + J_2 + J_3$, where

$$J_1 = \langle g(u(t))S(t)u'(t), u(t)\rangle = \|u'(t)\|^2,$$

$$J_2 = g(u(t)) \operatorname{Re}\langle S'(t)u(t), u(t)\rangle \geqq -\beta(t)\gamma^{-1}\|u'(t)\|^2,$$

$$J_3 = [g'(u(t))u'(t)]\langle S(t)u(t), u(t)\rangle \geqq 0 \quad \text{a.e.}$$

if either (i) or (ii) holds. Thus, the hypotheses of Theorem 1 hold with $\alpha(t) = \gamma^{-1}\beta(t) - 1$.

Example 2. Specialize Example 1 so that $g(x) = f(\|x\|^2)$, where $f:\mathbb{R} \to [\gamma, \infty)$, $\gamma > 0$. Then

$$g'(u(t))u'(t) = 2f'(\|u(t)\|^2)f(\|u(t)\|^2)\langle S(t)u(t), u(t)\rangle.$$

Therefore (i) holds if $S(t) \geqq 0$ for each t and f^2 is absolutely continuous and nondecreasing (so that $ff' \geqq 0$ a.e.), and (ii) holds if $S(t) \leqq 0$ for each t and f^2 is absolutely continuous and nonincreasing. For a specific example, take $\mathscr{H} = L^2(\mathbb{R}^n)$, $n \geqq 1$, $S(t) = \Delta$, the Laplacian ($\Delta \leqq 0$), and $f(x) = k_1(k_2 - \tan^{-1} x)$, where k_1 is a positive constant and $k_2 > \pi/2$. Then setting $\beta(t) = 0$, the hypotheses (including (ii)) hold, so there are no nonconstant (in time) strong solutions of the nonlinear heat equation $\partial u/\partial t = g(u)\Delta u$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ satisfying

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^n} |u(t,x)|^2\, dx < \infty,$$

where $g(u) = f(\|u\|^2)$, f as above.

3. Second order equations. Let $\mathscr{D} \subset \bigcap_{t\in\mathbb{R}} \mathscr{D}(A(t))$. A strong *$\mathscr{D}$-solution of*

$$u''(t) = A(t)u(t) \tag{4}$$

is a strongly absolutely continuous function $u:\mathbb{R} \to \mathscr{D}$ having a strongly absolutely continuous strong derivative such that (4) holds a.e.

THEOREM 2. *Let u be a strong $\mathscr{D}$-solution of* (4). *Suppose* $\operatorname{Re}\langle A(t)u(t), u(t)\rangle \geqq 0$ *a.e. Then either $u \equiv$ const. or else at least one of* $\lim_{t\to-\infty}\|u(t)\|$, $\lim_{t\to+\infty}\|u(t)\|$ *is* $+\infty$.

It is a conclusion of the theorem that $\lim_{t\to\pm\infty}\|u(t)\|$ both exist.

Proof. Let $F(t) = \|u(t)\|^2$. Then a straightforward computation shows that F and F' are absolutely continuous and

$$F''(t) = 2\|u'(t)\|^2 + 2\operatorname{Re}\langle A(t)u(t), u(t)\rangle \geqq 0 \quad \text{a.e.},$$

so F is convex. The theorem follows easily.

Example 3. Let $\mathscr{H} = L^2(\mathbb{R})$, $\mathscr{D} = H^2(\mathbb{R}) =$ the domain of the self-adjoint Laplacian $\Delta = d^2/dx^2$, and $A(t)v = Av = -g(v)\Delta v$. ($A$ is independent of t.) Let u be a strong $\mathscr{D}$-solution of the nonlinear equation

$$\partial^2 u/\partial t^2 = -g(u)\Delta u. \tag{5}$$

Then

$$\operatorname{Re}\langle Au(t), u(t)\rangle = \operatorname{Re}\left(\int_{-\infty}^{\infty} g(u)|u_x|^2\,dx + \int_{-\infty}^{\infty} g(u)_x u_x \bar{u}\,dx\right), \tag{6}$$

where $u_x = \partial u/\partial x$, etc. If $g(v) = f(\|v\|^2)$ and $f \geqq 0$, (6) reduces to

$$\operatorname{Re}\langle Au(t), u(t)\rangle = \int_{-\infty}^{\infty} f(\|u(t)\|^2)|u_x|^2\,dx \geqq 0,$$

and the hypotheses of Theorem 2 are satisfied. One can easily determine conditions on g which guarantee that the second integral on the right-hand side of (6) has nonnegative real part, even when $g(v)$ depends on v itself and not just on $\|v\|^2$.

Remark 2. By combining the ideas of Levine [3] and of this note, it is easy to obtain criteria so that the conclusions of Theorem 2 apply to the equation $Pu''(t) = A(t)u(t) + F(t, u(t), u'(t))$, with $A(t)$ nonlinear, so that Theorems 2 and 3 of [3] are subsumed as special cases. We omit the details.

Remark 3. It is possible to extend our results to an L^p situation. (Compare with [3, p. 250].) The resulting theorems are rather cumbersome to write down, so we omit precise statements. A useful tool is a lemma of T. Kato [2, p. 510]. We conclude with a nonlinear example in an L^p context.

Example 4. Let X be the real space $L^p(\mathbb{R})$, $2 \leqq p < \infty$. Let f be a nonnegative measurable function on $\mathbb{R}$, and let $g(v) = f(\|v\|_p^p)$ for $v \in X$. Let $u:\mathbb{R} \to X$ be a strong solution of (5). We claim that $u \equiv$ const. or else

$$\max\left\{\lim_{t\to-\infty} \|u(t)\|_p, \lim_{t\to+\infty} \|u(t)\|_p\right\} + \infty.$$

For $p = 2$ this follows from Example 3. For a proof, set

$$F(t) = \|u(t)\|_p^p = \int_{-\infty}^{\infty} |u(t,x)|^p\,dx.$$

A straightforward calculation using integration by parts and $g(u)_x = 0$ shows that

$$F''(t) = \int_{-\infty}^{\infty} p(p-1)|u|^{p-2}\left(g(u)\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right)dx,$$

and the result follows easily.

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FREE BOUNDARY PROBLEM*

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Abstract. Let $\mathscr{D}$ be a doubly connected region in the complex plane limited by the infinite point and a convex set Γ. If $\lambda > 0$, then we study the existence, uniqueness and geometry of annuli $\omega \subset \mathscr{D}$ having Γ as one boundary component and another boundary component γ, such that there exists a harmonic function V in ω satisfying: (a) $V = 0$ on Γ, (b) $V = 1$ on γ and (c) $|\text{grad } V| = \lambda$ on γ.

1. Introduction. Suppose $\mathscr{D}$ is a doubly connected region in the complex plane which is limited by a compact boundary component Γ and the point at infinity. If Q is a continuous positive function in $\mathscr{D}$, then in [1] Beurling considers the following problem: Can one find an annulus $\omega \subset \mathscr{D}$ having Γ as one boundary component and another boundary component γ, the "free boundary", such that there exists a harmonic function V in ω satisfying:

(a) $V = 0$ on Γ,

(b) $V = 1$ on γ,

(c) $|\text{grad } V| = Q$ on γ.

Beurling gives necessary and sufficient conditions for the above problem to have a solution. He also gives a sufficient condition for a unique solution to the problem.

In this paper, we shall be concerned with the qualitative properties of the free boundary in the special case when Γ is convex and Q is a constant $\lambda > 0$. From Beurling's work, it is easy to prove that in this case there do exist solutions. Furthermore, uniqueness in this case is actually shown by Beurling. A summary of our results is:

(i) The solution annuli as λ increases are monotone decreasing, and the free boundaries of the solution annuli exhaust all of $\mathscr{D}$ as λ tends to zero.

(ii) The free boundary of each solution annulus is a closed convex curve.

(iii) In a certain sense, the free boundaries are asymptotic to a family of circles as λ tends to zero.

In the next section, we summarize the results in [1].

2. Preliminaries. Suppose Q is a continuous positive function which is permitted to tend to infinity at Γ within the following limitation:

$$Q = O\left(\frac{|\text{grad } u|}{u}\right), \tag{1}$$

where u stands for any harmonic function which vanishes on Γ and is positive and regular in some annulus ω contained in $\mathscr{D}$ having Γ as one boundary component.

We shall use the following notation and definitions. $\mathscr{C}$ will denote the family of all finite subannuli of $\mathscr{D}$ having Γ as one boundary component. For $\omega \in \mathscr{C}$, the

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harmonic function in ω with boundary values (a) and (b) will be denoted V_ω and referred to as the stream function of ω.

As in [1], to give a proper definition of (c) we assign the following quantities to $\omega \in \mathscr{C}$:

$$a(\omega, Q) = a(\omega) = \liminf \left(\frac{|\text{grad } V_\omega|}{Q} \right),$$

$$b(\omega, Q) = b(\omega) = \limsup \left(\frac{|\text{grad } V_\omega|}{Q} \right),$$

where the limits are taken as $z \in \omega$ tends to the free boundary. By means of these limits we define the following three sets of annuli:

$$A(Q) = A = \{\omega \in \mathscr{C} : a(\omega) \geqq 1\},$$

$$B(Q) = B = \{\omega \in \mathscr{C} : b(\omega) \leqq 1\},$$

$$B_0(Q) = B_0 = \{\omega \in \mathscr{C} : b(\omega) < 1\}.$$

The intersection $A \cap B$ is the set of solutions.

The following three statements are consequences of the results in [1].

(I) If $\omega' \in A$ and $\omega'' \in B_0$, then there exists a solution Ω, $\omega' \subset \Omega \subset \omega''$.

(II) $B \neq \varnothing$ is necessary and sufficient for there to be a solution. Furthermore, if $\omega \in B$, then there exists a solution Ω, $\Omega \subset \omega$.

(III) If Γ is convex, then for $Q \equiv \lambda$, a positive constant, there cannot be more than one solution.

3. Existence and uniqueness. We shall make use of the following definition. If J is a closed convex curve and Δ is the region limited by J and the point at infinity, then $[z, J; \Delta]$ stands for the distance from $z \in \Delta$ to J. The following observation is easily shown:

(IV) If J is convex, then $[z, J; \Delta]$ is a subharmonic function of $z \in \Delta$.

We return to the case where $Q \equiv \lambda$, a preassigned positive constant. Using Beurling's results and (IV), we prove the following theorem.

THEOREM 1. *Given $\lambda > 0$, there exists a unique solution. Furthermore, if Ω_λ denotes the unique solution, then*

$$\Omega_\lambda \subset \{z \in \mathscr{D} : [z, \Gamma; \mathscr{D}] < \lambda^{-1}\}. \tag{2}$$

Proof. By (III) there cannot be more than one solution. From (II) we need only show $B(\lambda) \neq \varnothing$. To show this, let Γ' be any analytic closed convex curve $\subset \mathscr{D}$ which is homotopic to Γ. If $\mathscr{D}'$ is the region limited by Γ' and the point at infinity, then we define

$$\omega'_\lambda = \{z \in \mathscr{D}' : [z, \Gamma'; \mathscr{D}'] < \lambda^{-1}\} \tag{3}$$

and

$$\omega^*_\lambda = \mathscr{D} \cup \omega'_\lambda. \tag{4}$$

We claim that $\omega^*_\lambda \in B(\lambda)$. If V^*_λ is the stream function of ω^*_λ and

$$U(z) = \lambda[z, \Gamma'; \mathscr{D}'],$$

then for $z \in \omega'_\lambda$ we have

$$V^*_\lambda(z) \geqq U(z) \tag{5}$$

by (IV). We observe that equality holds in (5) on the free boundary of ω^*_λ. Therefore, on the free boundary of ω^*_λ, we have

$$0 \geqq \frac{\partial V^*_\lambda}{\partial \eta} \geqq \frac{\partial U}{\partial \eta} = -\lambda, \tag{6}$$

where the derivatives are taken in the inward normal direction. Hence $\omega^*_\lambda \in B(\lambda)$, and by (II) there exists a solution $\Omega_\lambda, \Omega_\lambda \subset \omega^*_\lambda$.

The relation (2) holds because Γ' is arbitrary.

4. Monotonicity. In the rest of this paper Ω_λ will stand for the unique solution annulus for the case when $Q \equiv \lambda$, a preassigned positive constant. We now prove the following theorem.

THEOREM 2. *If $\lambda_1 > \lambda_2$, then*

$$\Omega_{\lambda_1} \subset \Omega_{\lambda_2}. \tag{7}$$

Furthermore, if γ_λ is the free boundary of Ω_λ, then

$$\bigcup_{\lambda > 0} \gamma_\lambda = \mathscr{D}. \tag{8}$$

Proof. The relation (7) holds by uniqueness of the solution and because

$$\Omega_{\lambda_2} \in B(\lambda_1).$$

To show (8) we suppose $z_0 \in \mathscr{D}$ and show $z_0 \in \gamma_\lambda$ for some λ. We define

$$S_1 = \{\lambda : z_0 \notin \Omega_\lambda\}, \qquad S_2 = \{\lambda : z_0 \in \Omega_\lambda\}.$$

We know that $S_1 \neq \varnothing$. To show $S_2 \neq \varnothing$, choose $R > 0$ such that if E is the disk $|z| < R$, then $z_0 \in E$ and $\Gamma \subset E$. If U is the stream function of the annulus $E \cap \mathscr{D}$, then $|\text{grad } U|$ is bounded away from zero on the circle $|z| = R$. Hence, if

$$\delta = \min_{|z| = R} |\text{grad } U|,$$

then $E \cap \mathscr{D} \in A(\delta)$. Furthermore, for some $\delta_0 < \delta$, we have

$$E \cap \mathscr{D} \subset \{z \in \mathscr{D} : [z, \Gamma; \mathscr{D}] < \delta_0^{-1}\}.$$

Therefore, by (I), we have $z_0 \in \Omega_{\delta_0}$.

Since $S_i \neq \varnothing$ for $i = 1, 2$, we see that

$$\Omega_{\lambda_1} = \bigcup_{\lambda \in S_1} \Omega_\lambda, \qquad \Omega_{\lambda_2} = \bigcap_{\lambda \in S_2} \Omega_\lambda$$

for some λ_1 and λ_2. If $\lambda_1 > \lambda_2$, then there exists λ_3 such that $\lambda_1 > \lambda_3 > \lambda_2$. We then would have

$$\Omega_{\lambda_1} \subset \Omega_{\lambda_3} \subset \Omega_{\lambda_2} \tag{9}$$

which is impossible. Similarly, $\lambda_1 < \lambda_2$ leads to a contradiction. Hence, $\lambda_1 = \lambda_2$ and $z_0 \in \gamma_{\lambda_1}$.

5. Convexity of the free boundary. We introduce the following definitions and notation. If Γ' and Γ'' are closed convex curves, then we define

$$[\Gamma', \Gamma''] = \sup_{w' \in \Gamma'} \inf_{w'' \in \Gamma''} |w' - w''| \tag{10}$$

and observe that in general $[\Gamma', \Gamma''] \neq [\Gamma'', \Gamma']$. A sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of closed convex curves will be called increasing if Γ_n lies in the interior of Γ_{n+1} for $n = 1, 2, \cdots$.

We have the following theorem.

THEOREM 3. *If γ_λ is the free boundary of Ω_λ, then γ_λ is a closed convex curve.*

Proof. First suppose Γ is an analytic closed convex curve oriented in the positive sense with respect to $\mathscr{D}$ such that the curvature of Γ does not vanish. If V is the stream function of Ω_λ and U a conjugate of V, then the analytic function

$$f(z) = U + iV \tag{11}$$

maps Ω_λ minus a Jordan slit connecting Γ to γ_λ onto the rectangle, $0 < U < U_0$, $0 < V < 1$. Let

$$f'(z) = \exp[\rho(z) + i\theta(z)]. \tag{12}$$

At each point $z \in \partial\Omega_\lambda$, let η be the inward normal direction and s the tangent direction in the positive sense. By the Cauchy–Riemann equations, we have

$$\partial\rho/\partial s = \partial\theta/\partial\eta, \tag{13}$$

$$\partial\rho/\partial\eta = -\partial\theta/\partial s. \tag{14}$$

On Γ we have

$$\partial\rho/\partial\eta = -\partial\theta/\partial s < 0, \tag{15}$$

by convexity of Γ. Therefore, ρ attains its minimum on the free boundary where $\rho \equiv \log\lambda$. Hence, for $z \in \gamma_\lambda$ we have

$$0 \leqq [\partial\rho(z)]/\partial\eta = -[\partial\theta(z)]/\partial s$$

which implies γ_λ is convex.

Passing to the general case, let $\{\Gamma_n\}_{n=1}^{\infty}$ be an increasing sequence of analytic closed convex curves $\subset \mathbb{C} - \overline{\mathscr{D}}$, with nonvanishing curvature such that $[\Gamma, \Gamma_n]$ decreases to zero as $n \to \infty$. If $\Omega_\lambda^{(n)}$ is the solution annulus for the curve Γ_n, then it follows that

$$\Omega_\lambda = \bigcup_{n=1}^{\infty} (\Omega_\lambda^{(n)} \cap \mathscr{D}). \tag{16}$$

6. Asymptotic behavior of the free boundaries. To study the asymptotic behavior of the free boundaries, we transform $\mathscr{D}$ onto the exterior of a circle. For this purpose, let

$$z = f(w) = Cw + \alpha_0 + \sum_{n=-1}^{\infty} \alpha_n w^{-n} \tag{17}$$

be a schlicht mapping of $|w| > 1$ onto $\mathscr{D}$. If Γ^* denotes the circle $|w| = 1$, then the annulus $f^{-1}(\Omega_\lambda)$ is the unique solution to the free boundary problem for the curve Γ^* and the function

$$Q_\lambda(w) = \lambda |f'(w)|. \tag{18}$$

The annulus $f^{-1}(\Omega_\lambda)$ will be denoted by Ω_λ^*. We may assume C in (17) is positive.

We have the following theorem.

THEOREM 4. *For each $\lambda > 0$, there exists α_λ and $\beta_\lambda > 0$ such that*

$$\{w : 1 < |w| < \alpha_\lambda\} \subset \Omega_\lambda^* \subset \{w : 1 < |w| < \beta_\lambda\} \tag{19}$$

and $(\beta_\lambda - \alpha_\lambda) \to 0$ as $\lambda \to 0$.

Proof. If γ_λ^* is the free boundary of Ω_λ^*, and V_λ^* is the stream function of Ω_λ^*, then for $w \in \gamma_\lambda^*$ we have

$$|\text{grad } V_\lambda^*(w)| = \lambda |f'(w)|. \tag{20}$$

Let

$$\rho_\lambda = \inf\{|w| : w \in \gamma_\lambda^*\}. \tag{21}$$

For λ sufficiently small, there exists ε_λ' and ε_λ'' positive such that

$$0 < C - \frac{|\alpha_1|}{\rho_\lambda^2} - \varepsilon_\lambda'' \leqq |f'(w)| \leqq C + \frac{|\alpha_1|}{\rho_\lambda^2} + \varepsilon_\lambda', \tag{22}$$

where $\rho_\lambda^2 \varepsilon_\lambda'$ and $\rho_\lambda^2 \varepsilon_\lambda''$ tend to zero as $\lambda \to 0$. Let

$$E_\lambda = \lambda\left(C + \frac{|\alpha_1|}{\rho_\lambda^2} + \varepsilon_\lambda'\right), \qquad F_\lambda = \lambda\left(C - \frac{|\alpha_1|}{\rho_\lambda} - \varepsilon_\lambda''\right), \tag{23}$$

and define α_λ and β_λ by the equations

$$E_\lambda = \frac{1}{\alpha_\lambda \log \alpha_\lambda}, \qquad F_\lambda = \frac{1}{\beta_\lambda \log \beta_\lambda}. \tag{24}$$

If

$$\Omega_\lambda' = \{w : 1 < |w| < \alpha_\lambda\}, \qquad \Omega_\lambda'' = \{w : 1 < |w| < \beta_\lambda\}, \tag{25}$$

then we see that Ω_λ' and Ω_λ'' are the respective solutions to the free boundary problem for the curve Γ^* and the constants E_λ and F_λ. Hence by (22) and (I), we have

$$\Omega_\lambda' \subset \Omega_\lambda^* \subset \Omega_\lambda''. \tag{26}$$

We need only show that $(\beta_\lambda - \alpha_\lambda) \to 0$ as $\lambda \to 0$. However, for λ sufficiently small we have

$$
\begin{aligned}
\beta_\lambda - \alpha_\lambda &= \frac{1}{F_\lambda \log \beta_\lambda} - \frac{1}{E_\lambda \log \alpha_\lambda} \\
&\leqq \frac{1}{\log \alpha_\lambda}\left[\frac{1}{F_\lambda} - \frac{1}{E_\lambda}\right] \\
&= \frac{2}{\lambda \rho_\lambda \log \alpha_\lambda}\left[\frac{|\alpha_1|\rho_\lambda^{-1} + \rho_\lambda(\varepsilon'_\lambda + \varepsilon''_\lambda)}{(C - |\alpha_1|\rho_\lambda^{-2} - \varepsilon''_\lambda)(C + |\alpha_1|\rho_\lambda^{-2} + \varepsilon'_\lambda)}\right] \\
&= \frac{2\alpha_\lambda E_\lambda}{\lambda \rho_\lambda}\left[\frac{|\alpha_1|\rho_\lambda^{-1} + \rho_\lambda(\varepsilon'_\lambda + \varepsilon''_\lambda)}{(C - |\alpha_1|\rho_\lambda^{-2} - \varepsilon''_\lambda)(C + |\alpha_1|\rho_\lambda^{-2} + \varepsilon'_\lambda)}\right].
\end{aligned}
\tag{27}
$$

By (23), $\lambda^{-1}E_\lambda$ is bounded, and by (26), $\alpha_\lambda \rho_\lambda^{-1} < 1$. Hence $(\beta_\lambda - \alpha_\lambda) \to 0$ as $\lambda \to 0$.

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ERRATUM: SOME ISOPERIMETRIC INEQUALITIES FOR HARMONIC FUNCTIONS*

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Inequality (2.6a) and equation (2.7) are incomplete. Although the conclusions of the paper are correct (the two missing terms annihilate one another when (2.6a) and (2.7) are combined), the corrected versions should read as follows:

$$2\left[\frac{\partial H}{\partial n}\frac{\partial^2 H}{\partial n^2}+\frac{\partial H}{\partial s}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right)-\frac{\partial H}{\partial x_i}\frac{\partial H}{\partial x_j}\frac{\partial n_j}{\partial x_i}\right]>0, \tag{2.6a}$$

$$\frac{\partial^2 H}{\partial n^2}+K\frac{\partial H}{\partial n}-n_i\frac{\partial H}{\partial x_j}\frac{\partial n_j}{\partial x_i}+\frac{\partial^2 H}{\partial s^2}=0. \tag{2.7}$$

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ON THE ASYMPTOTIC BEHAVIOR OF THE BOUNDED SOLUTIONS OF A NONLINEAR VOLTERRA EQUATION*

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Abstract. This paper is concerned with the asymptotic behavior of the bounded solutions of the nonlinear Volterra equation $x(t) + \int_0^t a(t - \tau)g(x(\tau))\,d\tau = f(t)$. Conditions implying that the solutions are slowly varying are obtained. These conditions generalize earlier results by Levin and Shea and by the author.

Introduction. In this paper we investigate the asymptotic behavior of the bounded solutions of the nonlinear Volterra equation

$$x(t) + \int_0^t a(t - \tau)g(x(\tau))\,d\tau = f(t), \qquad 0 \leqq t < \infty, \tag{1.1}$$

where $a(t)$, $g(x)$, $f(t)$ are prescribed real functions. Specifically, our main result is the following.

THEOREM 1. *Let*

$$a(t) \geqq 0, \qquad 0 \leqq t < \infty, \tag{1.2}$$

$$a(t) \quad \textit{be nonincreasing on } [0, \infty), \qquad a(0) < \infty, \tag{1.3}$$

$$a(t) \in L^1(0, \infty), \tag{1.4}$$

$$g(x) \in C(-\infty, \infty), \tag{1.5}$$

$$f(t) \in L^\infty(0, \infty), \qquad \lim_{t \to \infty} f(t) = F. \tag{1.6}$$

Let $x(t)$ be a solution of (1.1) *on* $[0, \infty)$ *such that*

$$x(t) \in L^\infty(0, \infty). \tag{1.7}$$

Then $x(t)$ is slowly varying in the sense that for any positive constant T one has

$$\lim_{t \to \infty} \left[\sup_{t - T \leqq \tau \leqq t} x(\tau) - \inf_{t - T \leqq \tau \leqq t} x(\tau) \right] = 0. \tag{1.8}$$

Also

$$\lim_{t \to \infty} \operatorname{dist}(x(t), L) = 0, \tag{1.9}$$

where

$$L \overset{\text{def}}{=} \left\{ x \,|\, x + g(x) \int_0^\infty a(\tau)\,d\tau = F,\ \liminf_{t \to \infty} x(t) \leqq x \leqq \limsup_{t \to \infty} x(t) \right\}. \tag{1.10}$$

The equation (1.1) has recently been analyzed in [2], [3] and the present result generalizes the main result of [3], in the case of an integrable kernel $a(t)$,

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by significantly weakening the hypothesis on $f(t)$. More precisely, we observe that in [3] we postulated

$$f(t) \in C[0, \infty) \cap BV[0, \infty),$$

in order to obtain asymptotic results. In particular the condition $f \in BV[0, \infty)$, although rather technical and not absolutely intrinsic to the problem, was crucial to the proof in [3]. Our principal goal has therefore now been to construct a proof which does not make use of this condition.

While the main emphasis in the proof is thus on eliminating the hypothesis $f \in BV$, it turns out that by Lemma 1, to be discussed below, we are able to also abandon the assumption $f \in C$.

Observe that as only (1.6) is assumed on $f(t)$, it follows that Theorem 1 has immediate consequences on the asymptotic behavior of the bounded solutions of the delay equations

$$x(t) + \int_0^T g(x(t - \tau))a(\tau)\, d\tau = f(t), \qquad 0 \leqq t < \infty,$$

where $0 < T \leqq \infty$, where $a(t)$ satisfies only (1.2), (1.3) and (1.4), and where the initial function $x(t)$ is bounded on $-T \leqq t \leqq 0$.

Theorem 1 includes as special cases comparable results presented in [2] and [4] which make much stronger assumptions on both $a(t)$ and $g(x)$.

Discussing the proof we begin by observing that Theorem 1 is obtained by proving Theorem 2 and Lemma 1, which together imply Theorem 1.

Theorem 2 is concerned with (1.1) under identical assumptions on $a(t)$ and $g(x)$ as Theorem 1, but under strong smoothness hypotheses on $f(t)$. In particular, $f(t)$ absolutely continuous on $[0, \infty)$ and $\lim_{t\to\infty} \operatorname{ess\,sup}_{t \leqq \tau < \infty} |f'(\tau)| = 0$ are assumed. We show in Theorem 2 that (1.8) and (1.9) follow under this smoother hypothesis. Note however that $f \in BV$ is not postulated in Theorem 2.

Lemma 1 in turn simply states that if Theorem 2 holds, then so does Theorem 1. The proof of this statement, which we delegate to § 3, makes above all use of the smoothing nature of convolution, and of (1.4).

THEOREM 2. *Let*

(1.11) $$a(t) \geqq 0, \qquad 0 \leqq t < \infty,$$

(1.12) $a(t)$ *be nonincreasing on* $[0, \infty)$, $\qquad a(0) < \infty$,

(1.13) $$a(t) \in L^1(0, \infty),$$

(1.14) $$g(x) \in C(-\infty, \infty),$$

(1.15) $f(t)$ *be absolutely continuous on* $[0, \infty)$,

(1.16) $$f'(t) \in L^\infty(0, \infty),$$

(1.17) $$\lim_{t\to\infty} f(t) = F,$$

(1.18) $$\lim_{t\to\infty} \operatorname*{ess\,sup}_{t \leqq \tau \infty} |f'(\tau)| = 0.$$

Let $x(t)$ *be a solution of* (1.1) *on* $[0, \infty)$ *such that*

$$\sup_{0 \leqq t \leqq \infty} |x(t)| < \infty. \tag{1.19}$$

Then (1.8) *and* (1.9) *are satisfied.*

LEMMA 1. *Let Theorem* 2 *hold. Then Theorem* 1 *holds.*

We begin the proof of Theorem 2 by stating the uniform continuity of $g(x(t))$ and the absolute continuity of $x(t)$ on $[0, \infty)$. From (1.1) we then deduce, using (1.13) and (1.19), the equation (2.8) and the inequality (2.9). These relations are basic to the continuation of the proof.

The inequality (2.9) clearly gives an intuitive upper bound for the variation of $g(x(t))$ on any bounded interval. In Lemma 4 we make this intuitive bound exact. The proof of this lemma, see the last section of this paper, basically follows the method developed in the proof of [3, Lemma 1]. Of course, the conclusion of Lemma 4 is weaker than the conclusion of [3, Lemma 1] because we now only have (1.18) and not $f \in BV[0, \infty)$. We observe that the proof of Lemma 4 is notably simpler if $a(t)$ is a pure saltus function and does not contain any continuous part.

After Lemma 4 we turn our attention in the proof of Theorem 2 to analyzing the function $H(x(t))$, where $H(x)$ is defined in (2.6). The result of the discussion between (2.13) and (2.46) can be summarized by stating that $H(x(t))$ may essentially be taken as constant on $\cup_n \hat{\Gamma}_n$, where $\hat{\Gamma}_n$ are the intervals mentioned in Lemma 4. Note that if L (defined in (1.10)) consists of a finite number of points or of a finite number of disjoint closed intervals (possibly of both) then this part of the proof can be considerably simplified. In particular, if L consists of a single point or of a single closed interval, then this discussion reduces to a few lines.

From (2.46) onward the proof does not present any difficulties. We assume that (1.8) does not hold and show that this leads to a contradiction. The final statement (1.9) is then easily obtained from (1.1) and (1.8).

The results obtained clearly do not preclude the nonexistence of $\lim_{t \to \infty} x(t)$. However, as a byproduct of the proof of Theorem 2 we have that the possible oscillations necessarily are such that $\lim_{t \to \infty} H(x(t))$ exists.

Our method of proof also allows us to treat the case where $\lim_{t \to 0+} a(t) = \infty$. This is done in the following.

THEOREM 3. *Suppose* (1.13) *and* (1.17) *hold and let*

$$a(t) \geqq 0, \qquad 0 < t < \infty, \tag{1.20}$$

$$a(t) \quad \textit{be nonincreasing on } (0, \infty), \tag{1.21}$$

$$g(x) \quad \textit{be locally Lipschitzian for } |x| < \infty, \tag{1.22}$$

$$f(t) \in LBV[0, \infty), \tag{1.23}$$

$$\lim_{t \to \infty} V(f, [t - T, t]) = 0 \quad \textit{for any } T > 0. \tag{1.24}$$

Let $x(t)$ *be a solution of* (1.1) *on* $[0, \infty)$ *satisfying* (1.19). *Then* (1.8) *and* (1.9) *hold.*

In Theorem 3 we use the following notations:

$$V(f, I) \stackrel{\text{def}}{=} \text{total variation of } f \text{ on } I,$$

$$LBV[0,\infty) \stackrel{\text{def}}{=} \{f(t) \mid f(t) \text{ of bounded variation for any compact subinterval of } [0,\infty)\}.$$

Observe that the local variation of f has to be restricted. The condition (1.22) is added as we also need $g(x(t)) \in LBV[0, \infty)$. The proof of Theorem 3 (see § 4) consists of an application of the arguments of the proof of Theorem 2 to a certain family of functions $x_\varepsilon(t)$, $\varepsilon > 0$, which satisfy $\lim_{\varepsilon\to 0} \sup_{0 \leqq t < \infty} |x_\varepsilon(t) - x(t)| = 0$.

We note that the existence and boundedness of solutions of (1.1) under hypotheses related to those of Theorem 3 have recently been considered in [1].

To conclude this section we state two lemmas which will be frequently used. Lemma 2 states a well-known fact and the proof of Lemma 3 may be found in [3].

LEMMA 2. *Let* $u(t) \in L^\infty(0, \infty)$ *and let* $a(t)$ *satisfy* (1.4). *Define* $v(t)$ *by* $v(t) = \int_0^t a(t-\tau)u(\tau)\,d\tau$, $0 \leqq t < \infty$. *Then* $v(t)$ *is uniformly continuous on* $[0, \infty)$.

LEMMA 3. *Let* $u(t) \in C[0, \infty) \cap L^\infty(0, \infty)$ *and let* $a(t)$ *satisfy* (1.2) *and* (1.3). *Define* $v(t)$ *as in Lemma* 2. *Then* $v(t)$ *is absolutely continuous on* $[0, \infty)$, $v'(t) \in L^\infty(0, \infty)$, *and*

$$v'(t) = a(0)u(t) + \int_0^t u(t-\tau)\,da(\tau), \quad \textit{a.e. on } 0 \leqq t < \infty.$$

2. Proof of Theorem 2. From (1.14) and (1.19) it follows that

$$\sup_{0 \leqq t < \infty} |g(x(t))| = \overset{\circ}{M} < \infty. \tag{2.1}$$

Define $y(t)$ by

$$y(t) = \int_0^t a(t-\tau)g(x(\tau))\,d\tau, \qquad 0 \leqq t < \infty. \tag{2.2}$$

Then, by (1.1) and (2.2),

$$x(t) + y(t) = f(t), \qquad 0 \leqq t < \infty. \tag{2.3}$$

We claim that

$$g(x(t)) \quad \text{is uniformly continuous on } 0 \leqq t < \infty. \tag{2.4}$$

To realize that (2.4) holds, notice at first that from (1.13), (2.1) and Lemma 2, it follows that $y(t)$ is uniformly continuous on $0 \leqq t < \infty$. The conditions (1.15) and (2.3) then imply that the same is true for $x(t)$. But this, together with (1.14), gives (2.4).

Obviously (1.11), (1.12), (2.1), (2.4) and Lemma 3 yield that $y(t)$ is absolutely continuous on $[0, \infty)$ and so, by (1.15) and (2.3), $x(t)$ is absolutely continuous on $[0, \infty)$, and

$$x'(t) + a(0)g(x(t)) + \int_0^t g(x(t-\tau))\,da(\tau) = f'(t), \quad \text{a.e. on } 0 \leqq t < \infty. \tag{2.5}$$

Let $A = \int_0^\infty a(\tau)\,d\tau$ (without loss of generality, assume $A > 0$), and define $G(x)$, $H(x)$, $g(t)$, $H(t)$ by

$$G(x) = \int_0^x g(u)\,du, \quad H(x) = \frac{A}{2}g^2(x) + G(x), \qquad |x| < \infty, \tag{2.6}$$

$$g(t) = g(x(t)), \quad G(t) = G(x(t)), \quad H(t) = H(x(t)), \qquad 0 \leqq t < \infty. \tag{2.7}$$

If we expand $[g(\tau) - g(\tau - s)]^2$, integrate by parts, and use Fubini's theorem, we immediately find that (2.8) is equivalent to the equation obtained after multiplying (2.5) by $g(t)$ and then integrating.

$$\begin{aligned} &G(t) - G(0) + \frac{1}{2}\int_0^t g^2(t-\tau)a(\tau)\,d\tau + \frac{1}{2}\int_0^t g^2(\tau)a(\tau)\,d\tau \\ &\quad - \frac{1}{2}\int_0^t \left\{\int_0^\tau [g(\tau) - g(\tau - s)]^2\,da(s)\right\}d\tau = \int_0^t f'(\tau)g(\tau)\,d\tau, \qquad 0 \leqq t < \infty. \end{aligned} \tag{2.8}$$

Excepting the last term we have, by (1.13), (1.19), (2.1) and (2.6), that all the terms on the left side of (2.8) are bounded on $[0, \infty)$. Thus, again using (2.1), there certainly exists a constant K such that for any $\tilde{t}_1, \tilde{t}_2$, $0 \leqq \tilde{t}_1 < \tilde{t}_2 < \infty$,

$$-\int_{\tilde{t}_1}^{\tilde{t}_2}\left\{\int_0^\tau [g(\tau) - g(\tau - s)]^2\,da(s)\right\}d\tau \leqq 2M\int_{\tilde{t}_1}^{\tilde{t}_2} |f'(\tau)|\,d\tau + K. \tag{2.9}$$

Following the deduction of (2.4), (2.5), (2.8) and (2.9), our next step is to formulate Lemma 4.

LEMMA 4. *Let the hypothesis of Theorem 2 hold and let ν, $\hat{T}$ be arbitrary positive constants. There exist a positive constant $\tilde{T}$ and an integer $\hat{N}$ such that if*

$$\Gamma_n \overset{\text{def}}{=} \{t \mid n\tilde{T} \leqq t \leqq (n+1)\tilde{T}\}, \qquad n = 0, 1, \cdots, \tag{2.10}$$

then, for each $n \geqq \hat{N}$, there exists a closed interval $\hat{\Gamma}_n$ satisfying (m is the Lebesgue measure)

$$\sup_{t \in \hat{\Gamma}_n} g(t) - \inf_{t \in \hat{\Gamma}_n} g(t) \leqq \nu, \tag{2.11}$$

$$\hat{\Gamma}_n \subset \Gamma_n, \quad m(\hat{\Gamma}_n) \geqq \hat{T}. \tag{2.12}$$

For the proof of Lemma 4, see § 5. However, to convince oneself that Lemma 4 is plausible, it suffices to combine the first part of (1.12), (1.18) and (2.4) with (2.9).

To complement Lemma 4 we need (2.17) which says that the possible increase of $H(t)$ between two consecutive intervals $\hat{\Gamma}_{n-1}$ and $\hat{\Gamma}_n$ can be made arbitrarily small by taking n sufficiently large.

Let $\nu > 0$ be arbitrary and choose any $\hat{T}$ such that $\nu \geqq 2M^2\int_{\hat{T}}^\infty a(\tau)\,d\tau$. Then take any $\tilde{T}$, $\hat{N}$, $\{\hat{\Gamma}_n\}$ satisfying (2.10), (2.11) and (2.12), and let $\{t_n'\}$, $\{t_n\}$ be defined by $\hat{\Gamma}_n = \{t \mid t_n' \leqq t \leqq t_n\}$. Suppose (2.17) does not hold or equivalently, suppose there exists a subsequence $\{n_k\}$ of $\{n\}$ such that if $n \in \{n_k\}$, then

$$H(t_n) - H(t_{n-1}) > [A + 1]\nu. \tag{2.13}$$

Let $t = t_n, t_{n-1}$ in (2.8). From (1.11) and (1.12) now follows, after taking differences,

$$G(t_n) + \frac{1}{2}\int_0^{t_n} g^2(t_n - \tau)a(\tau)\,d\tau - G(t_{n-1}) - \frac{1}{2}\int_0^{t_{n-1}} g^2(t_{n-1} - \tau)a(\tau)\,d\tau \tag{2.14}$$
$$\leqq \int_{t_{n-1}}^{t_n} f'(\tau)g(\tau)\,d\tau.$$

By (2.11) one may clearly assume

$$\sup_{t\in\hat{\Gamma}_n} g^2(t) - \inf_{t\in\hat{\Gamma}_n} g^2(t) \leqq \nu, \quad \text{if } n \geqq \hat{N}. \tag{2.15}$$

But (2.14) gives, on account of (1.13), (2.1), the second part of (2.12), and (2.15),

$$H(t_n) - H(t_{n-1}) - A\nu - M^2\int_{\hat{T}}^{\infty} a(\tau)\,d\tau \leqq M\int_{t_{n-1}}^{t_n} |f'(\tau)|\,d\tau. \tag{2.16}$$

Note that $t_n - t_{n-1} < 2\tilde{T}$, a fixed number. Therefore, after using (1.18), (2.13) in (2.16) and remembering how $\hat{T}$ was chosen, one clearly obtains a contradiction, if n is sufficiently large. We conclude that

$$H(t_n) - H(t_{n-1}) \leqq [A + 1]\nu, \tag{2.17}$$

for any $\nu > 0$, if $\hat{T}$ satisfies $\nu \geqq 2M^2 \int_{\hat{T}}^{\infty} a(\tau)\,d\tau$ and n is taken sufficiently large.

Take any sequence $\{\nu_i\}$, $\nu_i > 0$, $i = 1, 2, \cdots$; $\nu_i \to 0$, $i \to \infty$, and then any sequence $\{\hat{T}_i\}$ satisfying

$$\nu_i \geqq 2M^2 \int_{\hat{T}_i}^{\infty} a(\tau)\,d\tau, \qquad i = 1, 2, \cdots. \tag{2.18}$$

By Lemma 4 there exist sequences $\{\tilde{T}_i\}$, $\{\hat{N}_i\}$ such that for $i = 1, 2, \cdots$, one has the following. If

$$\Gamma_{in} \stackrel{\text{def}}{=} \{t \mid n\tilde{T}_i \leqq t \leqq (n + 1)\tilde{T}_i\}, \qquad n = 0, 1, \cdots, \tag{2.19}$$

then, for any $n \geqq \hat{N}_i$, there exists a closed interval $\hat{\Gamma}_{in}$ satisfying

$$\sup_{t\in\hat{\Gamma}_{in}} g(t) - \inf_{t\in\hat{\Gamma}_{in}} g(t) \leqq \nu_i, \tag{2.20}$$

$$\hat{\Gamma}_{in} \subset \Gamma_{in}, \quad m(\hat{\Gamma}_{in}) \geqq \hat{T}_i. \tag{2.21}$$

Choose any such $\{\tilde{T}_i\}$, $\{\hat{N}_i\}$, $\{\{\hat{\Gamma}_{in}\}\}$, and define $\tilde{H}_i$, $\hat{H}_i$ by

$$\tilde{H}_i = \liminf_{n\to\infty} H(t), \quad \hat{H}_i = \limsup_{n\to\infty} H(t), \qquad i = 1, 2, \cdots, \tag{2.22}$$

where the inf and sup are taken over $\bigcup_{m\geqq n} \hat{\Gamma}_{im}$. There exists a subsequence $\{i_k\}$ of $\{i\}$ such that $\lim_{i_k\to\infty} \tilde{H}_{i_k} = \tilde{H}$ exists, and a subsequence $\{i_{k_1}\}$ of $\{i_k\}$ such that $\lim_{i_{k_1}\to\infty} \hat{H}_{i_{k_1}} = \hat{H}$ exists. Without loss of generality, let both these subsequences equal the original sequence.

Clearly $\tilde{H} \leqq \hat{H}$ and our next purpose is now to show that $\tilde{H} < \hat{H}$ cannot possibly hold. This will occupy us until (2.46).

Suppose

$$\tilde{H} < \hat{H}, \tag{2.23}$$

let L be as in (1.10), and write $H(L) = \{h | h = H(x) \text{ for some } x \in L\}$. As the first step in showing that (2.23) leads to a contradiction we assert that

$$[\tilde{H}, \hat{H}] \subset H(L). \tag{2.24}$$

To prove (2.24) we assume the opposite. Thus, suppose there exists H' such that $\tilde{H} \leqq H' \leqq \hat{H}$, $H' \notin H(L)$. By the continuity of $H(x)$ and as L is closed, there exists $\delta > 0$ such that

$$|H(x) - H'| \geqq \delta, \quad x \in L. \tag{2.25}$$

Take any such δ which also satisfies $4[\hat{H} - \tilde{H}] \geqq \delta$, and define $[H_1, H_2]$ by

$$[H_1, H_2] = \left[H' - \frac{\delta}{4}, H' + \frac{\delta}{4}\right] \cap [\tilde{H}, \hat{H}]. \tag{2.26}$$

Then

$$H_2 - H_1 \geqq \delta/4, \tag{2.27}$$

$$\operatorname{dist}([H_1, H_2], H(L)) \geqq 3\delta/4. \tag{2.28}$$

Define $\{t'_{in}\}$, $\{t_{in}\}$, by $\hat{\Gamma}_{in} = \{t | t'_{in} \leqq t \leqq t_{in}\}$, $i = 1, 2, \cdots$, and take i large enough so that

$$[A + 1]\nu_i \leqq \delta/8. \tag{2.29}$$

Note that by (2.17) and (2.18) we may assume for any i

$$H(t_{in}) - H(t_{i,n-1}) \leqq [A + 1]\nu_i, \quad \text{if } n \geqq \hat{N}_i,$$

which together with (2.29) gives

$$H(t_{in}) - H(t_{i,n-1}) \leqq \delta/8 \tag{2.30}$$

for i sufficiently large and $n \geqq \hat{N}_i$.

By (1.1), (1.13), (1.17), (2.1), (2.18) and (2.20) we may take

$$\sup_{t \in \hat{\Gamma}_{in}} x(t) - \inf_{t \in \hat{\Gamma}_{in}} x(t) \leqq \nu_i, \quad \text{if } n \geqq \hat{N}_i, \tag{2.31}$$

and combining this with the definitions (2.6), (2.7), and with (2.20) one has, without loss of generality, for any i,

$$\sup_{t \in \hat{\Gamma}_{in}} H(t) - \inf_{t \in \hat{\Gamma}_{in}} H(t) \leqq \nu_i, \quad \text{if } n \geqq \hat{N}_i. \tag{2.32}$$

From (2.32), the way $\tilde{H}_i$, $\hat{H}_i$ were defined, and the fact that $\nu_i \to 0$, it follows that for each sufficiently large i there exist $\{t_{i,n_{il}}\}$, $\{t_{i,n_{im}}\} \subset \{t_{in}\}$ such that

$$|H(t_{i,n_{il}}) - \tilde{H}_i| \leqq \frac{\delta}{32}, \qquad |H(t_{i,n_{im}}) - \hat{H}_i| \leqq \frac{\delta}{32}.$$

For i large enough we certainly also have

$$|\tilde{H}_i - \tilde{H}| \leqq \frac{\delta}{32}, \qquad |\hat{H}_i - \hat{H}| \leqq \frac{\delta}{32},$$

and so,

$$(2.33)\qquad |H(t_{i,n_{il}}) - \tilde{H}| \leqq \frac{\delta}{16}, \qquad |H(t_{i,n_{im}}) - \hat{H}| \leqq \frac{\delta}{16}.$$

By (2.26), (2.27) and (2.33),

$$(2.34)\qquad \begin{gathered} m\left(\left[H_1 + \frac{\delta}{16}, H_2 - \frac{\delta}{16}\right]\right) \geqq \frac{\delta}{8}, \\ \left[H_1 + \frac{\delta}{16}, H_2 - \frac{\delta}{16}\right] \subset [H(t_{i,n_{il}}), H(t_{i,n_{im}})]. \end{gathered}$$

By (2.30) the maximal possible increase of $H(t)$ between the (right) endpoints of two consecutive intervals $\hat{\Gamma}_{n-1}$, $\hat{\Gamma}_n$ can be taken sufficiently small. By the second part of (2.34) there exist endpoints on which H assumes values $\leqq H_1 + \delta/16$, and endpoints on which H assumes values $\geqq H_2 - \delta/16$. Combining these facts with the first part of (2.34) allows us to conclude that for each sufficiently large i there exists $\{t_{i,n_{ik}}\} \subset \{t_{in}\}$ such that

$$(2.35)\qquad H_1 + \frac{\delta}{16} \leqq H(t_{i,n_{ik}}) \leqq H_2 - \frac{\delta}{16}.$$

From (2.28) and (2.35), it follows that

$$(2.36)\qquad \operatorname{dist}(H(t_{in_{ik}}), H(L)) \geqq 3\delta/4,$$

and by (1.10), the continuity of $H(x)$, and (2.36), there exists $\mu > 0$ such that

$$(2.37)\qquad |x(t_{in_{ik}}) + g(t_{in_{ik}})A - F| \geqq \mu.$$

Choose any such μ. Then take $t = t_{in_{ik}}$ in (1.1) and use (1.13), (1.17), (2.1) and (2.20). This yields

$$(2.38)\qquad |x(t_{in_{ik}}) + g(t_{in_{ik}})A - F| \leqq A v_i + 2M \int_{\hat{T}_i}^{\infty} a(\tau)\, d\tau + \varepsilon_{in_{ik}},$$

where $\varepsilon_{in_{ik}} \to 0$, $n_{ik} \to \infty$. But combining (2.37) and (2.38), and recalling that $v_i \to 0$, $\hat{T}_i \to \infty$, when $i \to \infty$, clearly provides a contradiction if i, n_{ik} are taken sufficiently large. Hence (2.24) holds, assuming (2.23).

Before proceeding, observe the following which makes it intuitively obvious why (2.23) and (2.24) will ultimately lead to a contradiction. Elementary calculations show that $H(x)$ is constant on any closed interval $[x_a, x_b] \subset L$, $x_a < x_b$. Thus, by (2.23) and (2.24), L must contain an infinite (even nondenumerable) number of points (or an infinite number of intervals) such that for any two such points x'_a, x'_b, $x'_a < x'_b$, one has $[x'_a, x'_b] \not\subset L$ (and for any two such intervals $[x'_a, x'_b]$, $[x'_c, x'_d]$, $x'_b < x'_c$, one has $[x'_b, x'_c] \not\subset L$).

Define $\tilde{x}$, $\hat{x}$, $E(x)$ by

$$(2.39)\qquad \tilde{x} = \liminf_{t \to \infty} x(t), \quad \hat{x} = \limsup_{t \to \infty} x(t),$$

$$(2.40)\qquad E(x) = \frac{1}{2A}[F - x]^2 + \int_0^x g(u)\, du, \qquad \tilde{x} \leqq x \leqq \hat{x}.$$

Clearly (1.14) implies $E(x) \in C^1[\tilde{x}, \hat{x}]$. Also note that from (1.10), (2.6) and (2.40), it follows that

$$E(x) = H(x), \qquad x \in L,$$

and so $E(L) = H(L)$, which together with (2.24) yields

$$m(E(L)) \geqq \hat{H} - \tilde{H}, \tag{2.41}$$

(m is the Lebesgue measure). Differentiating $E(x)$ and using (1.10) gives

$$E'(x) = 0, \qquad x \in L. \tag{2.42}$$

Define λ by $\lambda = \hat{x} - \tilde{x}$ and take ε such that $0 < \varepsilon \leqq (1/2\lambda)[\hat{H} - \tilde{H}]$. By the uniform continuity of $E'(x)$ on $[\tilde{x}, \hat{x}]$, there exists $\hat{\delta} > 0$ such that if $E'(\bar{x}) = 0$ for some $\bar{x} \in [\tilde{x}, \hat{x}]$, then

$$|E'(x)| \leqq \varepsilon, \qquad \bar{x} - \hat{\delta} \leqq x \leqq \bar{x} + \hat{\delta}. \tag{2.43}$$

Take any such $\hat{\delta} > 0$. Let N_λ be any integer satisfying $N_\lambda \hat{\delta} \geqq \lambda$, and divide $[\tilde{x}, \hat{x}]$ in N_λ equal parts by x_j:

$$\tilde{x} = x_0 < x_1 < x_2 < \cdots < x_{N_\lambda} = \hat{x}, \qquad x_j - x_{j-1} = \frac{\lambda}{N_\lambda} \leqq \hat{\delta}, \qquad j = 1, 2, \cdots, N_\lambda. \tag{2.44}$$

Observe that if for some j_0, $1 \leqq j_0 \leqq N_\lambda$, one has $[x_{j_0-1}, x_{j_0}] \cap L \neq \varnothing$, ($\varnothing$ denotes the empty set) then, from (2.42), (2.43), and the second part of (2.44),

$$|E'(x)| \leqq \varepsilon, \qquad x_{j_0-1} \leqq x \leqq x_{j_0}. \tag{2.45}$$

Define J and S by

$$J = \{j | 1 \leqq j \leqq N_\lambda, [x_{j-1}, x_j] \cap L \neq \varnothing\}, \qquad S = \bigcup_{j \in J} [x_{j-1}, x_j]. \tag{2.46}$$

S may be written as the union of $\hat{N}_\lambda$ disjoint closed intervals $[\tilde{x}'_k, x'_k]$, $k = 1, 2, \cdots, \hat{N}_\lambda$. Note that $L \subset S$ and consequently

$$E(L) \subset E(S). \tag{2.47}$$

Recalling (2.45) and (2.46) we obtain $|E'(x)| \leqq \varepsilon$, $x \in S$, and so

$$\sum_{1 \leqq k \leqq \hat{N}_\lambda} \left[\sup_{\tilde{x}'_k \leqq x \leqq x'_k} E(x) - \inf_{\tilde{x}'_k \leqq x \leqq x'_k} E(x) \right] \leqq \sum_{1 \leqq k \leqq \hat{N}_\lambda} \varepsilon[x'_k - \tilde{x}'_k] \leqq \varepsilon\lambda,$$

which yields

$$m(E(S)) \leqq \varepsilon\lambda. \tag{2.48}$$

But by (2.47), (2.48), and the way ε was chosen,

$$m(E(L)) \leqq m(E(S)) \leqq \varepsilon\lambda \leqq \tfrac{1}{2}[\hat{H} - \tilde{H}], \tag{2.49}$$

which violates (2.41). From this contradiction we finally deduce that (2.23) cannot possibly hold. Thus let

$$H_0 \stackrel{\text{def}}{=} \tilde{H} = \hat{H}. \tag{2.50}$$

But this implies that there exists $\{\nu_i'\}$, $\nu_i' \to 0$, $i \to \infty$, such that without loss of generality we may assume

$$(2.51)\qquad |H(t) - H_0| \leqq \nu_i', \qquad t \in \bigcup_{n \geqq \hat{N}_i} \hat{\Gamma}_{in}, \quad i = 1, 2, \cdots.$$

Once we have (2.51) the remaining part of the proof is fairly straightforward. Suppose (1.8) is not satisfied. Then there exist constants T', ν' and sequences $\{\tau_p\}$, $\{\tau_p'\}$, $\tau_p, \tau_p' \to \infty$, $p \to \infty$, such that

$$(2.52)\qquad |x(\tau_p) - x(\tau_p')| \geqq 3\nu' > 0, \qquad 0 < \tau_p - \tau_p' \leqq T', \quad p = 1, 2, \cdots.$$

Choose $\hat{\nu}$ so that

$$(2.53)\qquad \hat{\nu}T'[a(0) + 2] \leqq \nu'.$$

From (2.20) it follows that we can take

$$(2.54)\qquad \sup_{t \in \hat{\Gamma}_{in}} g^2(t) - \inf_{t \in \hat{\Gamma}_{in}} g^2(t) \leqq \nu_i, \qquad i = 1, 2, \cdots, \quad n \geqq \hat{N}_i.$$

By (2.1), the second part of (2.6), (2.18), (2.21), (2.51) and (2.54),

$$\left| G(t_{in}) + \frac{1}{2}\int_0^{t_{in}} g^2(t_{in} - \tau)a(\tau)\,d\tau - G(t_{i,n-1}) - \frac{1}{2}\int_0^{t_{i,n-1}} g^2(t_{i,n-1} - \tau)a(\tau)\,d\tau \right|$$

$$(2.55)\qquad \leqq |H(t_{in}) - H(t_{i,n-1})| + A\nu_i + M^2 \int_{\hat{T}_i}^{\infty} a(\tau)\,d\tau$$

$$\leqq 2\nu_i' + A\nu_i + \frac{\nu_i}{2}, \qquad i = 1, 2, \cdots, \quad n \geqq \hat{N}_i.$$

Invoking (1.18) and (2.1) gives

$$(2.56)\qquad 2\int_{t_{i,n-1}}^{t_{in}} f'(\tau)g(\tau)\,d\tau \leqq 2M \int_{t_{i,n-1}}^{t_{in}} |f'(\tau)|\,d\tau \leqq \nu_i,$$

for $i = 1, 2, \cdots$, and $n \geqq \hat{N}_i$, if $\hat{N}_i$ is chosen sufficiently large. Combining (1.11), (1.12), (2.55) and (2.56) with (2.8) (where we let $t = t_{in}, t_{i,n-1}$ and then take differences) yields

$$(2.57)\qquad -\int_{t_{i,n-1}}^{t_{in}} \left\{ \int_0^{\tau} [g(\tau) - g(\tau - s)]^2\,da(s) \right\} d\tau \leqq 4\nu_i' + 2\nu_i[A + 1],$$

for $i = 1, 2, \cdots$, and $n \geqq \hat{N}_i$. By (1.12), (1.13), (2.1) and (2.5),

$$|x'(t)| \leqq \left| a(0)g(t) + \int_0^t g(t - \tau)\,da(\tau) \right| + |f'(t)|$$

$$(2.58)\qquad \leqq -\int_0^t |g(t) - g(t - \tau)|\,da(\tau) + \hat{\nu} + |f'(t)|$$

$$\leqq -\frac{1}{\hat{\nu}}\int_0^t |g(t) - g(t - \tau)|^2\,da(\tau) + \hat{\nu}a(0) + \hat{\nu} + |f'(t)|,$$

a.e. on $\tau \leqq t < \infty$, for any sufficiently large τ. From the second part of (2.52), (2.58), and the absolute continuity of x we obtain

$$\tag{2.59} \begin{aligned} |x(\tau_p) - x(\tau_p')| \leqq \int_{\tau_p'}^{\tau_p} |x'(s)|\, ds \leqq -\frac{1}{\hat{v}} \int_{\tau_p'}^{\tau_p} \left\{ \int_0^\tau [g(\tau - s) - g(\tau)]^2\, da(s) \right\} d\tau \\ + \hat{v}T'[a(0) + 1] + \int_{\tau_p'}^{\tau_p} |f'(s)|\, ds, \end{aligned}$$

if p is sufficiently large. Suppose that for some i_0, p_0 there exists $n_0 \geqq \hat{N}_{i_0}$ such that τ'_{p_0} is an interior point of $\hat{\Gamma}_{i_0,n_0-1}$. Recalling (2.31) and the fact that $v_i \to 0$, we immediately have that in such a case we may, without loss of generality, move τ'_{p_0} to t_{i_0,n_0-1} (i.e., let τ'_{p_0} equal the right endpoint of $\hat{\Gamma}_{i_0,n_0-1}$) and still have (2.52) for $p = p_0$. Hence, by the second part of (2.21), the second part of (2.52), and taking $T' \leqq \hat{T}_i$, we conclude that for any i, p there exists an n such that $[\tau'_p, \tau_p] \subset [t_{i,n-1}, t_{in}]$. But using this fact, together with (1.12), (1.18), the first part of (2.52), and (2.57) in (2.59) gives that if p is sufficiently large, then

$$\tag{2.60} 3v' \leqq \frac{1}{\hat{v}}[4v_i' + 2v_i[A + 1]] + \hat{v}T'[a(0) + 2].$$

However, combining (2.53) with (2.60) certainly provides a contradiction for sufficiently large i, as both $v_i, v_i' \to 0$, $i \to \infty$. Thus (1.8) follows.

To prove that (1.9) is satisfied we assume the opposite. Thus, suppose there exists $\tilde{\delta}$ and $\{s_p\}$, $s_p \to \infty$, such that $\operatorname{dist}(x(s_p), L) \geqq \tilde{\delta} > 0$. Take $\{s_p\}$ so that $\lim_{p\to\infty} x(s_p) = x_s$ exists and define $g_s = \lim_{p\to\infty} g(s_p)$. Evidently $x_s \notin L$ and so there exists $\tilde{\mu} > 0$ such that

$$\tag{2.61} |x_s + Ag_s - F| \geqq \tilde{\mu}.$$

By (1.8) and (1.14),

$$\tag{2.62} |x(t) - x_s| \leqq K\tilde{\delta}, \quad |g(t) - g_s| \leqq K\tilde{\delta}, \quad s_p - T_1 \leqq t \leqq s_p,$$

for any constants K, T_1 if p is sufficiently large. But making use of (1.13), (1.17), (2.1), (2.62) in (1.1), and taking K sufficiently small, T_1 sufficiently large, immediately produces a contradiction to (2.61). Hence (1.9) is satisfied.

This completes the proof of Theorem 2.

3. Proof of Lemma 1. Consider (1.1) under hypotheses (1.2)–(1.7). From (1.5) and (1.7) it follows that

$$\tag{3.1} g(x(t)) \in L^\infty(0, \infty).$$

Define $y(t)$ by

$$\tag{3.2} y(t) = \int_0^t a(t - \tau)g(x(\tau))\, d\tau, \qquad 0 \leqq t < \infty.$$

Clearly (1.1) and (3.2) imply

$$\tag{3.3} x(t) + y(t) = f(t), \qquad 0 \leqq t < \infty.$$

Define $\hat{x}(t)$, $\hat{y}(t)$ by

$$\hat{x}(t) = x(t) - f(t) + F, \qquad 0 \leqq t < \infty, \tag{3.4}$$

$$\hat{y}(t) = \int_0^t a(t-\tau)g(\hat{x}(\tau))\,d\tau, \qquad 0 \leqq t < \infty. \tag{3.5}$$

Combining (1.5), the first part of (1.6), (1.7) and (3.4) yields

$$\hat{x}(t) \in L^\infty(0, \infty), \tag{3.6}$$

$$g(\hat{x}(t)) \in L^\infty(0, \infty), \tag{3.7}$$

and from (1.4), (3.1), (3.7), and Lemma 2 one has

$$y(t) \in C[0, \infty), \qquad \hat{y}(t) \in C[0, \infty). \tag{3.8}$$

By (3.3) and (3.4) one may obviously write

$$\hat{x}(t) + \hat{y}(t) = \hat{f}(t), \qquad 0 \leqq t < \infty, \tag{3.9}$$

if

$$\hat{f}(t) \stackrel{\text{def}}{=} \hat{y}(t) - y(t) + F, \qquad 0 \leqq t < \infty. \tag{3.10}$$

We assert that

$$\hat{f}(t) \in C[0, \infty), \qquad \lim_{t\to\infty} \hat{f}(t) = F. \tag{3.11}$$

The first part of (3.11) follows from (3.8) and (3.10). For the second part we note that invoking (1.6) and (3.4) gives

$$\lim_{t\to\infty} [\hat{x}(t) - x(t)] = 0, \tag{3.12}$$

and by (1.5), (3.6) and (3.12),

$$\lim_{t\to\infty} [g(\hat{x}(t)) - g(x(t))] = 0. \tag{3.13}$$

But (1.4), (3.1), (3.2), (3.5), (3.7), (3.10) and (3.13) yield the second part of (3.11).

Clearly (1.5), the second part of (3.8), (3.9) and the first part of (3.11) imply

$$\hat{x}(t) \in C[0, \infty), \tag{3.14}$$

$$g(\hat{x}(t)) \in C[0, \infty). \tag{3.15}$$

Define $\tilde{x}(t)$, $\tilde{y}(t)$ by

$$\tilde{x}(t) = \hat{x}(t) - \hat{f}(t) + F, \qquad 0 \leqq t < \infty, \tag{3.16}$$

$$\tilde{y}(t) = \int_0^t a(t-\tau)g(\tilde{x}(\tau))\,d\tau, \qquad 0 \leqq t < \infty. \tag{3.17}$$

From (1.5), (3.6), (3.11), (3.14) and (3.16) one obtains

$$\sup_{0\leqq t<\infty} |\tilde{x}(t)| < \infty, \tag{3.18}$$

$$\sup_{0 \leqq t < \infty} |g(\tilde{x}(t))| < \infty. \tag{3.19}$$

Recalling (3.9) and (3.16) one easily verifies that

$$\tilde{x}(t) + \tilde{y}(t) = \tilde{f}(t), \qquad 0 \leqq t < \infty, \tag{3.20}$$

if

$$\tilde{f}(t) \stackrel{\text{def}}{=} \tilde{y}(t) - \hat{y}(t) + F, \qquad 0 \leqq t < \infty. \tag{3.21}$$

Note that by (3.17) one has that the equation (3.20) is of the same type as the original equation (1.1). However, as we show next, $\tilde{f}(t)$ is considerably smoother than the hypothesis (1.6) postulates. Thus we claim that

$$\tilde{f}(t) \quad \text{is absolutely continuous on } [0, \infty), \tag{3.22}$$

$$\tilde{f}'(t) \in L^\infty(0, \infty), \tag{3.23}$$

$$\lim_{t \to \infty} \operatorname*{ess\,sup}_{t \leqq \tau < \infty} |\tilde{f}'(\tau)| = 0, \tag{3.24}$$

$$\lim_{t \to \infty} \tilde{f}(t) = F. \tag{3.25}$$

To prove (3.22) and (3.23) we observe that (1.5), the first part of (3.11), (3.14) and (3.16) yield $g(\tilde{x}(t)) \in C[0, \infty)$. Combining this fact with (1.2), (1.3), (3.7), (3.15), (3.19), (3.21) and Lemma 3 immediately gives (3.22), (3.23) and

$$\tilde{f}'(t) = a(0)[g(\tilde{x}(t) - g(\hat{x}(t))] + \int_0^t [g(\tilde{x}(t - \tau)) - g(\hat{x}(t - \tau))] \, da(\tau), \tag{3.26}$$

a.e. on $0 \leqq t < \infty$. To prove (3.24) and (3.25) we notice that the second part of (3.11), and (3.16) imply

$$\lim_{t \to \infty} [\tilde{x}(t) - \hat{x}(t)] = 0, \tag{3.27}$$

and so, on account of (1.5) and (3.18),

$$\lim_{t \to \infty} [g(\tilde{x}(t)) - g(\hat{x}(t))] = 0. \tag{3.28}$$

But from (1.4), (3.26) and (3.28), one has (3.24), and from (1.4), (3.5), (3.17), (3.21) and (3.28), we obtain (3.25).

To conclude the proof we now observe that by (3.12), (3.17), (3.20), (3.22)–(3.25) and (3.27) one has Lemma 1.

4. Proof of Theorem 3. Assume at first that we also have

$$f(t) \in C[0, \infty), \tag{4.1}$$

$$x(t) \in LBV[0, \infty). \tag{4.2}$$

Clearly (2.1) is satisfied. Define $y(t)$ as in (2.2). By (1.13), (2.1) and Lemma 2 one obtains $y(t) \in C[0, \infty)$. Combining this with (2.3) and (4.1) gives $x(t) \in C[0, \infty)$. From (1.19), (1.22) and (4.2) it then follows that ($g(t)$ as in (2.7))

$$g(t) \in C[0, \infty) \cap LBV[0, \infty). \tag{4.3}$$

Define, for any $\varepsilon > 0$, a_ε, h_ε, x_ε by

(4.4) $$a_\varepsilon(t) = a(\varepsilon), \qquad 0 \leqq t < \varepsilon, \qquad a_\varepsilon(t) = a(t), \qquad \varepsilon \leqq t < \infty,$$

(4.5) $$h_\varepsilon(t) = \int_0^t [a(\tau) - a_\varepsilon(\tau)]g(t-\tau)\,d\tau, \qquad 0 \leqq t < \infty,$$

(4.6) $$x_\varepsilon(t) = x(t) + h_\varepsilon(t), \qquad 0 \leqq t < \infty.$$

By (1.13), (1.21), (2.1), (4.4), (4.5) and Lemma 2,

(4.7) $$\sup_{0 \leqq t < \infty} |h_\varepsilon(t)| < \infty, \qquad \text{where the bound is uniform for } \varepsilon > 0,$$

(4.8) $$\lim_{\varepsilon \to 0} \sup_{0 \leqq t < \infty} |h_\varepsilon(t)| = 0, \qquad h_\varepsilon(t) \in C[0, \infty).$$

Combining (1.13), (1.21), (4.3), (4.4), (4.5) and [5, Thm. 11.2b, p. 85] yields

(4.9) $$h_\varepsilon(t) \in LBV[0, \infty) \quad \text{for any } \varepsilon > 0,$$

and

(4.10) $$\lim_{\varepsilon \to 0} V(h_\varepsilon, I) = 0$$

for any compact interval $I \subset [0, \infty)$. From the continuity of x, (1.19), (4.2), (4.6), (4.7), (4.8), (4.9) and (4.10), it follows that

(4.11) $$\sup_{0 \leqq t < \infty} |x_\varepsilon(t)| < \infty, \qquad \text{uniformly for } \varepsilon > 0,$$

(4.12) $$\lim_{\varepsilon \to 0} \sup_{0 \leqq t < \infty} |x_\varepsilon(t) - x(t)| = 0, \qquad x_\varepsilon(t) \in C[0, \infty) \cap LBV[0, \infty),$$

and

(4.13) $$V(x_\varepsilon, I) \leqq V(x, I) + 1 < \infty,$$

for an arbitrary compact interval I if ε is sufficiently small.

Define g_ε for $\varepsilon > 0$ by $g_\varepsilon(t) = g(x_\varepsilon(t))$, $0 \leqq t < \infty$. Obviously (1.22), (4.11), (4.12) and (4.13) imply, for some constant M independent of ε,

(4.14) $$g_\varepsilon(t) \in C[0, \infty) \cap LBV[0, \infty),$$

(4.15) $$\sup_{0 \leqq t < \infty} |g_\varepsilon(t)| \leqq M < \infty,$$

(4.16) $$\lim_{\varepsilon \to 0} \sup_{0 \leqq t < \infty} |g_\varepsilon(t) - g(t)| = 0,$$

(4.17) $$V(g_\varepsilon, I) \leqq K_1[V(x, I) + 1],$$

for an arbitrary compact interval I if ε is sufficiently small. Clearly the constant K_1 depends neither on ε nor on I. Next define y_ε, $\hat{y}_\varepsilon$ for $\varepsilon > 0$ and $0 \leqq t < \infty$ by

(4.18) $$y_\varepsilon(t) = \int_0^t a_\varepsilon(t-\tau)g_\varepsilon(\tau)\,d\tau, \qquad \hat{y}_\varepsilon(t) = \int_0^t a_\varepsilon(t-\tau)g(\tau)\,d\tau,$$

and let $f_\varepsilon \overset{\text{def}}{=} y_\varepsilon - \hat{y}_\varepsilon$. By (1.13), (4.4), (4.16) and (4.18),

(4.19) $$\lim_{\varepsilon \to 0} \sup_{0 \leqq t < \infty} |f_\varepsilon(t)| = 0.$$

From (1.20), (1.21), (2.1), (4.3), (4.4), (4.14), (4.15), (4.18) and Lemma 3 it follows that

$$y_\varepsilon, f_\varepsilon \text{ are absolutely continuous on } [0, \infty), \tag{4.20}$$

and as one easily verifies (use (1.1), (4.5), (4.6) and (4.18)) that

$$x_\varepsilon + y_\varepsilon = f_\varepsilon + f, \tag{4.21}$$

one also has that $x_\varepsilon - f$ is absolutely continuous on $[0, \infty)$.

Invoking (1.24) and (4.1) gives that f is uniformly continuous on $[0, \infty)$ which together with (4.20) and (4.21) implies that the same is true for x_ε. Combining this last fact with (1.22) and (4.11) yields the uniform continuity of g_ε on $[0, \infty)$ for any $\varepsilon > 0$. However, using also (4.16) one immediately obtains somewhat more, namely:

$$\begin{gathered}\text{For any } \tilde{\varepsilon} > 0 \text{ there exist positive constants } \tilde{\delta}, \varepsilon_0 \text{ such that if} \\ |t_1 - t_2| \leqq \tilde{\delta},\ 0 < \varepsilon \leqq \varepsilon_0, \text{ then } |g_\varepsilon(t_1) - g_\varepsilon(t_2)| \leqq \tilde{\varepsilon}.\end{gathered} \tag{4.22}$$

Let A, $G(x)$, $H(x)$, $G(t)$, $H(t)$ be as in (2.6), (2.7) and define $G_\varepsilon(t)$, $H_\varepsilon(t)$ for $\varepsilon > 0$ and $0 \leqq t < \infty$ by $G_\varepsilon(t) = G(x_\varepsilon(t))$, $H_\varepsilon(t) = H(x_\varepsilon(t))$. From (4.20) and (4.21) one has, for any $\varepsilon > 0$,

$$[x_\varepsilon(t) - f(t)]' + a_\varepsilon(0)g_\varepsilon(t) + \int_0^t g_\varepsilon(t - \tau)\, da_\varepsilon(\tau) = f'_\varepsilon(t), \quad \text{a.e. on } 0 \leqq t < \infty,$$

which upon multiplication by $g_\varepsilon(t)$, integration, and after invoking (1.23), the second part of (4.12) and (4.14), yields the following analogue of (2.8):

$$\begin{aligned} G_\varepsilon(t) - G_\varepsilon(0) &+ \frac{1}{2}\int_0^t g_\varepsilon^2(t - \tau)a_\varepsilon(\tau)\, d\tau + \frac{1}{2}\int_0^t g_\varepsilon^2(\tau)a_\varepsilon(\tau)\, d\tau \\ &- \frac{1}{2}\int_0^t \left\{ \int_0^\tau [g_\varepsilon(\tau) - g_\varepsilon(\tau - s)]^2\, da_\varepsilon(s) \right\} d\tau \\ &= \int_0^t g_\varepsilon(\tau) f'_\varepsilon(\tau)\, d\tau + \int_0^t g_\varepsilon(\tau)\, df(\tau). \end{aligned} \tag{4.23}$$

By obvious reasoning one obtains a formula corresponding to (2.9):

$$\begin{aligned} &-\frac{1}{2}\int_{\tilde{t}_1}^{\tilde{t}_2} \left\{ \int_0^\tau [g_\varepsilon(\tau) - g_\varepsilon(\tau - s)]^2\, da_\varepsilon(s) \right\} d\tau \\ &\leqq K + \left| \int_{\tilde{t}_1}^{\tilde{t}_2} g_\varepsilon(\tau) f'_\varepsilon(\tau)\, d\tau \right| + \left| \int_{\tilde{t}_1}^{\tilde{t}_2} g_\varepsilon(\tau)\, df(\tau) \right| \\ &\leqq K + |f_\varepsilon(\tilde{t}_2)g_\varepsilon(\tilde{t}_2) - f_\varepsilon(\tilde{t}_1)g_\varepsilon(\tilde{t}_1)| + \left| \int_{\tilde{t}_1}^{\tilde{t}_2} f_\varepsilon(\tau)\, dg_\varepsilon(\tau) \right| + \left| \int_{\tilde{t}_1}^{\tilde{t}_2} g_\varepsilon(\tau)\, df(\tau) \right|, \end{aligned} \tag{4.24}$$

for some a priori K and any $\tilde{t}_1, \tilde{t}_2$; $0 \leqq \tilde{t}_1 < \tilde{t}_2 < \infty$. ((4.14) and (4.20) justify the integration by parts.) Observe the following facts concerning (4.24). By (1.24) and (4.15), the last term in (4.24) can be made arbitrarily small (independently of ε) by taking $\tilde{t}_1$, $\tilde{t}_2$ sufficiently large but requiring $\tilde{t}_2 - \tilde{t}_1 \leqq$ some fixed $T < \infty$.

From (4.15), (4.17) and (4.19) it follows that those terms in the last third of (4.24) which contain f_ε can be made arbitrarily small for any fixed interval $[\tilde{t}_1, \tilde{t}_2]$ by taking ε sufficiently small.

To continue our proof we need the following Lemma 5, the proof of which follows the proof of Lemma 4 fairly closely, except for evident alterations. The basic facts needed in the proof of Lemma 5 are (1.21), (4.22), (4.24) and the observations made in the preceding paragraph after (4.24).

LEMMA 5. *Let the hypothesis of Theorem* 3 *hold and let* v, $\hat{T}$ *be arbitrary positive constants. There exist a positive constant* $\tilde{T}$, *a sequence* $\{\varepsilon_n\}$, $\varepsilon_n > 0$, *and an integer* $\hat{N}$ *such that if*

$$\Gamma_n \overset{\text{def}}{=} \{t | n\tilde{T} \leqq t \leqq (n+1)\tilde{T}\}, \qquad n = 0, 1, \cdots, \tag{4.25}$$

then for each $n \geqq \hat{N}$ *there exists a closed interval* $\hat{\Gamma}_n$ *satisfying* $(g_0(t) \overset{\text{def}}{=} g(t))$

$$\sup_{t \in \hat{\Gamma}_n} g_\varepsilon(t) - \inf_{t \in \hat{\Gamma}_n} g_\varepsilon(t) \leqq v, \quad \textit{for } 0 \leqq \varepsilon \leqq \varepsilon_n, \tag{4.26}$$

$$\hat{\Gamma}_n \subset \Gamma_n, \quad m(\hat{\Gamma}_n) \geqq \hat{T}. \tag{4.27}$$

The remaining part of the proof of Theorem 3 very much parallels the corresponding part of the proof of Theorem 2 (except of course for the discussion below regarding the validity of (4.1), (4.2)). In what follows we consequently only indicate the necessary changes.

Repeating the arguments which gave (2.17) and using Lemma 5, one at first obtains the following. For any pair of positive constants v, $\hat{T}$ such that $v \geqq 2M^2 \int_{\hat{T}}^{\infty} a(\tau)\, d\tau$, there exist sequences $\{\hat{\Gamma}_n\}$, $\{\varepsilon_n\}$ satisfying (4.25), (4.26) and (4.27), such that for $0 < \varepsilon \leqq \varepsilon_n$ and sufficiently large n one has

$$H_\varepsilon(t_n) - H_\varepsilon(t_{n-1}) \leqq [A + 3/4]v, \tag{4.28}$$

where $\{t_n\}$ is defined by $\hat{\Gamma}_n = \{t | t'_n \leqq t \leqq t_n\}$. But clearly (4.28) and

$$\lim_{\varepsilon \to 0} \sup_{0 \leqq t < \infty} |H_\varepsilon(t) - H(t)| = 0, \tag{4.29}$$

(which holds) yield (2.17).

We next choose sequences $\{v_i\}$, $\{\hat{T}_i\}$ satisfying (2.18) and corresponding sequences $\{\tilde{T}_i\}$, $\{\hat{N}_i\}$, $\{\{\hat{\Gamma}_{in}\}\}$, $\{\{\varepsilon_{in}\}\}$ such that for $i = 1, 2, \cdots; 0 \leqq \varepsilon \leqq \varepsilon_{in}$, and $n \geqq \hat{N}_i$ one has

$$\sup_{t \in \hat{\Gamma}_{in}} g_\varepsilon^k(t) - \inf_{t \in \hat{\Gamma}_{in}} g_\varepsilon^k(t) \leqq v_i, \qquad k = 1, 2, \tag{4.30}$$

where $\hat{\Gamma}_{in}$ satisfies (2.19), (2.21). The definitions (2.22) carry over unchanged which entails that the arguments between (2.22) and (2.51) (where $a(0) < \infty$ is not needed) can be repeated verbatim. Consequently (2.51) holds. But by (2.51) and (4.29) there exist $\{\varepsilon_i\}$ such that for $i = 1, 2, \cdots$, and $0 < \varepsilon \leqq \varepsilon_i$ one has (H_0 as in (2.50))

$$|H_\varepsilon(t) - H_0| \leqq 2v'_i, \qquad t \in \bigcup_{n \geqq \hat{N}_i} \hat{\Gamma}_{in}. \tag{4.31}$$

Choose any such sequence $\{\varepsilon_i\}$ and without loss of generality let $\varepsilon_{in} \leqq \varepsilon_i$, for $i = 1, 2, \cdots$, and $n \geqq \hat{N}_i$.

Suppose (2.52) holds. By the first part of (4.12) this implies that there exists $\tilde{\eta} > 0$ such that if $0 < \varepsilon \leqq \tilde{\eta}$, then

$$|x_\varepsilon(\tau_p) - x_\varepsilon(\tau_p')| \geqq 2v' > 0. \tag{4.32}$$

Take any such $\tilde{\eta}$ and then choose $\hat{\eta} > 0$ such that for $\varepsilon > 0$,

$$\sup_{0 \leqq t < \infty} 4\int_{t-\hat{\eta}}^{t} a_\varepsilon(t - \tau)|g_\varepsilon(\tau)|\, d\tau \leqq v', \tag{4.33}$$

and in the remainder of the proof restrict ε to the interval $(0, \min(\hat{\eta}, \tilde{\eta})]$. Define α by

$$\alpha(t) = a(\hat{\eta}), \qquad 0 \leqq t < \hat{\eta}, \qquad \alpha(t) = a(t), \qquad \hat{\eta} \leqq t < \infty. \tag{4.34}$$

Then

$$a_\varepsilon(t) = \alpha(t), \qquad \hat{\eta} \leqq t < \infty. \tag{4.35}$$

Let $\hat{v}$ be any positive number satisfying

$$2\hat{v}T'[\alpha(0) + 2] \leqq v'. \tag{4.36}$$

Taking $k = 2$ in (4.30) and using (2.18), (2.21), (4.15) and (4.31) yields

$$\begin{aligned} &\left| G_\varepsilon(t_{in}) + \frac{1}{2}\int_0^{t_{in}} g_\varepsilon^2(t_{in} - \tau)a_\varepsilon(\tau)\, d\tau - G_\varepsilon(t_{i,n-1}) \right. \\ &\qquad \left. - \frac{1}{2}\int_0^{t_{i,n-1}} g_\varepsilon^2(t_{i,n-1} - \tau)a_\varepsilon(\tau)\, d\tau \right| \leqq 4v_i' + Av_i + \frac{v_i}{2}, \\ &\qquad\qquad i = 1, 2, \cdots, \quad n > \hat{N}_i, \quad 0 < \varepsilon \leqq \varepsilon_{in}, \end{aligned} \tag{4.37}$$

and where $\{t_{in}\}$ is defined as after (2.28). Consider the integrals

$$\left| 4\int_{t_{i,n-1}}^{t_{in}} g_\varepsilon(\tau)f_\varepsilon'(\tau)\, d\tau \right|, \qquad \left| 4\int_{t_{i,n-1}}^{t_{in}} g_\varepsilon(\tau)\, df(\tau) \right|, \tag{4.38}$$

for $i = 1, 2, \cdots$, $n > \hat{N}_i$ and $0 < \varepsilon \leqq \varepsilon_{in}$. From the observations made after (4.29) there exist $\{\varepsilon_i\}$ such that for $i = 1, 2, \cdots$, and $0 < \varepsilon \leqq \varepsilon_i$ one has (H_0 as integral can be made arbitrarily small (in particular $\leqq v_i$) by taking $\hat{N}_i$ sufficiently large. Remembering the integration by parts in the inequality (4.24), and still invoking the observations following (4.24), one also realizes that for each (i, n)-pair one may take the first integral in (4.38) $\leqq v_i$. (For this it suffices to take each ε_{in} sufficiently small.) Now let $t = t_{in}, t_{i,n-1}$ in (4.23), take differences, use (1.20), (1.21), (4.37), and the preceding estimates for the integrals in (4.38), and finally (4.34). This gives

$$-\int_{t_{i,n-1}}^{t_{in}} \left\{ \int_0^\tau [g_\varepsilon(\tau) - g_\varepsilon(\tau - s)]^2\, d\alpha(s) \right\} d\tau \leqq 8v_i' + 2Av_i + 2v_i, \tag{4.39}$$

for $i = 1, 2, \cdots, n > \hat{N}_i$, and $0 < \varepsilon \leqq \varepsilon_{in}$.

Define z_ε for $\varepsilon > 0$ by

$$z_\varepsilon(t) = \int_0^t \alpha(t - \tau) g_\varepsilon(\tau)\, d\tau, \qquad 0 \leqq t < \infty. \tag{4.40}$$

From (4.14), (4.34) and Lemma 3 the absolute continuity of $z_\varepsilon(t)$ follows. Differentiating (4.40), estimating and integrating over $[\tau'_p, \tau_p]$ therefore yields, (recall how (2.59) was deduced)

$$|z_\varepsilon(\tau_p) - z_\varepsilon(\tau'_p)| \leqq -\frac{1}{\hat{v}} \int_{\tau'_p}^{\tau_p} \left\{ \int_0^\tau [g_\varepsilon(\tau - s) - g_\varepsilon(\tau)]^2\, d\alpha(s) \right\} d\tau + \hat{v} T'[\alpha(0) + 1], \tag{4.41}$$

for $\varepsilon > 0$ and sufficiently large p. Define u_ε for $\varepsilon > 0$ by $u_\varepsilon = y_\varepsilon - z_\varepsilon$. By (1.21), (4.18), (4.33), (4.34), (4.35) and (4.40) one then has $4 \sup_{0 \leqq t < \infty} |u_\varepsilon(t)| \leqq v'$, and so $2|u_\varepsilon(\tau_p) - u_\varepsilon(\tau'_p)| \leqq v'$. Consequently,

$$2|y_\varepsilon(\tau_p) - y_\varepsilon(\tau'_p)| \leqq v' + 2|z_\varepsilon(\tau_p) - z_\varepsilon(\tau'_p)|. \tag{4.42}$$

From (1.24), (4.19) and (4.21) the existence of $\hat{\varepsilon} > 0$ follows such that if $0 < \varepsilon \leqq \hat{\varepsilon}$, then for all sufficiently large p,

$$|x_\varepsilon(\tau_p) - x_\varepsilon(\tau'_p)| \leqq |y_\varepsilon(\tau_p) - y_\varepsilon(\tau'_p)| + \hat{v} T'. \tag{4.43}$$

As in the proof of Theorem 2 we have that for any (i, p) there exists an n such that $[\tau'_p, \tau_p] \subset [t_{i,n-1}, t_{in}]$. Therefore, to obtain a contradiction it suffices to combine (1.21), (4.32), (4.36), (4.39), (4.41), (4.42) and (4.43), and to remember that $v_i, v'_i \to 0$, if $i \to \infty$. Thus we have (1.8) and (1.9), assuming (4.1) and (4.2) are satisfied.

To complete the proof we show that Theorem 3 may easily be reduced to a problem where in addition to the full hypothesis of Theorem 3 one also has (4.1), and that (4.2) follows from the hypothesis of Theorem 3.

Recall the first paragraph in the proof of Lemma 1. The statements in this paragraph clearly imply that the reduction mentioned above is equivalent to showing that ($\hat{f}$ as in (3.10))

$$\hat{f}(t) \in LBV[0, \infty), \tag{4.44}$$

$$\lim_{t \to \infty} V(\hat{f}, [t - T, t]) = 0 \quad \text{for any } T > 0. \tag{4.45}$$

By definition $\hat{f}$ may be written

$$\hat{f}(t) = \int_0^t a(t - \tau) \hat{g}(\tau)\, d\tau + F, \qquad 0 \leqq t < \infty, \tag{4.46}$$

if $\hat{g}(t) \stackrel{\text{def}}{=} g(\hat{x}(t)) - g(x(t))$. From (1.13), (1.19), (1.22), (1.23), (3.4), (3.6)—of course now $\sup_{0 \leqq t < \infty} |\hat{x}(t)| < \infty$—(4.46) and [5, Thm. 11.2b, p. 85],

$$V(\hat{f}, [0, T]) \leqq K V(f, [0, T]) + \tilde{K} < \infty, \tag{4.47}$$

for some a priori constants K, $\tilde{K}$ and any $T > 0$. Hence (4.44) follows.

We prove next that (4.45) holds. Suppose not. Then there exist $\delta > 0$, $\{t_n\}$, $t_n \to \infty$, and $T > 0$ such that

$$V(\hat{f}, [t_n - T, t_n]) \geqq \delta > 0, \qquad n = 1, 2, \cdots. \tag{4.48}$$

By (1.13) and (1.24) there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a sequence $\{T_{n_k}\}$ such that

$$T_{n_k} \to \infty, \quad a(T_{n_k}) \to 0, \quad n_k \to \infty, \qquad 0 \leqq t_{n_k} - T_{n_k} - T, \tag{4.49}$$

$$V(f, [t_{n_k} - T_{n_k} - T, t_{n_k}]) \to 0, \qquad n_k \to \infty. \tag{4.50}$$

Take any such $\{t_{n_k}\}$, $\{T_{n_k}\}$ and write $\hat{f}$ as follows:

$$\hat{f}(t) = \int_{t-T_{n_k}}^{t} + \int_0^{t-T_{n_k}} a(t-\tau)\hat{g}(\tau)\,d\tau + F, \qquad T_{n_k} \leqq t < \infty. \tag{4.51}$$

Call the first integral in (4.51) h_{n_k}, the second $\hat{h}_{n_k}$, and define $\hat{a}_{n_k}$, $\hat{g}_{n_k}$, b_{n_k} by

$$\hat{a}_{n_k}(\tau) = a(\tau), \qquad 0 < \tau \leqq T_{n_k}; \qquad \hat{a}_{n_k}(\tau) = 0, \qquad T_{n_k} < \tau \leqq t_{n_k}, \tag{4.52}$$

$$\begin{aligned} &\hat{g}_{n_k}(\tau) = \hat{g}(\tau) - \hat{g}(t_{n_k} - T_{n_k} - T), \qquad t_{n_k} - T_{n_k} - T \leqq \tau \leqq t_{n_k}, \\ &\hat{g}_{n_k}(\tau) = 0, \qquad 0 \leqq \tau < t_{n_k} - T_{n_k} - T, \end{aligned} \tag{4.53}$$

$$b_{n_k}(t) = \int_0^t \hat{a}_{n_k}(t-\tau)\hat{g}_{n_k}(\tau)\,d\tau, \qquad 0 \leqq t \leqq t_{n_k}. \tag{4.54}$$

Observe that these definitions imply

$$b_{n_k}(t) = h_{n_k}(t) - \hat{g}(t_{n_k} - T_{n_k} - T)A(T_{n_k}), \qquad t_{n_k} - T \leqq t \leqq t_{n_k},$$

$(A(t) \overset{\text{def}}{=} \int_0^t a(\tau)\,d\tau)$ which in turn gives

$$V(b_{nk}, [t_{n_k} - T, t_{n_k}]) = V(h_{nk}, [t_{n_k} - T, t_{n_k}]). \tag{4.55}$$

But clearly (4.52), (4.53), (4.54) and [5, Thm. 11.2b, p. 85], yield

$$\begin{aligned} V(b_{n_k}, [t_{n_k} - T, t_{n_k}]) &\leqq V(b_{n_k}, [0, t_{n_k}]) \leqq AV(\hat{g}_{n_k}, [0, t_{n_k}]) \\ &= AV(\hat{g}_{n_k}, [t_{n_k} - T_{n_k} - T, t_{n_k}]) = AV(\hat{g}, [t_{n_k} - T_{n_k} - T, t_{n_k}]) \\ &\leqq KAV(f, [t_{n_k} - T_{n_k} - T, t_{n_k}]), \end{aligned} \tag{4.56}$$

where the last inequality follows from (1.19), (1.22), (3.6) and the definition of $\hat{g}$. Hence, by (4.50), (4.55) and (4.56),

$$\lim_{n_k \to \infty} V(h_{n_k}, [t_{n_k} - T, t_{nk}]) = 0. \tag{4.57}$$

Our following step will be to show that (4.57) holds with h_{n_k} replaced by $\hat{h}_{n_k}$. Integrating by parts and using Fubini's theorem gives

$$\begin{aligned} \hat{h}_{n_k}(t) &= -\int_{T_{n_k}}^{t} a(\tau)\,d\left[\int_0^{t-\tau} \hat{g}(s)\,ds\right] \\ &= a(T_{n_k})\int_0^{t-T_{n_k}} \hat{g}(s)\,ds + \int_{T_{n_k}}^{t} \left\{\int_{T_{n_k}}^{s} \hat{g}(s-\tau)\,da(\tau)\right\} ds, \end{aligned} \tag{4.58}$$

which allows us to conclude that $\hat{h}_{n_k}(t)$ is absolutely continuous. (Note that the calculations in (4.58) are formally justified as $\hat{g}$ is locally of bounded variation and as $V(a, [T_{n_k}, \infty)) < \infty$.) However, combining (3.1), (3.7) and (4.49), the fact that $V(a, [T_{n_k}, \infty)) \to 0$, if $n_k \to \infty$, with (4.58) also gives

$$\lim_{n_k \to \infty} \operatorname*{ess\,sup}_{T_{n_k} \leqq t < \infty} |\hat{h}'_{n_k}(t)| = 0. \tag{4.59}$$

Therefore $V(\hat{h}_{n_k}, [t_{n_k} - T, t_{n_k}]) \to 0$, if $n_k \to \infty$. But this, together with (4.51) and (4.57), violates (4.48) and consequently (4.45) is obtained.

To demonstrate (4.2) we only need observe the following. Let $\hat{K}$ be such that $|g(x(t_1)) - g(x(t_2))| \leqq \hat{K}|x(t_1) - x(t_2)|$, $0 \leqq t_1, t_2 < \infty$, and choose $\varepsilon > 0$ such that $\hat{K}A(\varepsilon) < 1$. Then, by (1.1), (1.23) and [5, Thm. 11.2b, p. 85],

$$V(x, [0, \varepsilon]) \leqq [V(f, [0, \varepsilon]) + A(\varepsilon)|g(x(0))|][1 - \hat{K}A(\varepsilon)]^{-1} < \infty.$$

Continuing in steps of size ε yields (4.2).

5. Proof of Lemma 4. Let ν, $\hat{T}$ be arbitrary positive constants. By (1.11) and (1.12) we may write

$$a(t) = b(t) + c(t), \qquad 0 \leqq t < \infty, \tag{5.1}$$

where

$$b(t) \in C[0, \infty), \tag{5.2}$$

$$b(t) \geqq 0, \qquad 0 \leqq t < \infty, \tag{5.3}$$

$$b(t) \text{ nonincreasing on } [0, \infty), \qquad b(0) < \infty, \tag{5.4}$$

and (t_k are the discontinuity points of $a(t)$),

$$c(t) = \sum_{t \leqq t_k} [a(t_k) - a(t_k + 0)], \qquad 0 \leqq t < \infty, \tag{5.5}$$

$$c(t) \geqq 0, \qquad 0 \leqq t < \infty, \tag{5.6}$$

$$c(t) \text{ nonincreasing on } [0, \infty), \qquad c(0) < \infty. \tag{5.7}$$

As the proof of Lemma 4 is now essentially different depending on whether $b(t) \not\equiv 0$ or not, we consider the two cases separately, in I and II respectively.

I. Suppose

$$b(t) \not\equiv 0, \tag{5.8}$$

and define ε by $4\varepsilon = \nu$. From (2.4) the existence of $\delta > 0$ follows such that

$$|g(t) - g_0| \leqq \varepsilon \quad \text{on } t_0 - \delta \leqq t \leqq t_0 + \delta, \tag{5.9}$$

for any t_0, $\delta \leqq t_0 < \infty$. (We adopt the obvious notation $g_0 = g(t_0)$, $\hat{g}_n = g(\hat{t}_n)$, etc.) Choose any such $\delta > 0$. By (1.13), (5.4) and (5.8) there exists $[\eta_1, \eta_2]$ satisfying

$$0 < \eta_1 < \eta_2; \quad \eta_2 - \eta_1 \leqq \delta; \quad b(\eta_1) - b(\eta_2) > 0. \tag{5.10}$$

Let $[\eta_1, \eta_2]$ be any such interval.

In the next two paragraphs we work out some preliminaries and in particular we establish the existence of two disjoint intervals $[\alpha_1, \alpha_2]$, $[\beta_1, \beta_2]$ satisfying (5.17), such that $\alpha_2 - \alpha_1$ and $\beta_2 - \beta_1$ are sufficiently small compared to the distance between these intervals.

By (5.4), the last part of (5.10), and the uniform continuity of $b(t)$ on $[\eta_1, \eta_2]$, it is not hard to show that there exist two intervals $[\alpha_1', \alpha_2']$, $[\beta_1', \beta_2']$ such that

$$0 < \eta_1 \leqq \alpha_1' < \alpha_2' < \beta_1' < \beta_2' \leqq \eta_2, \tag{5.11}$$

$$b(\alpha_1') - b(\alpha_2') > 0, \quad b(\beta_1') - b(\beta_2') > 0. \tag{5.12}$$

Choose any two such intervals and define γ' by $2\gamma' = \beta_1' - \alpha_2'$. Then let N be any positive integer such that

$$\hat{T} \leqq 2N\gamma', \tag{5.13}$$

and let δ_0 be any positive constant which satisfies

$$N\delta_0 \leqq \gamma'. \tag{5.14}$$

By (5.4), (5.11) and (5.12) there exist intervals $[\alpha_1, \alpha_2]$, $[\beta_1, \beta_2]$ such that

$$\alpha_1' \leqq \alpha_1 < \alpha_2 \leqq \alpha_2', \qquad \beta_1' \leqq \beta_1 < \beta_2 \leqq \beta_2', \tag{5.15}$$

$$0 < \alpha_2 - \alpha_1 \leqq \delta_0, \qquad 0 < \beta_2 - \beta_1 \leqq \delta_0, \tag{5.16}$$

$$b(\alpha_1) - b(\alpha_2) > 0, \qquad b(\beta_1) - b(\beta_2) > 0. \tag{5.17}$$

Take any two such intervals and define α, β, γ, ω by

$$2\alpha = \alpha_1 + \alpha_2, \qquad 2\beta = \beta_1 + \beta_2, \tag{5.18}$$

$$2\gamma = \beta_2 - \alpha_1, \qquad \omega = \min(b(\beta_1) - b(\beta_2), b(\alpha_1) - b(\alpha_2)). \tag{5.19}$$

By the middle third of (5.10), (5.11), (5.14), (5.15), (5.18) and the first part of (5.19) one has

$$0 < 2N\delta_0 \leqq 2\gamma' \leqq \beta_1 - \alpha_2 < \beta - \alpha < 2\gamma \leqq \delta. \tag{5.20}$$

Thus (think about N as being a large integer) $\alpha_2 - \alpha_1$ and $\beta_2 - \beta_1$ have been chosen quite small compared to the distance between the intervals $[\alpha_1, \alpha_2]$, $[\beta_1, \beta_2]$. Also note that (5.9) and (5.20) give

$$|g(t) - g_0| \leqq \varepsilon \quad \text{on } t_0 - \gamma - N\delta_0 \leqq t \leqq t_0 + \gamma + N\delta_0, \tag{5.21}$$

for any t_0, $\delta \leqq t_0 < \infty$.

By (2.4) there exist positive constants δ_i, $i = 1, 2, \cdots, N$, satisfying

$$2\delta_i \leqq \delta_0, \tag{5.22}$$

$$|g(t) - g_0| \leqq \varepsilon 2^{-i-1}, \qquad t_0 - \delta_i \leqq t \leqq t_0, \tag{5.23}$$

for $i = 1, 2, \cdots, N$, and any $t_0, \delta_0 \leqq t_0 < \infty$. Choose any such δ_i.

After these preliminaries our goal will be to show that for each $i = 1, 2, \cdots, N$, there exist a sequence $\{t_{in}\}$ and a constant T_i such that

$$\begin{gathered} [t_{in} - i\beta, t_{in} - i\alpha] \subset [nT_i, (n+1)T_i], \\ |g(t) - g_{in}| < 2\varepsilon, \qquad t_{in} - i\beta \leqq t \leqq t_{in} - i\alpha, \end{gathered} \tag{5.24}$$

if n is sufficiently large. This will be done successively for increasing i, beginning with $i = 1$. Note that once we have (5.24) for $i = N$, then, as $N[\beta - \alpha] > 2N\gamma' \geqq \hat{T}$, and as $4\varepsilon = \nu$, we also have Lemma 4, assuming (5.8).

To attain our goal we repeatedly use (2.9), combining this inequality with (1.18) and (2.4) in the form of (5.21) and (5.23), the first part of (5.4), and with (5.8) in the form of (5.17).

Define S_1 by $S_1 = \{m\}$, $m = 1, 2, \cdots$. We claim that there exist a positive constant T_1, and an integer N_1 such that if

$$\Gamma_{1n} \overset{\text{def}}{=} \{t | nT_1 \leqq t \leqq (n+1)T_1, t \in S_1\}, \qquad n = 0, 1, 2, \cdots, \tag{5.25}$$

then for each $n \geqq N_1$ there exists at least one element of Γ_{1n} (call this element t_{1n}) such that for some points $\hat{t}_{1n}$, $\hat{\hat{t}}_{1n}$ satisfying

$$t_{1n} - \beta_2 - \delta_1 \leqq \hat{t}_{1n} \leqq t_{1n} - \beta_1, \qquad t_{1n} - \alpha_2 - \delta_1 \leqq \hat{\hat{t}}_{1n} \leqq t_{1n} - \alpha_1, \tag{5.26}$$

one has

$$|\hat{g}_{1n} - g_{1n}| \leqq \frac{\varepsilon}{2} \quad \text{and} \quad |\hat{\hat{g}}_{1n} - g_{1n}| \leqq \frac{\varepsilon}{2}. \tag{5.27}$$

Suppose not. Then, no matter how large T_1 is taken, we can find a subsequence $\{n_k\}$ of $\{n\}$ such that for every $k = 0, 1, \cdots$, the next sentence is valid. For any element $t_{1n_k} \in \Gamma_{1n_k}$ the inequality $|g(t) - g_{1n_k}| > \varepsilon/2$ holds on at least one of the intervals

$$t_{1n_k} - \beta_2 - \delta_1 \leqq t \leqq t_{1n_k} - \beta_1, \qquad t_{1n_k} - \alpha_2 - \delta_1 \leqq t \leqq t_{1n_k} - \alpha_1. \tag{5.28}$$

From (5.4), (5.17), (5.23) with $i = 1$ and $t_0 = t_{1n_k}$, and (5.28), one then obtains, for example,

$$\begin{aligned} -\int_0^t [g(t) - g(t-\tau)]^2\, db(\tau) &\geqq \frac{\varepsilon^2}{16}[b(t - t_{1n_k} + \beta_1) - b(t - t_{1n_k} + \beta_2 + \delta_1)] \\ &\geqq \frac{\varepsilon^2}{16}[b(\beta_1) - b(\beta_2)] > 0, \end{aligned} \tag{5.29}$$

$$t_{1n_k} - \delta_1 \leqq t \leqq t_{1n_k}, \quad k = 0, 1, \cdots.$$

Recalling (5.1), (5.4) and (5.7) we realize that (2.9) is still a true statement although $a(s)$ is replaced by $b(s)$. Therefore, taking in (2.9), $\tilde{t}_1 = n_k T_1$, $\tilde{t}_2 = \tilde{t}_1 + T_1$, and also invoking (5.4), the second part of (5.19) and (5.29) gives

$$\frac{\omega \varepsilon^2 \delta_1}{16} \rho_{1n_k} \leqq 2M \int_{n_k T_1}^{n_k T_1 + T_1} |f'(\tau)|\, d\tau + K, \qquad k = 0, 1, \cdots,$$

where ρ_{1n_k} = the number of elements $\in \Gamma_{1n_k}$. By (5.25) we may surely assume $2\rho_{1n_k} \geqq T_1$, and so

$$\frac{\omega \varepsilon^2 \delta_1}{32} \leqq 2M \operatorname*{ess\,sup}_{n_k T_1 \leqq \tau < \infty} |f'(\tau)| + K T_1^{-1}, \qquad k = 0, 1, \cdots. \tag{5.30}$$

But by (1.18) this provides a contradiction if T_1, k are taken sufficiently large. Consequently our claim is proved.

Take any T_1, N_1, $\{t_{1n}\}$, $\{\hat{t}_{1n}\}$, $\{\hat{\hat{t}}_{1n}\}$ satisfying (5.26) and (5.27), and define S_2 by $S_2 = \{(\hat{t}_{1n}, \hat{\hat{t}}_{1n})\}$. Note that concerning the location of the points $\hat{t}_{1n}$, $\hat{\hat{t}}_{1n}$ we can only infer that they satisfy (5.26). Analogously (see below) the points

t'_{2n}, t''_{2n}, t'''_{2n} are only known to satisfy (5.36). This limited knowledge, which of course becomes more troublesome with increasing i (i as in (5.24)), forms the reason for including the term $N\delta_0$ in (5.21). With the aid of this term we are able to circumvent the inconveniences caused by conditions like (5.26) and (5.36).

Clearly, by (5.21) and (5.27),

$$(5.31)\qquad |g(t) - g_{1n}| \leqq \frac{3\varepsilon}{2}, \qquad \begin{aligned} &\hat{t}_{1n} - \gamma - N\delta_0 \leqq t \leqq \hat{t}_{1n} + \gamma + N\delta_0, \\ &\hat{\hat{t}}_{1n} - \gamma - N\delta_0 \leqq t \leqq \hat{\hat{t}}_{1n} + \gamma + N\delta_0, \end{aligned}$$

where $n = 0, 1, \cdots$. But combining (5.26) and (5.31) gives $|g(t) - g_{1n}| \leqq 3\varepsilon/2$ on

$$(5.32)\qquad \begin{aligned} &t_{1n} - \beta_1 - \gamma - N\delta_0 \leqq t \leqq t_{1n} - \beta_2 + \gamma - \delta_1 + N\delta_0, \\ &t_{1n} - \alpha_1 - \gamma - N\delta_0 \leqq t \leqq t_{1n} - \alpha_2 + \gamma - \delta_1 + N\delta_0. \end{aligned}$$

By (5.16), (5.18), (5.22), and as $N \geqq 1$, one has that the intervals in (5.33) are subintervals of those in (5.32) and thus $|g(t) - g_{1n}| \leqq 3\varepsilon/2$ holds on

$$(5.33)\quad t_{1n} - \beta - \gamma \leqq t \leqq t_{1n} - \beta + \gamma, \qquad t_{1n} - \alpha - \gamma \leqq t \leqq t_{1n} - \alpha + \gamma.$$

From (5.20) we obtain that for any n the two intervals in (5.33) partially overlap and hence we may surely write

$$(5.34)\qquad |g(t) - g_{1n}| \leqq \frac{3\varepsilon}{2}, \qquad t_{1n} - \beta \leqq t \leqq t_{1n} - \alpha, \quad n = 0, 1, \cdots.$$

Finally note that $\beta - \alpha > 2\gamma'$ and so the length of any of the intervals in (5.34) is $> 2\gamma'$. Therefore, by (5.13), (5.34), and the fact that $4\varepsilon = v$, we have Lemma 4 assuming (5.8) and $N = 1$. (Or equivalently, we have (5.24) for $i = 1$.) Thus, in what remains of I, take $N \geqq 2$.

Continuing our construction we next assert that there exist a positive constant T_2, and an integer N_2, such that if

$$(5.35)\qquad \Gamma_{2n} \stackrel{\text{def}}{=} \{(s, \tau) | nT_2 \leqq s, \tau \leqq (n+1)T_2, (s, \tau) \in S_2\}, \qquad n = 0, 1, \cdots,$$

then for each $n \geqq N_2$ there exists at least one element of Γ_{2n} (call this element $(\hat{t}_{2n}, \hat{\hat{t}}_{2n})$) such that for some points $t'_{2n}, t''_{2n}, t'''_{2n}$ satisfying

$$(5.36)\qquad \begin{aligned} &\hat{t}_{2n} - \beta_2 - \delta_2 \leqq t'_{2n} \leqq \hat{t}_{2n} - \beta_1, \qquad \hat{t}_{2n} - \alpha_2 - \delta_2 \leqq t''_{2n} \leqq \hat{t}_{2n} - \alpha_1, \\ &\hat{\hat{t}}_{2n} - \alpha_2 - \delta_2 \leqq t'''_{2n} \leqq \hat{\hat{t}}_{2n} - \alpha_1, \end{aligned}$$

one has

$$(5.37)\qquad |g'_{2n} - \hat{g}_{2n}| \leqq \frac{\varepsilon}{4}, \quad |g''_{2n} - \hat{g}_{2n}| \leqq \frac{\varepsilon}{4}, \quad |g'''_{2n} - \hat{\hat{g}}_{2n}| \leqq \frac{\varepsilon}{4}.$$

Suppose not. Then, no matter how large T_2 is taken, we can find a subsequence $\{n_l\}$ of $\{n\}$ such that for $l = 0, 1, \cdots$, the next sentence holds. For any element $(\hat{t}_{2n_l}, \hat{\hat{t}}_{2n_l}) \in \Gamma_{2n_l}$ at least one of (5.38a)–(5.38c) is true:

$$(5.38a)\qquad |g(t) - \hat{g}_{2n_l}| > \frac{\varepsilon}{4}, \qquad \hat{t}_{2n_l} - \beta_2 - \delta_2 \leqq t \leqq \hat{t}_{2n_l} - \beta_1,$$

(5.38b) $$|g(t) - \hat{g}_{2n_l}| > \frac{\varepsilon}{4}, \qquad \hat{t}_{2n_l} - \alpha_2 - \delta_2 \leqq t \leqq \hat{t}_{2n_l} - \alpha_1,$$

(5.38c) $$|g(t) - \hat{\hat{g}}_{2n_l}| > \frac{\varepsilon}{4}, \qquad \hat{\hat{t}}_{2n_l} - \alpha_2 - \delta_2 \leqq t \leqq \hat{\hat{t}}_{2n_l} - \alpha_1.$$

By (5.4), (5.17), the second part of (5.19), (5.23) with $i = 2$ and $t_0 = \hat{t}_{2n_l}$ or $\hat{\hat{t}}_{2n_l}$ and (5.38), one then obtains, for any element $(\hat{t}_{2n_l}, \hat{\hat{t}}_{2n_l}) \in \Gamma_{2n_l}$,

(5.39) $$-\int_0^t [g(t) - g(t-\tau)]^2\, db(\tau) \geqq \frac{\omega\varepsilon^2}{64},$$

everywhere on at least one of the intervals

(5.40) $$[\hat{t}_{2n_l} - \delta_2, \hat{t}_{2n_l}], \quad [\hat{\hat{t}}_{2n_l} - \delta_2, \hat{\hat{t}}_{2n_l}], \qquad l = 0, 1, \cdots.$$

In (2.9) let $\tilde{t}_1 = n_l T_2$, $\tilde{t}_2 = \tilde{t}_1 + T_2$, and replace $a(s)$ by $b(s)$. Then, by (5.4), (5.39) and (5.40),

(5.41) $$\frac{\omega\varepsilon^2\delta_2}{64}\rho_{2n_l} \leqq 2MT_2 \operatorname*{ess\,sup}_{n_l T_2 \leqq \tau < \infty} |f'(\tau)| + K, \qquad l = 0, 1, \cdots,$$

where ρ_{2n_l} = the number of elements $\in \Gamma_{2n_l}$. Surely we may take $2\rho_{2n_l} \geqq T_2 T_1^{-1}$ and therefore, invoking also (1.18), one observes that (5.41) yields a contradiction if T_2, l are chosen sufficiently large. This proves our assertion.

Note that we demonstrated the truth of the assertion without recourse to anything in the paragraph containing (5.31)–(5.34). In fact, both this paragraph and the paragraph which contains (5.42)–(5.48) follow as by-products and are not necessary for the remaining part of the proof of Lemma 4. However, they do give clarifying interim results.

Take any T_2, N_2, $\{(\hat{t}_{2n}, \hat{\hat{t}}_{2n})\}$, $\{(t'_{2n}, t''_{2n}, t'''_{2n})\}$, satisfying (5.36) and (5.37).

By (5.35) we have that for each n there exists an integer $p(n)$ such that $\hat{t}_{2n} = \hat{t}_{1p(n)}$, $\hat{\hat{t}}_{2n} = \hat{\hat{t}}_{1p(n)}$. Take the sequence $\{t_{1p(n)}\}$ (clearly a subsequence of S_1) and call it $\{t_{2n}\}$. Then, from (5.26) for $n = 0, 1, \cdots$,

(5.42) $$t_{2n} - \beta_2 - \delta_1 \leqq \hat{t}_{2n} \leqq t_{2n} - \beta_1, \qquad t_{2n} - \alpha_2 - \delta_1 \leqq \hat{\hat{t}}_{2n} \leqq t_{2n} - \alpha_1,$$

and by (5.27),

(5.43) $$|\hat{g}_{2n} - g_{2n}| \leqq \frac{\varepsilon}{2}, \quad |\hat{\hat{g}}_{2n} - g_{2n}| \leqq \frac{\varepsilon}{2}, \qquad n = 0, 1, \cdots.$$

Using (5.42) to estimate $\hat{t}_{2n}$, $\hat{\hat{t}}_{2n}$ in (5.36) yields for $n = 0, 1, \cdots$,

(5.44) $$\begin{aligned} t_{2n} - 2\beta_2 - \delta_1 - \delta_2 &\leqq t'_{2n} \leqq t_{2n} - 2\beta_1, \\ t_{2n} - \alpha_2 - \beta_2 - \delta_1 - \delta_2 &\leqq t''_{2n} \leqq t_{2n} - \alpha_1 - \beta_1, \\ t_{2n} - 2\alpha_2 - \delta_1 - \delta_2 &\leqq t'''_{2n} \leqq t_{2n} - 2\alpha_1, \end{aligned}$$

and recalling (5.37) and (5.43) we obtain for $n = 0, 1, \cdots$,

(5.45) $$|g'_{2n} - g_{2n}| \leqq \frac{3\varepsilon}{4}, \quad |g''_{2n} - g_{2n}| \leqq \frac{3\varepsilon}{4}, \quad |g'''_{2n} - g_{2n}| \leqq \frac{3\varepsilon}{4}.$$

Taking in (5.21) successively $t_0 = t'_{2n}, t''_{2n}, t'''_{2n}$, and invoking also (5.44) and (5.45) readily gives $|g(t) - g_{2n}| \leqq 7\varepsilon/4$, $n = 0, 1, \cdots$, on

$$\begin{aligned} &t_{2n} - 2\beta_1 - \gamma - N\delta_0 \leqq t \leqq t_{2n} - 2\beta_2 + \gamma - \delta_1 - \delta_2 + N\delta_0. \\ (5.46)\quad &t_{2n} - \alpha_1 - \beta_1 - \gamma - N\delta_0 \leqq t \leqq t_{2n} - \alpha_2 - \beta_2 + \gamma - \delta_1 - \delta_2 + N\delta_0, \\ &t_{2n} - 2\alpha_1 - \gamma - N\delta_0 \leqq t \leqq t_{2n} - 2\alpha_2 + \gamma - \delta_1 - \delta_2 + N\delta_0. \end{aligned}$$

From (5.16), (5.18) and (5.22), and as $N \geqq 2$, we perceive that the intervals in (5.47) are subintervals of those in (5.46), and thus $|g(t) - g_{2n}| \leqq 7\varepsilon/4$ holds on

$$\begin{aligned} &t_{2n} - 2\beta - \gamma \leqq t \leqq t_{2n} - 2\beta + \gamma, \\ (5.47)\qquad &t_{2n} - \alpha - \beta - \gamma \leqq t \leqq t_{2n} - \alpha - \beta + \gamma, \\ &t_{2n} - 2\alpha - \gamma \leqq t \leqq t_{2n} - 2\alpha + \gamma. \end{aligned}$$

For any n we have by (5.20) that the three intervals in (5.47) partially overlap and so it is certainly true that

$$(5.48)\qquad |g(t) - g_{2n}| \leqq 7\varepsilon/4, \quad t_{2n} - 2\beta \leqq t \leqq t_{2n} - 2\alpha, \qquad n = 0, 1, \cdots.$$

Note that the length of any of the intervals in (5.48) is $> 4\gamma'$. Therefore, by (5.13), (5.48), and the fact that $4\varepsilon = v$, we arrive at Lemma 4 (assuming (5.8)) if $N = 2$. One may thus take $N \geqq 3$.

By now it is evident how to continue the proof. One simply repeats the arguments leading from (5.25) to (5.34) or those, basically the same, going from (5.35) to (5.48). Of course, the arguments have to be slightly modified at each step to allow for the increasing number of sequences one has to keep count of. After working through these arguments N times one observes that the result corresponding to (5.34) or to (5.48) is that there exist $\{t_{Nn}\}$ and a constant T_N such that

$$\begin{aligned} &[t_{Nn} - N\beta, t_{Nn} - N\alpha] \subset [nT_N, (n+1)T_N], \\ &|g(t) - g_{Nn}| \leqq [2 - 2^{-N}]\varepsilon < 2\varepsilon, \qquad t_{Nn} - N\beta \leqq t \leqq t_{Nn} - N\alpha, \end{aligned}$$

for n sufficiently large. Take T_N as the $\tilde{T}$ of Lemma 4. Then, as $N[\beta - \alpha] > 2N\gamma' \geqq \hat{T}$, and as $4\varepsilon = v$, we clearly have Lemma 4, assuming (5.8).

II. Suppose $b(t) \equiv 0$. The saltus function $c(t)$ may be written

$$(5.49)\qquad c(t) = \sum_{k=1}^{\infty} c_k(t), \qquad 0 \leqq t < \infty,$$

where

$$(5.50)\qquad c_k(t) = \begin{cases} a(t_k) - a(t_k + 0) \overset{\text{def}}{=} a_k > 0, & 0 \leqq t \leqq t_k, \\ 0, & t_k < t < \infty, \end{cases}$$

and where

$$(5.51)\qquad \sum_{k=1}^{\infty} a_k < \infty.$$

(Naturally, certain modifications are necessary if the series in (5.49) contains only a finite number of terms.) Relations (2.1) and (5.51) give the uniform convergence which justifies writing (2.2) as

$$y(t) = \sum_{k=1}^{\infty} \int_0^t c_k(t-\tau)g(\tau)\,d\tau, \qquad 0 \leqq t < \infty. \tag{5.52}$$

Define $f_k(t)$, $k = 1, 2, \cdots$, on $[0, \infty)$ by $f_k(t) = \int_0^t c_k(t-\tau)g(\tau)\,d\tau$. Conditions (2.1) and (5.50) yield that each $f_k(t)$ is absolutely continuous on $[0, \infty)$ and

$$f_k'(t) = \begin{cases} a_k g(t), & 0 \leqq t \leqq t_k, \\ a_k[g(t) - g(t-t_k)], & t_k \leqq t < \infty. \end{cases} \tag{5.53}$$

From (5.52) it follows that $y(t) = \sum_{k=1}^{\infty} f_k(t)$. By (2.1), (5.51) and (5.53) one has that $\sum_{k=1}^{\infty} f_k'(t)$ converges uniformly on $[0, \infty)$. Therefore

$$y(t) = \sum_{k=1}^{\infty} f_k(t) = \sum_{k=1}^{\infty} \int_0^t f_k'(\tau)\,d\tau = \int_0^t \sum_{k=1}^{\infty} f_k'(\tau)\,d\tau,$$

and so

$$y'(t) = \sum_{k=1}^{\infty} f_k'(t) \quad \text{a.e. on } [0, \infty). \tag{5.54}$$

From (2.9), (5.49), (5.50), and as $a(t) \equiv c(t)$ we obtain, for any $\tilde{t}_1$, $\tilde{t}_2$ such that $0 \leqq \tilde{t}_1 < \tilde{t}_2 < \infty$,

$$\int_{\tilde{t}_1}^{\tilde{t}_2} \sum_k \{a_k[g(\tau) - g(\tau - t_k)]^2\}\,d\tau \leqq 2M \int_{\tilde{t}_1}^{\tilde{t}_2} |f'(\tau)|\,d\tau + K, \tag{5.55}$$

where the sum extends over all k such that $\tau - t_k \geqq 0$.

By (1.14), (1.18), (1.19), (2.3), and the absolute continuity of x, y, f, there exists $\mu > 0$ such that if $|y'(t)| \leqq \mu$ a.e. on an interval $[t_a, t_b]$ satisfying $t_b - t_a = \hat{T}$, t_a sufficiently large, then

$$\sup_{[t_a,t_b]} g(t) - \inf_{[t_a,t_b]} g(t) \leqq \nu. \tag{5.56}$$

Take any such $\mu > 0$. Then choose k_0 so that

$$2\left|\sum_{k>k_0} f_k'(t)\right| \leqq \mu \quad \text{a.e. on } 0 \leqq t < \infty. \tag{5.57}$$

By (2.1), (5.51) and (5.53) this is possible. Let $T_0 = \max_{1 \leqq k \leqq k_0} t_k$. Recalling (5.53), (5.54), (5.57) gives

$$2|y'(t)| \leqq 2 \sum_{k=1}^{k_0} a_k |g(t) - g(t-t_k)| + \mu \quad \text{a.e. on } T_0 \leqq t < \infty. \tag{5.58}$$

As $a_k > 0$, $k = 1, 2, \cdots$, relation (5.55) implies

$$\int_{\tilde{t}_1}^{\tilde{t}_2} \sum_{k=1}^{k_0} a_k[g(\tau) - g(\tau - t_k)]^2\,d\tau \leqq 2M \int_{\tilde{t}_1}^{\tilde{t}_2} |f'(\tau)|\,d\tau + K, \tag{5.59}$$

for any $\tilde{t}_1, \tilde{t}_2$ satisfying $T_0 \leqq \tilde{t}_1 < \tilde{t}_2 < \infty$.

Suppose that, no matter how large $\tilde{T}$ is taken, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that in any subinterval (having length $\hat{T}$) of Γ_{n_i} (where Γ_{n_i} is defined by (2.10)) there exists at least one point $\tilde{t}$ having the property that for some k, $1 \leqq k \leqq k_0$,

$$|g(\tilde{t}) - g(\tilde{t} - t_k)| > \mu/(2a_k k_0). \tag{5.60}$$

By (2.4) and (5.60) there exists $\tilde{\delta} > 0$ such that

$$|g(t) - g(t - t_k)| > \mu/(4a_0 k_0), \qquad \tilde{t} - \tilde{\delta} \leqq t \leqq \tilde{t} + \tilde{\delta}, \tag{5.61}$$

where $a_0 = \max_{1 \leqq k \leqq k_0} a_k$. But using at first (1.18) and (5.61) in (5.59), then taking $\tilde{T}$, n_i sufficiently large, clearly gives a contradiction.

Thus there exist $\tilde{T}$, $\hat{N}$ such that if $n \geqq \hat{N}$, $1 \leqq k \leqq k_0$, then

$$|g(t) - g(t - t_k)| \leqq \mu/(2a_k k_0),$$

everywhere on an interval $\hat{\Gamma}_n$ satisfying $\hat{\Gamma}_n \subset \Gamma_n$, $m(\hat{\Gamma}_n) \geqq \hat{T}$, Γ_n defined by (2.10). By (5.58) this implies $|y'(t)| \leqq \mu$ a.e. on $\hat{\Gamma}_n$, and remembering in addition how μ was chosen, one immediately realizes that Lemma 4 holds if $b(t) \equiv 0$.

This completes the proof of Lemma 4.

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ANALYTICAL METHODS FOR A SINGULAR PERTURBATION PROBLEM IN A SECTOR*

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Abstract. From the exact solution of an elliptic boundary value problem in a sector, asymptotic approximations with respect to a small parameter are derived. The asymptotic expansion is uniformly valid in the boundary layers. Also the phenomena for the case of almost characteristic boundaries are discussed.

1. Introduction. In [4], the author considered a singular perturbation problem for an elliptic equation in a quarter-plane. The exact solution of the equation was represented as a contour integral and from this representation the asymptotic solution was derived by using saddle point methods.

In this paper we consider the same equation

$$\varepsilon\Delta\Phi_\varepsilon(x, y) - \frac{\partial}{\partial y}\Phi_\varepsilon(x, y) = 0, \tag{1.1}$$

the domain of definition now being an arbitrary sector shaped domain in the x, y-plane

$$A = \{r, \phi | r \geqq 0, 0 \leqq \phi \leqq \alpha\}. \tag{1.2}$$

In (1.1) ε is a small positive parameter and Δ is Laplace's operator; in (1.2) r and ϕ are polar coordinates, where

$$x = r\cos\phi, \quad y = r\sin\phi \tag{1.3}$$

and

$$0 < \alpha \leqq 2\pi. \tag{1.4}$$

The case $\alpha = \frac{1}{2}\pi$ (the quarter-plane) is discussed in [4].

Along the boundary of the sector A, the function Φ_ε is subjected to the following boundary conditions:

$$\Phi_\varepsilon(x, 0) = 0, \quad \Phi_\varepsilon(x, y) = 1 \quad \text{if } \phi = \alpha. \tag{1.5}$$

In order to investigate the asymptotic behavior for small values of ε, the exact solution of (1.1) is determined from which the asymptotic approximations are derived. Also the various types of boundary layers are discussed, for instance the "free" (i.e., internal) boundary layer in the case of an obtuse angle α. Finally, the case of an almost right angle will be considered.

2. The solution of the boundary value problem. We shall first remove the first order derivative in equation (1.1) by substituting

$$\Phi_\varepsilon(x, y) = 1 - e^{\omega y}F(x, y), \qquad \omega = 1/(2\varepsilon). \tag{2.1}$$

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Then the function F has to satisfy the following boundary value problem:

$$\begin{aligned} &\Delta F(x,y) - \omega^2 F(x,y) = 0, \\ &F(x,0) = 1, \quad F(x,y) = 0 \quad \text{if } \phi = \alpha. \end{aligned} \tag{2.2}$$

In general, the solution of an elliptic equation in an unbounded domain is not unique. But, by imposing a condition upon F concerning its growth at infinity, uniqueness can be ensured. We prove the following lemma. The function $I_0(\omega r)$ appearing in the lemma is a modified Bessel function satisfying $\Delta u - \omega^2 u = 0$, $I_0(\omega r) > 0$, $I_0(\omega r) = \exp(\omega r)/\sqrt{2\pi\omega r}\,(1 + O(r^{-1}))$ as $r \to \infty$.

LEMMA. *Assume that u is a regular function in A satisfying*: (i) $\Delta u - \omega^2 u = 0$, (ii) $u = 0$ *on the boundary of* A, (iii) $\lim_{r\to\infty} u(x,y)/I_0(\omega r) = 0$. *Then* $u = 0$ *in the whole domain* A.

Proof. Let $v = u/w$, with $w(x,y) = I_0(\omega r)$. The function v satisfies the elliptic equation

$$\Delta v + \frac{2}{w}(v_x w_x + v_y w_y) = 0,$$

and $v = 0$ on the boundary of A. Owing to (iii), for every positive number σ we can find R such that $r > R$ implies $|v(x,y)| < \sigma$. Consider the part Δ of A contained inside a circle with radius $R_1 > R$ and center at the origin. Then on the boundary of Δ we have $|v| < \sigma$. According to the maximum principle for elliptic equations in bounded domains, the inequality $|v| < \sigma$ holds in the whole set Δ. For an arbitrary point $(x_0, y_0) \in A$, R_1 can be chosen large enough for Δ to contain that point. Then $|v(x_0, y_0)| < \sigma$, and, since σ may be arbitrarily small, $v(x_0, y_0) = 0$ and hence $u(x_0, y_0) = 0$, which proves the lemma.

A formal solution of the Helmholtz equation in (2.2) may be written as

$$F(x,y) = \int e^{Aze^{-t} + B\bar{z}e^{t}} f(t)\,dt, \tag{2.3}$$

where $z = x + iy$, $\bar{z} = x - iy$ and A, B are constants to be specified. It can be easily verified that F in (2.3) satisfies the equation in (2.2) by writing Laplace's operator as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}.$$

Performing the differentiation in (2.3) we obtain $\Delta F = 4ABF$. Hence, if $4AB = \omega^2$, then F satisfies the equation in (2.2). Taking $A = \frac{1}{2}i\omega$, $B = -\frac{1}{2}i\omega$, $z = re^{i\phi}$, we have

$$F(x,y) = \int e^{-i\omega r \sinh t} f(t + i\phi)\,dt. \tag{2.4}$$

By changing (2.3) into

$$\int e^{A\bar{z}e^{-t} + Bze^{t}} g(t)\,dt,$$

we obtain a representation as in (2.4), but now with $g(t - i\phi)$ in the integrand. Hence, a formal solution of the Helmholtz equation can be represented by

$$F(x, y) = \int e^{-i\omega r \sinh t} \{f(t + i\phi) + g(t - i\phi)\}\, dt.$$

For suitable choices of f and g the expression $f(t + i\phi) + g(t - i\phi)$ becomes the real (or imaginary) part of a holomorphic function of the complex variable $\zeta = t + i\phi$ (with real t and ϕ). In that case this expression is a harmonic function of the variables t and ϕ.

To solve the boundary value problem (2.2) we choose a representation of the following kind:

$$F(x, y) = \int_{-\infty}^{\infty} e^{-i\omega r \sinh t}\, U(t, \phi)\, dt,$$

where U is harmonic (but not necessarily holomorphic) in the strip

$$B = \{t, \phi | -\infty < t < \infty, 0 < \phi < \alpha\}.$$

In view of the boundary conditions of F (see (2.2)) we obtain for U the following boundary value problem:

$$\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2 U}{\partial \phi^2} = 0 \quad \text{in } B, \qquad U(t, 0) = \delta(t), \quad U(t, \alpha) = 0. \tag{2.5}$$

Solutions of the Laplace equation in the strip B with Dirichlet boundary conditions can be obtained by using the conformal mapping $\eta = \exp(\pi\zeta/\alpha)$, which gives a potential problem in a half-plane. In the underlying case we choose a more direct method.

Suppose $U(t, \phi) = \operatorname{Re} f(\zeta)$, $\zeta = t + i\phi$. Then the singularity of U in $\zeta = 0$ may be represented by $(i/\pi)(1/\zeta)$ and f may be constructed by the principle of reflection.

$$f(\zeta) = \frac{i}{\pi}\left[\frac{1}{\zeta} + \sum_{k=1}^{\infty} \frac{2\zeta}{\zeta^2 + 4k^2\alpha^2}\right] = \frac{i}{\pi}\, \frac{\mu}{e^{\mu\zeta} - 1},$$

where

$$\mu = \pi/\alpha. \tag{2.6}$$

The real part of f is then given by

$$U(t, \phi) = \frac{1}{2\alpha}\, \frac{\sin \mu\phi}{\cosh \mu t - \cos \mu\phi}. \tag{2.7}$$

Hence

$$F(x, y) = \frac{\sin \mu\phi}{2\alpha} \int_{-\infty}^{\infty} e^{-i\omega r \sinh t} \frac{dt}{\cosh \mu t - \cos \mu\phi}.$$

This function is bounded by the expression

$$\frac{\sin \mu\phi}{2\alpha} \int_{-\infty}^{\infty} \frac{dt}{\cosh \mu t - \cos \mu\phi} = \frac{\alpha - \phi}{\alpha},$$

and hence the conditions for uniqueness are fulfilled. With this function F the solution of (1.1) is

$$\Phi_\varepsilon(x, y) = 1 - \frac{\sin \mu\phi}{2\alpha} e^{\omega r \sin \phi} \int_{-\infty}^{\infty} e^{-i\omega r \sinh t} \frac{dt}{\cosh \mu t - \cos \mu\phi}. \tag{2.8}$$

This representation of the solution of the singular perturbation problem (1.1) will be the starting point of the investigations on the asymptotic behavior of $\Phi_\varepsilon(x, y)$ for $\varepsilon \to 0$ (i.e., $\omega \to \infty$). The integral in (2.8) will be evaluated by saddle point methods. The saddle points of the function $e^{-i\omega r \sinh t}$ are located at the zeros of $\cosh t$, i.e., at $t_n = i(\frac{1}{2}\pi + n\pi)$, n being an integer. The steepest descent lines are lines parallel to the real t-axis through t_n. If convergence is not disturbed, the path of integration of the integral in (2.8) may be shifted towards a steepest descent line. With this condition, only the saddle point at $-\frac{1}{2}i\pi$ can be considered.

By shifting the path of integration of (2.8) downwards to the line $\operatorname{Im} t = -\frac{1}{2}\pi$, singularities of the integrand may be passed. The singularities in this case are poles due to the zeros of

$$\cosh \mu t - \cos \mu\phi = 2 \sin (\tfrac{1}{2}\mu(\phi + it)) \sin (\tfrac{1}{2}\mu(\phi - it)). \tag{2.9}$$

The zeros are $t_k = -i(\phi + 2\alpha k)$ and $\bar{t}_k$ (the complex conjugate of t_k) for integer values of k. The following poles are important in our problem:

$$\begin{aligned} t_k &= -i(\phi + 2\alpha k) \quad \text{for } k = 0, 1, 2, \cdots, \\ \bar{t}_k &= i(\phi + 2\alpha k) \qquad \text{for } k = -1, -2, -3, \cdots. \end{aligned} \tag{2.10}$$

Only these poles may be located in $[0, -\frac{1}{2}i\pi]$, the number of which is dependent on α. We consider two different cases: $\frac{1}{2}\pi < \alpha < 2\pi$ and $0 < \alpha < \frac{1}{2}\pi$. The first case is simpler than the second one, and will be considered first.

3. The case $\frac{1}{2}\pi < \alpha < 2\pi$. For $0 < \phi < \alpha$, only the pole $t_0 = -i\phi$ of (2.10) may be located in the interval $[0, -\frac{1}{2}i\pi]$. For values of ϕ close to $\frac{1}{2}\pi$, the pole t_0 lies close to the saddle point at $t = -\frac{1}{2}i\pi$. In order to obtain an asymptotic expansion of Φ_ε which holds uniformly for all values of ϕ in $[0, \alpha]$, we use the same method as in [4].

Essential to this method is the regularization of the integrand in (2.8) by an appropriate function. This will be done by determining a constant (i.e., independent of t) c such that the function

$$\frac{\sin \mu\phi}{2\alpha} \frac{1}{\cosh \mu t - \cos \mu\phi} - \frac{c}{\sinh \frac{1}{2}(t + i\phi)} \tag{3.1}$$

is regular at $t = -i\phi$. By calculating the residues at $t = -i\phi$ of both members of (3.1), we infer $c = i/4\pi$. The function Φ_ε of (2.8) can now be written as

$$(3.2)\quad \Phi_\varepsilon(x, y) = 1 + \frac{e^{\omega y}}{4\pi i}\int_{-\infty}^{\infty} e^{-i\omega r \sinh t}\frac{dt}{\sinh\frac{1}{2}(t + i\phi)} - e^{\omega y}\int_{-\infty}^{\infty} e^{-i\omega r\sinh t} g(t)\,dt,$$

where

$$(3.3)\qquad g(t) = U(t, \phi) + \frac{1}{4\pi i \sinh\frac{1}{2}(t + i\phi)},$$

and U is defined in (2.7).

The first integral in (3.2) can be evaluated by means of the following formula:

$$(3.4)\qquad \begin{gathered} F(r, \gamma) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-r\cosh u}\frac{du}{\sinh\frac{1}{2}(u - i\gamma)} = e^{-r\cos\gamma}\operatorname{erfc}(\sqrt{2r}\sin\tfrac{1}{2}\gamma), \\ 0 < \gamma < 2\pi, \end{gathered}$$

where

$$(3.5)\qquad \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}\,dt.$$

Formula (3.4) can be found in Lauwerier's papers [3] and is also used in [4]. A proof of (3.4) is easily obtained by verifying that

$$\frac{\partial}{\partial r}\{e^{r\cos\gamma} F(r, \gamma)\} = -2\sqrt{\frac{2\pi}{r}}\sin\tfrac{1}{2}\gamma\, e^{-r(1-\cos\gamma)}.$$

Now, letting $u = t + \frac{1}{2}i\pi$, $\gamma = 5\pi/2 - \phi$ and using

$$\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z),$$

we obtain

$$(3.6)\qquad 1 + \frac{e^{\omega r\sin\phi}}{4\pi i}\int_{-\infty}^{\infty} e^{-i\omega r\sinh t}\frac{dt}{\sinh\frac{1}{2}(t + i\phi)} = \tfrac{1}{2}\operatorname{erfc}(z),$$

where

$$(3.7)\qquad z = \sqrt{2\omega r}\sin\tfrac{1}{2}(\tfrac{1}{2}\pi - \phi).$$

Formula (3.6) holds for $\frac{1}{2}\pi < \phi < 2\pi$. But by considering complex values of ϕ and by using analytic continuation, (3.6) can be shown to hold for $0 < \operatorname{Re}\phi < 2\pi$.

The function g of (3.3) is regular for $t \in [0, -\frac{1}{2}i\pi]$ and $0 < \phi < \alpha$ $(\frac{1}{2}\pi < \alpha < 2\pi)$. Hence, by shifting the path of integration in the second integral of (3.2) downwards to the line $\operatorname{Im} t = -\frac{1}{2}\pi$, we obtain

$$(3.8)\qquad \Phi_\varepsilon(x, y) = \tfrac{1}{2}\operatorname{erfc}(z) - e^{\omega y}\int_{-\infty}^{\infty} e^{-\omega r\cosh t} g(t - \tfrac{1}{2}i\pi)\,dt.$$

So far, large values of ω (i.e., small values of ε) have not been considered. The representation (3.8) of the solution of the boundary value problem is the exact representation. In order to get an asymptotic expansion of Φ_ε, we expand g

into a series

$$g(t - \tfrac{1}{2}i\pi) = \cosh\tfrac{1}{2}t \sum_{k=0}^{\infty} c_k(\sinh\tfrac{1}{2}t)^k. \tag{3.9}$$

Substitution of this series in (3.8) and interchanging the order of summation and integration yields

$$\Phi_\varepsilon(x, y) \simeq \tfrac{1}{2}\operatorname{erfc}(z) - 2\,e^{-\omega r(1-\sin\phi)} \sum_{k=0}^{\infty} c_{2k}\frac{\Gamma(k+\frac{1}{2})}{(2\omega r)^{k+1/2}} \tag{3.10}$$

as $\omega r \to \infty$, uniformly with respect to ϕ, $0 \leqq \phi \leqq \alpha$.

The expansion in (3.10) breaks down if $\alpha \to \frac{1}{2}\pi$. Namely, the function $g(t - \frac{1}{2}i\pi)$ has a pole at $i(\phi + \frac{1}{2}\pi - 2\alpha)$. For $\phi = \alpha$, this pole is located at $i(\frac{1}{2}\pi - \alpha)$ and if $\alpha \to \frac{1}{2}\pi$ this pole approaches the origin. As a consequence, the coefficients c_k in (3.9) and (3.10) tend to infinity if $\alpha \to \frac{1}{2}\pi$. For this question the reader is referred to § 5.

The most important term in the asymptotic expansion (3.10) is

$$\Phi_\varepsilon^{(0)}(x, y) \equiv \tfrac{1}{2}\operatorname{erfc}(z)$$

with z defined in (3.7). This term exhibits the behavior of Φ_ε in the neighborhood of $\phi = \frac{1}{2}\pi$, i.e., along the y-axis in the x, y-plane. Just as in the quarter-plane case, this term leads to a parabolic boundary layer, situated along the positive y-axis. In this domain, for large values of ωr, the function $\Phi_\varepsilon^{(0)}$ (and hence Φ_ε) rapidly changes from the value $\frac{1}{2}$ to very small values ($x > 0$) or to values close to 1 ($x < 0$). This boundary layer is called a "free" or "internal" boundary layer, since it is not located along the boundary of the domain A for which the boundary value problem (1.1) is defined.

Along the boundary $\phi = \alpha$ boundary layers do not occur, as can be seen from (3.10). Namely, if $\phi \simeq \alpha$ ($\alpha > \frac{1}{2}\pi$),

$$\Phi_\varepsilon(x, y) - \Phi_\varepsilon^{(0)}(x, y) = O((\omega r)^{-N})$$

as $\omega r \to \infty$, for any positive N and all ϕ, $\frac{1}{2}\pi + \delta \leqq \phi \leqq \alpha$, where δ is a small positive constant. For these values of ϕ we also have

$$\Phi_\varepsilon^{(0)}(x, y) - 1 = O((\omega r)^{-N}),$$

as follows from

$$\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$$

and from the well-known asymptotic formula

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}z}e^{-z^2}(1 + O(z^{-2}))$$

as $z \to +\infty$.

4. The acute angle. In this section we consider values of ϕ and α in the range

$$0 \leqq \phi \leqq \alpha < \tfrac{1}{2}\pi. \tag{4.1}$$

First we determine the number of poles (2.14) located on the imaginary t-axis between 0 and $-\frac{1}{2}i\pi$.

We introduce

$$\lambda \equiv \frac{2\alpha}{\pi}\left(=\frac{2}{\mu}\right), \tag{4.2}$$

so that $0 < \lambda < 1$. Consequently, we can choose an integer $n \geqq 2$, satisfying

$$n - 1 < 1/\lambda \leqq n. \tag{4.3}$$

We distinguish two cases:

(a) If in (4.3) n is odd, then we have with $k_0 = \frac{1}{2}(n-1)$,

$$2\alpha k_0 < \tfrac{1}{2}\pi \leqq (2k_0 + 1)\alpha. \tag{4.4}$$

Therefore, the pole

$$t_{k_0} = -i(\phi + 2\alpha k_0) \tag{4.5}$$

passes through $-\frac{1}{2}i\pi$ when ϕ changes from 0 to α. If $\phi + 2\alpha k_0 < \frac{1}{2}\pi$, then $t_{k_0} \in [0, -\frac{1}{2}i\pi]$; if $\phi + 2\alpha k_0 > \frac{1}{2}\pi$, then $t_{k_0} \notin [0, -\frac{1}{2}i\pi]$. For all values of ϕ $(0 \leqq \phi \leqq \alpha)$, we have

$$t_k, \bar{t}_l \in [0, -\tfrac{1}{2}i\pi] \quad \text{for } k = 0, 1, \cdots, k_0 - 1, \quad l = 1, 2, \cdots, k_0.$$

(b) If in (4.3) n is even, we have with $l_0 = \frac{1}{2}n$,

$$(2l_0 - 1)\alpha < \tfrac{1}{2}\pi \leqq 2\alpha l_0. \tag{4.6}$$

Therefore, the pole

$$\bar{t}_{-l_0} = i(\phi - 2\alpha l_0) \tag{4.7}$$

passes the point $-\frac{1}{2}i\pi$ if ϕ changes from 0 to α. If $\phi + \frac{1}{2}\pi > 2\alpha l_0$, then $\bar{t}_{-l_0} \in [0, -\frac{1}{2}i\pi]$; if $\phi + \frac{1}{2}\pi < 2\alpha l_0$, then $\bar{t}_{-l_0} \notin [0, -\frac{1}{2}i\pi]$. For all values of ϕ $(0 \leqq \phi \leqq \alpha)$,

$$t_k, \bar{t}_{-l} \in [0, -\tfrac{1}{2}i\pi] \quad \text{for } k = 0, 1, \cdots, l_0 - 1, \quad l = 1, 2, \cdots, l_0 - 1.$$

As in § 3, the poles t_{k_0} and $\bar{t}_{-l_0}$ ((4.5) and (4.7) respectively) can be split off. In this way error functions are introduced. Afterwards the path of integration will be shifted downwards to the line $\operatorname{Im} t = -\frac{1}{2}\pi$. The residues of the poles being passed turn out to be exponential functions. A simple calculation gives the following results. (We distinguish again the two cases (a) and (b).)

(a)

$$\begin{aligned}\Phi_\varepsilon(x, y) = {} & \sum_{k=1}^{k_0} e^{-\omega r\{\sin(2k\alpha - \phi) - \sin\phi\}} - \sum_{k=1}^{k_0 - 1} e^{-\omega r\{\sin(2k\alpha + \phi) - \sin\phi\}} \\ & - \tfrac{1}{2}\operatorname{erfc}(z)\, e^{-\omega r\{\sin(2\alpha k_0 + \phi) - \sin\phi\}} - e^{\omega r\sin\phi}\int_{-\infty}^{\infty} e^{-\omega r\cosh t}\, g(t)\, dt,\end{aligned} \tag{4.8}$$

where k_0 is specified in (4.4), $z = \sqrt{2\omega r}\sin\frac{1}{2}\gamma$,

$$g(t) = U(t - \tfrac{1}{2}i\pi, \phi) + \frac{1}{4\pi i \sinh\frac{1}{2}(t - i\gamma)}, \tag{4.9}$$

and $\gamma = \frac{1}{2}\pi - \phi - 2\alpha k_0$.

(b)

$$\Phi_\varepsilon(x, y) = \sum_{k=1}^{l_0-1} e^{-\omega r\{\sin(2\alpha k-\phi)-\sin\phi\}} - \sum_{k=1}^{l_0-1} e^{-\omega r\{\sin(2\alpha k+\phi)-\sin\phi\}}$$

(4.10)

$$+ \tfrac{1}{2}\operatorname{erfc}(z)\, e^{-\omega r\{\sin(2\alpha l_0-\phi)-\sin\phi\}} - e^{\omega r\sin\phi}\int_{-\infty}^{\infty} e^{-\omega r\cosh t} g(t)\,dt,$$

where l_0 is specified in (4.6), $z = \sqrt{2\omega r}\sin\frac{1}{2}\gamma$,

$$g(t) = U(t - \tfrac{1}{2}i\pi, \phi) - \frac{1}{4\pi i\sinh\frac{1}{2}(t - i\gamma)} \qquad (4.11)$$

and $\gamma = \frac{1}{2}\pi + \phi - 2\alpha l_0$.

The representation (4.8) (resp. (4.10)) of $\Phi_\varepsilon(x, y)$ is the exact solution. In order to get an asymptotic expansion of Φ_ε for small values of ε, the function g in (4.9) (resp. (4.11)) may be expanded in the same way as was done for g in (3.9). The asymptotic expansion obtained by interchanging the order of summation and integration (cf. (3.10)) is uniformly valid in $0 \leqq \phi \leqq \alpha$. Just as in the foregoing section, if $\alpha \to \frac{1}{2}\pi$, the expansion must be reconsidered (see § 5).

We conclude this section with some remarks concerning the boundary layer. If $\phi \simeq \alpha$, the asymptotic behavior of Φ_ε is determined by the first term of the first finite series in (4.8) (resp. (4.10)); the other terms are of lower order. Hence

$$\Phi_\varepsilon(x, y) \simeq e^{-\omega r\{\sin(2\alpha-\phi)-\sin\phi\}} \qquad (4.12)$$

as $\omega r \to \infty$, $\phi \to \alpha$, $\phi \leqq \alpha$. If $\phi < \alpha$, the right-hand side of (4.12) is very small, explicitly

$$\Phi_\varepsilon(x, y) = O((\omega r)^{-N}),$$

where N is an arbitrary positive number. This estimate, however, is not uniformly valid in ϕ. If $\phi \to \alpha$, the exponential function in (4.12) may not be small at all. We can determine the locus in the x, y-plane on which the argument of the exponential function in (4.12) is constant. We infer from

$$-\omega r\{\sin(2\alpha - \phi) - \sin\phi\} = -c \qquad (c > 0)$$

that the locus is a straight line

$$y = x\tan\alpha - c/\omega, \qquad (4.13)$$

which is parallel to the boundary $y = x\tan\alpha$ of the sector A. From these aspects we conclude that along the line $y = x\tan\alpha$ a boundary layer of thickness $O(\varepsilon)$ is located.

The term with the error function in (4.8) (resp. 4.10)) is asymptotically of lower order than the term in (4.12). The error function part, however, is of great importance. The error function changes rapidly at $\phi = \frac{1}{2}\pi - 2\alpha k_0$ (resp. $2\alpha l_0 - \frac{1}{2}\pi$), but the effect is damped by the exponential function contained in this term. This term is gaining in influence if $\alpha \to \frac{1}{2}\pi$ and the (hidden) internal boundary layer due to the error function comes to light if $\alpha \to \frac{1}{2}\pi$ (see § 5).

5. The almost right angle. In §§ 3 and 4 we discussed the asymptotic behavior of Φ_ε ($\varepsilon \to 0$) for values of α larger, respectively smaller than $\frac{1}{2}\pi$. However, the expansion in (3.9) and the expansions which can be derived from (4.8) (resp. (4.10)) in an analogous way, are not valid if ϕ, $\alpha \to \frac{1}{2}\pi$, since the function $g(t)$ has a singularity, which tends to zero for ϕ, $\alpha \to \frac{1}{2}\pi$. In this section we shall give the asymptotic representation of Φ_ε holding for all $\alpha \in [\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta]$, where $0 < \delta < \frac{1}{4}\pi$.

Suppose first that $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$. In this case the results of § 4 can be used. In (4.2) we have $\frac{1}{2} < \lambda < 1$ and in (4.3) we have $n = 2$. Hence, the (b)-case applies and from (4.6) it follows that $l_0 = 1$. Thus (4.8) becomes

$$(5.1) \qquad \Phi_\varepsilon(x, y) = \tfrac{1}{2}\operatorname{erfc}(z)\, e^{-\omega r\{\sin(2\alpha-\phi)-\sin\phi\}} + e^{\omega r \sin\phi}\int_{-\infty}^{\infty} e^{-\omega r\cosh t} g(t)\, dt,$$

where $z = \sqrt{2\omega r}\sin\frac{1}{2}\gamma$, $\gamma = \phi + \frac{1}{2}\pi - 2\alpha$ and g is defined in (4.9). The function g has a pole in $i(\frac{1}{2}\pi - \phi)$, corresponding to t_k in (2.14) with $k = 0$. In (4.8) this pole has no influence since α is constant. If, alternatively, $\alpha \simeq \frac{1}{2}\pi$, this singularity is close to the origin for values of ϕ close to α. This pole can be split off and so another error function is introduced.

Suppose next $\frac{1}{2}\pi < \alpha < 2\pi$. The function g in (3.3) has a pole in $i(\phi + \frac{1}{2}\pi - 2\alpha)$, corresponding to $\bar{t}_k$ in (2.14) with $k = -1$. Again, for $\alpha \simeq \frac{1}{2}\pi$, this pole is close to the origin for values of ϕ close to α.

Combining the two cases, we have

$$(5.2) \qquad \begin{aligned} \Phi_\varepsilon(x, y) &= \tfrac{1}{2}\operatorname{erfc}(\zeta) + \tfrac{1}{2}\operatorname{erfc}(z)\, e^{-\omega r\{\sin(2\alpha-\phi)-\sin\phi\}} \\ &\quad - e^{\omega r\sin\phi}\int_{-\infty}^{\infty} e^{-\omega r\cosh t} h(t)\, dt, \end{aligned}$$

where

$$h(t) = U(t - \tfrac{1}{2}i\pi, \phi) - \frac{1}{4\pi i \sinh\frac{1}{2}(t - i\gamma)} + \frac{1}{4\pi i\sinh\frac{1}{2}(t - i(\frac{1}{2}\pi - \phi))},$$
$$\gamma = \phi + \tfrac{1}{2}\pi - 2\alpha, \quad z = \sqrt{2\omega r}\sinh\tfrac{1}{2}\gamma, \quad \zeta = \sqrt{2\omega r}\sin\tfrac{1}{2}(\tfrac{1}{2}\pi - \phi).$$

The asymptotic expansion of Φ_ε for large ωr may be derived by expanding h in the same way as was done for g in (3.8). The asymptotic expansion so obtained holds uniformly in $0 \leqq \phi \leqq \alpha$, $\frac{1}{2}\pi - \delta \leqq \alpha \leqq \frac{1}{2}\pi + \delta$, where $0 < \delta < \frac{1}{4}\pi$. If $\alpha = \frac{1}{2}\pi$, (the quarter-plane, see [4]) we have $\gamma = \phi - \frac{1}{2}\pi$ and

$$\Phi_\varepsilon(x, y) = \operatorname{erfc}(\sqrt{2\omega r}\sin\tfrac{1}{2}(\tfrac{1}{2}\pi - \phi)) - e^{\omega r\sin\phi}\int_{-\infty}^{\infty} e^{-\omega r\cosh t} h(t)\, dt,$$

where the integral equals the corresponding integral of reference [4] (formula (4.6)).

The significant terms of (5.2) are the two terms with the error functions. For $\alpha < \frac{1}{2}\pi$, the second term may be connected with the "linear" boundary layer along $\phi = \alpha$ and the (hidden) internal layer at $\phi = 2\alpha - \frac{1}{2}\pi$. The first one may be connected with an external parabolic boundary layer outside the sector A. This boundary layer has no influence since it is situated outside the domain of definition. If however $\alpha \to \frac{1}{2}\pi$ ($\alpha < \frac{1}{2}\pi$) this boundary layer enters the domain A, and coalesces (in the limit $\alpha = \frac{1}{2}\pi$) with both the "linear" boundary layer and the (hidden) internal layer at $\phi = 2\alpha - \frac{1}{2}\pi$ (see Fig. 1).

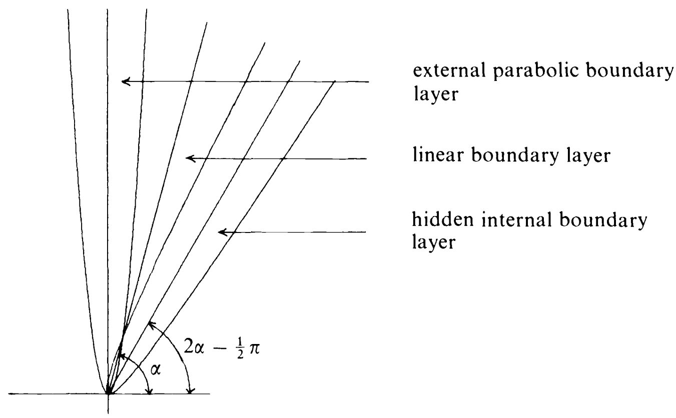

FIG. 1. $\alpha < \frac{1}{2}\pi$

For $\alpha > \frac{1}{2}\pi$, the first term in (5.2) may be associated with the internal parabolic boundary layer inside the sector at $\phi = \frac{1}{2}\pi$. The second one may be connected with a boundary layer outside A, which is situated at $\phi = 2\alpha - \frac{1}{2}\pi$ and which enters the domain if $\alpha \to \frac{1}{2}\pi$. In the limit ($\alpha = \frac{1}{2}\pi$) the two types of boundary layers pass into the parabolic boundary layer along $\phi = \alpha = \frac{1}{2}\pi$ (see Fig. 2).

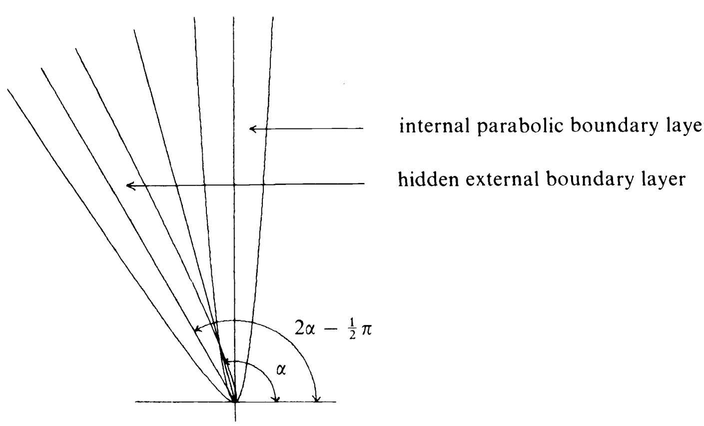

FIG. 2. $\alpha > \frac{1}{2}\pi$

From our remarks on the coincidence of the boundary layers it may be established that the parabolic boundary layer of the quarter-plane is a particular case of parabolic boundary layers for the almost right angle. For other aspects, the reader is referred to Grasman [2], where the case of almost characteristic boundaries is treated with coordinate-stretching techniques.

6. An analogous problem. An analogous, but much simpler, problem is encountered in looking for the asymptotic expansion of the solutions of the boundary value problem

$$\varepsilon\Delta\Phi_\varepsilon(x, y) - \mu\frac{\partial}{\partial x}\Phi_\varepsilon(x, y) - \lambda\frac{\partial}{\partial y}\Phi_\varepsilon(x, y) = 0 \tag{6.1}$$

in the quarter-plane $A = \{x, y | x \geqq 0, y \geqq 0\}$ with boundary conditions

$$\Phi_\varepsilon(0, y) = 1, \quad \Phi_\varepsilon(x, 0) = 0.$$

In (6.1), μ and λ are numbers independent of x and y. The characteristics of the reduced equation ($\varepsilon = 0$ in (6.1))

$$\mu\frac{\partial\phi}{\partial x} + \lambda\frac{\partial\phi}{\partial y} = 0 \tag{6.2}$$

are the lines $y = (\lambda/\mu)x + c$. For small values of μ, the characteristics of (6.2) are nearly parallel to the boundary line $x = 0$. Therefore, for small values of μ it is expected that again two error functions appear in the asymptotic expansion of Φ_ε (for $\varepsilon \to 0$). As can be verified by the methods of § 2, the function Φ_ε can be written down as follows:

$$\Phi_\varepsilon(x, y) = e^{\omega r \sin(\phi+\beta)}\int_{-\infty}^{\infty} e^{-i\omega r \sinh t}\, U(t, \phi)\, dt,$$

where

$$x = r\cos\phi, \quad y = r\sin\phi, \quad \lambda = \rho\cos\beta, \quad \mu = \rho\sin\beta, \quad \omega = \rho/(2\varepsilon),$$

$$\begin{aligned} U(t, \phi) &= \frac{1}{\pi}\operatorname{Re}\{\tan\tfrac{1}{2}(it + \phi + \beta) + \tan\tfrac{1}{2}(it + \phi - \beta)\} \\ &= \frac{1}{\pi}\frac{\sin(\phi+\beta)}{\cosh t + \cos(\phi+\beta)} + \frac{1}{\pi}\frac{\sin(\phi-\beta)}{\cosh t + \cos(\phi-\beta)}. \end{aligned}$$

7. Concluding remarks. In this paper we used analytical methods which only can be applied on singular perturbation problems with simple differential operators, boundary values and suitable domains of definition. The methods cannot easily be generalized for other problems. In treating the relatively simple problems, however, we have a different aim.

For instance, our approach of the problem gives results which are not easily noticed by using the usual singular perturbation techniques. We allude to the existence of the hidden boundary layer along the line $\phi = 2\alpha - \frac{1}{2}\pi$ (see Fig. 1 and the conclusion of § 4). This aspect is not discussed in boundary layer techniques, since the function Φ_ε is asymptotically of order zero in the neighborhood of this internal layer. In order to obtain a uniform asymptotic expansion with respect to ϕ (in $0 \leqq \phi \leqq \alpha < \frac{1}{2}\pi$), the error function corresponding with this layer cannot be omitted.

Further we shall point to the case of an almost characteristic boundary (see § 5). In a clear and simple way the asymptotic behavior of Φ_ε can be described, using our methods. Also the way in which the various boundary layers pass into each other is apparent.

An important disadvantage of our methods is the following. The asymptotic expansions are derived for large values of ωr. Hence, the results of our paper do not hold in an ε-neighborhood of the origin. This domain is very small but it is very interesting, since in this part of the x, y-plane the boundary layers arise. It is possible to give expansions which represent the behavior of Φ_ε close to the origin, but it seems better to us to tackle this problem with coordinate stretching techniques. This aspect, however, falls outside the scope of this paper.

Our results can successfully be applied in general singular perturbation problems, which yield reduced problems with relatively simple differential operators, boundary values and domains of definition. With these reduced problems the local behavior of the solutions are investigated.

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DISTRIBUTIONAL WATSON TRANSFORMS*

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Abstract. For all Watson transforms W in $L^2(\mathbb{R}_+)$ a triple of Hilbert space $\mathscr{L}_G \subset L^2(\mathbb{R}_+) \subset \mathscr{L}'_G$ is constructed such that W may be extended to $\mathscr{L}'_G$. These results allow the construction of a triple $\mathscr{L} \subset L^2(\mathbb{R}_+) \subset \mathscr{L}'$, where $\mathscr{L}$ is a Gelfand–Fréchet space. This leads to a theory of distributional Watson transforms.

Introduction. Guinand [2], [3], Miller [5], [6], [7] and Goldberg [1] have considered linear manifolds, dense in $L^2(\mathbb{R}_+)$, which are invariant under Watson transforms and equipped with a suitable inner product form a Hilbert space. These manifolds were generalized in [11].

Miller [7], [8], [9] has constructed the dual of such linear manifolds, which is of course a Hilbert space, containing a copy of $L^2(\mathbb{R}_+)$. By means of this dual, Miller [7], [8] extended Watson transforms beyond $L^2(\mathbb{R}_+)$.

In this note we wish to show that the results in [11] make it possible to construct a linear manifold in $L^2(\mathbb{R}_+)$, which is invariant under Watson transforms; however, this linear manifold is not a Hilbert space, but a Gelfand–Fréchet space. This leads to a theory of distributions and Watson transforms defined on them. For all Watson transforms we may choose the same Gelfand–Fréchet space. Specialization of the Gelfand–Fréchet space in $L^2(\mathbb{R}_+)$ leads to a theory analogous to the distributional Fourier transform on the space of tempered distributions.

1. Preliminaries. For $K \in L^\infty(\mathbb{R})$ the linear operator $M[K]$ on $L^2(\mathbb{R})$ is defined by $M[K]f = Kf$. The operator P on $L^2(\mathbb{R}_+)$ is given by $(Pf)(x) = (1/x)f(1/x)$. By $\mathscr{M}$ we denote the Mellin transform, which is an isometry from $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R})$, cf. [13]. In [10] it was shown that every Watson transform on $L^2(\mathbb{R}_+)$ can be written as

$$W = \mathscr{M}^{-1}M[K]\mathscr{M}P, \qquad K \in L^\infty(\mathbb{R}). \tag{1.1}$$

To denote the dependence on K we shall also write W_K instead of W. In [11] the operator $V\,(=V_G)$ on $L^2(\mathbb{R}_+)$ was introduced by

$$V = \mathscr{M}^{-1}M[G]\mathscr{M}, \qquad G \in L^\infty(\mathbb{R}).$$

In the sequel we shall consider a sequence of functions $G_i \in L^\infty(\mathbb{R})$, $i = 1, 2, \cdots$. We therefore introduce the following notation:

$$V_i = V_{G_i},$$
$$\overline{V}_i = V_{\overline{G}_i},$$
$$V_{ji} = V_{G_j/G_i} \quad \text{if } G_j/G_i \in L^\infty(\mathbb{R}),$$

and

$$V'_G = V_{G'},$$

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where

$$G'(x) = G(-x).$$

In [11] it was shown that $V'W = WV$ and hence W maps L_G into $L_{G'}$, where L_G and $L_{G'}$ are the ranges of V_G and $V_{G'}$ respectively.

If $G \neq 0$ a.e., then $V\ (= V_G)$ is injective and equipped with the inner product

$$(Vf, Vg)_{L_G} = (f, g)_{L^2(\mathbb{R}_+)}, \qquad f, g \in L^2(\mathbb{R}_+).$$

L_G then is a Hilbert space, isometrically isomorphic to $L^2(\mathbb{R}_+)$. In this case W maps L_G continuously into $L_{G'}$. Instead of the notation $(\cdot,\cdot)_{L_G}$, we shall also use $(\cdot,\cdot)_G$.

PROPOSITION 1. *Let $G_1, G_2 \in L^\infty(\mathbb{R})$ satisfy $G_i \neq 0$ a.e., $i = 1, 2$, and $G_2/G_1 \in L^\infty(\mathbb{R})$. Then*

$$L_{G_2} \subset L_{G_1}, \tag{1.2}$$

(1.3) *the injection $I: L_{G_2} \to L_{G_1}$ is continuous and has a dense range.*

Proof. If $g \in L_{G_2}$, then for some $f \in L^2(\mathbb{R}_+)$, $g = V_2 f$ and $g = V_1 h$ with $h = V_{21} f \in L^2(\mathbb{R}_+)$. Hence (1.2) holds. Using the same notation, we obtain

$$\begin{aligned}\|g\|_{G_1} = \|h\|_{L^2(\mathbb{R}_+)} = \left\| \frac{G_2}{G_1}\mathscr{M}f \right\|_{L^2(\mathbb{R})} &\leqq \left\| \frac{G_2}{G_1} \right\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R}_+)} \\ &= \left\| \frac{G_2}{G_1} \right\|_{L^\infty(\mathbb{R})} \cdot \|g\|_{G_2}.\end{aligned}$$

This implies the continuity of I in (1.3). The remainder of (1.3) follows from

$$(V_1 u, g)_{G_1} = (u, h)_{L^2(\mathbb{R}_+)} = (\mathscr{M}u, (G_2/G_1)\mathscr{M}f)_{L^2(\mathbb{R})}, \quad u \in L^2(\mathbb{R}_+), \quad g \in L_{G_2}.$$

Since $M[G_2/G_1]$ has a dense range in $L^2(\mathbb{R})$, it follows that if the left-hand side equals zero for fixed $u \in L^2(\mathbb{R}_+)$ and arbitrary $g \in L_{G_2}$, then the right-hand side shows that $\mathscr{M}u = 0$ and thus $V_1 u = 0$. This completes the proof.

For $G \in L^\infty(\mathbb{R})$, $G \neq 0$ a.e., denote by L'_G the dual of L_G.

PROPOSITION 2. *Let $G \in L^\infty(\mathbb{R})$ satisfy $G \neq 0$ a.e. Then L'_G is (isomorphic to) the completion of $L^2(\mathbb{R}_+)$ under the norm $\|\overline{V}f\|_{L^2(\mathbb{R}_+)}$.*

Proof. Denote by A and B the antilinear isometries from L_G onto L'_G and $L^2(\mathbb{R}_+)$ onto $L^{2\prime}(\mathbb{R}_+)$ respectively. Then the restriction of Bf to L_G belongs to L'_G and $A^{-1}Bf = V\overline{V}f, f \in L^2(\mathbb{R}_+)$, for

$$|(Bf)(Vg)| = |(Vg, f)_{L^2(\mathbb{R}_+)}| = |(g, \overline{V}f)_{L^2(\mathbb{R}_+)}| \leqq \|\overline{V}f\|_{L^2(\mathbb{R}_+)} \|Vg\|_G$$

and

$$(Vg, A^{-1}Bf)_G = (Bf)(Vg) = (Vg, f)_{L^2(\mathbb{R}_+)} = (Vg, V\overline{V}f)_G,$$

where $g \in L^2(\mathbb{R}_+)$.

If $w \in L'_G$ and $A^{-1}w = Vh$, $h \in L^2(\mathbb{R}_+)$, then with $f \in L^2(\mathbb{R}_+)$,

$$(Bf, w)_{L'_G} = (A^{-1}Bf, A^{-1}w)_G = (V\overline{V}f, Vh)_G = (\overline{V}f, h)_{L^2(\mathbb{R}_+)}.$$

From this it follows that $BL^2(\mathbb{R}_+)$ is dense in L'_G. Also,

$$(Bf, Bf)_{L'_G} = (\bar{V}f, \bar{V}f)_{L^2(\mathbb{R}_+)}.$$

If we identify Bf with f in $L^2(\mathbb{R}_+)$, the theorem follows.

2. The space $\mathscr{L}$ and its dual $\mathscr{L}'$. Let $(G_j)_{j=1}^\infty$ be a sequence of functions on $\mathbb{R}$ such that

(2.1) $$G_j \in L^\infty(\mathbb{R}), \qquad j = 1, 2, \cdots,$$

(2.2) $$G_j \neq 0 \quad \text{a.e.}, \qquad j = 1, 2, \cdots,$$

(2.3) $$G_{j+1}/G_j \in L^\infty(\mathbb{R}), \qquad j = 1, 2, \cdots,$$

and

(2.4) G_k/G_j is locally bounded on $\mathbb{R}$, $k = 1, 2, \cdots, j-1; j = 2, 3, \cdots$.

The conditions (2.1), (2.2) and (2.3) imply that we have obtained a descending chain of Hilbert spaces:

$$L^2(\mathbb{R}_+) \supset L_{G_1} \supset \cdots \supset L_{G_j} \supset L_{G_{j+1}} \supset \cdots.$$

Put $\mathscr{L} = \bigcap_{j=1}^\infty L_{G_j}$.

PROPOSITION 3. *$\mathscr{L}$ is dense in all L_{G_k}, $k = 1, 2, \cdots$.*

Proof. Denote by $C_0(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ having a compact support. For each $\phi \in C_0(\mathbb{R})$, let $u_\phi = G_k\phi$. Then

$$u_\phi = G_j(G_k/G_j)\phi, \qquad j = 1, 2, \cdots.$$

By (2.3) and (2.4), $(G_k/G_j)\phi \in L^2(\mathbb{R}), j = 1, 2, \cdots$. Thus

$$\mathscr{M}^{-1}u_\phi = V_j\mathscr{M}^{-1}(G_k/G_j)\phi, \qquad j = 1, 2, \cdots.$$

Consequently, $\mathscr{M}^{-1}u_\phi \in \mathscr{L}$. Since for all $g \in L^2(\mathbb{R}_+)$ and $\phi \in C_0(\mathbb{R})$,

(2.5) $$(\mathscr{M}^{-1}u_\phi, V_kg)_{G_k} = (\phi, \mathscr{M}g)_{L^2(\mathbb{R})},$$

and since $C_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it follows that if the left-hand side of (2.5) equals zero for a fixed $g \in L^2(\mathbb{R}_+)$ and arbitrary $\phi \in C_0(\mathbb{R})$, then the right-hand side of (2.5) implies that $\mathscr{M}g = 0$ and consequently $V_kg = 0$. Hence $\mathscr{L}$ is dense in L_{G_k}, $k = 1, 2, \cdots$.

PROPOSITION 4. *$\mathscr{L}$ is dense in $L^2(\mathbb{R}_+)$.*

Proof. We use the notation of the proof of Proposition 3. Proposition 4 follows from

$$(f, \mathscr{M}^{-1}u_\phi)_{L^2(\mathbb{R}_+)} = (\bar{G}_k\mathscr{M}f, \phi)_{L^2(\mathbb{R}_+)}, \qquad f \in L^2(\mathbb{R}_+).$$

We equip $\mathscr{L}$ with the initial topology, that is the coarsest topology such that all identity mappings $\mathscr{L} \to L_{G_j}$ are continuous. A consequence of Proposition 3 is that $\mathscr{L}$ is complete. Hence $\mathscr{L}$ is a Fréchet–Gelfand space. The dual $\mathscr{L}'$ of $\mathscr{L}$ is given by

$$\mathscr{L}' = \bigcup_{j=1}^\infty L'_{G_j}.$$

Here equality means that if $f \in L'_{G_j}$, then f restricted to $\mathscr{L}$ belongs to $\mathscr{L}'$ and, conversely, that if $g \in \mathscr{L}'$, then for some integer j, g can be extended to the whole space L_{G_j} and thus belongs to L'_{G_j}.

We equip $\mathscr{L}'$ with the strong topology $\beta(\mathscr{L}', \mathscr{L})$. Since $\mathscr{L}$ is bornological, $\mathscr{L}'$ is complete (see [4, p. 223]).

3. The distributional Watson transform. Let

$$\mathscr{H} = \bigcap_{j=1}^{\infty} L_{G'_j},$$

equip $\mathscr{H}$ with the initial topology, and give the dual $\mathscr{H}'$ of $\mathscr{H}$ the strong topology.

The Watson transform $W = W_K$ given by (1.1) is a mapping from $\mathscr{L}$ into $\mathscr{H}$. For, if $g \in \mathscr{L}$, then there exists a sequence $(f_j)_{j=1}^{\infty}$ in $L^2(\mathbb{R}_+)$, such that $g = V_j f_j$ and $Wg = WV_j f_j = V'_j W f_j$. Consequently, $Wg \in \mathscr{H}$. Furthermore, $W: \mathscr{L} \to \mathscr{H}$ is continuous. For, with the same notation,

$$\|Wg\|_{G'_j} = \|Wf_j\|_{L^2(\mathbb{R}_+)} \leqq \|W\|_{L^2(\mathbb{R}_+)} \|f_j\|_{L^2(\mathbb{R}_+)} = \|W\|_{L^2(\mathbb{R}_+)} \|g\|_{G_j}.$$

The adjoint W^* of W is a Watson transform given by

$$W^* = \mathscr{M}^{-1} M[\bar{K}'] \mathscr{M} P.$$

Let $\mathscr{W} = {}^t W^*$ be the transpose of W^*. It maps $\mathscr{H}'$ continuously into $\mathscr{L}'$ (see [4, p. 256]). We call $\mathscr{W}$ the distributional Watson transform. The following proposition says why.

PROPOSITION 5. $\mathscr{W}|_{L^2(\mathbb{R}_+)} = W$.

Proof. Let B be the antilinear isometry from $L^2(\mathbb{R}_+)$ onto its dual. Then $Bf \in \mathscr{H}'$, $f \in L^2(\mathbb{R}_+)$. For each $g \in \mathscr{L}$ we have

$$\langle {}^t W^* Bf, g \rangle = \langle Bf, W^* g \rangle = (W^* g, f)_{L^2(\mathbb{R}_+)} = (g, Wf)_{L^2(\mathbb{R}_+)} = \langle BWf, g \rangle.$$

Hence ${}^t W^* B = BW$. If we identify the elements of $L^2(\mathbb{R}_+)$ and $BL^2(\mathbb{R}_+)$ the proposition follows.

Remark 1. If with $K \in L^\infty(\mathbb{R})$, also $1/K \in L^\infty(\mathbb{R})$, then W_K maps $\mathscr{L}$ onto $\mathscr{H}$ and hence $\mathscr{W}$ maps $\mathscr{H}'$ onto $\mathscr{L}'$.

Remark 2. If $K(x)K(-x) = 1$ and if $L_{G_j} = L_{G'_j}$ for all $j = 1, 2, \cdots$, then W_K is involutory and hence $\mathscr{W}$ is involutory.

4. Remarks. Let ϕ be a measurable function on $\mathbb{R}_+$ for which the integral

$$\int_0^\infty t^{-1/2} |\phi(t)|\, dt$$

is finite. Let G be defined by

$$G(x) = \int_0^\infty t^{-1/2 + ix} \phi(t)\, dt, \qquad x \in \mathbb{R}.$$

Then $G \in L^\infty(\mathbb{R})$ and

$$(V_G f)(x) = \int_0^\infty \phi(xt^{-1}) t^{-1} f(t)\, dt$$

(see [12]).

For each integer j, we choose ϕ to be the function ϕ_j defined by

$$\phi_j(t) = \begin{cases} \dfrac{1}{\Gamma(j)}(1-t)^{j-1}, & 0 < t < 1, \\ 0, & t \geqq 1. \end{cases}$$

Then

$$G_j(x) = \frac{\Gamma(\frac{1}{2} + ix)}{\Gamma(\frac{1}{2} + ix + j)}, \qquad x \in \mathbb{R},$$

and the sequence $(G_j)_{j=1}^{\infty}$ satisfies the conditions (2.1), (2.2), (2.3) and (2.4). Furthermore we have

$$L_{G_j} = \left\{ f \in L^2(\mathbb{R}_+) | f(x) = \frac{1}{\Gamma(j)} \int_x^{\infty} (t-x)^{j-1} f^{(j)}(t)\, dt \right.$$
$$\left. \text{for some function } f^{(j)}(t) \text{ defined on } \mathbb{R}_+ \text{ with } t^j f^{(j)}(t) \in L^2(\mathbb{R}_+) \right\}.$$

These spaces have been considered by Guinand, Miller and Goldberg. Since ϕ_j is real, we have $L_{G_j} = L_{G'_j}$ and therefore $\mathscr{L} = \mathscr{H}$. Hence Watson transforms map $\mathscr{L}$ into $\mathscr{L}$.

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BROUWER'S FIXED-POINT THEOREM: AN ALTERNATIVE PROOF*

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Abstract. This paper gives an alternative proof of Brouwer's fixed-point theorem. The expected preknowledge on the part of the reader in following the proof is the continuity of the roots of polynomial equations with respect to the coefficients, and the standard compactness argument.

This note gives an alternative proof of the following theorem.

BROUWER'S FIXED-POINT THEOREM [3]. *If f is a continuous mapping from the simplex $S_{n-1} = \{x \mid \sum_{i=1}^n x_i = 1, x \geqq 0\}$ of the Euclidean n-space into itself, then there is a point $x^* \in S_{n-1}$ such that $f(x^*) = x^*$.*

The fixed-point theorem is now an indispensable tool for the study of mathematical economics. Its proof requires, however, a number of preliminary concepts and related lemmas, and is fairly complicated. As several attempts[1] have been made to simplify the proof, an additional effort will be meaningful only when it gives a much simpler and self-contained account. The author assumes this has been done in the following Steps I–III.

Step I. Extension of Bernstein's polynomial approximation to S_{n-1}.[2]

Let N be the set of n-dimensional nonnegative integer vectors of which elements add up to ν, that is,

$$N = \{a_1, a_2, \cdots, a_\mu\},$$

where

$$a_i = [a_{i1}, \cdots, a_{in}], \quad \mu = n(n+1)\cdots(n+\nu-1)/\nu!, \quad \sum_{j=1}^n a_{ij} = \nu$$

and the a_{ij}'s are nonnegative integers.

Let

$$g(x; a_i) = \frac{\nu!}{a_{i1}!a_{i2}!\cdots a_{in}!} x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}.$$

Then $g(x; a_i)$ satisfies the following relations:

$$(1) \qquad \sum_{a_i \in N} g(x; a_i) = 1, \qquad x \in S_{n-1},$$

$$(2) \qquad \nu x_j = \sum_{a_i \in N} a_{ij} \cdot g(x; a_i), \qquad j = 1, \cdots, n, \quad x \in S_{n-1},$$

$$(3) \qquad \nu(\nu-1)(x_j)^2 = \sum_{a_i \in N} a_{ij}(a_{ij}-1) g(x; a_i), \qquad j = 1, \cdots, n, \quad x \in S_{n-1}.$$

(1) is the multinomial expansion of $(\sum_{i=1}^n x_i)^\nu$ on S_{n-1}; (2) and (3) can be derived by differentiation of the expansion formula.

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[1] See Dunford and Schwartz [4, pp. 467–470], Scarf [6], Arrow and Hahn [1, Appendix].

[2] For the case of one variable, see Bartle [2, Chap. IV].

We shall show that if $f(x)$ is continuous on S_{n-1}, then given an $\varepsilon > 0$,

$$\left\| f(x) - \sum_{a_i \in N} f(a_i/\nu)\cdot g(x;a_i)\right\| < 2\varepsilon \tag{4}$$

holds uniformly on S_{n-1} for a sufficiently large integer ν.[3]

It is noted that we may set $\|f(x)\| \leqq K$ for $x \in S_{n-1}$, and that there exists a $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ holds uniformly for $\|x - y\| < \delta, x, y \in S_{n-1}$.

By (1), the l.h.s. of (4) becomes

$$\left\| \sum_{a_i \in N} [f(x) - f(a_i/\nu)]\cdot g(x;a_i)\right\|.$$

We shall evaluate this expression by partitioning N into N_1 and N_2 such that

$$N_1 = \{a_i \mid \|x - (a_i/\nu)\| < \delta, a_i \in N\}$$

and

$$N_2 = \{a_i \mid \|x - (a_i/\nu)\| \geqq \delta, a_i \in N\}.$$

For N_1, we have

$$\left\| \sum_{a_i \in N_1} [f(x) - f(a_i/\nu)]\cdot g(x;a_i)\right\| \leqq \varepsilon \sum_{a_i \in N_1} g(x;a_i) \leqq \varepsilon.$$

For N_2, as $\|f(x)\| \leqq K$,

$$\begin{aligned}
&\left\| \sum_{a_i \in N_2} [f(x) - f(a_i/\nu)]\cdot g(x;a_i)\right\| \\
&\leqq 2K \sum_{a_i \in N_2} g(x;a_i) \\
&\leqq 2K \sum_{a_i \in N_2} \frac{\sum_{j=1}^n (x_j - a_{ij}/\nu)^2}{\delta^2} g(x;a_i) \\
&\leqq \frac{2K}{\nu^2\delta^2} \sum_{a_i \in N} \sum_{j=1}^n (\nu x_j - a_{ij})^2 g(x;a_i) \\
&= \frac{2K}{\nu^2\delta^2}\left[\nu^2 \sum_{j=1}^n (x_j)^2 - 2\nu \sum_{j=1}^n x_j \sum_{a_i \in N} a_{ij}\cdot g(x;a_i) + \sum_{j=1}^n \sum_{a_i \in N} (a_{ij})^2 g(x;a_i)\right] \\
&= 2K \left[1 - \sum_{j=1}^n (x_j)^2\right]\Big/\nu\delta^2 \qquad \text{(by (2) and (3))} \\
&\leqq \varepsilon \quad \text{if} \quad \nu \geqq 2K(1 - 1/n)/\delta^2\varepsilon.
\end{aligned}$$

Step II. Let $f^\nu(x) = \sum_{a_i \in N} f(a_i/\nu) g(x;a_i)$, the polynomial approximation obtained in Step I. Since $f(a_i/\nu)$ belongs to S_{n-1}, $f^\nu(x)$ is also a continuous mapping from S_{n-1} into itself by (1).

If the theorem is proved for $f^\nu(x)$, then for each ν, there is a point x^ν such that $f^\nu(x^\nu) = x^\nu \in S_{n-1}$. Since S_{n-1} is compact, some subsequence $\{x^{\nu_i}\}$ converges to a point x^* in S_{n-1}. As $\lim_{i\to\infty} f^{\nu_i}(x) = f(x)$ uniformly on S_{n-1}, $f(x^*) = \lim_{i\to\infty} f^{\nu_i}(x^{\nu_i}) = \lim_{i\to\infty} x^{\nu_i} = x^*$. Therefore it suffices to obtain a proof for the case of $f^\nu(x)$.

[3] $\|z\| = [\sum (z_i)^2]^{1/2}$.

Step III. We can prove the fixed-point theorem for S_1 by using the intermediate value theorem on continuous functions. We shall prove the general case by induction.

Let $f(x): S_{n-1} \to S_{n-1}$ be continuous and $f^\nu(x)$ be the polynomial obtained in Step I. In case $f_n^\nu(0, \cdots, 0, 1) = 1$, then $(0, \cdots, 0, 1)$ is the fixed point for f^ν since $f^\nu(0, \cdots, 0, 1)$ is in S_{n-1}, and therefore we may assume that

$$f_n^\nu(0, \cdots, 0, 1) < 1. \tag{5}$$

Now consider, for given $\xi = (\xi_1, \cdots, \xi_{n-1}) \in S_{n-2}$, the following polynomial equation in x_n:

$$f_n^\nu(\xi, x_n) = x_n(1 + x_n)^{\nu - 1}. \tag{6}$$

We obtain ν roots as solutions to (6), and designate the real and imaginary parts of these roots by $R_l(\xi)$ and $I_l(\xi)$, $l = 1, \cdots, \nu$. Let us then put

$$\rho(\xi, t) = \max_{1 \leqq l \leqq \nu} [R_l(\xi) - t|I_l(\xi)|],$$

which is continuous for $\xi \in S_{n-2}$ and $t \geqq 0$. By (5) and the intermediate value theorem for univariant continuous functions, equation (6) yields a real and non-negative root for any $\xi \in S_{n-2}$, and hence $\rho(\xi, t) \geqq 0$.

Studies on mathematical models of general economic equilibrium have been centered around the continuity of excess demand functions, say $E(p) = [E_1(p), \cdots, E_n(p)]$, with prices $p = (p_1, \cdots, p_n)$ as the variables, and the so-called Walras' law [8] $\sum_{i=1}^n p_i E_i(p) \equiv 0$ that holds identically on price space. Among others, Uzawa [7] established that Brouwer's fixed-point theorem [3] is equivalent to stating that for continuous excess demand function $E(p)$ defined on S_{n-1} there exists an equilibrium price constellation $p^* \in S_{n-1}$ such that $E(p^*) \leqq 0$, using a transformation $E_i(p) = \phi_i(p) - p_i(\sum p_j \phi_j / \sum p_j^2)$; $\phi(p): S_{n-1} \to S_{n-1}$. Motivated by this observation, let us now introduce for given value of $t \geqq 0$,

$$h_i(\xi; t) = f_i^\nu\left(\frac{\xi}{1 + \rho(\xi, t)}, \frac{\rho(\xi, t)}{1 + \rho(\xi, t)}\right) - \lambda \xi_i, \quad i = 1, \cdots, n - 1, \tag{7}$$

where

$$\lambda = \sum_{j=1}^{n-1} \xi_j f_j^\nu\left(\frac{\xi}{1 + \rho(\xi, t)}, \frac{\rho(\xi, t)}{1 + \rho(\xi, t)}\right) \Big/ \sum_{j=1}^{n-1} (\xi_j)^2.$$

This mapping satisfies Walras' law, that is, the identity

$$\sum_{i=1}^{n-1} \xi_i \cdot h_i(\xi; t) \equiv 0 \quad \text{for} \quad \xi \in S_{n-2}. \tag{8}$$

Then the following mapping used by Nikaidô [5] in establishing the existence theorem of a general equilibrium price:

$$d_i(\xi; t) = \frac{\xi_i + \max(h_i, 0)}{1 + \sum_{j=1}^{n-1} \max(h_j, 0)}, \qquad i = 1, \cdots, n - 1,$$

is a continuous mapping from S_{n-2} into itself, given t. Therefore, by the induction hypothesis, it admits a fixed point ξ^t for given $t \geqq 0$ such that

$$d(\xi^t, t) = \xi^t \in S_{n-2}. \tag{9}$$

Equation (9) implies

$$\max(h_i(\xi^t; t), 0) = \xi_i^t \sum_{j=1}^{n-1} \max(h_j(\xi^t; t), 0), \quad i = 1, \cdots, n-1,$$

and hence together with (8),

$$\begin{aligned}\sum_{i=1}^{n-1} [\max(h_i(\xi^t; t), 0)]^2 &= \sum_{i=1}^{n-1} h_i(\xi^t; t) \cdot \xi_i^t \cdot \sum_{j=1}^{n-1} \max(h_j(\xi^t; t), 0) \\ &= 0.\end{aligned}$$

This leads to

$$h_i(\xi^t; t) \leqq 0, \qquad i = 1, \cdots, n-1. \tag{10}$$

In view of (8), (10) is subject to a restriction

$$h_i(\xi^t; t) = 0 \quad \text{if} \quad \xi_i^t > 0. \tag{11}$$

Since the f_i^ν's are nonnegative, (10) and (11) must lead to

$$f_i^\nu\left(\frac{\xi^t}{1 + \rho(\xi^t, t)}, \frac{\rho(\xi^t, t)}{1 + \rho(\xi^t, t)}\right) = \lambda^t \xi_i^t, \qquad i = 1, \cdots, n-1, \tag{12}$$

where λ^t is defined by the formula in (7) with $\xi = \xi^t$.

We now take a nonnegative sequence $\{t_\eta\}$ such that $\lim_{\eta\to\infty} t_\eta = +\infty$, and the corresponding fixed point ξ^{t_η}. By the standard compactness argument, we may assume $\lim_{\eta\to\infty} \xi^{t_\eta} = \xi^* \in S_{n-2}$.

On the other hand, there is an index θ such that $\rho(\xi^{t_\eta}, t_\eta)$ is attained infinitely many times by $R_\theta(\xi^{t_\eta}) - t_\eta|I_\theta(\xi^{t_\eta})|$. Again by taking a subsequence $\{t_{\eta_\iota}\}$ of $\{t_\eta\}$ and $\{\xi^{t_{\eta_\iota}}\}$ of $\{\xi^{t_\eta}\}$, we may put

$$\rho(\xi^{t_{\eta_\iota}}, t_{\eta_\iota}) = R_\theta(\xi^{t_{\eta_\iota}}) - t_{\eta_\iota}|I_\theta(\xi^{t_{\eta_\iota}})| \quad \text{for all } t_{\eta_\iota}.$$

Since $\rho(\xi, t) \geqq 0$ and $R_\theta(\xi)$ is bounded on S_{n-2}, $R_\theta(\xi^{t_{\eta_\iota}}) \geqq t_{\eta_\iota}|I_\theta(\xi^{t_{\eta_\iota}})|$ implies

$$I_\theta(\xi^*) = 0. \tag{13}$$

As (6) has a real nonnegative solution for $\xi \in S_{n-2}$, there is another index $\tilde\theta$ such that

$$R_{\tilde\theta}(\xi^{t_{\eta_\iota}}) = \max_{1 \leqq l \leqq \nu, I_l(\xi^{t_{\eta_l}}) = 0} R_l(\xi^{t_{\eta_l}})$$

infinitely many times. Taking again a subsequence $\{\xi^{t_{\eta_{\iota'}}}\}$ of $\{\xi^{t_{\eta_\iota}}\}$, we may put

$$R_\theta(\xi^{t_{\eta_{\iota'}}}) \geqq R_\theta(\xi^{t_{\eta_{\iota'}}}) - t_{\eta_{\iota'}}|I_\theta(\xi^{t_{\eta_{\iota'}}})| \geqq R_{\tilde\theta}(\xi^{t_{\eta_{\iota'}}}) \quad \text{for all } t_{\eta_{\iota'}}.$$

This, with (13), leads to

$$\lim_{\iota'\to\infty} t_{\eta_{\iota'}}|I_\theta(\xi^{t_{\eta_{\iota'}}})| = 0. \tag{14}$$

Thus by (6), (12), (13), and (14), we have

$$
(15)\qquad
\begin{aligned}
&f_i^{\nu}\left(\frac{\xi^*}{1+R_\theta(\xi^*)}, \frac{R_\theta(\xi^*)}{1+R_\theta(\xi^*)}\right) = \lambda^*\xi_i^*, \qquad i = 1, \cdots, n-1,\\
&f_n^{\nu}(\xi^*, R_\theta(\xi^*)) = R_\theta(\xi^*)\cdot[1 + R_\theta(\xi^*)]^{\nu-1},
\end{aligned}
$$

where $\lambda^* = \lim_{\iota'\to\infty} \lambda^{t_{n_{\iota'}}}$.

As f_n^{ν} is homogeneous of degree ν with respect to (ξ, x_n), we have

$$
f_n^{\nu}\left(\frac{\xi^*}{1+R_\theta(\xi^*)}, \frac{R_\theta(\xi^*)}{1+R_\theta(\xi^*)}\right) = \frac{R_\theta(\xi^*)}{1+R_\theta(\xi^*)}.
$$

This with (1) and (15) yields $\lambda^* = 1/[1 + R_\theta(\xi^*)]$. We have thus proved that

$$
x^{\nu} = \left[\frac{\xi^*}{1+R_\theta(\xi^*)}, \frac{R_\theta(\xi^*)}{1+R_\theta(\xi^*)}\right]
$$

is a fixed point for f^{ν} on S_{n-1}, that is, $f^{\nu}(x^{\nu}) = x^{\nu} \in S_{n-1}$.

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BRANCHING SOLUTIONS OF EQUATIONS CONTAINING SEVERAL PARAMETERS*

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Abstract. Let $\mathscr{B}$ be a complex (or real) Banach space and

$$F(\boldsymbol{\lambda}, \boldsymbol{\beta}, u, w) = \sum_{j=1}^{m} \lambda_j[L_j u + N_j(u, w)] + \sum_{i=1}^{m} \beta_i S_i(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)$$

be a bounded map from $C^{m+n} \times \mathscr{B}^2$ (or $R^{m+n} \times \mathscr{B}^2$) to $\mathscr{B}$ where the L_j are linear and the N_j are higher order in u. Assuming $|\beta|$ and $\|w\|$ small a generalization of the method of Lyapunov and Schmidt is used to establish the existence of solutions u to $u = F(\boldsymbol{\lambda}, \boldsymbol{\beta}, u, w)$ near those $\lambda_0 = (\lambda_{01}, \cdots, \lambda_{0m}) \in C^m$ (or R^m) for which 1 is a simple eigenvalue of $L_0 = \sum_{j=1}^m \lambda_{0j} L_j$. The branching equation is examined by employing the Weierstrass preparation theorem. An example of a physical problem which leads to such equations is given.

Introduction. In this paper we use a generalization of the method of Lyapunov and Schmidt to establish the existence of solutions of a class of operator equations. Let $\mathscr{B}$ be a complex (or real) Banach space. Let F be a function which maps $C^{m+n} \times \mathscr{B}^2$ (or $R^{m+n} \times \mathscr{B}^2$) into $\mathscr{B}$.

$$F(\lambda_1, \cdots, \lambda_m, \beta_1, \cdots, \beta_n, u, w) = \boldsymbol{\lambda} \cdot \mathbf{L}u + \boldsymbol{\lambda} \cdot \mathbf{N}(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, w), \text{ where}$$

$$\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_m); \quad \boldsymbol{\beta} = (\beta_1, \cdots, \beta_n);$$

$$\boldsymbol{\lambda} \cdot \mathbf{L}u = \sum_{j=1}^{m} \lambda_j L_j u; \quad \boldsymbol{\lambda} \cdot \mathbf{N}(u, w) = \sum_{j=1}^{m} \lambda_j N_j(u, w)$$

and $\boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, w) = \sum_{i=1}^n \beta_i S_i(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)$. Explicit hypotheses will be given later, but for now we require each of the operators L_j to be linear and completely continuous while each N_j and S_i is to be analytic and, in some sense, bounded. We seek values of $\boldsymbol{\lambda}, \boldsymbol{\beta}$ and w for which the equation $u = F(\boldsymbol{\lambda}, \boldsymbol{\beta}, u, w)$ has solutions u in $\mathscr{B}$. It will be shown that near any $\boldsymbol{\lambda}_0$ for which 1 is an algebraically simple eigenvalue of $\boldsymbol{\lambda}_0 \cdot \mathbf{L}$ there exist u in $\mathscr{B}$ satisfying

$$u = \boldsymbol{\lambda} \cdot \mathbf{L}u + \boldsymbol{\lambda} \cdot \mathbf{N}(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, w). \tag{1}$$

Such equations arise quite naturally when one attempts to establish the existence of solutions of certain boundary value problems of mathematical physics. The terms $\boldsymbol{\beta} \cdot \mathbf{S}$ usually account for forcing terms or may arise when the geometry of the problem does not quite fit a standard coordinate system. The w is included here so that when $m = 1$ and $\boldsymbol{\beta} = 0$ we have an equation of the type Sather considers in [3] near eigenvalues of multiplicity greater than one. An example of a problem connected with the steady state Navier–Stokes equation which leads to (1) with $m = n = 1$ can be found in [8].

The following example, although interesting in its own right, is presented primarily to further motivate the study of equations like (1).

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1. An example. Consider a simple pendulum with unit length and unit mass which is free to swing in the xy-plane. Suppose that the pendulum is supported at the origin, that the gravitational force $\mathbf{g} = -g\mathbf{j}$ and that $u = u(t)$ measures the deflection from the negative y-axis in radians. Further assume that fixed on the lines $x = \pm d$, $d > 1$, are two wires of infinite length, that the wires carry c units of positive charge per unit length and that the bob of the pendulum carries 1 unit of positive charge. It is known [2] that $2c/r$ is the magnitude of the force exerted on a unit point charge r units away from an infinite wire whose linear charge density is c. Thus the force E on the pendulum due to the field set up by the two wires is given by

$$\mathbf{E} = \frac{-4c \sin u}{d^2 - \sin^2 u}\mathbf{i}.$$

Assume, in addition to the gravitational and electrostatic forces, there is a forcing term whose tangential component is

$$f(t) = -\beta_1 \sin t - \beta_2 \sin 2t.$$

If we equate the tangential component of the acceleration $\ddot{u}$ to the sum of $f(t)$ and the tangential components of $\mathbf{g}$ and $\mathbf{E}$ we have

$$\ddot{u} + \lambda_1 \sin u + \frac{\lambda_2 \sin 2u}{1 - d^{-2}\sin^2 u} + \beta_1 \sin t + \beta_2 \sin 2t = 0, \tag{2}$$

where $\lambda_1(g) = g$ and $\lambda_2(c) = 2cd^{-2}$. In connection with the problem of finding periodic solutions to (2) we seek values of $\lambda_1, \lambda_2, \beta_1, \beta_2$ for which (2), subject to the boundary conditions

$$u(0) = u(\pi) = 0, \tag{3}$$

has solutions. If $\beta_1 = \beta_2 = 0$ we will be interested only in nontrivial solutions to (2)–(3).

Recalling that the Green's function for $-d^2/dt^2$ on the manifold defined by (3) is

$$g(t, t') = \frac{t + t'}{2} - \frac{|t - t'|}{2} - \frac{tt'}{\pi},$$

we have the following equivalent formulation in $C[0, \pi]$ of our boundary value problem (2)–(3):

$$u(t) = \int_0^\pi g(t, t')\left[\lambda_1 \sin u(t') + \frac{\lambda_2 \sin 2u(t')}{1 - d^{-2}\sin^2 u(t')} + \beta_1 \sin t' + \beta_2 \sin 2t'\right] dt'. \tag{4}$$

If we let

$$L_n u = \int_0^\pi g(t, t') n u(t')\, dt', \qquad n = 1, 2,$$

$$N_1(u) = \int_0^\pi g(t, t')(-u(t') + \sin u(t'))\, dt',$$

$$N_2(u) = \int_0^\pi g(t,t')\left(-2u(t') + \frac{\sin 2u(t')}{1 - d^{-2}\sin^2 u(t')}\right) dt',$$

$$S_n = \int_0^\pi g(t,t') \sin nt' \, dt', \qquad n = 1, 2,$$

then (4) is equivalent to

$$u = \lambda_1 L_1 u + \lambda_2 L_2 u + \lambda_1 N_1(u) + \lambda_2 N_2(u) + \beta_1 S_1 + \beta_2 S_2, \tag{5}$$

which is (1) with $m = n = 2$ and $\mathscr{B} = C[0, \pi]$ with the uniform norm. In the next two sections we examine the general equation (1) and then later apply some of our results to (5).

2. The complex case. We now consider (1) in the case when $F:C^{m+n} \times \mathscr{B}^2 \to \mathscr{B}$ and $\mathscr{B}$ is a complex Banach space. Regarding the operators in (1) we make the following hypotheses:

H1. L_j, $j = 1, 2, \cdots, m$, is linear and completely continuous.

H2. $\mathbf{N}(u, w) = (N_1(u, w), N_2(u, w), \cdots, N_m(u, w))$ is higher order in u and analytic at $(0, 0)$, i.e., $\mathbf{N}(u, w) = \mathbf{A}w + \sum_{r+s \geqq 2} \mathbf{A}^{rs}(u, w)$, where $\mathbf{A}w = (A_1 w, A_2, \cdots, A_m w)$ and each $A_j:\mathscr{B} \to \mathscr{B}$ is bounded and linear; $\mathbf{A}^{rs}(u, w) = (A_1^{rs}(u, w), A_2^{rs}(u, w), \cdots, A_m^{rs}(u, w))$ and each $A_j^{rs}(u, w)$ is a homogeneous polynomial of degree r in u and s in w.

H3. There exist functions $Q_j:R^3 \to R^+$ such that $\|N_j(u_1, w_1) - N_j(u_2, w_2)\| \leqq Q_j(\|u_1\|, \|u_2\|, \|w\|)\|u_1 - u_2\|$ and $\lim_{|x| \to 0} Q_j(x_1, x_2, x_3) = 0$, $j = 1, 2, \cdots, m$.

H4. For every bounded subset of $C^{m+n} \times \mathscr{B}$ there is a constant s_i such that $\|S_i(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)\| \leqq s_i$, $i = 1, 2, \cdots, n$.

H5. $S_i(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)$ is analytic, $i = 1, 2, \cdots, n$.

Suppose $\boldsymbol{\lambda}_0 = (\lambda_{01}, \lambda_{02}, \cdots, \lambda_{0m}) \in C^m$ is such that 1 is an algebraically simple eigenvalue of the linear operator L_0 defined by $L_0 = \sum_{j=1}^m \lambda_{0j} L_j = \boldsymbol{\lambda}_0 \cdot \mathbf{L}$. By simple we mean $\dim \mathscr{N}(I - L_0)^k = 1$, $k = 1, 2, \cdots$, where $\mathscr{N}(I - L_0)^k$ is the null space of $(I - L_0)^k$. To simplify the notation, set $\mathscr{N} = \mathscr{N}(I - L_0)$ and let $\mathscr{R}$ denote the range of $I - L_0$. Since the linear operator L_0 is completely continuous and 1 is a simple eigenvalue, we know [6, § 5.4] $\mathscr{B}$ can be expressed as the direct sum of the invariant subspace $\mathscr{N}$ and $\mathscr{R}$. Let P be the projection defined by $Pu = u_1$, where $u = u_1 + u_2$, $u_1 \in \mathscr{N}$, $u_2 \in \mathscr{R}$, is the direct sum decomposition of u. Since $\mathscr{N}$ and $\mathscr{R}$ are closed, P is continuous.

We now introduce some quantities which are relevant to our study of equation (1). Let $u_0 \in \mathscr{N}$ be such that $\|u_0\| = 1$. Any u in $\mathscr{B}$ can be written unambiguously in the form $u = \zeta u_0 + v$, where $\zeta \in C$ and $v \in \mathscr{R}$ so that $Pu = \zeta u_0$ for any $u \in \mathscr{B}$. In our later work it is sometimes helpful to look at $|\zeta|$ as a measure of how much of u is in $\mathscr{N}$. Since we are interested in solutions to (1) for $\boldsymbol{\lambda}$ near $\boldsymbol{\lambda}_0$, we introduce the vector $\boldsymbol{\tau} = (\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) \in C^m$. Finally, as a convenience, we normalize w by setting $w = \sigma w_0$, where $\sigma \in C$ and $\|w_0\| = 1$.

Remark 2.1. If $\mathscr{B}$ happens to be a Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$, we can define the projection P as follows. Let $\mathscr{N}^*$ be the null space of $I - L_0^*$, where L_0^* is the adjoint of L_0. From the Riesz theory for completely continuous linear operators [4] recall $\dim \mathscr{N} = \dim \mathscr{N}^*$ and $\mathscr{R} = \mathscr{N}^{*\perp}$. Moreover, since 1 is a simple eigenvalue of L_0, $\mathcal{N}$ and $\mathcal{R}$ are complementary subspaces and we can select $u_0 \in \mathcal{N}$ and $v_0 \in \mathcal{N}^*$ such that $\|u_0\| = 1$ and $\langle u_0, v_0 \rangle = 1$. We can then define $Pu = \langle u, v_0 \rangle u_0$, $u \in \mathcal{H}$.

To establish the existence of solutions of (1) using our generalized Lyapunov and Schmidt method, it is necessary to invert $I - L_0$ on $\mathcal{R}$. One way to do this is by constructing an invertible linear operator $I - \hat{L}_0$ which agrees with $I - L_0$ on $\mathcal{R}$. Following E. Schmidt [7, p. 29], we set

$$I - \hat{L}_0 = \begin{cases} I - L_0, & u \in \mathcal{R}, \\ I, & u \in \mathcal{N}, \end{cases}$$

which holds if we define $\hat{L}_0$ by

$$\hat{L}_0 u = L_0 u - Pu, \qquad u \in \mathcal{B}.$$

Since $I - L_0$ maps $\mathcal{R}$ onto $\mathcal{R}$ in a one-to-one manner [6, § 5.4], $I - \hat{L}_0$ is a one-to-one mapping from $\mathcal{B}$ onto itself, so from the open mapping theorem we have the following lemma.

LEMMA 2.1. *$T = (I - \hat{L}_0)^{-1}$ exists and is a bounded linear operator.*

Using Lemma 2.1 we can put (1) in a form which admits an application of the contraction mapping principle. We begin by rewriting (1) as

$$(I - L_0)u = \boldsymbol{\tau} \cdot \mathbf{L}u + \boldsymbol{\lambda} \cdot \mathbf{N}(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, e), \tag{6}$$

which in turn is equivalent to

$$(I - \hat{L}_0)u = Pu + \boldsymbol{\tau} \cdot \mathbf{L}u + \boldsymbol{\lambda} \cdot \mathbf{N}(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, w). \tag{7}$$

If we let $\mathbf{L}' = T\mathbf{L}$, $\mathbf{N}' = T\mathbf{N}$, $\mathbf{S}' = T\mathbf{S}$ and note that $TPu = Pu$, then applying T to both sides of (7) we have

$$u = Pu + \boldsymbol{\tau} \cdot \mathbf{L}'u + \boldsymbol{\lambda} \cdot \mathbf{N}'(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}'(\boldsymbol{\lambda}, \boldsymbol{\beta}, w). \tag{8}$$

Recalling $Pu = \zeta u_0$ we see from (8) that (1) is equivalent to the pair of equations

$$u = \zeta u_0 + \boldsymbol{\tau} \cdot \mathbf{L}'u + \boldsymbol{\lambda} \cdot \mathbf{N}'(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}'(\boldsymbol{\lambda}, \boldsymbol{\beta}, w), \tag{9}$$

$$Pu = \zeta u_0 \tag{10}$$

in the sense that if $u^* = u^*(\zeta, \boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ is a fixed point of (9) and $\zeta = \zeta^*(\boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ satisfies $Pu^* = \zeta^* u_0$, then $u^*(\zeta^*, \boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ is a solution of (1).

LEMMA 2.2. *There exist positive numbers ζ_0, τ_0, β_0, σ_0, δ_0 such that for fixed ζ, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, σ satisfying $|\zeta| < \zeta_0$, $|\boldsymbol{\tau}| < \tau_0$, $|\boldsymbol{\beta}| < \beta_0$, $|\sigma| < \sigma_0$, the right side of* (9) *defines a contraction map on the ball $\|u\| < \delta_0$.*

Proof. The norm of the right side of (9) is clearly bounded by

$$|\zeta| + \|T\| \sum_{i=1}^{n} |\beta_i| s_i + \|T\| \sum_{j=1}^{m} [\tau_j \|L_j\| + \lambda_j Q_j(\|u\|, 0, \sigma)] \|u\|. \tag{11}$$

In view of the boundedness of the operators L_j and H3, we can choose τ_0, σ_0, δ_0 such that if $|\tau| < \tau_0$, $|\sigma| < \sigma_0 \|u_i\| < \delta_0$, $i = 1, 2$, then

$$\|T\| \sum_{j=1}^{m} [\tau_j \|L_j\| + \lambda_j Q_j(\|u_1\|, \|u_2\|, \sigma)] < c, \tag{12}$$

where $0 < c < 1$. Next take ζ_0, β_0 so that if $|\zeta| < \zeta_0$ and $|\beta| < \beta_0$,

$$(13) \qquad |\zeta| + \|T\| \sum_{i=1}^{m} |\beta_i| s_i < (1 - c)\delta_0.$$

Now if ζ, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, σ and $\|u\|$ satisfy (12) and (13), it follows from (11) that the right side of (9) maps $B(\delta_0) = \{u : \|u\| < \delta_0\}$ into itself.

If we let $u_1, u_2 \in B(\delta_0)$ and use (12) we see

$$\begin{aligned} &\|(\zeta u_0 + \boldsymbol{\tau} \cdot \mathbf{L}' u_1 + \boldsymbol{\lambda} \cdot \mathbf{N}'(u_1, w) + \boldsymbol{\beta} \cdot \mathbf{S}'(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)) \\ &\quad - (\zeta u_0 + \boldsymbol{\tau} \cdot \mathbf{L}' u_2 + \boldsymbol{\lambda} \cdot \mathbf{N}(u_2, w) + \boldsymbol{\beta} \cdot S'(\boldsymbol{\lambda}, \boldsymbol{\beta}, w))\| \\ &\leqq \|T\| \sum_{j=1}^{m} [\tau_j \|L_j\| + \lambda_j Q_j(\|u_1\|, \|u_2\|, \sigma)] \|u_1 - u_2\| < c\|u_1 - u_2\|, \end{aligned}$$

so that the right side of (9) is a contraction map on $B(\delta_0)$ for fixed ζ, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, σ satisfying (12) and (13).

From the contraction mapping principle and the analyticity assumptions H2 and H5, it follows that if we choose ζ, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, σ, δ_0 satisfying (12) and (13), then (9) has a unique fixed point u^* in $B(\delta_0)$ and u^* is expressible as a convergent power series in $\mathscr{B}$.

$$(14) \qquad u^*(\zeta, \boldsymbol{\tau}, \boldsymbol{\beta}, \sigma) = \sum u_{j\mathbf{p}k\mathbf{q}} \zeta^j \boldsymbol{\tau}^{\mathbf{p}} \sigma^k \boldsymbol{\beta}^{\mathbf{q}}, \qquad u_{j\mathbf{p}k\mathbf{q}} \in \mathscr{B},$$

where the summation is over $j + |\mathbf{p}| + k + |\mathbf{q}| > 0$ and

$$\boldsymbol{\tau}^{\mathbf{p}} = \prod_{j=1}^{m} \tau_j^{p_j}, \quad \boldsymbol{\beta}^{\mathbf{q}} = \prod_{i=1}^{n} \beta_i^{q_i}, \quad |\mathbf{p}| = \sum_{j=1}^{m} p_j \quad \text{and} \quad |\mathbf{q}| = \sum_{i=1}^{n} q_i.$$

Remark 2.2. If ζ, $\boldsymbol{\tau}$, σ, $\boldsymbol{\beta}$ and δ_0 satisfy (12) and (13), then u^* is the limit of the sequence $u^0 = u_0$ and, for $k = 0, 1, \cdots$,

$$u^{k+1} = \zeta u_0 + \boldsymbol{\tau} \cdot \mathbf{L}' u^k + \boldsymbol{\lambda} \cdot \mathbf{N}'(u^k, w) + \boldsymbol{\beta} \cdot \mathbf{S}'(\boldsymbol{\lambda}, \boldsymbol{\beta}, w).$$

We now turn to the problem of determining the number of small solutions to (1). Since, for any choice of ζ, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, σ, δ_0 satisfying (12) and (13), equation (9) has a unique fixed point, it follows that the number of solutions of (1) in $B(\delta_0)$ coincides with the number of small solutions $\zeta = \zeta(\boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ of

$$(15) \qquad Pu^*(\zeta, \boldsymbol{\tau}, \boldsymbol{\beta}, \sigma) = \zeta u_0.$$

Before proceeding we rewrite (15) in a form which displays its dependence on the operators L, N and S. From equations (9) and (15) we have

$$P[\zeta u_0 + \boldsymbol{\tau} \cdot \mathbf{L}' u^* + \boldsymbol{\lambda} \cdot \mathbf{N}'(u, w) + \boldsymbol{\beta} \cdot \mathbf{S}'(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)] = \zeta u_0,$$

which is equivalent to

$$T[P(\boldsymbol{\tau} \cdot \mathbf{L} u^*) + P(\boldsymbol{\lambda} \cdot \mathbf{N}(u, w) + P(\boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, w))] = 0.$$

Since T is invertible we see (15) is equivalent to

$$(16) \qquad P(\boldsymbol{\tau} \cdot \mathbf{L} u^*) + P[(\boldsymbol{\lambda}_0 + \boldsymbol{\tau}) \cdot \mathbf{N}(u^*, w)] + P(\boldsymbol{\beta} \cdot \mathbf{S}(\boldsymbol{\lambda}, \boldsymbol{\beta}, w)) = 0.$$

Thus, to determine the number of small solutions to (1) we examine equation (16) which can be viewed as a generalization of the usual branching equation in the method of Lyapunov and Schmidt. A discussion of the branching equation can be found in [5, § 9].

LEMMA 2.3. *If*

$$P(\boldsymbol{\lambda}_0 \cdot \mathbf{A}^{r0}(u_0, 0)) = 0, \qquad r = 2, 3, \cdots, p-1, \tag{17}$$

and

$$P(\boldsymbol{\lambda}_0 \cdot \mathbf{A}^{p0}(u_0, 0)) \neq 0,$$

then, if $|\boldsymbol{\tau}|$, $|\boldsymbol{\beta}|$ *and* $|\sigma|$ *are sufficiently small,* (16) *has p small solutions* $\zeta = \zeta(\boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ *vanishing at* $(0, 0, 0)$ *and each* $\zeta(\boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ *is continuous in* $\boldsymbol{\tau}, \boldsymbol{\beta}$ *and* σ.

Proof. H2 and H5 imply that (16) can be put in the form $(\sum a_{j\mathbf{r}k\mathbf{s}}\zeta^j \boldsymbol{\tau}^{\mathbf{r}} \sigma^k \boldsymbol{\beta}^{\mathbf{s}})u_0 = 0$ or

$$\sum a_{j\mathbf{r}k\mathbf{q}}\zeta^j \boldsymbol{\tau}^{\mathbf{r}} \sigma^k \beta^{\mathbf{q}} = 0, \tag{18}$$

where the $a_{j\mathbf{r}k\mathbf{q}}$ are calculable scalars and the summation is over $j + |\mathbf{r}| + k + |\mathbf{q}| > 0$ [7]. From the form of the right side of (9) it follows that $u^* = \zeta u_0 + \text{h.o.t.}$, where h.o.t. stands for any term in (14) which contains ζ to a power greater than one or vanishes with $\boldsymbol{\tau}$, σ and $\boldsymbol{\beta}$. Since

$$N_j(u^*, 0) = N_j(\zeta u_0 + \text{h.o.t.}, 0) = \sum_{r \geqq 2} A_j^{r0}(\zeta u_0 + \text{h.o.t.}, 0) = \sum_{r \geqq 2} \zeta^r A_j^{r0}(u_0, 0) + \text{h.o.t.},$$

we see from (16) and (17) that $a_{j000} = 0$ for $j = 2, 3, \cdots, p-1$ and $a_{p000} \neq 0$. The Weierstrass preparation theorem [1] implies that for $|\zeta|$, $|\boldsymbol{\tau}|$, $|\boldsymbol{\beta}|$ and $|\sigma|$ sufficiently small, (18) is equivalent to

$$(\zeta^p + A_{p-1}\zeta^{p-1} + \cdots + A_1\zeta + A_0)D(\zeta, \boldsymbol{\tau}, \boldsymbol{\beta}, \sigma) = 0, \tag{19}$$

where $A_j = A_j(\boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$, $j = 0, 1, \cdots, p-1$, is analytic and vanishes at $(0, 0, 0)$, while $D(\zeta, \boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ is analytic and nonvanishing in a neighborhood of $(0, 0, 0)$. Since the zeros of the "polynomial" part of (19) give the small solutions to (16), this completes the proof.

Remark 2.3. The coefficients of the power series for A_j and D can be determined from certain recurrence relations. From (16) we see the coefficient of $\zeta\tau_j$ in (18) is given by PL_ju_0. Since

$$u_0 = L_0u_0 = PL_0u_0 - P\sum_{j=1}^{m} \lambda_{0j}L_ju_0 = \sum_{j=1}^{m} \lambda_{0j}PL_ju_0,$$

$PL_ju_0 \neq 0$ for some values of j and (18) contains terms of the form $\zeta\tau_j$. Thus in (19) there is some $A_j \not\equiv 0$.

Collecting the results in the above lemmas we have the following result when $\mathscr{B}$ is a complex Banach space and P is the projection of $\mathscr{B}$ onto $\mathscr{N}$ determined by the decomposition $\mathscr{B} = \mathscr{N} \oplus \mathscr{R}$.

THEOREM 2.1. *Suppose* H1–H5 *hold and that* $\boldsymbol{\lambda}_0 \in C^m$ *is such that* 1 *is an algebraically simple eigenvalue of* $L_0 = \boldsymbol{\lambda}_0 \cdot \mathbf{L}$ *with* $u_0 = L_0u_0$, $\|u_0\| = 1$. *If* (17) *holds, then for all sufficiently small values of* $|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0|$, $|\boldsymbol{\beta}|$ *and* $\|w\|$, *equation* (1) *has p solutions counting multiplicities. Moreover, each solution is continuous in* $\boldsymbol{\lambda} - \boldsymbol{\lambda}_0, \boldsymbol{\beta}$ *and* w.

If we set $\sigma = |\boldsymbol{\beta}| = 0$ equation (1) becomes

$$u = \boldsymbol{\lambda} \cdot \mathbf{L}u + \boldsymbol{\lambda} \cdot \mathbf{N}(u, 0). \tag{20}$$

$u = 0$ is a solution of (20) for any $\boldsymbol{\lambda} \in C^m$, but, according to Theorem 2.1 and Remark 2.3, every neighborhood of $\boldsymbol{\lambda}_0$ contains $\boldsymbol{\lambda}$ for which (20) admits nontrivial solutions. We make the following definition for use here and in the next section.

DEFINITION 2.1. If every neighborhood in $C^m \times \mathscr{B}$ (or $R^m \times \mathscr{B}$) of $(\boldsymbol{\lambda}_0, 0)$ contains a pair $(\boldsymbol{\lambda}, u)$ with $\|u\| \neq 0$ satisfying (20), then $\boldsymbol{\lambda}_0$ is said to be a *bifurcation point* of (20).

Roughly stated then, Theorem 2.1 says that if $|\boldsymbol{\beta}|$ and $\|w\|$ are small, equation (1) has solutions near certain bifurcation points of (20). We conclude this section with some immediate consequences of Theorem 2.1.

COROLLARY 2.1. *If λ_{0j} is a simple characteristic value of L_j, then $(\delta_{1j}\lambda_{01}, \delta_{2j}\lambda_{02}, \cdots, \delta_{mj}\lambda_{0m})$ is a bifurcation point of* (20).

COROLLARY 2.2. *Suppose k of the linear operators, which without loss of generality we take to be $L_1, L_2, \cdots, L_k$, are related by $L_j = \alpha_j L_1, j = 1, 2, \cdots, k$. If λ_{01} is a simple characteristic value of L_1, then each point $(\lambda_1, \cdots, \lambda_k, 0, \cdots, 0)$ in R^m such that*

$$\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 + \cdots + \alpha_k\lambda_k + 0 \cdot \lambda_{k+1} + \cdots + 0 \cdot \lambda_m = \lambda_{01}$$

is a bifurcation point of (20).

3. The real case. In this section we take $\mathscr{B}$ to be a real Banach space and the parameters λ_j, β_i, $j = 1, 2, \cdots, m$, $i = 1, 2, \cdots, n$, in (1) to be real numbers. In Theorem 3.1 we give conditions which are sufficient to insure that (20) has a real bifurcation point. As a corollary we will see that on certain lines in R^m can be found values of $\boldsymbol{\lambda}$ for which (20) has a nontrivial solution. Finally we indicate how the methods of § 2 can be used to obtain results regarding the existence and multiplicity of solutions to (1) for $\mathscr{B}$ real.

THEOREM 3.1. *Suppose* H1–H3 *hold. If $\boldsymbol{\lambda}_0 = (\lambda_{01}, \lambda_{02}, \cdots, \lambda_{0m}) \in R^m$ is such that* 1 *is an algebraically simple eigenvalue of $L_0 = \boldsymbol{\lambda}_0 \cdot \mathbf{L}$, then $\boldsymbol{\lambda}_0$ is a bifurcation point of* (20).

Proof. Without loss of generality we can assume $\lambda_{0j} \neq 0$ if $j = 1, 2, \cdots, k$, while $\lambda_{0j} = 0$ if $j = k + 1, \cdots, m$, where $1 \leqq k \leqq m$. Let Γ be the line passing through the origin and λ_0.

$$\Gamma : \boldsymbol{\lambda} = \mu\boldsymbol{\lambda}_0, \qquad \mu \in R^1.$$

On the line Γ, (20) can be written as

$$u = \mu L_0 u + \mu N_0(u), \tag{21}$$

where we have set $N_0(u) = \sum_{j=1}^k \lambda_{0j} N_j(u, 0)$ for convenience. From H3 we see that $L_0 u + N_0(u)$ is strictly differentiable at 0 [5, p. 310]. Since $\mu = 1$ is a simple eigenvalue of L_0, it follows from Theorem 6 [5, p. 311] that $\mu = 1$ is a bifurcation point of (21) so that $\boldsymbol{\lambda}_0$ is a bifurcation point of (20).

COROLLARY 3.1. *If the hypotheses of Theorem* 3.1 *are satisfied, then on the line $\Gamma : \boldsymbol{\lambda} = \mu\boldsymbol{\lambda}_0, \mu \in R^1$, there are values of $\boldsymbol{\lambda}$ for which* (20) *admits a nontrivial solution.*

If H1–H5 hold and we again assume $\boldsymbol{\lambda}_0 \in R^m$ is such that $\boldsymbol{\lambda}_0 \cdot \mathbf{L}$ has 1 as an algebraically simple eigenvalue, we can apply the methods of § 2 to the case that $\mathscr{B}$ is real as follows. Lemmas 2.1 and 2.2 can be repeated for $\mathscr{B}$ real to establish the existence of a unique fixed point u^* of (9) with a power series representation (14). In analyzing the branching equation we must now ask, how many real solutions does (18) have when the a_{jrkq} are real? If we assume (17) holds, we can apply the Weierstrass preparation theorem as before to show (18) equivalent to (19). The analysis of the branching equation can thus be reduced to finding the real zeros $\zeta = \zeta(\boldsymbol{\tau}, \boldsymbol{\beta}, \sigma)$ of the polynomial part of (19) when $|\boldsymbol{\tau}|$, $|\boldsymbol{\beta}|$ and $|\sigma|$ are small. In this context we have the following theorem.

THEOREM 3.2. *Suppose the hypotheses of Theorem* 2.1 *hold with* $\boldsymbol{\lambda}_0 \in R^m$. *If* (17) *holds, and p is odd, then for all sufficiently small values of* $|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0|$, $|\beta|$ *and* $\|w\|$, *equation* (1) *has at least one real solution.*

Since the complex roots of the polynomial part of (19) appear in conjugate pairs, we have the following.

COROLLARY 3.2. *Under the hypotheses of Theorem* 3.2, *equation* (1) *has q solutions counting multiplicities where q is one of the numbers* $p - 2k$, $k - 0, 1, \cdots [P/2]$.

4. Remarks on the example. We now return to the example of § 1 to illustrate some of the above results. It can be shown that H1–H5 are satisfied by the operators in (5). Since $n^2, n = 1, 2, \cdots$, is a simple characteristic value of L_1 and $L_2 = 2L_1$, it follows that 1 is a simple eigenvalue of $L_0 = \lambda_{01}L_1 + \lambda_{01}L_2$, whenever $\boldsymbol{\lambda}_0 = (\lambda_{01}, \lambda_{02})$ is on the line $\Gamma_n : \lambda_1 + 2\lambda_2 = n^2, n = 1, 2, \cdots$. From Corollary 2.2 it now follows that any point on one of the lines Γ_n is a bifurcation point of (5) with $\beta_1 = \beta_2 = 0$. In what follows we take $\boldsymbol{\lambda}_0$ to be on Γ_n and, for physical reasons, assume λ_{01} and λ_{02} are nonnegative. With this choice of $\boldsymbol{\lambda}_0$ it is clear that $u_n(t) = \sin nt$ is a basis for $\mathscr{N}$, the null space of $I - L_0$. If we introduce the notation $[u, v] = (2/\pi)\int_0^\pi u(t)v(t)\,dt$, we can define the projection $P: C^1[0, \pi] \to \mathscr{N}$ by $Pu = [u, u_n]u_n$.

Recall from § 1,

$$N_1(u) = \int_0^\pi g(t, t')[\sin u(t') - u(t')]\,dt'$$

and

$$N_2(u) = \int_0^\pi g(t, t')\left[\frac{\sin 2u(t')}{1 - d^{-2}\sin^2 u(t')} - 2u(t')\right] dt.$$

By slightly modifying the notation introduced in H2 we can write

$$\mathbf{N}(u) = \sum_{n=2}^{\infty} \mathbf{A}^r(u),$$

where $\mathbf{A}^r(u) = A_1^r(u), A_2^r(u))$ and $A_j^r(u), j = 1, 2$, is a homogeneous polynomial in u of degree r. If we expand in a Taylor series about zero the bracketed terms in the integrands of N_1 and N_2, we see $\mathbf{A}^2(u) = (0, 0)$ and

$$\mathbf{A}^3(u) = \left(-\frac{1}{6}\int_0^\pi g(t, t')u^3(t')\,dt', \left(\frac{2}{d^2} - \frac{4}{3}\right)\int_0^\pi g(t, t')u^3(t')\,dt'\right).$$

We now show that if $0 \leqq \lambda_{02} \leqq n^2/3$, then, in (17), $p = 3$, while if $n^2/3 < \lambda_{02} \leqq n^2/2$, then $p = 3$ for all but one value of $d > 1$. To establish this we note that

$$P(\boldsymbol{\lambda}_0 \cdot \mathbf{A}^3(u_n)) = \left\{ -\frac{1}{6}\lambda_{01} + \left(\frac{2}{d^2} - \frac{4}{3}\right)\lambda_{02} \right\} \left[\int_0^\pi g(t, t') u_n^3(t')\, dt', u_n(t) \right] u_n.$$

Using $u_n'' = -n^2 u_n$ and $-g_{tt}(t, t') = \delta(t - t')$, we see, after integrating by parts, that

$$\left[\int_0^\pi g(t, t') u_n^3(t')\, dt', u_n(t) \right] = \frac{2}{\pi n^2} \int_0^\pi u_n^4(t)\, dt.$$

Thus $p = 3$ unless

$$-\frac{1}{6}\lambda_{01} + \left(\frac{2}{d^2} - \frac{4}{3}\right)\lambda_{02} = 0,$$

which holds on Γ_n when

$$d^2 = \frac{12\lambda_{02}}{n^2 + 6\lambda_{02}}.$$

The above equation yields a unique $d > 1$ if and only if $n^2/3 < \lambda_{02}$, which establishes our claim regarding p. If $p = 3$, Corollary 3.2 implies that for small β_1 and β_2, (5) has either one or three small solutions for any $\boldsymbol{\lambda}$ near one of the lines Γ_n.

We conclude with some qualitative remarks about the branching solutions to (5). For simplicity, take $\beta_1 = \beta_2 = 0$. The parameter $|\zeta|$ can be viewed as a measure of the maximum deflection of the pendulum from rest so that $\zeta = 0$ corresponds to the static case, $u \equiv 0$. The sign of ζ serves to determine the initial direction of the pendulum. Consider the plane S in $\lambda_1\lambda_2\zeta$-space which passes through λ_0 and is orthogonal to Γ_n. If $\boldsymbol{\lambda}$ near $\boldsymbol{\lambda}_0$ is in S, then we must have $\tau_2 = 2\tau_1$. From (16) we see that for $\boldsymbol{\lambda} \in S$, if we include only the lowest order terms, the branching equation is

$$5\tau_1\zeta + \left[-\frac{1}{6}\lambda_{01} + \left(\frac{2}{d^2} - \frac{4}{3}\right)\lambda_{02} \right]\zeta^3 = 0. \tag{22}$$

In [5, p. 313] it is shown that the sign of the coefficient of ζ^3 in (22) determines the direction of the branching. Applying this result we have:

(i) If $n^2/3 < \lambda_{02} \leqq n^2/2$ and $d^2 > 12\lambda_{02}/(n^2 + 6\lambda_{02})$ or $0 \leqq \lambda_{02} < n^2/3$, then in S $|\zeta|$ increases with τ_1, i.e., branching from Γ_n is away from the origin.

(ii) If $n^2/3 < \lambda_{02} \leqq n^2/2$ and $d^2 < 12\lambda_{02}/(n^2 + 6\lambda_{02})$, then in S, $|\zeta|$ increases with $-\tau_1$, i.e., branching from Γ_n is toward the origin.

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ON A SEQUENCE OF LINEAR TRIGONOMETRIC POLYNOMIAL OPERATORS*

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Abstract. A sequence of linear trigonometric polynomial operators is constructed with the help of the operators L_{np-p}, which are generalizations of Jackson operators, and is shown to have the order of approximation given by Jackson's theorems. With the help of these operators, Jackson's theorems may be given a direct proof, without resorting to the relation $E_n(f) \leqq CE_n(f')/n$.

1. Introduction. In this paper we give an explicit construction of a sequence of linear trigonometric polynomial operators of the best approximation order as given by Jackson's theorems. P. P. Korovkin [4] has shown that the order of approximation by linear positive trigonometric polynomial operators does not exceed $1/n^2$ where n is the degree of the polynomial. Seemingly this has had a retarding effect on the studies of linear positive operators. Nevertheless, as will be clear from the technique employed in the sequel, there is a way out. Implicitly, this technique has already been exploited by P. L. Butzer [1] for Bernstein polynomials.

F. Schurer [6] has studied the approximation of functions belonging to $C_{2\pi}$ by means of the operators L_{np-p} (n, p positive integers) defined by

$$L_{np-p}(f;x) = \frac{1}{A_{np-p}} \int_{-\pi}^{\pi} f(x+t)\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^{2p} dt, \tag{1}$$

where

$$A_{np-p} = \int_{-\pi}^{\pi} \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^{2p} dt.$$

For 2π-periodic functions L_{np-p} is a trigonometric polynomial of degree at most $np - p$. Schurer proves the following results.

THEOREM I. *The sequence $\{L_{np-p}(f;x)\}$, $n = 1, 2, \cdots$, converges uniformly on $[-\pi, \pi]$ to the function $f(x) \in C_{2\pi}$.*

THEOREM II. *If $f(x) \in C_{2\pi}$ and if $f(x)$ is twice differentiable at the point $t = x_0 \in [-\pi, \pi]$, then*

$$L_{np-p}(f;x_0) - f(x_0) = \left(1 - \frac{\rho_1^{(np-p)}}{\rho_0^{(np-p)}}\right) f''(x_0) + o\left(\frac{1}{n^2}\right), \tag{2}$$

where $\rho_0^{(np-p)}$ and $\rho_1^{(np-p)}$ are defined by

$$\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^{2p} = \frac{1}{6}\rho_0^{(np-p)} + \frac{1}{3}\sum_{k=1}^{np-p} \rho_k^{(np-p)} \cos kt. \tag{3}$$

Theorem II has given a partial motivation for our investigations. In the next section we derive some properties of the operators L_{np-p}.

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2. Basic results. In [6] Schurer has shown that

$$A_{np-p} = \int_{-\pi}^{\pi} \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^{2p} dt \geqq C_p n^{2p-1}, \tag{4}$$

where C_p is a positive constant depending on p.

LEMMA 1. *Let m be a positive number and $2p > m + 1$; then*

$$L_{np-p}(|t|^m; 0) \leqq C_{m,p} n^{-m}, \tag{5}$$

where $C_{m,p}$ is a constant independent of n.

Proof. Using $|\sin nt| \leqq n|\sin t|$, $-\infty < t < \infty$, and $\sin t \geqq 2t/\pi$, $0 \leqq t \leqq \pi/2$, we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} |t|^m \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^{2p} dt \\
&= 2^{m+2}\left[\int_0^{\pi/2n} t^m \left(\frac{\sin nt}{\sin t}\right)^{2p} dt + \int_{\pi/2n}^{\pi/2} t^m \left(\frac{\sin nt}{\sin t}\right)^{2p} dt\right] \\
&\leqq 2^{m+2}\left[\int_0^{\pi/2n} t^m n^{2p}\, dt + \int_{\pi/2n}^{\pi/2} \left(\frac{\pi}{2t}\right)^{2p} t^m (\sin nt)^{2p}\, dt\right] \\
&\leqq 2^{m+2}\left[\left(\frac{\pi}{2n}\right)^{m+1} \frac{n^{2p}}{m+1} + \left(\frac{\pi}{2}\right)^{2p} \frac{1}{m-2p+1}\left\{\left(\frac{\pi}{2}\right)^{m-2p+1} - \left(\frac{\pi}{2n}\right)^{m-2p+1}\right\}\right] \\
&\leqq 2\pi^{m+1}\left(\frac{1}{m+1} + \frac{1}{2p-m-1}\right) n^{2p-m-1},
\end{aligned}$$

and using (4) the inequality (5) follows.

LEMMA 2. *For each fixed k the coefficients $\rho_k^{(np-p)}$ in the expansion* (3) *can be written as polynomials in n of degree $2p - 1$.*

Proof. We know that

$$\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 = n + 2\sum_{k=1}^{n-1} (n-k)\cos kt.$$

Hence the result is true for $p = 1$.

Let us assume that $\rho_k^{(np-p)} = O(n^{2p-1})$, $k = 0, 1, \cdots, np - p$. Multiplying the respective sides of (3) and the above identity and bearing in mind how the terms in $\cos kt$ are formed, we see that

$$\rho_k^{((p+1)(n-1))} = O(n^{2p-1} \cdot n \cdot n) = O(n^{2(p+1)-1}).$$

Thus by induction it follows that the above assumption is true for all p. Now, following the method of Schurer [6, Chap. 3, § 2.1] for calculating $\rho_k^{(3n-3)}$ from $\rho_k^{(2n-2)}$ and using induction over p, it is easily proved that the $\rho_k^{(np-p)}$ are polynomials in n and k in the ranges $r(n-1) \leqq k \leqq (r+1)(n-1)$, $r = 0, 1, 2, \cdots, p-1$. Thus in particular for each fixed k it follows that $\rho_k^{(np-p)}$ is a polynomial in n of degree not exceeding $2p - 1$. As from (4), $\rho_0^{(np-p)} \geqq C_p n^{2p-1}$; hence it is indeed a polynomial of degree $2p - 1$ in n. Since

$$\frac{\rho_k^{(np-p)}}{\rho_0^{(np-p)}} \to 1, \qquad n \to \infty,$$

for each fixed k, it follows that $\rho_k^{(np-p)}$ is a polynomial of degree $2p - 1$ in n. This completes the proof of the lemma.

It is clear that in (3),

$$|\rho_k^{(np-p)}/\rho_0^{(np-p)}| \leqq 1$$

for all n, k and p, and so by Lemma 2 we conclude that

$$\frac{\rho_k^{(np-p)}}{\rho_0^{(np-p)}} = \sum_{\lambda=0}^{\infty} \frac{a_\lambda}{n^\lambda} \tag{6}$$

for each fixed value of k, where the a_λ do not depend on n. It is clear that

$$L_{np-p}(f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + t)u_n(t)\,dt, \tag{7}$$

where

$$u_n(t) = \frac{1}{2} + \sum_{k=1}^{np-p} \frac{\rho_k^{(np-p)}}{\rho_0^{(np-p)}} \cos kt.$$

Let us consider the determinant $\Delta^{(k)}(t, x)$, $k = 0, 1, \cdots, m + 1$, of $(m + 1)$th order defined by

$$\Delta^{(k)}(t, x) \equiv |a_{ij}^{(k)}|,$$

where

$$a_{1j}^{(k)} = \begin{cases} 1, & j = 1, \\ \sin [j/2]t, & j \text{ even}, \\ \cos [j/2]t, & j \text{ odd}, \end{cases}$$

$$a_{k+1\,j}^{(k)} = \begin{cases} 1, & j = 1, \\ \sin [j/2]x, & j \text{ even}, \\ \cos [j/2]x, & j \text{ odd}, \quad k \neq 0, \end{cases}$$

$$a_{ij}^{(k)} = \frac{d^{i-1}}{dx^{i-1}} a_{k+1\,j}^{(k)}, \quad 1 \neq i \neq k + 1.$$

Here $[j/2]$ denotes the largest integer not greater than $j/2$.

Let us define the quantities $\Omega^{(k)}$ by

$$\Omega^{(k)} = -\frac{\Delta^{(k)}(t, x)}{\Delta^{(0)}(x, x)}.$$

It is readily verified that the expansion of $\Omega^{(k)}$ about the point $t = x$ has the form

$$\Omega^{(k)} = (t - x)^k + O(|t - x|^{m+1}). \tag{8}$$

Thus it is clear from Lemma 1 that

$$L_{np-p}(\Omega^{(k)}; x) = O(n^{-k}) \tag{9}$$

provided that $m < 2p - 2$. Since $\Omega^{(k)}$ is a trigonometric polynomial of degree

$[(m + 1)/2]$ in t, we conclude from (6) and (7) that

$$L_{np-p}(\Omega^{(k)}; x) = \sum_{\lambda=k}^{\infty} \frac{b_\lambda^{(k)}(x)}{n^\lambda}, \tag{10}$$

where $b_\lambda^{(k)}(x)$ are functions in x independent of n.

In the sequel, by the 2π-neighborhood of x we mean the closed interval $[-\pi + x, \pi + x]$. Let $f(t)$ be a function bounded in the 2π-neighborhood of x and having rth derivative at the point $t = x$ where $r \leqq m < 2p - 2$. From (8) we then deduce that

$$f(t) - f(x) = \sum_{k=1}^{r} f^{(k)}(x)\frac{\Omega^{(k)}}{k!} + h_x(t)(t - x)^r, \tag{11}$$

where with $h_x(x) = 0$, $h_x(t)$ is bounded and is continuous at $t = x$. From Lemma 1 and the linearity and monotony of L_{np-p}, it can easily be shown that

$$L_{np-p}(h_x(t)(t - x)^r; x) = \frac{\varepsilon_n}{n^r},$$

where $|\varepsilon_n| \to 0$ as $n \to \infty$. Hence denoting $L_{np-p}(\Omega^{(k)}; x)$ by $\mu_{n;k}$ we have

$$L_{np-p}(f; x) - f(x) = \sum_{k=1}^{r} \frac{f^{(k)}(x)}{k!}\mu_{n,k} + \frac{\varepsilon_n}{n^r}.$$

Thus from (10) we have the following lemma.

LEMMA 3. *Let $f^{(r)}(t)$ exist at $t = x$ and let f be bounded in the 2π-neighborhood of x. If $r \leqq m < 2p - 2$, then*

$$L_{np-p}(f; x) - f(x) = \sum_{k=1}^{r} \frac{\alpha_k(x)}{n^k} + \frac{\varepsilon_n}{n^r}, \tag{12}$$

where $\alpha_k(x)$ are functions in x independent of n and $\varepsilon_n \to 0$ as $n \to \infty$.

Using $(t - x)$ in place of $\Omega^{(1)}$ in (11) we find that $\alpha_1(x) \equiv 0$.

3. Linear combinations of L_{np-p}. Let

$$\Delta_{m+1} \equiv |a_{ij}|,$$

where

$$a_{ij} = \frac{1}{i^{j-1}}, \qquad i, j = 1, 2, \cdots, m + 1,$$

and denote by $\Delta_{m+1}(L_{np-p})$ the determinant obtained from Δ_{m+1} by replacing a_{i1} by $L_{inp-p}(f; x)$, $i = 1, \cdots, m + 1$.

We shall consider the following combinations:

$$\mathscr{L}_{n,p}^{[m]}(x) = \frac{\Delta_{m+1}(L_{np-p})}{\Delta_{m+1}}. \tag{13}$$

It is clear that $\mathscr{L}_{n,p}^{[m]}$ is a polynomial operator of degree $(m + 1)np - p$. We note that by utilizing the fact that $\alpha_1(x) \equiv 0$ in (12) we could have used another combination, obtained by deleting the second columns and the last rows from both determinants in the definition of $\mathscr{L}_{n,p}^{[m]}(x)$, which would then have been of degree $mnp - p$. In the sequel it is immaterial which definition we choose, except for minor changes in the proofs.

Obviously $\mathscr{L}_{n,p}^{[m]}$ inherits the property of L_{np-p} given in Theorem I.

The following is our main result giving approximation of r-times differentiable functions by the operators $\mathscr{L}_{n,p}^{[m]}(x)$.

THEOREM 1. *Let $f(t)$ be bounded in the 2π-neighborhood of x possessing an r-th derivative at $t = x$, $r \leqq m < 2p - 2$. Then*

$$|\mathscr{L}_{n,p}^{[m]}(x) - f(x)| = o(n^{-r}) \quad \text{and} \quad |\mathscr{L}_{n,p}^{[r-1]}(x) - f(x)| = O(n^{-r}). \tag{14}$$

Proof. From Lemma 3 and the definition of $\mathscr{L}_{n,p}^{[m]}$ we have

$$\Delta_{m+1}[\mathscr{L}_{n,p}^{[m]}(x) - f(x)] = \Delta_{m+1}^{(1)},$$

where $\Delta_{m+1}^{(1)}$ is obtained from Δ_{m+1} by replacing a_{i1} by

$$\sum_{k=1}^{r} \frac{\alpha_k(x)}{(in)^k} + \frac{\varepsilon_{in}}{n^r}, \qquad i = 1, 2, \cdots, m + 1.$$

Multiplying the $(j + 1)$th column of $\Delta_{m+1}^{(1)}$ by $\alpha_j(x)/n^j$, $j = 1, 2, \cdots, r$, and subtracting from the first column we have

$$\Delta_{m+1}[\mathscr{L}_{n,p}^{[m]}(x) - f(x)] = \Delta_{m+1}^{(2)},$$

where $\Delta_{m+1}^{(2)}$ is obtained by replacing a_{i1} in Δ_{m+1} by ε_{in}/n^r. The first relation of (14) follows. The second assertion can also be proved in a like manner.

4. Approximation of functions with continuous r-th derivative. Applying (12) to the function $(t - x)^r$, $r \leqq m < 2p - 2$, we have

$$L_{np-p}((t - x)^r; x) = \sum_{k=r}^{n} \frac{\beta_k(x)}{n^k} + \frac{\varepsilon_n}{n^m}, \tag{15}$$

where the $\beta_k(x)$ are independent of n and $\varepsilon_n \to 0$ as $n \to \infty$. It is also clear that if r is odd, then the left-hand side of (15) is identically equal to zero.

In the sequel $\langle a, b\rangle$ denotes an open neighborhood of $[a, b]$.

THEOREM 2. *Let $f^{(r)}(x)$, $r \leqq m < 2p - 2$, exist and be continuous on $\langle a, b\rangle$ having modulus of continuity $\omega(\delta)(\not\equiv 0)$. Further, let f be bounded on the 2π-neighborhoods of a and b. Then*

$$|\mathscr{L}_{n,p}^{[m]}(x) - f(x)| \leqq \frac{A}{n^r}\omega(n^{-1}) \tag{16}$$

for $x \in [a, b]$, where $A = A(r)$ is independent of n.

Proof. With the given hypothesis on f we can write

$$\begin{aligned} f(t) - f(x) = {} & \sum_{k=1}^{r} (t - x)^r \frac{f^{(k)}(x)}{k!} \\ & + \frac{(t - x)^r}{r!}(f^{(r)}(\eta) - f^{(r)}(x))\lambda(t) + (t - x)^{m+1}h(t, x) \end{aligned} \tag{17}$$

for $t \in [-\pi + a, b + \pi]$, $x \in [a, b]$, where $\eta = \eta(t, x)$ lies between t and x, $\lambda(t)$ is the characteristic function of $\langle a, b\rangle$, and $h(t, x)$ is bounded, by M, say.

We can write

$$\begin{aligned}\mathscr{L}^{[m]}_{n,p}(x) - f(x) &= \sum_{j=1}^{m+1} \alpha_j (L_{jnp-p}(f; x) - f(x)) \\ &= \sum_{j=1}^{m+1} \frac{\alpha_j}{A_{jnp-p}} \int_{-\pi}^{\pi} (f(t+x) - f(x)) \left(\frac{\sin \frac{1}{2} jnt}{\sin \frac{1}{2} t} \right)^{2p} dt \\ &= \sum_{j=1}^{m+1} \frac{\alpha_j}{A_{jnp-p}} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{r} t^k \frac{f^{(k)}(x)}{k!} + \frac{t^r}{r!} (f^{(r)}(\eta(t+x, x)) \right. \\ &\qquad \left. - f^{(r)}(x)) \lambda(t+x) + t^{m+1} h(t+x, x) \right) \left(\frac{\sin \frac{1}{2} jnt}{\sin \frac{1}{2} t} \right)^{2p} dt.\end{aligned}$$

Denoting the three terms of the sum by Σ_1, Σ_2 and Σ_3 respectively and using Theorem 1, Lemma 1, Lemma 3 and the boundedness of $h(t, x)$ we easily have

$$\Sigma_1 = O(n^{-m-1}), \qquad \Sigma_3 = O(n^{-m-1}).$$

Now for an arbitrary $\delta > 0$,

$$\begin{aligned}|f^{(r)}(\eta(t+x, x)) - f^{(r)}(x)| &\leqq \omega(\delta) \left(1 + \frac{|\eta - x|}{\delta} \right) \\ &\leqq \omega(\delta) \left(1 + \frac{|t|}{\delta} \right).\end{aligned}$$

Hence by Lemma 1 we have

$$\begin{aligned}\Sigma_2 &\leqq \omega(\delta) \sum_{j=1}^{m+1} |\alpha_j| \left(\frac{C_{r,p}}{n^r} + \frac{C_{r+1,p}}{\delta n^{r+1}} \right) \\ &\leqq \omega(n^{-1}) \frac{C}{n^r},\end{aligned}$$

where C is a number independent of n. Since the order of $\omega(n^{-1})$ cannot exceed $1/n$, the inequality (16) follows from these estimates of Σ_1, Σ_2 and Σ_3.

It is clear that if $\omega(\delta) \equiv 0$ then we have

$$|\mathscr{L}^{[m]}_{n,p}(x) - f(x)| = O\left(\frac{1}{n^{m+1}} \right).$$

Immediately we have the following corollary.

COROLLARY. *Let $f^{(r)}(x) \in \operatorname{Lip} \alpha$ on $\langle a, b \rangle$, where $0 < \alpha \leqq 1$ and $r \leqq m < 2p - 2$, and let f be bounded on the 2π-neighborhood of a and b. Then*

$$|\mathscr{L}^{[m]}_{n,p}(x) - f(x)| \leqq M n^{-r-\alpha}, \qquad x \in [a, b],$$

where M is a constant independent of x and n.

The following theorem gives the convergence of L_{np-p} and consequently that of $\mathscr{L}^{[m]}_{n,p}$ for continuous functions.

THEOREM 3. *Let f be continuous on $\langle a, b \rangle$ having modulus of continuity $\omega(\delta)$ ($\not\equiv 0$). Further let f be bounded on 2π-neighborhoods of a and b. Then for*

$x \in [a, b]$,

$$|L_{np-p}(f; x) - f(x)| \leqq A\omega(n^{-1})$$

and

$$|\mathscr{L}_{n,p}^{[m]}(x) - f(x)| \leqq B\omega(n^{-1}),$$

where A and B are positive numbers independent of x and n.

The proof can be given on lines somewhat similar to those for Theorem 2, and we omit it. We note that in the case $\omega(\delta) \equiv 0$ both moduli in Theorem 3 will be $O(1/n^{2p-2})$. The following corollary is evident.

COROLLARY. *Let $f \in \operatorname{Lip} \alpha$ on $\langle a, b \rangle$, where $0 < \alpha \leqq 1$; then for $x \in [a, b]$,*

$$|L_{np-p}(f; x) - f(x)| \leqq Mn^{-\alpha}$$

and

$$|\mathscr{L}_{n,p}^{[m]}(x) - f(x)| \leqq Nn^{-\alpha},$$

where M and N are constants independent of x and n.

5. Approximation of functions of class $C_{2\pi}$. So far we have considered the approximation of a general function without specializing it to belong to the class $C_{2\pi}$. If this is done, however, there is an additional advantage of enabling the constants in various error estimates to be independent of the functions concerned.

To prove this we shall use the following lemma, which G. Freud [2] proved for the case $m = 2$ and conjectured for arbitrary integer m which was subsequently proved by G. I. Sunouchi [8, Remark, p. 183]. The continuity modulus ω_m of mth order of smoothness occurring in the sequel is defined by

$$\omega_m(f; \delta) = \max_{|h| \leqq \delta} \sum_{\nu=0}^{m} C_\nu^m (-1)^\nu f(x + \nu h).$$

LEMMA I. *Let B be a linear operator $C_{2\pi} \to C_{2\pi}$, $B^{(m)}$ the restriction of B to the space $C_{2\pi}^{(m)}$ of 2π-periodic functions having a continuous 2π-periodic mth derivative; then for an arbitrary $f \in C_{2\pi}$ and an arbitrary integer ν we have for all x,*

$$\|B(f)\|_{C_{2\pi}} = \max |B(f; x)| \leqq K_m(\|B\| + \nu^m \|B^{(m)}\|)\omega_m(f; \nu^{-1}),$$

where the norms of the linear operators B, respectively $B^{(m)}$, are defined as

$$\|B\| = \sup_{\|f\|_{C_{2\pi}} \leqq 1} \|B(f)\|_{C_{2\pi}}, \qquad \|B^{(m)}\| = \sup_{f \in C_{2\pi}^{(m)}, \|f^{(m)}\| \leqq 1} \|B^{(m)}(f)\|_{C_{2\pi}}.$$

For $f \in C_{2\pi}^{(k)}$, $k \geqq 2$, G. Freud [3] proved the relation

$$\|f^{(k-1)}\| \leqq 2\pi \|f^{(k)}\|.$$

By repeated use we have for $f \in C_{2\pi}^{(k)}$,

$$\|f^{(i)}\| \leqq (2\pi)^{k-i} \|f^{(k)}\|, \qquad i = 1, 2, \cdots, k - 1.$$

Let $f \in C_{2\pi}^{(k)}$. Then

$$f(t) - f(x) = (t - x)f'(x) + \frac{(t - x)^2}{2!} f''(x) + \cdots + \frac{(t - x)^{k-1}}{(k - 1)!} f^{(k-1)}(x) + \frac{(t - x)^k}{k!} f^{(k)}(\xi),$$

where ξ lies between t and x. Denoting $L_{np-p}((t-x)^i; x)$ by $\mu_i^{(n)}$ we then have

$$|L_{np-p}(f; x) - f(x) - \mu_1^{(n)} f'(x) - \frac{\mu_2^{(n)}}{2!} f''(x) - \cdots - \frac{\mu_{k-1}^{(n)}}{(k-1)!} f^{(k-1)}(r)| \leqq \frac{*\mu_k^{(n)}}{k!} M_k,$$

where $M_k = \max_x |f^{(k)}(x)|$ and $*\mu_k^{(n)} = L_{np-p}(|t-x|^k; x)$.

Since we have

$$\mathscr{L}_{n,p}^{[m]}(x) = \sum_{i=1}^{m+1} \alpha_i L_{inp-p},$$

where $\sum_{i=1}^{m+1} \alpha_i = 1$, we find that

$$|\mathscr{L}_{n,p}^{[m]}(x) - f(x) - f'(x) \sum_{i=1}^{m+1} \alpha_i \mu_1^{(ni)} - \frac{f''(x)}{2!} \sum_{i=1}^{m+1} \alpha_i \mu_2^{(ni)} - \cdots - \frac{f^{(k-1)}(x)}{(k-1)!} \sum_{i=1}^{m+1} \alpha_i \mu_{k-1}^{(ni)}| \leqq \frac{M_k}{k!} \sum_{i=1}^{m+1} \alpha_i {}^*\mu_k^{(ni)}.$$

Thus by (13) and (15) it easily follows that

$$\begin{aligned} |\mathscr{L}_{n,p}^{[m]}(x) - f(x)| &= |f'(x)| O\left(\frac{1}{n^{m+1}}\right) + |f''(x)| O\left(\frac{1}{n^{m+1}}\right) \\ &\quad + \cdots + |f^{(k-1)}(x)| O\left(\frac{1}{n^{m+1}}\right) + M_k O\left(\frac{1}{n^k}\right) \\ &= M_k O\left(\frac{1}{n^k}\right), \end{aligned}$$

because of norm inequalities for the derivatives of f. Here the O-term holds uniformly in x.

Hence we can write

$$|\mathscr{L}_{n,p}^{[m]}(x) - f(x)| \leqq A_k M_k / n^k,$$

where A_k is an absolute constant. Thus taking $B \equiv \mathscr{L}_{n,p}^{[m]}(x) - f(x)$, we find that

$$\begin{aligned} \|B^{(k)}\| &= \sup_{f \in C_{2\pi}^{(k)}, \|f^{(k)}\| \leqq 1} \|B^{(k)}(f)\|_{C_{2\pi}} \\ &\leqq \frac{A_k}{n^k}. \end{aligned}$$

Also

$$\|B\| = \sup_{\|f\|_{C_{2\pi}} \leqq 1} \|B(f)\|_{C_{2\pi}} \leqq \sum_{i=1}^{m+1} |\alpha_i| 2 = B, \quad \text{say}.$$

Then by Lemma I for an arbitrary $f \in C_{2\pi}$ and $m+1 \geqq k$ we have

$$\begin{aligned} \|\mathscr{L}_{n,p}^{[m]}(x) - f(x)\| &= \max |\mathscr{L}_{n,p}^{[m]}(x) - f(x)| \\ &\leqq K_k \left(B + b^k \frac{A_k}{n^k}\right) \omega_k(f; n^{-1}), \end{aligned}$$

that is,

$$\|\mathscr{L}_{n,p}^{[m]}(x) - f(x)\| \leqq C_k\omega_k(f; n^{-1}),$$

where C_k is an absolute constant independent of f and n.

Using

$$\omega_{r+1}(f; t) \leqq t^r\omega(f^{(r)}; t)$$

[7, p. 56], we have for $f \in C_{2\pi}^{(k)}$,

$$\|\mathscr{L}_{n,p}^{[m]}(x) - f(x)\| \leqq \frac{C_{k+1}}{n^k}\omega\left(f^{(k)}; \frac{1}{n}\right).$$

Thus the operators $\mathscr{L}_{n,p}^{[m]}(x)$ provide us with a direct proof of Jackson's theorems.

6. Derivatives of $\mathscr{L}_{n,p}^{[m]}(x)$. Let $f^{(k)}$ be continuous in $[x - \pi, x + \pi]$. Differentiating (1) k-times using Theorem 1 in [9, p. 59], we have

$$L_{np-p}^{(k)}(f; x) = L_{np-p}(f^{(k)}; x).$$

By linearity it follows that

$$(\mathscr{L}_{n,p}^{[m]}(x))^{(k)} = \mathscr{L}_{n,p}^{[m]}(f^{(k)}; x).$$

Hence the derivatives of $\mathscr{L}_{n,p}^{[m]}(x)$ approximate the continuous derivatives of f in the same way that $\mathscr{L}_{n,p}^{[m]}(x)$ approximates the derivatives of f, enabling the results of the preceding sections to be applied. We note that this technique is also applicable to the partial sums of Fourier series, etc.

In summary, the linear operators we have used give the best approximation order for Jackson's theorems [4], [5]. By S. N. Bernstein's theorems [4], [5] it is clear that approximation orders given by Theorems 2, 3 and the corollaries thereof cannot be improved. Without proof we remark that there is an infinity of linear combinations of L_{np-p} which give the same order of approximation as $\mathscr{L}_{n,p}^{[m]}$, but none gives a better order.

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SOLUTION OF A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS†

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Abstract. A class of partial differential equations with initial conditions is considered. Conditions and a method for obtaining the exact solution, when it exists, are presented.

Denote by I the set of integers and by $\mathscr{N}$ the nonnegative integers, and let Ω be a finite subset of $I \times I \times \mathscr{N} \times \mathscr{N}$. We consider the partial differential equation

$$\frac{\partial \pi(w, z, t)}{\partial t} = \sum w^u z^v \frac{\partial^{l+m} \pi(w, z, t)}{\partial w^l \partial z^m}, \tag{1}$$

where the summation is over all $(u, v, l, m) \in \Omega$ and

$$\frac{\partial^{l+m}\pi}{\partial w^l \partial z^m} = \begin{cases} \pi, & l = m = 0 \\ \dfrac{\partial^l \pi}{\partial w^l}, & l > 0 \text{ and } m = 0 \\ \dfrac{\partial^m \pi}{\partial z^m}, & l = 0 \text{ and } m > 0. \end{cases}$$

We let N be a finite positive integer and define the set

$$S_N = \{(i, j) \in \mathscr{N} \times \mathscr{N} : i + j \leqq N\}.$$

We shall present conditions and subsequently a method for obtaining the exact solution, when it exists, of (1) subject to the initial condition

$$\pi(w, z, 0) = \sum_{(i,j) \in S_N} w^i z^j a_{ij}, \tag{2}$$

where the a_{ij} are constants.

The method of solution will lead to a triangular system of differential-difference equations if one of the following conditions holds.

Condition A. For each $(u, v, l, m) \in \Omega$, we have

$$2(N + 1)(m - v) + 2(l - u) - (m - v)(2r - 1 + m - v) \geqq 0$$

for $r = 0, 1, \cdots, N$.

Condition A*. Same as Condition A with $\geqq$ replaced by $\leqq$.

For example, Condition A holds for

$$\frac{\partial \pi}{\partial t} = (w^2 - zw)\frac{\partial^2 \pi}{\partial w \partial z} + (1 - w)\frac{\partial \pi}{\partial w},$$

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representing the familiar general stochastic epidemic (see [1] for the solution of this equation using the methods of this paper); but neither Condition A nor Condition A* holds for

$$\frac{\partial \pi}{\partial t} = (w^2 - z^2)\frac{\partial^2 \pi}{\partial w \partial z} + (1 - w)\frac{\partial \pi}{\partial w}.$$

To obtain the exact solution of (1) subject to (2), we begin by setting

$$\pi(w, z, t) = \sum_{(i,j) \in S_N} w^i z^j f_{ij}(t).$$

Then substituting $\pi(w, z, t)$ in (1) enables the partial differential equation in $\pi(w, z, t)$ to be expressed as a system of differential-difference equations in $f_{ij}(t)$, for $(r,s) \in S_N$,

$$\begin{aligned} f'_{rs}(t) = \sum &(s + l - u)(s + l - u - 1) \cdots (s - u + 1)(r + m - v) \\ &\cdot (r + m - v - 1) \cdots (r - v + 1) f_{r+m-v,s+l-u}(t), \end{aligned} \tag{3}$$

where the sum is over $(u, v, l, m) \in \Omega$ and $f_{ij}(t) \equiv 0$ whenever $(i, j) \notin S_N$, subject to $f_{ij}(0) = a_{ij}$ for $(i, j) \in S_N$.

Using the approach of [2] we represent each pair $(i, j) \in S_N$ by the positive integer

$$k \equiv k(i, j; N) = (N + 1)(N + 2)/2 - (N + 1)i - j + (i - 1)i/2.$$

Setting $f_{ij}(t) = \varepsilon(i)\varepsilon(j)\varepsilon(N - i - j)x_k(t)$, where $\varepsilon(y) = 1$ for $y \geqq 0$ and 0 for $y < 0$, enables us to write (3) as the system: for $(r, s) \in S_N$,

$$\begin{aligned} x'_k(t) = \sum [&(s + l - u)(s + l - u - 1) \cdots (s - u + 1)(r + m - v) \\ &\cdot (r + m - v - 1) \cdots (r - v + 1)\varepsilon(r + m - v)\varepsilon(s + l - u) \\ &\cdot \varepsilon(N - r - m + v - s - l + u)] x_{k(r+m-v,s+l-u;N)}(t), \end{aligned} \tag{4}$$

again the sum being over $(u, v, l, m) \in \Omega$, subject to $x_k(0) = a_{ij}$.

We now note that the system (4) is lower triangular whenever $k(r + m - v, s + l - u; N) \leqq k(r, s; N)$ for all $(r, s) \in S_N$ and $(u, v, l, m) \in \Omega$; this, it is easy to check, is equivalent to Condition A. Obviously Condition A* is equivalent to the system (4) being upper triangular. Thus the solution for the $x_k(t)$ readily follows by, say, application of [3, Thm. 1]. Consequently the solution for $f_{ij}(t)$ and hence $\pi(w, z, t)$ is given.

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A DIFFERENTIAL TRANSFORM*

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Abstract. This paper contains a definition of a linear operator T, referred to as a differential transform. The differential transform T commutes with differentiation and as a result, T can be applied for the solving of some linear partial differential equations. Two illustrative examples for the application of T are introduced.

1. Introduction. In a previous paper [1], a linear operator T has been defined and shown to be related in some sense to the Laplace transform operator. The operator T transforms functions of a specific type of the variables $x = (x_1, x_2, \cdots, x_n)$, λ and s into functions of the variables x and μ and satisfies the following two relations: $T\partial_x^p = \partial_x^p T$ and $Ts^n = \partial_\mu^n T$. These relations make T useful in solving some boundary value problems. Usually, the operator T is defined in terms of differentiation rather than integration, and hence it is referred to as a *differential transform.* In this paper the definition of T is generalized, and its properties are proved by using integrals in the complex plane. The need for a generalization of the definition of T has been pointed out in [1]. For the definition of T we need the notion of "continued integrals", which are defined in § 2. These integrals are analytic continuations of ordinary ones. The definition of T, its properties and applications are given in § 3. Two illustrative examples for the application of T are given in § 4, and an "inverse" of T is introduced in § 5.

2. Continued integrals. In this section we define the "continued integrals" which are analytic continuations of ordinary ones.

DEFINITION 1. Let $F(\omega, s)$ be an analytic function of ω, regular on a domain D for each $s \geqq 0$. Let

$$\int_0^\infty F(\omega, s)\, e^{-sa}\, ds = f(a, \omega), \quad a > 0, \tag{2.1}$$

be uniformly and absolutely convergent in every compact subset of D. Then, for an analytic function $h(\omega)$ of ω, regular in D and satisfying

$$|h(\omega)| > a \tag{2.2}$$

there, we define the *continued integral*

$$\mathop{\mathrm{C}\!\!\!\int}_0^\infty F(\omega, s)\, e^{-sh(\omega)}\, ds = f(h(\omega), \omega). \tag{2.3}$$

Obviously, if the range $h(D)$ of h cuts the positive real axis, then (2.3) is the analytic continuation of

$$\int_0^\infty F(u, s)\, e^{-sh(u)}\, ds,$$

where u is the restriction of ω to the curve in D. for which $h(u)$ is positive. The letter C on the integral sign in (2.3) refers to analytic continuation.

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3. A differential transform. Consider the triplet $\Gamma = \{X, D_\mu, D_\lambda(x)\}$ where X is a domain of the n-dimensional complex space, D_μ is a simply connected domain of the complex μ-plane whose boundary is ∂D_μ, and $D_\lambda(x)$ is a simply connected domain of the complex λ-plane whose boundary is $\partial D_\lambda(x)$. $D_\lambda(x)$ is dependent on the elements x of X. Let M and N be two sets of analytic functions related to Γ as follows. M is the set of all analytic functions μ^* regular on $X \times D_\lambda(x)$ onto D_μ such that $\dot{\mu}^* = \partial\mu^*/\partial\lambda \neq 0$ and μ^* maps $D_\lambda(x)$ onto D_μ for every x in X. N is the set of all analytic functions regular on $X \times D_\mu$ onto $D_\lambda(x)$ such that $\dot{\lambda}^* = \partial\lambda^*/\partial\mu \neq 0$ and λ^* maps D_μ onto $D_\lambda(x)$ for every x in X. There exists a one-to-one correspondence between elements of M and elements of N correlating μ^* in M to the element λ^* in N such that $\mu = \mu^*(x, \lambda^*(x, \mu))$ and $\lambda = \lambda^*(x, \mu^*(x, \lambda))$ are identities on D_μ and on $D_\lambda(x)$, respectively, for every x in X. In this sense, μ^* and λ^* are regarded as inverses of each other with respect to the triplet Γ.

For the triplet Γ we assign two linear spaces A, B and a linear operator T from A onto B.

DEFINITION 2. A is the linear space generated by all functions of type $F(x, \lambda, s)\, e^{-s\mu^*}$ such that

(i) $x \in X$, $\lambda \in D_\lambda(x)$ and $s \in S$, where S is a domain in the complex s-plane including all positive values of s;

(ii) $\mu^* \in M$;

(iii) F is regular on $X \times D_\lambda(x) \times S$ ($X \times D_\lambda(x)$ is the set of all points (x, λ) such that $x \in X$ and $\lambda \in D_\lambda(x)$);

(iv) the integral $\int_0^\infty F(x, \lambda, s)\, e^{-sa}\, ds$ is uniformly and absolutely convergent in every compact subset of $X \times D_\lambda$ for every $a > 0$.

B is the linear space generated by all functions of type

$$\frac{1}{2\pi i} \int_{\partial D_\mu} \dot{\lambda}^*\, d\mu' \oint_0^\infty F(x, \lambda^*, s)\, e^{-s(\mu' - \mu)}\, ds,$$

where

(i′) $x \in X$, $\mu \in D_\mu$ and $s \in S$;

(ii′) $\lambda^* \in N$;

(iii′, iv′) F satisfies (iii) and (iv).

DEFINITION 3. Let T be a linear operator from A onto B given by

$$T\{F(x, \lambda, s)\, e^{-s\mu^*}\} = \frac{1}{2\pi i} \int_{\partial D_\mu} \dot{\lambda}^*\, d\mu' \oint_0^\infty F(x, \lambda^*, s)\, e^{-s(\mu' - \mu)}\, ds, \tag{3.1}$$

where μ^* and λ^* are inverses of each other.

The operator T will be redefined in Definition 4 by means of differentiation rather than integration for the particular case in which F is entire in s. In Definition 3 one can transform to the variable of integration λ through the subsitution $\lambda = \lambda^*(x, \mu')$, obtaining

$$T\{F(x, \lambda, s)\, e^{-s\mu^*}\} = \frac{1}{2\pi i} \int_{\partial D_\lambda(x)} d\lambda \oint_0^\infty F(x, \lambda, s)\, e^{-s(\mu^* - \mu)}\, ds \tag{3.2}$$

as an equivalent definition of T, where $\partial D_\lambda(x)$ is the image $\lambda^*(x, \partial D_\mu)$ of ∂D_μ in the λ-plane for $x \in X$.

From (3.2) it follows that T is well-defined, that is, if $F(x, \lambda, s)\, e^{-s\mu^*}$ and $G(x, \lambda, s)\, e^{-s\nu^*}$ are two representations of the same element of A, i.e., if

$$F(x, \lambda, s)\, e^{-s\mu^*} = G(x, \lambda, s)\, e^{-s\nu^*},$$

then their images under T in B coincide.

For an index exponent $p = (p_1, p_2, \cdots, p_n)$ of order n whose components are nonnegative integers, $|p| = p_1 + p_2 + \cdots + p_n$. Let ∂_x^p be the differential operator on A and on B

$$\partial_x^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}.$$

From (3.2) the following relations are immediately deduced.

THEOREM 1. *$T\partial_x^p = \partial_x^p T$ and $Ts^k = \partial_\mu^k T$ for $k = 0, 1, 2, \cdots$.*

The relations in Theorem 1 make the operator T useful in solving some linear partial differential equations, as we shall see in two forthcoming examples.

Formally expanding $F(x, \lambda, s)$ in (3.1) in power series in s, let

$$F(x, \lambda, s) = \sum_{k=0}^{\infty} s^k f_k(x, \lambda).$$

Applying the operator T, we get

$$\begin{aligned} T\{F(x, \lambda, s)\, e^{-s\mu^*}\} &= T\left\{\sum_{k=0}^{\infty} s^k f_k(x, \lambda)\, e^{-s\mu^*}\right\} \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\partial D_\mu} f_k(x, \lambda^*)\dot{\lambda}^*\, d\mu' \oint_0^{\infty} s^k\, e^{-s(\mu' - \mu)}\, ds \\ &= \sum_{k=0}^{\infty} \frac{k!}{2\pi i} \int_{\partial D_\mu} \frac{f_k(x, \lambda^*)\dot{\lambda}^*}{(\mu' - \mu)^{k+1}}\, d\mu' \\ &= \sum_{k=0}^{\infty} \partial_\mu^k \{f_k(x, \lambda^*)\dot{\lambda}^*\} \\ &\stackrel{\text{def}}{=} F(x, \lambda^*, \partial_\mu)\dot{\lambda}^*. \end{aligned} \tag{3.3}$$

The formal operations in (3.3) can be justified in some cases. For example, they are justified when $F(x, \lambda, s)$ is entire for s and

$$F(x, \lambda, s) = O(e^{\varepsilon|s|}), \qquad |s| \to \infty,$$

for every positive ε. This is a consequence of [7, Chap. II, Cor. 14c]. Here λ^* is a function of x and μ' inside the integral sign or a function of x and μ after integration is performed. From (3.3) the differential nature of the operator T is evident, transforming s into ∂_μ. Hence, T is referred to as a *differential transform*. It should be noticed that T is not one-to-one, as nonzero functions can be transformed into zero. In the definition in (3.3) we treat the operational symbol ∂_μ for differentiation, as if it were a variable throughout a calculation. We give the symbol ∂_μ its original meaning after expanding its function in power series in ∂_μ. Then ∂_μ^k performs differentiation with respect to μ of order k of the whole expression in which it appears.

Adopting the notation in (3.3), we may rewrite the definition of T in the following form.

DEFINITION 4. For $F(x, \lambda, s)\, e^{-s\mu^*}$ in A,

$$T\{F(x, \lambda, s)\, e^{-s\mu^*}\} = F(x, \lambda^*, \partial_\mu)\dot{\lambda}^*,$$

where λ^* is the inverse of μ^* with respect to the triplet Γ.

The source under T is regular on $X \times D_\lambda$ as a function of x and λ and its image is regular on $X \times D_\mu$. We note that usually the image of T is given as an infinite series as in (3.3). When the series does not converge, we use the integral representation (3.1) for the image of T. From Definition 4 we see that the independent variable λ is transformed into the dependent one λ^*, the variable s is transformed into the operational symbol ∂_μ, and $e^{-s\mu^*}$ is transformed into $\dot{\lambda}^*$. The "operational calculus" as presented in Definition 4 and Theorem 1 can be extended to a more general one which considers s, λ and μ as vectors of L variables. This generalization, which has been introduced in [1] for a particular case, can be treated by the present method with no difficulties, though details are not given here.

As a particular case, let $D_\lambda = D_\mu$ be a domain in the complex plane and let $\mu^*(x, \lambda) = \lambda$ and $\lambda^*(x, \mu) = \mu$. Then $\dot{\lambda}^* = 1$ and

$$\begin{aligned} T_0\{F(x, \lambda, s)\} &\stackrel{\text{def}}{=} T\{F(x, \lambda, s)\, e^{-s\lambda}\} = F(x, \mu, \partial_\mu) \\ &= F(x, \lambda, \partial_\lambda), \end{aligned}$$

since the variables λ and μ are regarded here as two notations for the same variable. Here $F(x, \lambda, s)$ is entire in s, and both $F(x, \lambda, s)$ and $F(x, \lambda, \partial_\lambda)$ are regular on $X \times D_\lambda$. Obviously, T_0 also possesses the relations in Theorem 1, and we have the next definition and theorem.

DEFINITION 5. For $F(x, \lambda, s)\, e^{-s\lambda}$ in A,

$$T_0\{F(x, \lambda, s)\} = F(x, \lambda, \partial_\lambda).$$

THEOREM 2. *$T_0\partial_x^p = \partial_x^p T_0$ and $T_0 s^k = \partial_\lambda^k T_0$ for $k = 0, 1, 2, \cdots$,*

4. Examples.

Example 1. Let X be the complex plane. The function $\phi = F(\lambda)\cosh(\sqrt{s}\, x)$ satisfies the Helmholtz equation $\partial^2\phi/\partial x^2 = s\phi$ for an arbitrary analytic function F of λ regular in some domain D, where $s \in S$ and $x \in X$. By Theorem 2, the image ψ of ϕ,

$$\psi = T_0\{\phi\} = F(\mu)\cosh(\sqrt{\partial_\mu}\, x),$$

satisfies the heat equation

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi}{\partial\mu}.$$

Expanding ψ in power series in ∂_μ, we get

$$\psi = \sum_{n=0}^{\infty} \frac{1}{(2n)!} F^{(n)}(\mu) x^{2n}.$$

This solution of the heat equation has been studied by Widder [6].

Example 2. Consider the function

$$t^*(r, \theta, z, \lambda) = r \sinh \alpha \sinh (\lambda + i\theta) + z \cosh \alpha, \qquad \alpha > 0,$$

and its inverse [4], [5],

$$\lambda^*(r, \theta, z, t) = \sinh^{-1} \left(\frac{t - z \cosh \alpha}{r \sinh \alpha} \right) - i\theta.$$

The derivative $\dot{\lambda}^*$ of λ^* with respect to "time" t,

$$\dot{\lambda}^* = [(t - z \cosh \alpha)^2 + r^2 \sinh^2 \alpha]^{-1/2},$$

is finite and different from zero for $-\infty < t < \infty$, $-\infty < z < \infty$ and $0 < r < \infty$. Hence, there exists a domain X of the 3-dimensional complex space of the variables r, θ, z and a domain D_t of the complex t-plane such that (a) X includes all real values of θ and z and all positive values of r, (b) D_t includes all real values of t, and (c) $\dot{\lambda}^* \neq 0$ on $X \times D_t$. Let $D_\lambda(x)$ be the range $\lambda^*(x \times D_t)$ of λ^* for x in X. Thus we have a triplet $\Gamma = \{X, D_t, D_\lambda(x)\}$ for which t^* and λ^* are elements of M and N, respectively. Let $\phi = F(\lambda)\, e^{-st^*}$ be an element of A. ϕ satisfies the reduced wave equation $\Delta\phi = s^2\phi$, where Δ denotes the Laplacian operator in cylindrical coordinates:

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2.$$

Hence, by Theorem 1,

$$\psi = T\{F(\lambda)\, e^{-st^*}\} = F(\lambda^*)\dot{\lambda}^*$$

satisfies the wave equation

$$\Delta\psi = \partial_t^2\psi.$$

A typical application of the differential transform T to the theory of wave propagation is the representation of a progressing wave $\dot{\lambda}^*$ in terms of a plane progressing wave e^{-st^*} by $\dot{\lambda}^* = T\{e^{-st^*}\}$. A boundary value problem is solved first for the plane wave e^{-st^*} and then, by applying T, for the progressing wave $\dot{\lambda}^*$ (see [2]–[5]). In a simple case, assume that the solution includes the expression

$$T\left\{\frac{f(\lambda)}{1 - k\, e^{-sh}}\, e^{-st^*}\right\}, \qquad h > 0, \quad 0 < k < 1, \quad \operatorname{Re}(s) \geqq 0.$$

This expression represents a sum of infinite reflections:

$$T\left\{f(\lambda) \sum_{n=0}^{\infty} k^n\, e^{-s(t^* + nh)}\right\} = \sum_{n=0}^{\infty} k^n f(\lambda_n^*)\dot{\lambda}_n^*,$$

where

$$\lambda_n^*(r, \theta, z, t) = \lambda^*(r, \theta, z, t - nh).$$

5. The inverse of *T*. From (3.2) it can be seen that T is not one-to-one, as nonzero elements of A can be transformed into the zero of B. As an example, let us consider the two elements of M: $\mu_1^* = x \sinh \lambda$ and $\mu_2^* = (e^\lambda - x^2 e^{-\lambda})/2$, where

x is a single variable. Their inverses, respectively, are these elements of N: $\lambda_1^* = \sinh^{-1}(\mu/x)$ and $\lambda_2^* = \ln(\mu + \sqrt{\mu^2 + x^2})$, and we have

$$T\{e^{-s\mu_1^*}\} = T\{e^{-s\mu_2^*}\} = \dot{\lambda}_1^* = \dot{\lambda}_2^* = (\mu^2 + x^2)^{-1/2}.$$

However, it is obvious that $T\{f(\lambda, s)\, e^{-s\mu_1^*}\}$ and $T\{f(\lambda, s)\, e^{-s\mu_2^*}\}$ are not equal for an arbitrary function f of λ and s. Hence, we make the following definition.

DEFINITION 6. $T\{F(x, \lambda, s)\, e^{-s\mu^*}\}$ and $T\{G(x, \lambda, s)\, e^{-s\nu^*}\}$ are said to be *equal*, in symbol

$$T\{F(x, \lambda, s)\, e^{-s\mu^*}\} \equiv T\{G(x, \lambda, s)\, e^{-s\nu^*}\},$$

if

$$T\{f(\lambda, s)F(x, \lambda, s)\, e^{-s\mu^*}\} = T\{f(\lambda, s)G(x, \lambda, s)\, e^{-s\nu^*}\}$$

for an arbitrary function $f(\lambda, s)$ for which the operand of T is an element of A.

From (3.2) it is obvious that T is one-to-one under the equality relation $\equiv$, and hence the inverse T^{-1} of T can be defined. When all functions of s are entire in s, the definition takes the following form.

DEFINITION 7. For $F(x, \lambda^*, \partial_\mu)\dot{\lambda}^*$ in B,

$$T^{-1}\{F(x, \lambda^*, \partial_\mu)\dot{\lambda}^*\} \equiv F(x, \lambda, s)\, e^{-s\mu^*},$$

where μ^* is the inverse of λ^*.

For T^{-1}, Theorem 1 implies Theorem 4 as follows.

THEOREM 3. *$T^{-1}\partial_x^p = \partial_x^p T^{-1}$ and $T^{-1}\partial_\mu^k = s^k T^{-1}$.*

As an application of the definition of T^{-1}, we shall prove the following theorem. Let D be a linear differential operator performing differentiations with respect to x_i ($i = 1, 2, \cdots, n$) and let $x = (x_1, x_2, \cdots, x_n)$.

THEOREM 4. *Let $\mu^*(x, \lambda)$ be an analytic function of x and λ such that $\dot{\mu}^* = \partial\mu^*/\partial\lambda$ is different from zero in a region. For an arbitrary analytic function g of μ^*, regular over the range of μ^*, $Dg(\mu^*) = 0$ if and only if $De^{-s\mu^*} = 0$ in the region for $0 \leqq s < \infty$.*

Proof. By assumption, $Df(\lambda, s)\, e^{-s\mu^*} = 0$ for an arbitrary function $f(\lambda, s)$ (this arbitrariness is understood to be restricted to those functions for which the appropriate expressions are elements of A). By Theorem 1, T commutes with D, and hence

$$DT\{f(\lambda, s)\, e^{-s\mu^*}\} = Df(\lambda^*, \partial_\mu)\dot{\lambda}^* = 0.$$

This implies that

$$Dg(\mu)f(\lambda^*, \partial_\mu)\dot{\lambda}^* \equiv 0.$$

The arbitrariness of f permits one to apply T^{-1} to the last partial differential equation to obtain

$$DT^{-1}g(\mu)f(\lambda^*, \partial_\mu)\dot{\lambda}^* \equiv 0$$

on using the commutative relation between D and T^{-1}. Hence

$$Dg(\mu^*)f(\lambda, s)\, e^{-s\mu^*} = 0.$$

This equation reduces to the statement of the theorem for $s = 0$ and $f(\lambda, s) = 1$, and hence the result is obtained.

Example 3. Let $\square$ be the wave operator in cylindrical coordinates (r, θ, z) given by

$$\square = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2 - \partial_t^2$$

and let

$$a = \sinh^{-1}\left(\frac{t - z\cosh\alpha}{r\sinh\alpha}\right), \qquad \alpha > 0, \quad r > 0,$$

and $b = \theta$. Then $\square e^{-s(a+ib)} = 0$. Hence, by Theorem 4, $\square g(a + ib) = 0$ for an arbitrary analytic function g of $a + ib$.

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A LINEAR VOLTERRA EQUATION IN HILBERT SPACE*

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Abstract. We determine the limit, as $t \to \infty$, of the solution of the equation

$$\mathbf{x}(t) + \int_0^t \mathbf{A}(t-s)\mathbf{x}(s)\,ds = \mathbf{x}_0 + t\mathbf{k}$$

in Hilbert space. Here $\mathbf{A}(t) = \int_{-\infty}^{\infty} A(t,\lambda)\,d\mathbf{E}_\lambda$, where the spectral family $\{\mathbf{E}_\lambda\}$ corresponds to a fixed self-adjoint linear operator $\mathbf{L}$ and $A(t,\lambda)$ has certain monotonicity properties as a function of t. The results generalize our earlier work on the special case $\mathbf{A}(t) = A(t)\mathbf{L}$ with $A(t)$ scalar. The main new step is a continuity result for a related scalar equation depending on the parameter λ.

1. Introduction. We consider the equation

$$\mathbf{x}(t) + \int_0^t \mathbf{A}(t-s)\mathbf{x}(s)\,ds = \mathbf{x}_0 + t\mathbf{k}, \qquad t \geqq 0, \tag{1.1}$$

where $\mathbf{A}(t)$ is a positive self-adjoint linear operator with domain containing the (fixed) dense subspace D of a separable Hilbert space H. The initial data are the prescribed elements $\mathbf{x}_0$ and $\mathbf{k}$ of D. A solution of (1.1) is a continuous function $\mathbf{x}: R^+ \to H$ ($R^+ = [0,\infty)$) with values in D such that $\mathbf{A}(t-s)\mathbf{x}(s)$ is continuous in s and (1.1) holds. We find conditions under which $\lim_{t\to\infty} \mathbf{x}(t)$ exists.

We assume that

$$\mathbf{A}(t)\mathbf{y} = \int_{-\infty}^{\infty} A(t,\lambda)\,d\mathbf{E}_\lambda\mathbf{y}, \qquad \mathbf{y} \in D. \tag{1.2}$$

Here $\{\mathbf{E}_\lambda\}$ is the spectral family [10] corresponding to a fixed self-adjoint linear operator $\mathbf{L}: D \to H$ with spectrum $\Lambda \subset (-\infty,\infty)$; $A(t,\lambda)$ is continuous in λ for fixed t, and $A(t,\lambda) = \int_0^t a(s,\lambda)\,ds$ with $a(t,\lambda)$ nonnegative ($\lambda \in \Lambda$). (All integrands in formulas like (1.2) will be interpreted as zero when $\lambda \notin \Lambda$.)

We represent solutions of (1.1) as linear combinations

$$\mathbf{y}(t) = \mathbf{U}(t)\mathbf{x}_0 + \mathbf{W}(t)\mathbf{k}, \tag{1.3}$$

where $\mathbf{U}$ and $\mathbf{W}$ respectively are formal solutions of the resolvent equations

$$\mathbf{U}(t) + \int_0^t \mathbf{A}(t-s)\mathbf{U}(s)\,ds = \mathbf{I}\ (=\text{identity}), \tag{1.4}$$

$$\mathbf{W}(t) + \int_0^t \mathbf{A}(t-s)\mathbf{W}(s)\,ds = t\mathbf{I}. \tag{1.5}$$

If $H = R = \{\text{reals}\}$, $\mathbf{A} = A$, (1.4) and (1.5) reduce to scalar equations with solutions $U(t)$ and $W(t) = \int_0^t U(s)\,ds$. In this case, recent results of D. F. Shea and S. Wainger [11] imply that if $A' = a \in L^1(0,1) \cap C^1(0,\infty)$ and a is nonnegative,

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nonincreasing, and convex, then $U \in L^1(0, \infty) \cap C^1[0, \infty)$ and $U'(t) \to 0$ $(t \to \infty)$. It is then easy to use resolvent formulas to study the asymptotic behavior of solutions of variants of (1.1) with more general forcing terms.

Our results in the vector case are less complete; we do not determine whether $\|\mathbf{U}(t)\|$ is in L^1 or obtain limits in the norm topology as $t \to \infty$ for $\mathbf{U}(t)$ and $\mathbf{W}(t)$. We do, however, find conditions under which $\mathbf{x}(\infty)$ exists and $\|\mathbf{U}(t)\|$ and $\|\mathbf{W}(t)\|$ are bounded. These results, combined with suitable resolvent formulas, yield some results for variants of (1.1) (see Remark (iv) following Theorem 1).

In Theorem 1 we use spectral decomposition to reduce the problem to consideration of the scalar equation (2.2) with parameter. This reduction is possible because of the special form (1.2) of $\mathbf{A}(t)$. Theorems 2, 3 and 4 show how the solution of (2.2) depends on λ; they give conditions on $a(t, \lambda)$ ensuring that the hypotheses (2.5), (2.6) and (2.7) of Theorem 1 hold.

Our results apply, for example, to the equation

$$(1.6) \qquad u_t(t, y) + \int_0^t [\alpha(t - s)\mathbf{L} + \beta(t - s)\mathbf{L}^{1/2}]u(s, y)\, ds = g(y)$$

with initial condition $u(0, y) = f(y)$, where $\mathbf{L}$ is a positive Sturm–Liouville operator and $a(t, \lambda) = \alpha(t)\lambda + \beta(t)\lambda^{1/2}$ satisfies appropriate conditions, including (2.11) below (integration puts (1.6) in the form (1.1)). More generally,

$$a(t, \lambda) = \sum_{j=1}^{\infty} \alpha_j(t)\lambda^{\beta_j}$$

(with suitable α_j, β_j) is a possible kernel for (1.6). The unusual kernel,

$$(1.7) \qquad a(t, \lambda) = (t + 1)^{-1}\lambda + \lambda^{(t+1)/(3t+2)} - \lambda^{1/3},$$

is admissible if $\mathbf{L} \geqq \mathbf{I}$ (see the last paragraph of § 2).

The particular concern of the present work is illustrated by the kernel

$$\mathbf{A}(t) = (1 + t)^{1/2}\mathbf{L} + (1 + t)^{1/3}\mathbf{L}^{1/2} - \mathbf{L} - \mathbf{L}^{1/2},$$

where $a(t, \lambda)$ tends to zero $(t \to \infty)$ but is not in $L^1(0, \infty)$. Even in the scalar case, this situation leads to analytic difficulties, studied most successfully so far by means of the Fourier transform. Levin and Nohel [8] used this method; later versions appear in [4], [11] and in our proofs of Theorems 3 and 4 below.

In [7] we determined an asymptotic formula

$$\|\mathbf{x}(t) - \mathbf{\Omega}(t)\| \to 0 \qquad (t \to \infty),$$

for the special case of (1.1) where $a(t, \lambda) = \lambda a(t)(\mathbf{A}(t) = A(t)\mathbf{L}, \mathbf{L} \geqq \lambda_0\mathbf{I} > 0)$ and $a(t)$ is nonnegative, nonincreasing, and convex on $(0, \infty)$, $\int_0^1 a(t)\, dt < \infty$, and $a(t) \not\equiv a(\infty)$. Here $\mathbf{\Omega}(t) \equiv A^{-1}(\infty)\mathbf{L}^{-1}\mathbf{k}$, except when $a(t)$ is piecewise linear and of a special form. Earlier [5], [6] we developed boundedness and continuity results analogous to Theorems 3 and 4, but with $a(t, \lambda) = \lambda a(t)$; generalizing these results to the present case is the main analytic task in this paper. Our hypotheses will exclude the possibility that $a(t, \lambda)$ is a piecewise linear function of t.

A. Friedman and M. Shinbrot [3] study the operator equation

$$\mathbf{R}(t) + \mathbf{A}\int_0^t h(t - s)\mathbf{R}(s)\, ds = \mathbf{I}$$

in Banach space. They use complex spectral decomposition to determine conditions under which, for example, $\|\mathbf{R}(t)\| \in L^{\rho}(0, \infty)(1 \leqq \rho < \infty)$. They also discuss the existence (in various senses) of solutions of equations with variable operator kernels.

C. M. Dafermos [1] studies equations like (1.1) (but with kernel $\mathbf{A}(t, s)$ not necessarily of convolution type) by means of Lyapunov functionals and discusses applications to linear viscoelasticity.

R. M. MacCamy and J. S. W. Wong [9] study the equation

$$u_t(t, y) + \int_0^t a(t - s)\mathbf{L}u(s, y) = f(t, y),$$

where $\mathbf{L}$ is a strongly elliptic partial differential operator with discrete spectrum and a is a strongly positive (scalar) kernel. They, too, use certain functionals in H.

2. Statement of results. Following Friedman [2], we shall study (1.4) by means of the spectral representation

$$\mathbf{U}(t)\mathbf{y} = \int_{-\infty}^{\infty} u(t, \lambda)\, d\mathbf{E}_\lambda \mathbf{y}, \tag{2.1}$$

where $u(t, \lambda)$ is the solution of the scalar resolvent equation, which we write in the differentiated form

$$u'(t, \lambda) + \int_0^t a(t - s, \lambda)u(s, \lambda)\, ds = 0, \qquad u(0, \lambda) = 1. \tag{2.2}$$

(Primes denote differentiation with respect to the first variable, t in this case.) We let $w(t, \lambda) = \int_0^t u(s, \lambda)\, ds$ and

$$\mathbf{W}(t)\mathbf{y} = \int_0^t \mathbf{U}(s)\mathbf{y}\, ds = \int_{-\infty}^{\infty} w(t, \lambda)\, d\mathbf{E}_\lambda \mathbf{y}. \tag{2.3}$$

Then

$$w'(t, \lambda) + \int_0^t a(t - s, \lambda)w(s, \lambda)\, ds = 1, \qquad w(0, \lambda) = 0. \tag{2.4}$$

Throughout this paper we write $A^{-1}(\infty, \lambda) = (\int_0^\infty a(t, \lambda)\, dt)^{-1}$; if the integral is infinite, $A^{-1}(\infty, \lambda) = 0$.

THEOREM 1. *With* $\mathbf{A}(t)$ *as in* (1.2), *suppose*

$$\lim_{t \to \infty} u(t, \lambda) = 0, \qquad \lim_{t \to \infty} w(t, \lambda) = A^{-1}(\infty, \lambda), \qquad \lambda \in \Lambda, \tag{2.5}$$

and there exists a $C < \infty$ *such that*

$$|u(t, \lambda)| + |w(t, \lambda)| \leqq C, \qquad 0 \leqq t < \infty, \quad \lambda \in \Lambda. \tag{2.6}$$

Assume further that

$$u(t, \mu) \to u(t, \lambda), \qquad w(t, \mu) \to w(t, \lambda) \qquad (\mu \to \lambda, \quad \mu \in \Lambda), \tag{2.7}$$

uniformly in $0 \leqq t < \infty$.

Then (2.1) *and* (2.3) *define operators* $\mathbf{U}(t)$, $\mathbf{W}(t): H \to H$ *with* $\|\mathbf{U}(t)\| \leqq C$, $\|\mathbf{W}(t)\| \leqq C$. $\mathbf{U}$ *and* $\mathbf{W}$ *are strongly continuous in t, and both map D into D.*

If $\mathbf{x}_0$ *and* $\mathbf{k}$ *belong to* H, *and if* $\mathbf{y}(t)$ *is the function defined by* (1.3), *then*

$$\mathbf{y}(t) \to \int_{-\infty}^{\infty} A^{-1}(\infty, \lambda)\, d\mathbf{E}_\lambda \mathbf{k} \qquad (t \to \infty). \tag{2.8}$$

Suppose in addition that

$$A(t, \lambda) \leqq (1 + |\lambda|)\alpha(t) \qquad (\lambda \in \Lambda), \tag{2.9}$$

where $\alpha(t)$ *is bounded on compact subsets of* R^+. *If* $\mathbf{x}_0$ *and* $\mathbf{k}$ *belong to* D, *then* $\mathbf{y}(t)$ *is the unique solution of* (1.1) *and*

$$\lim_{t\to\infty} \mathbf{L}\mathbf{y}(t) = \mathbf{L}[\lim_{t\to\infty} \mathbf{y}(t)]. \tag{2.10}$$

Remarks. (i) If $\mathbf{L}^{-1}$ exists and is compact, hypothesis (2.7) may be omitted; see the remark following the proof of Theorem 1.

(ii) If $\mathbf{x}_0$ (or $\mathbf{k}$) belongs to $H \sim D$, $\mathbf{y}(t)$ is a weak solution of (1.1), in the sense that

$$\lim_{n\to\infty} \int_0^t \mathbf{A}(t - s)\mathbf{y}_n(s)\, ds = \lim_{n\to\infty} [\mathbf{x}_n + t\mathbf{k}_n - \mathbf{y}_n(t)] = \mathbf{x}_0 + t\mathbf{k} - \mathbf{y}(t),$$

where $\mathbf{x}_n, \mathbf{k}_n \in D$, $\mathbf{x}_n \to \mathbf{x}_0$, $\mathbf{k}_n \to \mathbf{k}$, $\mathbf{y}_n = \mathbf{U}(t)\mathbf{x}_n + \mathbf{W}(t)\mathbf{k}_n$.

(iii) Equation (2.10) tells us that $\mathbf{y}(t) \to \mathbf{y}(\infty)$ in the topology on D determined by the norm $(\|\mathbf{y}\|^2 + \|\mathbf{L}\mathbf{y}\|^2)^{1/2}$.

(iv) If a forcing term $h(t)\mathbf{g}$ is added to (1.1), the resolvent formulas

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{y}(t) + \int_0^t \mathbf{U}(t - s)\mathbf{g}h(s)\, ds \\ &= \mathbf{y}(t) + h(0)\mathbf{g} + \int_0^t \mathbf{W}(t - s)\mathbf{g}\, dh(s) \end{aligned}$$

can be used to study the solution.

The next three theorems give sufficient conditions for (2.5), (2.6) and (2.7). For each $\lambda \in \Lambda$ we assume that as a function of t, $a(t, \lambda)$ satisfies the following condition:

$$\begin{aligned} &a(t, \lambda) \in C^1(0, \infty) \cap L^1(0, 1); \\ &a \text{ is nonnegative, nonincreasing and convex on } \{0 < t < \infty\}, \\ &0 < a(0+, \lambda) \leqq \infty \quad \text{and} \quad a(\infty, \lambda) = 0. \end{aligned} \tag{2.11}$$

Our first result merely restates some earlier results.

THEOREM 2. *If* (2.11) *holds*, $|u(t, \lambda)| \leqq \sqrt{2}$ $(0 \leqq t < \infty)$; *moreover*, $u(t, \lambda) \to 0$ *and* $w(t, \lambda) \to A^{-1}(\infty, \lambda)$ $(t \to \infty)$.

Proof. The first conclusion is Theorem 2 of [5]; the second is included in the main result of [4].

THEOREM 3. *Suppose* (2.11) *holds, and let there exist positive numbers* T, M, η *such that* (*for a particular* λ)

$$A(T, \lambda) \geqq 4M; \tag{2.12}$$

suppose with $\rho = \min\{M, \pi/3T\}$, $a(t, \lambda)$ *is twice differentiable* (*in* t) *on a closed interval* J *of length* $4\pi/\rho$, *and*

$$0 < \eta a(0+, \lambda) \leqq a''(t, \lambda) < \infty, \qquad t \in J. \tag{2.13}$$

Then there exists a constant $C = C(T, M, \eta)$ *such that*

$$|w(t, \lambda)| \leqq C, \qquad 0 \leqq t < \infty. \tag{2.14}$$

In [7] we show that when $a(t, \lambda) = \lambda a(t)$, $\lambda \geqq \lambda_0 > 0$, (2.11) alone implies (2.14), even if $a \in C^1$ is replaced by $a \in C$ and the condition $a(\infty, \lambda) = 0$ is dropped. We conjecture that $a(\infty, \lambda) = 0$ is unnecessary here, too; it appears more difficult to determine what happens when $a(t, \lambda)$ is merely continuous in t.

Our proof of Theorem 3 follows the method used in [5] for the special case $a(t, \lambda) = \lambda a(t)$, $\lambda \geqq \lambda_0 > 0$ (to prove a weaker result than that in [7]). (2.12) and (2.13) replace the following condition of [5]:

(2.13a) $a(t) \in L^1(0, \infty)$; or $a(t)$ is twice differentiable, $a''(t)$ is bounded away from zero on finite intervals $(0, R]$, and $a(0+) < \infty$.

Note that when $a(t, \lambda) = \lambda a(t)$, the present result introduces one technical improvement: the fixed interval J replaces arbitrary intervals $(0, R]$. Moreover, by introducing the constants T, M, η, we obtain uniform estimates for kernels $a(t) = a(t, \lambda)$ having graphs with different shapes. From this point of view, the result may hold interest independent of Theorem 1.

The results of [5] permitted the following alternative to (2.13a): $a''(t)$ is nonincreasing and either (i) $a(t) = O(t^{-\beta})$, $[a'(t)]^{-1} = O(t^{\beta})$ $(t \to 0)$ for some β, $0 < \beta < 1$, or (ii) $[a'(t)]^{-1} = O(t)$ $(t \to 0)$. A version of this alternative (made uniform in λ) would suffice in place of (2.13) in Theorem 3.

THEOREM 4. *For each* $\lambda \in \Lambda$, *suppose* (2.11) *holds. Assume that*

$$\int_0^t |a(s, \lambda) - a(s, \mu)|\, ds \to 0 \qquad (\mu \to \lambda, \quad t > 0), \tag{2.15}$$

$$A^{-1}(\infty, \lambda) \text{ is continuous on } \Lambda. \tag{2.16}$$

Then (2.7) *holds.*

In [6] we proved that in the special case $a(t, \lambda) = \lambda a(t)$, $\lambda \geqq \lambda_0 > 0$, the maps $\lambda \to u(\cdot, \lambda)$, $\lambda \to w(\cdot, \lambda)$ (from $[\lambda_0, \infty)$ to the space of bounded continuous functions on $0 \leqq t < \infty$ with the uniform topology) are differentiable if (2.11) holds; the proof is much simpler than the one we give below for Theorem 4.

We can now verify that the kernel (1.7) satisfies the hypotheses of Theorem 1 if $\mathbf{L} \geqq \mathbf{I}$ (so that $\lambda \geqq 1$, $\lambda \in \Lambda$). Direct computation shows that (2.11) holds; so by Theorem 2, (2.5) and $|u| \leqq C$ are satisfied. For (2.6) we take

$$T = 1, \quad M = \int_0^1 (1 + t)^{-1}\, dt/4 = (\log 2)/4, \quad J = [\alpha, \beta]$$

any interval of length $4\pi/M$.

Now $a(0+, \lambda) = O(\lambda)$ $(\lambda \to \infty)$, while $a''(t, \lambda) \geqq 2(1 + \beta)^{-3}\lambda$ on J; thus η can be chosen so that (2.13) holds. By Theorem 3, hypothesis (2.6) is satisfied. (2.15) and (2.16) are easy to check, so (2.7) holds, by Theorem 4.

3. Proof of Theorem 1. The estimates on $\|\mathbf{U}\|$ and $\|\mathbf{W}\|$ follow immediately from (2.1), (2.3) and (2.6). For strong continuity, we have

$$\|[\mathbf{U}(t) - \mathbf{U}(s)]\mathbf{y}\|^2 = \int_{-\infty}^{\infty} [u(t,\lambda) - u(s,\lambda)]^2 d(\mathbf{E}_\lambda \mathbf{y}, \mathbf{y});$$

this tends to zero ($s \to t$) since u is bounded by C and continuous in t. Next,

$$\|\mathbf{L}\mathbf{U}(t)\mathbf{y}\| = \left\| \int_{-\infty}^{\infty} \lambda u(t,\lambda)\, d\mathbf{E}_\lambda \mathbf{y} \right\| \leqq C\|\mathbf{L}\mathbf{y}\|,$$

so $\mathbf{U}(t): D \to D$. $\mathbf{W}$ is handled similarly. Let $\varepsilon > 0$, and choose R so large that

$$C^2(\|\mathbf{F}_R \mathbf{x}_0\|^2 + 4\|\mathbf{F}_R \mathbf{k}\|^2) < \varepsilon, \tag{3.1}$$

where $\mathbf{F}_R = \mathbf{I} - \mathbf{E}_R + \mathbf{E}_{-R}$. By (2.5) and (2.7), there is a $T > 0$ such that

$$u^2(t,\lambda) + |w(t,\lambda) - A^{-1}(\infty,\lambda)|^2 < \varepsilon(\|\mathbf{x}_0\|^2 + \|\mathbf{k}\|^2)^{-1} \tag{3.2}$$

whenever $t \geqq T$ and $|\lambda| \leqq R$. For such t,

$$\begin{aligned} \left\| \mathbf{y}(t) - \int_{-\infty}^{\infty} A^{-1}(\infty,\lambda)\, d\mathbf{E}_\lambda \mathbf{k} \right\|^2 &\leqq \int_{-R}^{R} u^2(t,\lambda) d(\mathbf{E}_\lambda \mathbf{x}_0, \mathbf{x}_0) \\ &+ \int_{-R}^{R} |w(t,\lambda) - A^{-1}(\infty,\lambda)|^2 d(\mathbf{E}_\lambda \mathbf{k}, \mathbf{k}) + \varepsilon \leqq 2\varepsilon, \end{aligned} \tag{3.3}$$

and (2.8) holds.

Now suppose (2.9) holds. Without loss of generality, we assume that α is an increasing function. If $\mathbf{x}_0, \mathbf{k} \in D$, we have

$$\begin{aligned} &\|\mathbf{A}(t-\sigma)\mathbf{U}(\sigma)\mathbf{x}_0 - \mathbf{A}(t-s)\mathbf{U}(s)\mathbf{x}_0\|^2 \\ &= \int_{-\infty}^{\infty} |A(t-\sigma,\lambda)u(\sigma,\lambda) - A(t-s,\lambda)u(s,\lambda)|^2 d(\mathbf{E}_\lambda \mathbf{x}_0, \mathbf{x}_0) \to 0 \quad \text{as } \sigma \to s, \end{aligned}$$

since the integrand tends to zero for each λ and is dominated by $4(1 + |\lambda|)^2 C^2 \alpha^2(t)$. Thus $\mathbf{A}(t-s)\mathbf{U}(s)\mathbf{x}_0$ is continuous in s; a similar argument holds for $\mathbf{W}(s)\mathbf{k}$.

Next we verify that $\mathbf{y}(t)$ solves (1.1):

$$\begin{aligned} &\mathbf{U}(t)\mathbf{x}_0 + \int_0^t \mathbf{A}(t-s)\mathbf{U}(s)\mathbf{x}_0\, ds \\ &= \mathbf{U}(t)\mathbf{x}_0 + \int_0^t \left\{ \int_{-\infty}^{\infty} A(t-s,\lambda)u(s,\lambda)\, d\mathbf{E}_\lambda \mathbf{x}_0 \right\} ds \\ &= \mathbf{U}(t)\mathbf{x}_0 + \int_0^t \lim_{R\to\infty} \int_{-R}^{R} \{\cdots\}\, ds \\ &= \int_{-\infty}^{\infty} \left[u(s,\lambda) + \int_0^t A(t-s,\lambda)u(s,\lambda)\, ds \right] d\mathbf{E}_\lambda \mathbf{x}_0 = \mathbf{x}_0, \end{aligned}$$

where (2.2) (integrated), Fubini's theorem, and the estimate

$$\begin{aligned}
&\left\| \left[\int_{-\infty}^{-R} + \int_{R}^{\infty} \right] A(t-s,\lambda) u(s,\lambda)\, d\mathbf{E}_\lambda \mathbf{x}_0 \right\|^2 \\
&\quad \leqq \left[\int_{-\infty}^{-R} + \int_{R}^{\infty} \right] \alpha^2(t)(1+|\lambda|)^2 C^2 d(\mathbf{E}_\lambda \mathbf{x}_0, \mathbf{x}_0) \\
&\quad \leqq 2\alpha^2(t) C^2 (\|\mathbf{F}_R \mathbf{x}_0\|^2 + \|\mathbf{F}_R \mathbf{L} \mathbf{x}_0\|^2) \\
&\quad \to 0 \qquad (R \to \infty, \text{uniformly in } 0 \leqq s \leqq t)
\end{aligned}$$

have been used. Similarly,

$$\mathbf{W}(t)\mathbf{k} + \int_0^t A(t-s)\mathbf{W}(s)\mathbf{k}\, ds = t\mathbf{k}.$$

In view of (1.3), this shows that $\mathbf{y}(t)$ is a solution of (1.1).

If $\mathbf{y}_1(t)$ is another solution of (1.1), set $\mathbf{z} = \mathbf{y} - \mathbf{y}_1$, $\xi = \|(\mathbf{I} - \mathbf{F}_\mu)\mathbf{z}\|$ $(\mu > 0)$. Then

$$\begin{aligned}
\|(\mathbf{I} - \mathbf{F}_\mu)\mathbf{A}(t-s)\mathbf{z}(s)\|^2 &= \int_{-\mu}^{\mu} A^2(t-s,\lambda) d(\mathbf{E}_\lambda \mathbf{z}(s), \mathbf{z}(s)) \\
&\leqq (|\mu| + 1)^2 \alpha^2(t) \xi^2(s), \qquad 0 \leqq s \leqq t.
\end{aligned}$$

But $(\mathbf{I} - \mathbf{F}_\mu)\mathbf{z}(t) = -\int_0^t (\mathbf{I} - \mathbf{F}_\mu)\mathbf{A}(t-s)\mathbf{z}(s)\, ds$, so $0 \leqq \xi(t) \leqq (|\mu| + 1)\alpha(t) \int_0^t \xi(s)\, ds$, and Gronwall's inequality shows that $\xi \equiv 0$. Thus $\mathbf{y} \equiv \mathbf{y}_1$.

Finally, the proof of (2.10) is similar to that of (2.8), except that we estimate $\|\mathbf{L}\mathbf{y}(t) - \int_{-\infty}^{\infty} \lambda A^{-1}(\infty, \lambda)\, d\mathbf{E}_\lambda \mathbf{k}\|^2$ and a factor of λ^2 appears in the integrals. This proves Theorem 1.

Remark. Hypothesis (2.7) was used only for (3.2). If $\mathbf{L}^{-1}$ is compact, the integrals $\int_{-R}^{R}$ in (3.3) are finite sums and can be made $< \varepsilon$ without (2.7).

4. Preliminaries for Theorems 3 and 4. In this section we develop integral representations for the solutions to be estimated in §§ 5 and 6. We first collect some facts proved in [4] concerning the Laplace transform

$$\alpha^*(\zeta, \lambda) = \lim_{R \to \infty} \int_0^R e^{-\zeta t} a(t, \lambda)\, dt$$

of a. In this section we regard λ as fixed.

LEMMA 4.1. *If* (2.11) *holds, then* $a^*(\zeta, \lambda)$ *is analytic in* $\{\operatorname{Re} \zeta > 0\}$ *and can be extended as a continuous function to* $Z = \{\operatorname{Re} \zeta \geqq 0, \zeta \neq 0\}$. *Moreover,* $[\alpha^*(\zeta, \lambda)]^{-1} \to A^{-1}(\infty, \lambda)$ $(\zeta \to 0,\ \zeta \in Z)$, $a^*(\sigma + i\tau) \to 0$ $(|\tau| \to \infty)$ *uniformly in* $0 \leqq \sigma < \infty$, *and* $\operatorname{Re} a^*(\zeta, \lambda) > 0$ $(\zeta \in Z)$.

Proof. Each of these results is either well known or is contained in Lemmas 3 or 5 of [4].

We shall write $a^*(i\tau, \lambda) = \varphi(\tau, \lambda) - i\psi(\tau, \lambda)$, and we define

$$\varphi_1(\tau, \lambda) = \int_0^{\pi/2\tau} a(t, \lambda) \cos \tau t\, dt, \qquad \varphi_2 = \varphi - \varphi_1.$$

LEMMA 4.2. *If* (2.11) *holds, we have the following inequalities*:

$$0 \leqq -\varphi_2 \leqq \psi \leqq 4\varphi_1, \tag{4.1}$$

$$\varphi_1(\tau, \lambda) \leqq A(\pi/3\tau, \lambda) \leqq 4|a^*(i\tau, \lambda)| \leqq 18A(\pi/3\tau, \lambda) \leqq 36\varphi_1(\tau, \lambda). \tag{4.2}$$

Proof. The last inequality is obvious. The others were proved in [4, Lemma 3 and lines (3.22)–(3.24)].

We shall write $D(\tau, \lambda) = |a^*(i\tau, \lambda) + i\tau|$. Since $\varphi(\tau, \lambda) = \operatorname{Re} a^*(i\tau, \lambda) > 0$,

$$0 < \varphi/D \leqq 1, \qquad \tau > 0. \tag{4.3}$$

LEMMA 4.3. *If* (2.11) *holds, then* $\varphi_1(\tau, \lambda) = O(D(\tau, \lambda))$ $(t \to 0^+)$.

Proof. As $\tau \to 0^+$, $a^*(i\tau, \lambda) \to A(\infty, \lambda) > 0$, so $D(\tau, \lambda) > |a^*(i\tau, \lambda)|/2$ for small τ, and our result follows from (4.2).

LEMMA 4.4. *Suppose* (2.11) *holds. Then*

$$v(t, \lambda) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \operatorname{Re}\left\{\frac{e^{i\tau t} Q(\tau, \lambda)}{i\tau[a^*(i\tau, \lambda) + i\tau]}\right\} d\tau, \qquad t > 0, \tag{4.4}$$

where

$$Q(\tau, \lambda) = 1 - A^{-1}(\infty, \lambda) a^*(i\tau, \lambda), \qquad v(t, \lambda) = \pi[w(t, \lambda) + A^{-1}(\infty, \lambda)(u(t, \lambda) - 1)].$$

Remark. In the following we shall use without explicit mention the obvious fact that $|Q(\tau, \lambda)| \leqq 2$ $(\tau > 0)$.

Proof. Integrating (2.4), we see that

$$w(t) + \int_0^t A(t - s) w(s)\, ds = t, \qquad t \geqq 0.$$

(Here and below, we suppress λ.) By a standard result on Volterra equations, $w(t)$ satisfies an inequality $|w(t)| \leqq b_1 \exp(b_2 t)(b_1, b_2 > 0)$. Taking Laplace transforms, we obtain

$$w^*(\zeta) = [\zeta(\zeta + a^*(\zeta))]^{-1}, \qquad \operatorname{Re}\zeta > b_2.$$

A similar argument starting from (2.2) shows that for some $b \geqq b_2$, $|u(t)| \leqq b' \exp(bt)$ and

$$u^*(\zeta) = [\zeta + a^*(\zeta)]^{-1}, \qquad \operatorname{Re}\zeta > b.$$

Then

$$\begin{aligned} v^*(\zeta) &= \pi[w^*(\zeta) + A^{-1}(\infty, \lambda)(u^*(\zeta) - \zeta^{-1})] \\ &= \pi[1 - A^{-1}(\infty) a^*(\zeta)][\zeta(\zeta + a^*(\zeta))]^{-1}, \qquad \operatorname{Re}\zeta > b. \end{aligned} \tag{4.5}$$

By Lemma 4.1, (4.5) defines $v^*(\zeta)$ as a continuous function in Z, analytic in $\{\operatorname{Re}\zeta > 0\}$. Because $v(t)$ is continuously differentiable, the complex inversion formula

$$2\pi i v(t) = \lim_{R \to \infty} \int_{\sigma - iR}^{\sigma + iR} e^{\zeta t} v^*(\zeta)\, d\zeta, \qquad t > 0, \quad \sigma > b \tag{4.6}$$

holds. By Lemma 4.1 and (4.5), $v^*(\sigma + i\tau) = O(\tau^{-2})$ ($|\tau| \to \infty$), uniformly in $\sigma \geqq 0$. Moreover, if $A^{-1}(\infty) = 0$, $|a^*(\zeta)| \to \infty$ ($\zeta \to 0, \zeta \in Z$), so $v^*(\zeta) = o(\zeta^{-1})$; if $A^{-1}(\infty) > 0$, $1 - A^{-1}(\infty)a^*(\zeta) \to 0$ ($\zeta \to 0, \zeta \in Z$), so again $v^*(\zeta) = o(\zeta^{-1})$. Therefore we can shift the contour in (4.6) to obtain

$$2\pi v(t) = \lim_{\varepsilon \to 0^+} \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right] e^{i\tau t} v^*(i\tau)\, d\tau,$$

and a change of variable $\tau \to -\tau$ in the first of these integrals, together with (4.5), yields (4.4).

We can represent $u(t, \lambda)$ similarly.

LEMMA 4.5. *If* (2.11) *holds, then*

$$\pi u(t, \lambda) = \lim_{R \to \infty} \int_0^R \operatorname{Re}\left\{\frac{e^{i\tau t}}{a^*(i\tau, \lambda) + i\tau}\right\} d\tau, \qquad t > 0. \tag{4.7}$$

This was proved in [4]; see line (3.3) of that paper.

Finally, we prove an important consequence of Lemma 4.4.

LEMMA 4.6. *Suppose* (2.11) *holds and* $0 < R < \infty$. *Then*

$$\int_0^R \frac{\psi(\tau, \lambda)\, d\tau}{\tau D^2(\tau, \lambda)} = \int_0^R \frac{\operatorname{Re} Q(\tau, \lambda)}{D^2(\tau, \lambda)} d\tau - I_2(0, \lambda), \tag{4.8}$$

where

$$I_2(t, \lambda) = \int_R^\infty \operatorname{Re}\left\{\frac{e^{i\tau t} Q(\tau, \lambda)}{i\tau[a^*(i\tau, \lambda) + i\tau]}\right\} d\tau, \qquad t \geqq 0.$$

Proof. Taking real and imaginary parts in (4.4), we obtain

$$\begin{aligned} v(t) = {} & \lim_{\varepsilon \to 0^+} \int_\varepsilon^R \frac{\cos \tau t\, \psi(\tau)}{\tau D^2(\tau)} d\tau - \int_0^R \cos \tau t \frac{\operatorname{Re} Q(\tau)}{D^2(\tau)} d\tau \\ & + \int_0^R \frac{\sin \tau t}{\tau D^2(\tau)} [\varphi(\tau) \operatorname{Re} Q(\tau) - (\psi(\tau) - \tau) \operatorname{Im} Q(\tau)]\, d\tau + I_2(t). \end{aligned} \tag{4.9}$$

Here we know the second integral exists, because $D^{-2}(\tau)$ is continuous at $\tau = 0$ (Lemma 1). The third integrand is bounded by a constant times $tD^{-1}(\tau)$, by Lemmas 4.2 and 4.3; thus the third integral exists (and tends to zero as $t \to 0^+$). These facts imply that the indicated limit in (4.9) exists. Moreover, for small t, the first integrand is nonnegative, and it increases to $\psi(\tau)/\tau D^2(\tau)$ as $t \downarrow 0$. Thus we may pass to the limit in (4.9) (using dominated convergence in the second and fourth terms on the right) and obtain (4.8). This proves Lemma 4.6.

We note for later reference that the third integral in (4.9) can be written

$$\begin{aligned} & \int_0^R \frac{\sin \tau t}{\tau D^2(\tau, \lambda)} \{\cdots\}\, d\tau \\ & = \int_0^R \frac{\sin \tau t}{\tau} [\varphi_1^{-1}(\tau, \lambda) - A^{-1}(\infty, \lambda)]\, d\tau + \int_0^R E(\tau, \lambda) \sin \tau t\, d\tau, \end{aligned} \tag{4.10}$$

where

$$E(\tau, \lambda) = [\varphi_2(\operatorname{Re} Q - \varphi_1 A^{-1}(\infty)) - (\psi - \tau)\operatorname{Re} Q]/\tau\varphi_1^2$$
$$+ [2\varphi_1\varphi_2 + \varphi_2^2 + (\psi - \tau)^2][(\psi - \tau)\operatorname{Im} Q - \varphi \operatorname{Re} Q]/\tau\varphi_1^2 D^2.$$

5. Proof of Theorem 3. Throughout this section, C denotes a constant depending only on T, M and η. We shall suppress λ.

By Lemma 4.4 we must show that the integral in (4.4) is bounded by C. We use the representation (4.9) with $R = \rho$. Consider first $I_2(t)$. If

$$\omega = \max\{\rho, [2\pi a(0)]^{1/2}\},$$

then $\tau \geqq \omega$ implies $0 \leqq \psi(\tau) \leqq a(0)\pi/\tau \leqq \omega^2/2\tau \leqq \tau/2$, so

$$D(\tau) \geqq |\operatorname{Im}[a^*(i\tau) + i\tau]| = |\tau - \psi(\tau)| \geqq \tau/2.$$

Then

$$\left|\int_\omega^\infty \operatorname{Re}\left\{\frac{e^{i\tau t}Q(\tau)}{i\tau[a^*(i\tau) + i\tau]}\right\} d\tau\right| \leqq 4\int_\rho^\infty \tau^{-2}\, d\tau = C. \tag{5.1}$$

For $\rho \leqq \tau \leqq \omega$, let $T_j = 2j\pi/\tau$, $j = 0, 1, \cdots$, where $T_\nu, T_{\nu+1}, \cdots, T_{\nu+N}$ denote the particular T_j which belong to the interval J of (2.13). Then $N = N(\tau) \geqq \tau/\rho$. Since $a(t)$ is convex, $a'(t)$ is nondecreasing, and integration by parts shows that

$$\begin{aligned}\tau\varphi(\tau) &= -\int_0^\infty a'(t)\sin \tau t\, dt\\ &= \sum_{j=0}^\infty \int_0^{\pi/\tau} \sin \tau t[a'(t + T_j + \pi/\tau) - a'(t + T_j)]\, dt\\ &\geqq \sum_{j=\nu}^{\nu+N-1} \int_0^{\pi/\tau} \sin \tau t[\eta a(0)\pi/\tau]\, dt,\end{aligned}$$

by (2.13). Thus $\tau\varphi(\tau) \geqq 2\pi a(0)\eta N/\tau^2 \geqq \omega^2\eta/\tau\rho$ $(\rho \leqq \tau \leqq \omega)$, so

$$\left|\int_\rho^\omega \operatorname{Re}\left\{\frac{e^{i\tau t}Q(\tau)}{i\tau[a^*(i\tau) + i\tau]}\right\} d\tau\right| \leqq \int_\rho^\omega \frac{2\, d\tau}{\tau\varphi(\tau)} \leqq \frac{\rho}{\eta} = C.$$

Together with (5.1), this establishes

$$|I_2(t)| \leqq C. \tag{5.2}$$

When $0 < \tau \leqq \rho$,

$$\varphi_1(\tau) \geqq \tfrac{1}{2}A(\pi/3\rho) \geqq \tfrac{1}{2}A(T) \geqq 2M \geqq 2\tau \tag{5.3}$$

by (2.12). But the modulus of a complex number is at least $2^{-1/2}$ times the sum of its real and imaginary parts, so (using (4.1)) we see that

$$\begin{aligned}D(\tau) &\geqq 2^{-1/2}[\varphi_1(\tau) + \varphi_2(\tau) + \psi(\tau) - \tau]\\ &\geqq 2^{-1/2}[\varphi_1(\tau) - \tau] \geqq 2^{-3/2}\varphi_1(\tau) \geqq 2^{-1/2}M, \qquad 0 < \tau \leqq \rho.\end{aligned} \tag{5.4}$$

Then

$$\left| \int_0^\rho \frac{(\cos \tau t) \operatorname{Re} Q(\tau)}{D^2(\tau)} \, d\tau \right| \leqq 4\rho/M^2 = C. \tag{5.5}$$

Now (5.2), (5.5) and (4.8) imply that

$$\int_0^\rho \frac{\psi(\tau) \, d\tau}{\tau D^2(\tau)} \leqq C. \tag{5.6}$$

Our proof will be complete once we find a bound for the third integral in (4.9). Here we refer to (4.10). Note that by (4.1), (5.3), (5.4) and the fact that $\varphi_1(\tau) A^{-1}(\infty) \leqq 1$,

$$E(\tau) \leqq C \frac{\psi(\tau) + \tau}{\tau D^2(\tau)}, \qquad 0 < \tau \leqq \rho.$$

By (5.6) and (5.4), $|\int_0^\rho E(\tau) \sin \tau t \, d\tau| < C$.

Finally, since $\varphi_1'(\tau) = -\int_0^{\pi/2\tau} ta(t) \sin \tau t \, dt \leqq 0$, $\varphi_1^{-1}(\tau) - A^{-1}(\infty) \downarrow 0$ as $\tau \downarrow 0$. By the second law of the mean,

$$\int_0^\rho \frac{\sin \tau t}{\tau} [\varphi_1^{-1}(\tau) - A^{-1}(\infty)] \, d\tau = [\varphi_1^{-1}(\rho) - A^{-1}(\infty)] \int_\xi^\rho \frac{\sin \tau t}{\tau} \, d\tau,$$

where $0 < \xi < \rho$. Since $\varphi_1(\rho) \geqq \frac{1}{2} A(T) \geqq 2M$ and

$$\left| \int_\xi^\rho \frac{\sin \tau t}{\tau} \, d\tau \right| < C,$$

we have

$$\int_0^\rho \frac{\sin \tau t}{\tau} [\varphi_1^{-1}(\tau) - A^{-1}(\infty)] \, d\tau < C$$

and our proof is complete.

6. Proof of Theorem 4. This proof follows the same outline as the preceding one, except that in each step we establish continuity instead of boundedness. Throughout this section we fix $\lambda \in \Lambda$ and let C_λ and δ_λ denote constants depending only on λ. We write $|\mu - \lambda| < \delta_\lambda$ or $\mu \to \lambda$, tacitly requiring $\mu \in \Lambda$. Our main tool is the following convergence lemma.

LEMMA 6.1. *Suppose that for each $t > 0$ and $|\mu - \lambda| < \delta_\lambda$, $f(\tau, \mu)$ and $g(\tau, \mu)$ are integrable as functions of τ on $\{0 < \tau < \tau_1\}$ with $f \geqq 0$. Suppose*

$$\int_0^{\tau_1} f(\tau, \mu) \, d\tau \to \int_0^{\tau_1} f(\tau, \lambda) \, d\tau \qquad (\mu \to \lambda), \tag{6.1}$$

$$f(\tau, \mu) \to f(\tau, \lambda) \quad (\mu \to \lambda) \quad \textit{uniformly on any set } 0 < \tau_0 \leqq \tau \leqq \tau_1, \tag{6.2}$$

$$|g(\tau, \mu)| \leqq C_\lambda \quad (|\mu - \lambda| < \delta_\lambda), \quad g(\tau, \mu) \to g(\tau, \lambda) \qquad (\mu \to \lambda). \tag{6.3}$$

Then

$$\int_0^{\tau_1} |f(\tau, \mu) g(\tau, \mu) - f(\tau, \lambda) g(\tau, \lambda)| \, d\tau \to 0 \qquad (\mu \to \lambda).$$

Proof. First assume $g \equiv 1$. Let $\varepsilon > 0$ and choose $\tau_0 > 0$ such that $\int_0^{\tau_0} f(\tau, \lambda)\, d\tau < \varepsilon$. Choose δ such that $|\mu - \lambda| < \delta$ implies $|f(\tau, \mu) - f(\tau, \lambda)| < \varepsilon/\tau_1$ $(\tau_0 \leqq \tau \leqq \tau_1)$ and $|\int_0^{\tau_1} [f(\tau, \mu) - f(\tau, \lambda)]\, d\tau| < \varepsilon$. Two applications of the triangle inequality show that if $|\mu - \lambda| < \delta$, $\int_0^{\tau_1} |f(\tau, \mu) - f(\tau, \lambda)|\, d\tau < 5\varepsilon$. For arbitrary g we note that

$$|f(\tau, \mu)g(\tau, \mu) - f(\tau, \lambda)g(\tau, \lambda)| \leqq |f(\tau, \mu) - f(\tau, \lambda)|C_\lambda + f(\tau, \lambda)|g(\tau, \mu) - g(\tau, \lambda)|$$

and use Lebesgue's dominated convergence theorem. This proves Lemma 6.1.

Choose $\tau_1 > 0$ so that $A(\pi/3\tau_1, \lambda) > 10\tau_1$. We fix this value of τ_1 for the remainder of this section. By (2.15) and estimates like (5.3), (5.4),

$$A(\pi/3\tau_1, \mu) > 8\tau_1, \qquad |\mu - \lambda| < \delta_\lambda, \tag{6.4}$$

$$D(\tau, \mu) \geqq 2^{-3/2}\varphi_1(\tau) \geqq \tau_1, \qquad |\mu - \lambda| < \delta_\lambda, \quad 0 < \tau \leqq \tau_1. \tag{6.5}$$

Similarly, by Lemma 4.2,

$$D(\tau, \mu)\varphi_1^{-1}(\tau, \mu) < C_\lambda, \quad |\mu - \lambda| < \delta_\lambda, \quad 0 < \tau \leqq \tau_1. \tag{6.6}$$

LEMMA 6.2. *$a^*(i\tau, \mu) \to a^*(i\tau, \lambda)$, uniformly for $0 < \tau_0 \leqq \tau \leqq \tau_1$. The same conclusion holds for $\varphi(\tau, \mu)$, $\varphi_1(\tau, \mu)$, and $\psi(\tau, \mu)$.*

Proof (for φ; the others are similar). Let $\varepsilon > 0$. Choose $T > 0$ such that

$$\int_T^{T + 2\pi/\tau_0} a(t, \lambda)\, dt < \varepsilon,$$

and choose $\delta > 0$ such that $|\mu - \lambda| < \delta$ implies

$$\int_0^{T + 2\pi/\tau_0} |a(t, \lambda) - a(t, \mu)|\, dt < \varepsilon.$$

Then if $\tau \geqq \tau_0$, choose an integer ν such that $\omega = (2\nu + 1)\pi/2\tau$ is in $[T, T + \pi/\tau_0]$. Then if $|\mu - \lambda| < \delta$, $|\varphi(\tau, \mu) - \int_0^\omega a(t, \mu) \cos \tau t\, dt| < \int_\omega^{\omega + \pi/\tau} a(t, \mu)\, dt < 2\varepsilon$, so

$$|\varphi(\tau, \mu) - \varphi(\tau, \lambda)| < 3\varepsilon + \int_0^\omega |a(t, \mu) - a(t, \lambda)|\, dt < 4\varepsilon.$$

The estimate is uniform in $\tau \geqq \tau_0$, so our proof is complete.

Turning to Theorem 4 itself, we express (4.7) as

$$\pi u(t, \lambda) = \int_0^{\tau_1} + \lim_{R \to \infty} \int_{\tau_1}^R = J_1(t, \mu) + J_2(t, \mu).$$

Using (6.5) and Lemma 6.2, we see that J_1 is continuous in μ, uniformly in t. Now

$$J_2(t, \lambda) - J_2(t, \mu) = \int_{\tau_1}^\infty \mathrm{Re} \left\{ \frac{e^{i\tau t}[a^*(i\tau, \mu) - a^*(i\tau, \lambda)]}{[a^*(i\tau, \lambda) + i\tau][a^*(i\tau, \mu) + i\tau]} \right\} d\tau.$$

Here, for $|\mu - \lambda| < \delta_\lambda$, the integrand is dominated by $2D^{-2}(\tau, \lambda) = O(\tau^{-2})$ $(\tau \to \infty)$, since $a^*(i\tau, \lambda) \to 0$ $(\tau \to \infty)$ and $i\tau + a^*(i\tau, \lambda) \neq 0$. (Lemma 4.1). By Lemma 6.2, the integrand tends to zero as $\mu \to \lambda$. By Lebesgue's dominated convergence theorem, $J_2(t, \mu) \to J_2(t, \lambda)$ $(\mu \to \lambda)$, and the convergence is evidently uniform in t. This establishes the first conclusion of Theorem 4.

Since $A^{-1}(\infty, \mu)$ and $u(t, \mu)$ are continuous in μ, our proof will be complete if we show that

$$v(t, \mu) \to v(t, \lambda) \qquad (\mu \to \lambda, \text{ uniformly in } t > 0). \tag{6.7}$$

where v is the function defined in Lemma 4.4. Define $I_2(t, \mu)$ as in Lemma 4.6, with $R = \tau_1$. Then

$$|I_2(t, \lambda) - I_2(t, \mu)| \leqq \int_{\tau_1}^{\infty} \frac{1}{\tau} \left| \frac{Q(\tau, \lambda)}{[a^*(i\tau, \lambda) + i\tau]} - \frac{Q(\tau, \mu)}{[a^*(i\tau, \mu) + i\tau]} \right| d\tau.$$

But by Lemmas 4.1 and 6.2, the latter integrand is dominated by the integrable function $6/\tau D(\tau, \lambda)$ ($|\lambda - \mu| < \delta_\lambda$) and is pointwise convergent to zero ($\mu \to \lambda$). Therefore

$$I_2(t, \mu) \to I_2(t, \lambda) \qquad (\mu \to \lambda, \text{ uniformly in } t \geqq 0). \tag{6.8}$$

It is likewise clear that

$$\int_0^{\tau_1} \cos \tau t \frac{\operatorname{Re} Q(\tau, \mu)}{D^2(\tau, \mu)} \to \int_0^{\tau_1} \cos \tau t \frac{\operatorname{Re} Q(\tau, \lambda)}{D^2(\tau, \lambda)} d\tau \quad (\mu \to \lambda, \text{ uniformly in } t \geqq 0). \tag{6.9}$$

Referring to (4.8), we reach the important conclusion that the function $f(t, \mu) = \psi(\tau, \mu)/\tau D^2(\tau, \mu) \geqq 0$ satisfies hypotheses (6.1) and (6.2) of Lemma 6.1. Then

$$\int_0^{\tau_1} \frac{\cos \tau t \psi(\tau, \mu)}{\tau D^2(\tau, \mu)} d\tau \to \int_0^{\tau_1} \frac{\cos \tau t \psi(\tau, \lambda)}{\tau D^2(\tau, \lambda)} d\tau \tag{6.10}$$

as $\mu \to \lambda$, and the convergence is uniform in t.

Finally, consider (4.10) with μ in place of λ and $R = \tau_1$. Using (6.6) and Lemmas 4.2 and 6.2, one sees that

$$E(\tau, \mu) = \frac{\psi(\tau, \mu)}{\tau D^2(\tau, \mu)} g_1(\tau, \mu) + \frac{g_2(\tau, \mu)}{\varphi_1^2(\mu)},$$

where g_1 and g_2 satisfy hypothesis (6.3) of Lemma 6.1. But each of $\psi/\tau D^2$ and φ_1^{-2} satisfies the first two hypotheses of that lemma (the latter obviously since $\varphi_1^{-2} \leqq 8/\tau_1^2$ on $(0, \tau_1]$, by (6.5)). Therefore

$$\left| \int_0^{\tau_1} [E(\tau, \mu) \sin \tau t - E(\tau, \lambda) \sin \tau t] \, d\tau \right| \leqq \int_0^{\tau_1} |E(\tau, \mu) - E(\tau, \lambda)| \, d\tau \to 0 \tag{6.11}$$

as $\mu \to \lambda$, with convergence again uniform in t.

To treat the other integral in (4.10), recall that $\varphi_1^{-1}(\tau, \mu) - A^{-1}(\infty, \mu) \downarrow 0$ as $\tau \downarrow 0$. Let $\varepsilon > 0$. Choose $\theta > 0$ such that $\varphi_1^{-1}(\theta, \lambda) - A^{-1}(\infty, \lambda) < \varepsilon$, and choose $\delta > 0$ so that $|\mu - \lambda| < \delta$ implies

$$|[\varphi_1^{-1}(\tau, \mu) - A^{-1}(\infty, \mu)] - [\varphi_1^{-1}(\tau, \lambda) - A^{-1}(\infty, \lambda)]| < \varepsilon\theta$$

($\tau \geqq \theta$). This is possible, by Lemma 6.2 and (2.16). By the second law of the mean, with $\Phi(\tau, \mu) = \varphi_1^{-1}(\tau, \mu) - A^{-1}(\infty, \mu)$,

$$\int_0^{\tau_1} \frac{\sin \tau t}{\tau}[\Phi(\tau, \lambda) - \Phi(\tau, \mu)]\, d\tau$$

$$= \left| \Phi(\theta, \lambda) \int_{\xi}^{\theta} \frac{\sin \tau t}{\tau}\, d\tau - \Phi(\theta, \mu) \int_{\xi'}^{\theta} \frac{\sin \tau t}{\tau}\, d\tau + \int_{\theta}^{\tau_1} \frac{\sin \tau t}{\tau}[\Phi(\tau, \lambda) - \Phi(\tau, \mu)]\, d\tau \right|$$

$$\leqq 3\varepsilon \sup \int \frac{\sin x}{x}\, dx + \varepsilon \tau_1 .$$

Hence

$$\int_0^{\tau_1} \frac{\sin \tau t}{\tau} \Phi(\tau, \mu)\, d\tau \to \int_0^{\tau_1} \frac{\sin \tau t}{\tau} \Phi(\tau, \lambda)\, d\tau \qquad (\mu \to \lambda, \text{ uniformly in } t > 0.)$$

Assembling (6.8), (6.9), (6.10) and (6.11) and comparing these to (4.9) and (4.10), we see that (6.7) holds, and the proof is complete.

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ON AVERAGES OF A FUNCTION AND ITS APPLICATION TO BOUNDARY VALUE PROBLEMS WITH INSUFFICIENT DATA*

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Abstract. We prove that a periodic function is uniquely determined by its averages over some equally spaced points. Also we show how to recapture a function from its averages and give some application to boundary value problems with insufficient data.

1. Introduction and results. Let f be a continuous periodic function on the real line with period one. For any α, $0 \leqq \alpha \leqq 1$, we define the *α-means of f* to be

$$S_{\alpha,n}(f) = \frac{1}{n}\sum_{k=1}^{n} f\left(\alpha + \frac{k}{n}\right),$$

$n = 1, 2, \cdots$. We denote

$$\begin{aligned} R_{\alpha,n}(f) &= S_{\alpha,n}(f) - \lim_{n\to\infty} S_{\alpha,n}(f) \\ &= S_{\alpha,n}(f) - S_{\alpha,\infty}(f) \\ &= \frac{1}{n}\sum_{k=1}^{n} f\left(\alpha + \frac{k}{n}\right) - \int_0^1 f(t)\,dt. \end{aligned}$$

Some properties of $S_{0,n}(f)$ have been discussed in [6], and $\{R_{0,n}(f)\}$ are defined to be the Riemann coefficients of f in [3]. For any $s > 0$, a periodic function with period one is defined to be in B^s (cf. [4]) if the Fourier coefficients of f satisfy the following condition:

$$a_n(f) = \int_0^1 f(t)\,e^{-i2\pi nt}\,dt = O\left(\frac{1}{n^s}\right).$$

THEOREM 1. *Let α and β be any two real numbers in* $[0, 1]$.

(i) *Suppose $\alpha - \beta$ is irrational and f is in B^s with $s > 1$ such that $S_{\alpha,n}(f) = S_{\beta,n}(f) = 0$ for all $n > 0$. Then f is the zero function.*
(ii) *Suppose $\alpha - \beta$ is rational. Then there is a trigonometric function $g \neq 0$ such that $S_{\alpha,n}(g) = S_{\beta,n}(g) = 0$.*
(iii) *Suppose $\alpha - \beta$ is irrational. Then there exist trigonometric polynomials $P_{\alpha,m}$ such that*

$$\begin{aligned} R_{\beta,n}(P_{\alpha,m}) &= S_{\beta,n}(P_{\alpha,m}) = 0, \\ R_{\alpha,n}(P_{\alpha,m}) &= S_{\alpha,n}(P_{\alpha,m}) = \delta_{m,n}. \end{aligned}$$

for all positive integers m and n.

We remark that if $f \in B^s$, $s > 1$, and f is an even function, i.e., $f(t) = f(-t)$, then the function f is uniquely determined by $S_{0,n}(f)$ (cf. [6]). On the other hand, we can extend any function f in $[0, \frac{1}{2})$ to be an even periodic function and then apply theorems in [6] to f. This procedure has been carried out (cf. [1]). The defects

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of such "reflection" methods are that the extended function is no longer smooth and the first derivative of the extended function cannot be an even function which gives considerable difficulty in application (e.g., Proposition 2). Our representation formula for an even function f in terms of $S_{0,n}(f)$ holds under the condition that f is in $B^{1+\varepsilon}$, $\varepsilon > 0$, yet the extended function is usually in B^1. Furthermore, $S_{0,n}(f')$ vanishes for all n since the extended f' is odd even when f' is smooth.

We let $\mu(k)$ be the Möbius function (cf. [8]) defined on positive integers, i.e.,

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } p^2|k \text{ for some } p > 1, \\ (-1)^m & \text{if } k = P_1 \cdots P_m \text{ with distinct primes } P_j. \end{cases}$$

(Here $j|n$ means that j is a factor of n.)

THEOREM 2. *Let α and β be any two reals in* $[0, 1]$.

(i) *Suppose $\alpha - \beta$ is irrational. Let f be in B^s with $s > 1$. Then*

$$a_0(f) = \lim_{n\to\infty} S_{\alpha,n}(f) = \lim_{n\to\infty} S_{\beta,n}(f),$$

$$a_m(f) = \sum_{k=1}^{\infty} \frac{\mu(k)[e^{-i2\pi m\beta}R_{\alpha,|mk|}(f) - e^{-i2\pi m\alpha}R_{\beta,|mk|}(f)]}{2i\sin 2\pi m(\alpha - \beta)}$$

for all $m = \pm 1, \pm 2, \cdots$.

(ii) *Let $\alpha - \beta$ be an algebraic number of degree $q \geqq 2$ and f be in B^s with $s > \sqrt{2q}$. Then*

$$f(t) = R_{\alpha,\infty}(f) + \sum_{n=1}^{\infty} [R_{\alpha,n}(f)P_n(\alpha, \beta, t) + R_{\beta,n}(f)P_n(\beta, \alpha, t)],$$

where the series converges uniformly with the rate N^{x-s}, $\sqrt{2q} < x < s$, and

$$P_n(\alpha, \beta, t) = \sum_{j|n} \mu\left(\frac{n}{j}\right)\frac{\sin 2\pi j(t - \beta)}{\sin 2\pi j(\alpha - \beta)}.$$

Finally, we would like to mention some applications of the above theorems to boundary value problems with insufficient data.

PROPOSITION 1. *Let $u(r, \theta)$ be a solution of the following boundary value problem in R^2:*

$$\Delta u = 0 \quad \text{for } r < 1;$$

$$\frac{1}{kn}\sum_{j=1}^{kn} u\left(1, \frac{2\pi j}{kn} + 2\pi\alpha\right) = b_{\alpha,nk},$$

$$\frac{1}{kn}\sum_{j=1}^{kn} u\left(1, \frac{2\pi j}{kn} + 2\pi\beta\right) = b_{\beta,nk},$$

for some fixed positive integer k and all $n = 1, 2, 3, \cdots$. Suppose $\alpha - \beta$ is an *algebraic number of degree* $q \geqq 2$ *and* $u(1, 2\pi\theta) \in B^s$ *with* $s > \sqrt{2q}$. *Then*

$$S_\infty(u(r, 2\pi t)) = \lim_{n\to\infty} b_{\alpha,nk},$$

$$R_{nk}(u(r, 2\pi t)) = \sum_{m=1}^{\infty} \left[b_{\alpha,mnk} \left(\sum_{l|m} \mu\left(\frac{m}{l}\right) r^{lnk} \right) \right]$$

for all $n = 1, 2 \cdots$.

It follows from Theorem 1 that for $k \neq 1$, the data $\{b_{\alpha,nk}\}$ and $\{b_{\beta,nk}\}$ fail to determine $u(1, \theta)$, and hence the solution $u(r, \theta)$, uniquely. However, the corresponding averages of the solutions are uniquely determined when the data are smooth. Of course we can use Theorem 2 and Proposition 1 to construct the solution u from $\{b_{\alpha,nk}\}$ and $\{b_{\beta,nk}\}$ for the case $k = 1$.

PROPOSITION 2. *Let* α *and* β *be any two real numbers such that* $\alpha - \beta$ *is an algebraic number of degrees* $q \geqq 2$. *Let* $u(t) \in B^s$, $s > \sqrt{2q} - 1$, *be the solution of the following boundary value problem*:

$$au'' + bu' + cu = f; \quad f \in B^p, \quad p > 1; \quad u(0) = u(1), \quad u'(0) = u'(1),$$

where a, b and c are constants satisfying

$$c + 2\pi inb - 4\pi^2 n^2 a \neq 0 \tag{1}$$

for all integers $n = 0, \pm 1, \pm 2, \cdots$. *Then*

$$R_{\alpha,\infty}(u) = R_{\beta,\infty}(u) = B_{\alpha,\infty}(f)/c,$$

$$R_{\alpha,n}(u) = \sum_{k=1}^{\infty} R_{\alpha,kn}(f) \left[\sum_{l|k} \mu\left(\frac{k}{l}\right) \frac{(c - 4\pi^2 l^2 n^2 a) - 2\pi lnb \cot 2\pi ln(\alpha - \beta)}{c^2 - 8\pi^2 l^2 n^2 a + 4\pi^2 l^2 n^2 b^2 + 16\pi^4 l^4 b^4 a^2} \right]$$
$$+ \sum_{k=1}^{\infty} R_{\beta,kn}(f) \left[\sum_{l|k} \mu\left(\frac{k}{l}\right) \frac{2\pi lnb \csc 2\pi ln(\alpha - \beta)}{c^2 - 8\pi^2 l^2 n^2 a + 4\pi^2 l^2 n^2 b^2 + 16\pi^4 l^4 n^4 a^2} \right]$$

for all $k = 1, 2 \cdots$.

We remark that the condition (1) is necessary in order to guarantee the existence of the solution for the problem.

2. Proofs of theorems and propositions. To prove Theorem 1, we let $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{-i2\pi nt}$. It is easy to verify that

$$S_{\alpha,m}(f) = a_0 + \sum_{k=1}^{\infty} (a_{km} e^{i2\pi km\alpha} + a_{-km} e^{-i2\pi km\alpha}) \tag{2}$$

and

$$R_{\alpha,m}(f) = \sum_{k=1}^{\infty} (a_{km} e^{i2\pi km\alpha} + a_{-km} e^{-i2\pi km\alpha}). \tag{3}$$

Applying the Möbius inversion formula to (3), we obtain

$$a_m e^{i2\pi m\alpha} + a_{-m} e^{-i2\pi m\alpha} = \sum_{k=1}^{\infty} \mu(k) R_{\alpha,mk}(f), \tag{4}$$

(for details, see [6, Thm. 1]). From the hypothesis on f, we have $a_m e^{i2\pi m\alpha} + a_{-m} e^{-i2\pi m\alpha} = 0$ and $a_m e^{i2\pi m\beta} + a_{-m} e^{-i2\pi m\beta} = 0$, which imply that $a_m = a_{-m} = 0$ for all $m = 1, 2 \cdots$, as $\alpha - \beta$ is irrational. Furthermore, we have

$$a_0 = \int_0^1 f(t)\,dt = \lim_{n\to\infty} S_{\alpha,n}(f) = 0.$$

So f is the zero function.

If $\alpha - \beta$ is equal to q/p for some integers p and q, we let $g(t) = \sin 2\pi pj(t - \beta)$, where j is any nonzero integer. Then $S_{\alpha,n}(g) = S_{\beta,n}(g) = 0$ for all $n = 0, \pm 1, \pm 2 \cdots$. If $\alpha - \beta$ is irrational, we let

$$P_{\alpha,m} = \sum_{j|m} \mu\left(\frac{m}{j}\right) \frac{\sin 2\pi j(t - \beta)}{\sin 2\pi j(\alpha - \beta)}.$$

Then we have

$$\begin{aligned}
S_{\beta,n}(P_{\alpha,m}) &= \sum_{j|m} \mu\left(\frac{m}{j}\right)\left(\frac{1}{n}\sum_{l=1}^{n} \frac{\sin 2\pi j(l/n)}{\sin 2\pi j(\alpha - \beta)}\right) = 0,\\
S_{\alpha,n}(P_{\alpha,m}) &= \sum_{j|m} \mu\left(\frac{m}{j}\right)\left(\frac{1}{n}\sum_{l=1}^{n} \frac{\sin 2\pi j(\alpha - \beta + l/n)}{\sin 2\pi j(\alpha - \beta)}\right)\\
&= \sum_{j|m} \mu\left(\frac{m}{j}\right)\left(\frac{1}{n}\sum_{l=1}^{n} \cos\frac{2\pi jl}{n}\right)\\
&= \sum_{n|j}^{j|m} \mu\left(\frac{m}{j}\right) = \delta_{m,n}.
\end{aligned}$$

Here we have used the identity

$$\sum_{k|n} \mu(k) = \begin{cases} 0 & \text{if } n \neq 1,\\ 1 & \text{if } n = 1.\end{cases}$$

To prove Theorem 2, we use (4) to obtain

$$\begin{aligned}
a_m e^{i2\pi m\alpha} + a_{-m} e^{-i2\pi m\alpha} &= \sum_{k=1}^{\infty} \mu(k) B_{\alpha,mk}(f)\\
a_m e^{i2\pi m\beta} + a_{-m} e^{-i2\pi m\beta} &= \sum_{k=1}^{\infty} \mu(k) R_{\beta,mk}(f).
\end{aligned}$$

Thus we have

$$\begin{aligned}
a_m &= \sum_{k=1}^{\infty} \mu(k) \frac{e^{-i2\pi m\beta} R_{\alpha,mk}(f) - e^{-i2\pi m\alpha} R_{\beta,mk}(f)}{2i \sin 2\pi m(\alpha - \beta)}\\
a_{-m} &= \sum_{k=1}^{\infty} \mu(k) \frac{e^{i2\pi m\alpha} R_{\beta,mk}(f) - e^{i2\pi m\beta} R_{\alpha,mk}(f)}{2i \sin 2\pi m(\alpha - \beta)}.
\end{aligned}$$

This completes the proof of (i).

If $\alpha - \beta$ is an algebraic number of degree $q \geqq 2$, we can deduce easily from Dyson's theorem (cf. [8]) that for some constant C,

$$|\alpha - \beta - a/b| \geqq C/b^x \tag{5}$$

for all integers a and b and any real number $x > \sqrt{2q}$. From now on, we use C to denote various constants. It follows from (5) that

$$|\sin 2\pi j(\alpha - \beta)| \geqq C/j^{x-1}. \tag{6}$$

From (3), we can easily obtain

$$R_{\alpha,j}(f) = O(1/j^s), \tag{7}$$

$$R_{\beta,j}(f) = O(1/j^s) \tag{8}$$

for $f \in B^s$, $s > 1$. Now, for any x, $\sqrt{2q} < x < s$, we can prove from (6), (7) and (8) that

$$\begin{aligned}
&\Big| f(t) - R_{\alpha,\infty}(f) - \sum_{n=1}^{N} R_{\alpha,n}(f)P_n(\alpha,\beta;t) + R_{\beta,n}(f)P_n(\beta,\alpha;t) \Big| \\
&= \left| \sum_{n\neq 0} a_n(f)\, e^{i2\pi nt} - \sum_{n=1}^{N} R_{\alpha,n}(f) \sum_{j|n} \mu\left(\frac{n}{j}\right) \frac{\sin 2\pi j(t-\beta)}{\sin 2\pi j(\alpha-\beta)} \right. \\
&\qquad \left. - \sum_{n=1}^{N} R_{\beta,n}(f) \sum_{j|n} \mu\left(\frac{n}{j}\right) \frac{\sin 2\pi j(t-\alpha)}{\sin 2\pi j(\beta-\alpha)} \right| \\
&= \left| \sum_{n\neq 0} a_n(f)\, e^{i2\pi nt} - \sum_{n\neq 0} e^{i2\pi nt} \sum_{|n|/m}^{m\leqq N} \mu\left(\frac{m}{|n|}\right) \frac{e^{-i2\pi n\beta}R_{\alpha,m} - e^{i2\pi n\alpha}R_{\beta,m}}{2i\sin 2\pi n(\alpha-\beta)} \right| \\
&= \left| \sum_{n\neq 0} e^{i2\pi nt} a_n(f) - \sum_{m=1}^{[N/|n|]} \frac{\mu(m)\, e^{-i2\pi n\beta}R_{\alpha,m|n|} - e^{-i2\pi n\alpha}R_{\beta,m|n|}}{2i\sin 2\pi n(\alpha-\beta)} \right| \\
&= \left| \sum_{n\neq 0} e^{i2\pi nt} \left(\sum_{m>[N/|n|]} \frac{\mu(m)(e^{-i2\pi n\beta}R_{\alpha,|mn|} - e^{-i2\pi n\alpha}R_{\beta,|mn|})}{2i\sin 2\pi n(\alpha-\beta)} \right) \right| \\
&\leqq 2\left(\sum_{n\leqq N} \sum_{m>[N/n]} \frac{|R_{\alpha,mn}| + |R_{\beta,mn}|}{|\sin 2\pi n(\alpha-\beta)|} + \sum_{n\geqq N} \sum_{m=1}^{\infty} \frac{|B_{\alpha,mn}| + |R_{\beta,mn}|}{|\sin 2\pi n(\alpha-\beta)|} \right) \\
&\leqq C\left[\sum_{n=1}^{N} \frac{1}{N^{s-1}} n^{x-2} + \sum_{n=N+1}^{\infty} \frac{1}{n^{s+1-x}} \right] \\
&\leqq \frac{C}{N^{s-x}}.
\end{aligned}$$

We remark that the proof of Theorem 2(ii) works for any α and β which satisfy the estimate (6). However, it is well known that such estimate does not hold for any Liouville number (cf. [10]), for example, $\alpha - \beta = \sum_{m=1}^{\infty} 10^{-m!}$.

To prove Proposition 1, we observe from Theorem 2(ii) that

$$\begin{aligned}
u(r,\theta) = u_{\alpha,\infty}(1,2\pi t) &+ \sum_{m=1}^{\infty} R_{\alpha,m}(u(1,2\pi t)) \sum_{l|m} \mu\left(\frac{m}{l}\right) \frac{r^l \sin 2\pi l(t-\beta)}{\sin 2\pi l(\alpha-\beta)} \\
&+ \sum_{m=1}^{\infty} R_{\beta,m}(u(1,2\pi t)) \sum_{l|m} \mu\left(\frac{m}{l}\right) \frac{r^l \sin 2\pi l(t-\alpha)}{\sin 2\pi l(\beta-\alpha)}.
\end{aligned}$$

From the mean value property of harmonic functions, we have $u_{\alpha,\infty}(r, 2\pi t) = u_{\alpha,\infty}(1, 2\pi t)$. Thus we obtain

$$\begin{aligned} R_{\alpha,kn}u(r, 2\pi t) &= \sum_{m=1}^{\infty} R_{\alpha,m}(u(1, 2\pi t)) \sum_{l|m} \mu\left(\frac{m}{l}\right) R_{\alpha,kn}\left(\frac{r^l \sin 2\pi l(t-\beta)}{\sin 2\pi l(\alpha-\beta)}\right) \\ &= \sum_{m=1}^{\infty} R_{\alpha,m}(u(1, 2\pi t)) \sum_{l|m}^{kn/l} \mu\left(\frac{m}{l}\right) r^l \\ &= \sum_{m=1}^{\infty} R_{\alpha,mkn}(u(1, 2\pi t)) \sum_{l|m} \mu\left(\frac{m}{l}\right) r^{lkn} \\ &= \sum_{m=1}^{\infty} b_{\alpha,mnk} \sum_{l|m} \mu\left(\frac{m}{l}\right) r^{lkn}. \end{aligned}$$

We remark that a similar formula holds for analytic functions in the unit disk (cf. [5]).

To prove Proposition 2, we observe that

$$a_n(u) = \frac{a_n(f)}{c + 2\pi inb - 4\pi^2 n^2 a^2}. \tag{9}$$

Using Theorem 2, (2) and (9), we have

$$\begin{aligned} R_{\alpha,n}(u) &= \sum_{k\neq 0} a_{kn}(u)\, e^{i2\pi kn\alpha} \\ &= \sum_{k\neq 0} \frac{e^{i2\pi kn\alpha}}{c + 2\pi ikn b - 4\pi^2 n^2 a^2} \\ &\quad \cdot \left[\sum_{l=1}^{\infty} \mu(l) \frac{e^{-2\pi kn\beta} R_{\alpha,|kn|l}(f) - e^{-i2\pi kn\alpha} R_{\beta,|kn|l}(f)}{2i \sin 2\pi kn(\alpha-\beta)}\right] \\ &= \sum_{k,l=1}^{\infty} \frac{[(c - 4\pi^2 n^2 k^2 a) - 2\pi knb \cot 2\pi kn(\alpha-\beta)]\mu(l) R_{\alpha,kln}}{c^2 + 4\pi^2 k^2 n^2 b^2 - 8\pi^2 k^2 n^2 a + 16\pi^4 k^4 n^4 a^2} \\ &\quad + \sum_{k,l=1}^{\infty} \frac{\mu(l) 2\pi knb \csc 2\pi kn(\alpha-\beta) R_{\beta,kln}}{c^2 + 4\pi^2 k^2 n^2 b^2 - 8\pi^2 k^2 n^2 a + 16\pi^4 h^4 n^4 a^2} \\ &= \sum_{k=1}^{\infty} \frac{R_{\alpha,kn}(f) \sum_{l|k} \mu\left(\frac{k}{l}\right)(c - 4\pi^2 l^2 n^2 a - 2\pi lnb[\cot 2\pi ln(\alpha-\beta))]}{c^2 + 4\pi^2 k^2 n^2 b^2 - 8\pi^2 k^2 n^2 a + 16\pi^4 n^4 a^2} \\ &\quad + \sum_{k=1}^{\infty} \frac{R_{\beta,kn}(f) \sum_{l|k} \mu\left(\frac{k}{l}\right) 2\pi lnb[\csc 2\pi ln(\alpha-\beta)]}{c^2 + 4\pi^2 k^2 n^2 b^2 - 8\pi^2 k^2 n^2 a + 16\pi^4 n^4 a^2} \end{aligned}$$

3. Final remark. It can be seen from the above proof that the double translation is quite essential, as the above series will be meaningless if α is equal to β. That is why we introduce two sequences of means instead of just one as in [1] or [6]. Mean boundary value problems arise from consideration of boundary value problems with discrete data instead of continuous data, and the solution here can be readily used in computation. Also due to the fluctuation of observation under various conditions, the errors in measuring the average data of input (e.g., the forcing term in Proposition 2) or the temperature on the edge of a circular plate in the equilibrium state is in general less than that at some single points, thus we can hope to obtain a more accurate solution. Of course, from the physical point of view, it is more desirable to consider averages over an arbitrary sequence of points than just the equally spaced points; the difficulty for this rather general choice of points is that we do not know any other "good" choice of points except the conformal images as mentioned in [1]. Also it would be interesting to find some analogous results for some boundary value problems whose data is not compatible at a few points on the boundary.

Using a result of Davenport (cf. [7]) and the proof of Theorem 1(i), we can prove a similar uniqueness theorem for periodic functions of bounded variation in its period (cf. [2]). It is easy to see from the proof of Proposition 2 that we can find the α- and β-means of f' in terms of the α- and β-means of f for smooth f:

$$\begin{aligned} R_{\alpha,m}(f') &= 2\pi m \sum_{k=1}^{\infty} R_{\alpha,km}(f)\left[\sum_{l|k} \mu\left(\frac{k}{l}\right) l \cot 2\pi lm(\alpha - \beta)\right] \\ &\quad - 2\pi m \sum_{k=1}^{\infty} R_{\beta,km}\left[\sum_{l|k} \mu\left(\frac{k}{l}\right) l \csc 2\pi lm(\alpha - \beta)\right]; \\ R_{\alpha,m}(f'') &= 4\pi^2 m^2 \sum_{k=1}^{\infty} R_{\alpha,km}(f)\left[\sum_{l|k} \mu\left(\frac{k}{l}\right) l^2\right]. \end{aligned}$$

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THE ENERGY EQUATION FOR THE NAVIER-STOKES SYSTEM*

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Abstract. It is shown that a weak solution of the Navier–Stokes equations lying in a space $L^q(0, T; L^p)$, where $1/p + 1/q \leqq 1/2$ and $p \geqq 4$, satisfies an energy equality rather than the usual energy inequality. This is true regardless of the dimension of the underlying space.

1. Introduction. Let V be a domain in R^n. Write $L^p = L^p(V) \times \cdots \times L^p(V)$ (n times). A weak solution of the Navier–Stokes equations is a function $s: t \to s(t)$ with values in L^2 and satisfying a certain functional equation. It is known [1], [2], [4], [5] that a weak solution s exists satisfying the energy inequality

$$\|s(t)\|_2^2 + 2\nu \int_0^t \|\nabla s(\tau)\|_2^2 \, d\tau \leqq \|s(0)\|_2^2, \tag{1.1}$$

whenever $s(0) \in L^2$. Here $\|s(t)\|_2^2$ denotes the norm of $s(t)$ in L^2, $\|\nabla s(t)\|_2^2$ the Dirichlet integral of $s(t)$, and ν a certain constant that appears in the equations (the kinematic viscosity).

Formally, solutions of the Navier–Stokes equations satisfy, not merely (1.1), but the stronger energy *equality*

$$\|s(t)\|_2^2 + 2\nu \int_0^t \|\nabla s(\tau)\|_2^2 \, d\tau = \|s(0)\|_2^2 \tag{1.2}$$

and it is of interest, therefore, to ask when a weak solution satisfies (1.2), not least because (1.2) implies the continuity of s as an L^2-valued function [3], [5].

Naturally, s satisfies (1.2) if it is a classical solution of the equations. The question is how badly behaved s can be in order that it still must satisfy (1.2). The best result is due to Serrin [3], who showed that if $s \in L^q(0, T; L^p)$, where

$$n/p + 2/q \leqq 1, \tag{1.3}$$

then s satisfies (1.2) for $0 \leqq t \leqq T$.

In this paper, we derive the same conclusion if $s \in L^q(0, T; L^p)$, where

$$2/p + 2/q \leqq 1 \tag{1.4}$$

and

$$p \geqq 4. \tag{1.5}$$

Notice that the hypothesis (1.4) is weaker than (1.3) in all cases, and that both (1.4) and (1.5) are weaker than (1.3) if $n \geqq 4$, since then (1.3) implies $p \geqq n$.

What is perhaps most interesting about the conditions (1.4)–(1.5) is not that they are weaker than (1.3), but that *they do not depend on n*. This means that it is no harder for a weak solution to achieve the degree of smoothness measured by (1.2) in three dimensions (for instance) than in two. Conditions (1.4)–(1.5) are the first smoothness criterion for weak solutions with this property.

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2. Notation. Let V be a domain in R^n, bounded or not, but with a sufficiently smooth boundary. The set of all infinitely differentiable functions having compact support in V and taking values in R^n is denoted by C_0^∞. Let ∇ be the usual symbol for the gradient. We define

$$\mathbf{C}_0^\infty = \{\phi \in C_0^\infty : \nabla \cdot \phi = 0\}.$$

If $\phi \in C_0^\infty$, we denote its Euclidean length by $|\phi|$. For $\phi \in C_0^\infty$, define

$$\|\phi\|_p = \left(\int_V |\phi|^p \, dx\right)^{1/p} \tag{2.1}$$

and

$$\|\nabla\phi\|_p = \left(\int_V |\nabla\phi|^p \, dx\right)^{1/p}. \tag{2.2}$$

If $1 \leqq p < \infty$, we denote the completion of C_0^∞ with respect to the norms (2.1) and (2.2) py L^p and $H_0^{1,p}$, respectively. The completions of $\mathbf{C}_0^\infty$ with respect to these norms are denoted by $\mathbf{L}^p$ and $\mathbf{H}_0^{1,p}$.

The spaces L^2 and $H_0^{1,2}$ are clearly Hilbert spaces. We denote the scalar product of two elements, ϕ and ψ, of L^2 or $H_0^{1,2}$ by (ϕ, ψ) and $(\nabla\phi, \nabla\psi)$, respectively. Orthogonal projection of L^2 onto the subspace $\mathbf{L}^2$ is denoted by P.

If B is a Banach space with norm $\|\cdot\|_B$ and $\phi : t \to \phi(t)$ is a measurable function defined on an interval I taking values in B and if

$$\int_I \|\phi(t)\|_B^q \, dt < \infty, \qquad 1 \leqq q < \infty,$$

we write $\phi \in L^q(I; B)$. The set of essentially bounded, measurable functions from I to B is denoted by $L^\infty(I; B)$.

If $\phi \in L^q(0, T; L^p)$, while T is fixed and its value understood, we often write

$$\|\phi\|_{p,q} = \left(\int_0^T \|\phi(t)\|_p^q \, dt\right)^{1/q}, \qquad 1 \leqq q < \infty,$$

and

$$\|\phi\|_{p,\infty} = \operatorname*{ess\,sup}_{0<t<T} \|\phi(t)\|_p.$$

Similarly, if $\phi \in L^q(0, T; H_0^{1,p})$, we write

$$\|\nabla\phi\|_{p,q} = \left(\int_0^T \|\nabla\phi(t)\|_p^q \, dt\right)^{1/q}, \qquad 1 \leqq q < \infty.$$

Finally, if ϕ is infinitely differentiable with respect to t, has compact support in the interval I, and takes values in $\mathbf{C}_0^\infty$, we write $\phi \in C_0^\infty(I; \mathbf{C}_0^\infty)$.

3. The Navier–Stokes equations. The Navier–Stokes equations are

$$\dot{s}(t) + Ps(t) \cdot \nabla s(t) - \nu P \nabla^2 s(t) = 0, \tag{3.1}$$

$$\nabla \cdot s(t) = 0. \tag{3.2}$$

These equations must, of course, be supplemented by boundary and initial conditions. The usual boundary condition can be written in the form

$$s(t) \in \mathbf{H}_0^{1,2}, \qquad t > 0, \tag{3.3}$$

which implies that $s(t) = 0$ on the boundary of V if $s(t)$ is smooth. In addition, one wants

$$s(0) = s_0, \tag{3.4}$$

where s_0 is given. We refer to the problem of solving (3.1)–(3.4) as the *Navier–Stokes initial-value problem.*

The problem is hard and, indeed, it is likely that in general no classical solution exists for all $t > 0$. Because of this, one often considers weak solutions of the problem. A function $s: t \to s(t)$ is called a *weak solution* of the Navier–Stokes system (3.1)–(3.4) if

$$s \in L^\infty(0, \infty; \mathbf{L}^2) \cap L^2(0, \infty; \mathbf{H}_0^{1,2}), \tag{3.5}$$

and if

$$(s_0, \phi(0)) + \int_0^\infty [(s(t), \dot{\phi}(t)) - (s(t) \cdot \nabla s(t), \phi(t)) - \nu(\nabla s(t), \nabla \phi(t))]dt = 0 \tag{3.6}$$

for all $\phi \in C_0^\infty([0, \infty); \mathbf{C}_0^\infty)$.

E. Hopf, in a most important paper [1], was the first to show that (3.1)–(3.4) has a weak solution whenever $s_0 \in \mathbf{L}^2$. (For an alternative proof, see [4] where, although the proof of existence is correct, there is an error in the later § 5.) It is also shown in [1] and [4] that the weak solution constructed satisfies the energy inequality (1.1).

The following result is also due to Hopf [1] (see also [3]). A proof based on the methods of [4] can be found in [5].

THEOREM 3.1. *Let $s_0 \in \mathbf{L}^2$, and let s be a weak solution of* (3.1)–(3.4). *Then, after suitable redefinition of s on a set of values of t of one-dimensional measure zero, we have*

$$\begin{aligned}(s(t), \phi(t)) \\ = (s_0, \phi(0)) + \int_0^t [(s(\tau), \dot{\phi}(\tau)) - (s(\tau) \cdot \nabla s(\tau), \phi(\tau)) - \nu(\nabla s(\tau), \nabla \phi(\tau))]\, d\tau\end{aligned} \tag{3.7}$$

for all $\phi \in C_0^\infty([0, \infty); \mathbf{C}_0^\infty)$ and all $t \geqq 0$.

In all that follows, whenever we speak of a weak solution, we assume the adjustment of the theorem has been made, so that (3.7) holds for all $t \geqq 0$.

In addition to Theorem 3.1, we need the following corollary.

COROLLARY 3.2. *Let $s_0 \in \mathbf{L}^2$, and let s be a weak solution of* (3.1)–(3.4). *Then, as a function of t, s is continuous in the weak topology of $\mathbf{L}^2$.*

Proof. Let $\psi \in \mathbf{C}_0^\infty$. Let $\theta: t \to \theta(t)$ be a real-valued function equal to unity on a sufficiently large interval and having compact support. Writing $\phi = \theta\psi$ in (3.7), we find

$$(s(t), \psi) = (s_0, \psi) - \int_0^t [(s(\tau) \cdot \nabla s(\tau), \psi) + \nu(\nabla s(\tau), \nabla \psi)]\, d\tau.$$

The right side is obviously continuous. Since $s(t)$ is bounded in $\mathbf{L}^2$ (cf. (3.5)), the corollary follows.

4. The energy equality. Our main result is the energy equality (1.2). The proof follows closely one of Serrin's [3], with a different treatment of the nonlinear term in (3.1). This treatment depends on three easy lemmas.

LEMMA 4.1. *Let $\phi \in L^p$, $\psi \in H_0^{1,2}$, $\chi \in L^2 \cap L^p$, where $p \geqq 4$. Then*

$$|(\phi \cdot \nabla\psi, \chi)| \leqq \|\phi\|_p \|\nabla\psi\|_2 \|\chi\|_2^{2-q/2} \|\chi\|_p^{q/2-1}, \tag{4.1}$$

where q is defined by

$$2/p + 2/q = 1. \tag{4.2}$$

Proof. If $\chi \in L^q$, then Hölder's inequality and (4.2) give

$$|(\phi \cdot \nabla\psi, \chi)| \leqq \|\phi\|_p \|\nabla\psi\|_2 \|\chi\|_q.$$

But (4.2) entails $2 \leqq q \leqq p$ if $p \geqq 4$. Therefore, $\chi \in L^q$ since $\chi \in L^2 \cap L^p$, and another application of Hölder's inequality gives (4.1).

LEMMA 4.2. *Let*

$$\phi \in L^q(0, T; L^p), \psi \in L^2(0, T; H_0^{1,2}), \chi \in L^\infty(0, T; L^2) \cap L^q(0, T; L^p),$$

where $p \geqq 4$ and p and q are related by (4.2). *Then*

$$\left| \int_0^T (\phi(t) \cdot \nabla\psi(t), \chi(t))\, dt \right| \leqq \|\phi\|_{p,q} \|\nabla\psi\|_{2,2} \|\chi\|_{2,\infty}^{2-q/2} \|\chi\|_{p,q}^{q/2-1}. \tag{4.3}$$

Proof. Replace ϕ, ψ and χ by $\phi(t)$, $\psi(t)$ and $\chi(t)$ in (4.1). Integrating with respect to t and using Hölder's inequality once again, we obtain (4.3).

If the functions involved lie in the appropriate spaces, the roles of ϕ and χ can be interchanged in the proof of Lemma 4.1 and, therefore, in Lemma 4.2. Thus we have the following lemma.

LEMMA 4.3. *Let*

$$\phi \in L^\infty(0, T; L^2) \cap L^q(0, T; L^p), \psi \in L^2(0, T; H_0^{1,2}), \chi \in L^q(0, T; L^p),$$

where $p \geqq 4$ and p and q are related by (4.2). *Then*

$$\left| \int_0^T (\phi(t) \cdot \nabla\psi(t), \chi(t))\, dt \right| \leqq \|\phi\|_{2,\infty}^{2-q/2} \|\phi\|_{p,q}^{q/2-1} \|\nabla\psi\|_{2,2} \|\chi\|_{p,q}$$

We are now in a position to prove our main result.

THEOREM 4.4. *Let $s_0 \in \mathbf{L}^2$, and let s be a weak solution of* (3.1)–(3.4). *If $s \in L^q(0, T; L^p)$, where $p \geqq 4$ and*

$$2/p + 2/q \leqq 1, \tag{4.4}$$

then s satisfies the energy equality

$$\|s(t)\|_2^2 + 2\nu \int_0^t \|\nabla s(\tau)\|_2^2\, d\tau = \|s_0\|_2^2, \qquad 0 \leqq t \leqq T. \tag{4.5}$$

Proof. First, notice that we may assume $T < \infty$. Then, we may reduce q (if that is necessary) to obtain a pair (p, q) satisfying (4.2) rather than (4.4).

When $t = 0$, (4.5) is true because of (3.7). Let t_0 be fixed, $0 < t_0 \leqq T$. Let

$$(k_\varepsilon * \phi)(t) = \int_0^{t_0} k_\varepsilon(t - \tau)\phi(\tau)\,d\tau$$

be a mollifier, so that k_ε is a C^∞, real-valued, nonnegative function, supported in $[-\varepsilon, \varepsilon]$, and integrating to unity. Let $\{s^i\} \subset C_0^\infty([0, \infty); \mathbf{C}_0^\infty)$ be a sequence converging to s in $L^2(0, T; \mathbf{L}^2) \cap L^2(0, T; \mathbf{H}_0^{1,2}) \cap L^q(0, T; L^p)$. Such a sequence exists because of (3.5) and the assumption $T < \infty$. Set $t = t_0$ and $\phi = k_\varepsilon * s^i$ in (3.7). One obtains

$$\begin{aligned}
&\int_0^{t_0} k_\varepsilon(t_0 - t)(s(t_0), s^i(t))\,dt \\
&\quad = \int_0^{t_0} k_\varepsilon(-t)(s_0, s^i(t))\,dt + \int_0^{t_0}\int_0^{t_0} \dot{k}_\varepsilon(t - \tau)(s(t), s^i(\tau))\,d\tau\,dt \\
&\quad\quad - \int_0^{t_0}\int_0^{t_0} k_\varepsilon(t - \tau)[(s(t)\cdot\nabla s(t), s^i(\tau)) + \nu(\nabla s(t), \nabla s^i(\tau))]\,d\tau\,dt.
\end{aligned}$$

In this formula, we want to send i to infinity. The only term that gives any trouble is the nonlinear one, but it can be estimated using Lemma 4.1. Indeed, since k_ε is bounded,

$$\begin{aligned}
&\left|\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t - \tau)(s(t)\cdot\nabla s(t), s^i(\tau) - s(\tau))\,d\tau\,dt\right| \\
&\quad \leqq c\int_0^{t_0} \|s(t)\|_p\|\nabla s(t)\|_2\,dt \int_0^{t_0} \|s_i(\tau) - s(\tau)\|_2^{2-q/2}\|s_i(\tau) - s(\tau)\|_p^{q/2-1}\,d\tau
\end{aligned}$$

by (4.1), where c is a positive constant. The first integral here is bounded by $\|s\|_{p,2}\|\nabla s\|_{2,2}$, which is finite since $q \geqq 2$ and $T < \infty$. The second integral does not exceed a constant (depending on t_0) times

$$\|s_i - s\|_{2,2}^{2-q/2}\|s_i - s\|_{p,q}^{q/2-1},$$

and this goes to zero. Thus, we obtain

$$\begin{aligned}
&\int_0^{t_0} k_\varepsilon(t - t_0)(s(t_0), s(t))\,dt \\
&\quad = \int_0^{t_0} k_\varepsilon(-t)(s_0, s(t))\,dt + \int_0^{t_0}\int_0^{t_0} \dot{k}_\varepsilon(t - \tau)(s(t), s(\tau))\,d\tau\,dt \qquad (4.6)\\
&\quad\quad - \int_0^{t_0}\int_0^{t_0} k_\varepsilon(t - \tau)[(s(t)\cdot\nabla s(t), s(\tau)) + \nu(\nabla s(t), \nabla s(\tau))]\,d\tau\,dt.
\end{aligned}$$

The term here involving the derivative of k_ε vanishes if k_ε is chosen to be even. We send ε to zero in the remaining terms. Because of the usual properties of mollifiers,

$$\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t - \tau)(\nabla s(t), \nabla s(\tau))\,d\tau\,dt \to \int_0^{t_0} \|\nabla s(t)\|_2^2\,dt.$$

Also,

$$\int_0^{t_0} k_\varepsilon(t_0 - t)(s(t_0), s(t))\,dt = \int_0^{\varepsilon} k_\varepsilon(t)(s(t_0), s(t_0 - t))\,dt$$

if $\varepsilon < t_0$, since k_ε is supported in $[-\varepsilon, \varepsilon]$. By Corollary 3.2, then,

$$\int_0^{t_0} k_\varepsilon(t_0 - t)(s(t_0), s(t))\,dt = \int_0^{\varepsilon} k_\varepsilon(t)[\|s(t_0)\|_2^2 + O(1)]\,dt \to \frac{1}{2}\|s(t_0)\|_2^2$$

by the Lebesgue convergence theorem and the fact that k is even and integrates to unity. A similar argument shows that

$$\int_0^{t_0} k_\varepsilon(-t)(s_0, s(t))\,dt \to \frac{1}{2}\|s_0\|_2^2.$$

Finally, we consider the nonlinear term in (4.6). We have

$$\begin{aligned}(4.7)\qquad &\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)(s(t)\cdot\nabla s(t), s(\tau))\,d\tau\,dt - \int_0^{t_0}(s(t)\cdot\nabla s(t), s(t))\,dt\\ &= \int_0^{t_0}(s(t)\cdot\nabla s(t), (k_\varepsilon * s)(t) - s(t))\,dt.\end{aligned}$$

By Lemma 4.2, this is bounded by

$$\|s\|_{p,q}\|\nabla s\|_{2,2}\|k_\varepsilon * s - s\|_{2,\infty}^{2-q/2}\|k_\varepsilon * s - s\|_{p,q}^{q/2-1}.$$

This goes to zero because of the usual properties of mollifiers. Thus, the expression (4.7) goes to zero. On the other hand, $\int_0^{t_0}(s(t)\cdot\nabla s(t), s(t))\,dt$ vanishes. To see this, notice that Lemma 4.3 says that the function F defined by

$$F(\psi, \chi) = \int_0^{t_0}(s(t)\cdot\nabla\psi(t), \chi(t))\,dt$$

is continuous on $L^2(0, t_0; H_0^{1,2}) \times L^q(0, t_0; L^p)$. On the other hand, integration by parts shows that $F(\psi, \psi) = 0$ if ψ is smooth. Let $\{s^i\}$ be a sequence from $C_0^\infty([0,\infty); \mathbf{C}_0^\infty)$ converging to s in the appropriate spaces. Then we find $0 = F(s^i, s^i) \to F(s, s)$. All of this shows that

$$\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)(s(t)\cdot\nabla s(t), s(\tau))\,d\tau\,dt \to 0$$

as ε goes to zero.

Now let ε go to zero in (4.6). What we have proved is that in the limit,

$$\frac{1}{2}\|s(t_0)\|_2^2 + \nu\int_0^{t_0}\|\nabla s(t)\|_2^2\,dt = \frac{1}{2}\|s_0\|_2^2,$$

which is (4.5) for $t = t_0$. Since t_0 is arbitrary, the proof of Theorem 4.4 is complete.

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ON FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND WITH CONVEX CONSTRAINTS*

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Abstract. A Fredholm integral equation of the first kind is an incorrectly posed problem. In order to make it correctly posed, we lay constraints on f of the form $f \in B$, where B is convex and closed and where $B - B$ (or sometimes B itself) is boundedly compact.

Introduction. Consider a Fredholm integral equation of the first kind:

$$g_0(t) = \int_a^b K(t,s)f(s)\,ds = Kf, \qquad -\infty < a < b < +\infty,$$

which we know has a unique solution f_0 for fixed g_0, but where we know g_0 only approximately. For instance, g can be a measurement of g_0 with some error. The equation $g = Kf$ is not always solvable. However, we may look for f_0 among those f which make the difference small, and among those we surely find f_0 if the error $\|g - g_0\|$ is small enough. But among those are also, for instance, $f_0(t) + k \cdot \sin nt$ for arbitrary k if n is large enough.

If it is desired to construct from g an f near to f_0, it is evident that further information about f_0 or $g - g_0$ is needed. If we require f to belong to some subset B of possible solutions, are there natural and reasonable conditions on B which guarantee that $\|Kf - g\| = \min$, $f \in B$, implies that f approximates f_0, if g approximates g_0?

We will show that if K is one-to-one and $\|\cdot\|$ stands for L^2-norm, then the problem "minimize $\|g - Kf\|$ with constraint $f \in B$" has a unique solution $\hat{K}g$, and $\hat{K}$ is continuous on $L^2(a, b)$ if and only if B is convex and $B \cap \{\|f\| \leqq r\}$ is compact for every $r > 0$. If we want $\hat{K}$ to be uniformly continuous, then B has to be convex, closed and such that $(B - B) \cap \{\|f\| \leqq r\}$ is compact for every $r > 0$.

Observe that the condition "$\hat{K}$ is continuous on $L^2(a, b)$" is somewhat strong. In order to solve the integral equation problem, it would be sufficient to know that $\hat{K}$ is continuous on KB.

Finally, we show that almost all results are valid in $L^p(a, b)$, where $1 \leqq p \leqq \infty$, or in other Banach spaces.

1. Proof of the main theorem. Let $K: H \to H$ be a one-to-one compact linear map on the Hilbert space H. (H is evidently separable.) We put $E_r = \{x \in H : \|x\| \leqq r\}$ and say that a set $B \subset H$ is *boundedly compact* if $B \cap E_r$ is compact for every $r > 0$.

Our main theorem is as follows.

MAIN THEOREM. *If $\hat{K}: H \to B \subset H$ is defined by $\hat{K}y =$ the element x in B which minimizes $\|y - Kx\|$, then $\hat{K}$ is well-defined and continuous (uniformly continuous) if and only if B is convex and boundedly compact (B is convex and closed and $B - B$ is boundedly compact).*

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For the proof we need some lemmas and theorems.

LEMMA 1. *If B is convex and boundedly compact, then to every $s > 0$ there is an $r > 0$ so that $K(B \cap E_r) \subset (KB) \cap E_s$.*

Proof. Assume the contrary: there is an $s_0 > 0$ and a sequence $b_i \in B$ such that $\|b_i\| > i$ but $\|Kb_i\| \leqq s_0$. Put

$$c_i = \frac{1}{\|b_i\|} b_i + \left(1 - \frac{1}{\|b_i\|}\right) b_1 .$$

We have that $c_i \in B \cap E_t$ for $t = 1 + \|b_i\|$, and therefore there is a subsequence c_{i_j} which converges to $c \neq b_1$ since $\|c_i - b_1\|$ tends to 1. But $Kc = \lim_{j \to \infty} Kc_{i_j} = Kb_1$, which is impossible since K was one-to-one.

LEMMA 2. *If B is convex and boundedly compact, then KB is closed.*

Proof. Let Kx_i converge to y with $x_i \in B$. Since Kx_i is bounded, Lemma 1 gives that x_i is bounded, too. Hence there is a subsequence converging to x and $Kx = y$.

LEMMA 3. *If B is closed and $B - B$ is boundedly compact, then so is B.*

Proof. Let $b \in B$. The set $B - b$ is a closed subset of $B - B$, and therefore $B - b$ is boundedly compact. So is B since it is a translate of $B - b$.

LEMMA 4. *K maps every bounded, convex and closed set M onto a compact set.*

Proof. Since every convex and closed set M is the intersection of all closed half-spaces $\{x \in H : (x - b, a) \geqq 0, a, b \in H\}$ which contain it, a bounded convex and closed set is the intersection of all "half-balls" $\{x \in H : (x - b, a) \geqq 0, \|x - b\| \leqq r, a, b \in H, r > 0\}$ which contain it. Further, K can be written $K = QS$, where Q is isometric and S is self-adjoint (see, for example, [1]). Thus it is sufficient to prove the lemma for K self-adjoint and $M = \{x \in H : (x, a) \geqq 0, \|x\| \leqq 1\}$.

Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal base such that $Ke_i = \lambda_i e_i$ and let $\{x_k\}_{k=1}^{\infty}$ be a sequence in M. Since K is compact we may assume that $y_k = Kx_k$ converges to y. Thus $\lim_{k \to \infty} (y_k, e_i) = (y, e_i)$ and $\lim_{k \to \infty} (x_k, e_i) = (y, e_i)/\lambda_i = \xi_i$. For n fixed we have

$$\sum_{i=1}^{n} |\xi_i|^2 = \lim_{k \to \infty} \sum_{i=1}^{n} |(x_k, e_i)|^2 \leqq \|x_k\| \leqq 1 .$$

Hence $\sum_{i=1}^{\infty} |\xi_i|^2 \leqq 1$. If we put $x = \sum_{i=1}^{\infty} \xi_i e_i$, then $\|x\| \leqq 1$ and

$$Kx = \sum_{i=1}^{\infty} \xi_i K e_i = \sum_{i=1}^{\infty} (y, e_i) e_i = y.$$

It remains to prove that $(x, a) \geqq 0$, i.e., if $\alpha_i = (a, e_i)$, then $\sum_{i=1}^{\infty} \xi_i \alpha_i \geqq 0$. This follows from the inequalities $\sum_{i=1}^{\infty} (x_k, e_i)\alpha_i \geqq 0$ for all k and

$$\left| \sum_{i=n}^{\infty} (x_k, e_i) \cdot \alpha_i \right|^2 \leqq \sum_{i=n}^{\infty} |(x_k, e_i)|^2 \cdot \sum_{i=n}^{\infty} |\alpha_i|^2 \leqq \sum_{i=n}^{\infty} |\alpha_i|^2,$$

since the last term is smaller than any positive number as long as n is large enough.

THEOREM 1. *Let B be a convex and closed subset of H. Then $K^{-1}|_{KB}$ is continuous if and only if B is boundedly compact.*

Proof. Necessity. Since $B \cap E_r$ is bounded, convex and closed, Lemma 4 gives that $K(B \cap E_r)$ is compact. The continuity of $K^{-1}|_{KB}$ implies that $K^{-1}|_{KB}(K(B \cap E_r)) = B \cap E_r$ is compact.

Sufficiency. Since $B \cap E_r$ is compact, and K is one-to-one and continuous, $K^{-1}|_{K(B\cap E_r)}$ is continuous, too, and so is $K^{-1}|_{(KB)\cap E_s}$ if $(KB) \cap E_s \subset K(B \cap E_r)$. Because of Lemma 1, there is to every $s > 0$ such an r, and hence $K^{-1}|_{KB}$ is continuous.

THEOREM 2. *Let B be a convex and closed subset of H. Then $K^{-1}|_{KB}$ is uniformly continuous if and only if $B - B$ is boundedly compact.*

Proof. Necessity. We show that $K^{-1}|_{K(B-B)}$ is continuous, from which the assertion follows by virtue of Theorem 1.

Let $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$, where x_1', x_1'', x_2' and $x_2'' \in B$. Put $y_1 = Kx_1$, $y_2 = Kx_2$, and so on. Since $K^{-1}|_{KB}$ is uniformly continuous there is to each $\varepsilon > 0$ a δ_ε such that

$$\|x_1 - x_2\| = 2\|\tfrac{1}{2}(x_1' + x_2'') - \tfrac{1}{2}(x_1'' + x_2')\| < 2\varepsilon$$

if

$$\|\tfrac{1}{2}(y_1' + y_2'') - \tfrac{1}{2}(y_1'' + y_1')\| = \tfrac{1}{2}\|y_1 - y_2\| < \delta_\varepsilon.$$

Thus $K^{-1}|_{K(B-B)}$ is uniformly continuous.

Sufficiency. Put $N(\alpha) = \inf\{\|Kx\| : x \in B - B, \|x\| = \alpha\}$ for $0 \leqq \alpha \leqq \operatorname{diam}(B)$. Since the infimum is attained and K is one-to-one, $N(\alpha) > 0$ if $\alpha > 0$. Further, $N(\alpha)$ is nondecreasing since $B - B$ is star-shaped. The definition of N now gives that $\|Kx - Ky\| = \|K(x - y)\| \geqq N(\|x - y\|)$ for $x, y \in B$. From this follows that to every $\varepsilon > 0$ there is $\delta = N(\varepsilon) > 0$ such that $\|x - y\| < \varepsilon$ if $Kx, Ky \in KB$ and $\|Kx - Ky\| < \delta$, i.e., $K^{-1}|_{KB}$ is uniformly continuous.

THEOREM 3. *Let M be a subset of H. If $P_M : H \to M$ is defined by $y = P_M(x)$ = the $y \in M$ which minimizes $\|y - x\|$, then P_M is well-defined and (uniformly) continuous if and only if M is convex and closed.*

Proof. That the condition is sufficient is a well-known property of Hilbert spaces. Asplund [2] has shown that it is also necessary.

Proof of the main theorem. Suppose that B is convex and boundedly compact. Then KB is convex, too, and according to Lemma 2, KB is closed. Theorem 3 gives that P_{KB} is well-defined and continuous, and Theorem 1 gives that $K^{-1}|_{KB}$ is continuous, too. The assertion now follows from $\hat{K} = K^{-1}|_{KB} \circ P_{KB}$.

Suppose that B is convex and closed and that $B - B$ is boundedly compact. According to Lemma 3, B is boundedly compact, too. Hence P_{KB} is well-defined and uniformly continuous as above. Theorem 2 gives that $K^{-1}|_{KB}$ is uniformly continuous, and the assertion now follows from $\hat{K} = K^{-1}|_{KB} \circ P_{KB}$.

Conversely, suppose that $\hat{K}$ is well-defined and (uniformly) continuous. Then $P_{KB} = K \circ \hat{K}$ is well-defined and continuous, and Theorem 3 gives that KB is convex and closed. Since K is continuous B is convex and closed. $K^{-1}|_{KB} = \hat{K}|_{KB}$ shows that $K^{-1}|_{KB}$ is (uniformly) continuous, and according to Theorem 1 (resp., Theorem 2), B (resp., $B - B$) is boundedly compact.

2. Properties of $N(\alpha)$. The function N which was used in the proof of Theorem 2 gives a quantitative description of the uniform continuity of $\hat{K}$ when $B - B$ is boundedly compact. We shall now investigate some properties of N and we shall also give a theorem on how a convex, symmetric and boundedly compact set can be represented. As before, we assume that K is a one-to-one and compact linear map on H. B is a convex and closed subset such that $B - B$ is boundedly compact. The function $N:[0, \operatorname{diam} B] \to R^+$ is defined by $N(\alpha) = \inf\{\|Kx\| : x \in B - B, \|x\| = \alpha\}$.

PROPOSITION. *N has the following properties*:

(a) $N(\alpha) \leqq \alpha \cdot \|K\|$;

(b) *$N(\alpha)/\alpha$ is nondecreasing*;

(c) *$N(\alpha)$ is continuous from the left.*

Proof. Part (a) is trivially clear. Part (b) follows from $(N(\alpha)/\alpha) = \inf\{\|Kx\| : \alpha x \in B - B, \|x\| = 1\}$ and the fact that $B - B$ is star-shaped. For the proof of (c) we choose a sequence $\{\alpha_k\}_{k=1}^{\infty}$ which increases towards α_0 and $x_k \in B - B$ such that $\|x_k\| = \alpha_k$ and $N(\alpha_k) = \|Kx_k\|$. Since $B - B$ is boundedly compact, we may assume that x_k converges to $x_0 \in B - B$ and $\|x_0\| = \alpha_0$. We have

$$N(\alpha_0) \leqq \|Kx_0\| = \lim_{k\to\infty} \|Kx_k\| = \lim_{\alpha\to\alpha_0 - 0} N(\alpha) \leqq N(\alpha_0),$$

and the continuity from the left is proved.

Let us define $B_0 = \{x \in H : nx \in B - B \text{ for all } n \in Z^+\}$ and $B_\infty = \{x \in H:$ there is an $n \in Z^+$ such that $(x/n) \in B - B\}$.

LEMMA 5. *B_0 and B_∞ are linear spaces, and B_0 is of finite dimension and $B_0 = \{0\}$ if and only if* $\operatorname{diam} B < \infty$.

Proof. The first statement follows from the fact that $B - B$ is convex and symmetric. Since $B_0 \subset B - B$, which is boundedly compact, the unit ball of B_0 is compact, which implies that B_0 is finite-dimensional. The last assertion follows from the following lemma.

LEMMA 6. *If $\{y_k\}_{k=1}^{\infty}$ is a sequence in $B - B$ such that $\|y_k\| = 1$ and $ky_k \in B - B$, then there is a subsequence $\{y_{k_j}\}$ which converges to $y \in B_0$.*

Proof. We have only to prove that $y \in B_0$. Choose $m \in Z^+$. We have that $my_{k_j} \in B - B$ when $k_j \geqq m$, and $\lim_{j\to\infty} my_{k_j} = my$ gives that $my \in B - B$. Since m was choosen arbitrarily, $y \in B_0$.

LEMMA 7. $(B - B) + B_0 = B - B$.

Proof. Let $x \in B - B$ and $e \in B_0$. Then

$$x + e = \lim_{n\to\infty} \frac{1}{n+1}(nx + (ne)) \in B - B.$$

PROPOSITION. *N has the following properties*:

(d) $\lim_{\alpha\to 0}(N(\alpha)/\alpha) = \inf\{\|Kx\| : x \in B_\infty, \|x\| = 1\}$;

(e) $\lim_{\alpha\to 0}(N(\alpha)/\alpha) > 0$ *if and only if B_∞ is finite-dimensional*;

(f) $\operatorname{diam} B = \infty$ *implies that N is continuous*;

(g) $\operatorname{diam} B = \infty$ *implies that* $\lim_{\alpha\to\infty}(N(\alpha)/\alpha) = \inf\{\|Kx\| : x \in B_0, \|x\| = 1\}$.

Proof. Part (d) follows from the fact that $(N(\alpha)/\alpha) = \inf\{\|Kx\| : \alpha x \in B - B, \|x\| = 1\}$ and the definition of B_∞. If B_∞ is finite-dimensional, then the unit ball is compact. Therefore the infimum is attained for some x_0 and $\|Kx_0\| > 0$ since K was one-to-one. If, on the other hand, B_∞ is infinite-dimensional, then choose an orthonormal system $\{e_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} Ke_k = b$, and if $b \neq 0$, put $f_k = (e_k - e_{k-1})/\sqrt{2}$ and $\lim_{k\to\infty} Kf_k = 0$.

In order to prove (f) we take $x_0 \in B - B$ such that $\alpha_0 = \|x_0\|$ and $N(\alpha_0) = \|Kx_0\|$. According to Lemma 5, there is an $e \in B_0$, $e \neq 0$. Put $\alpha_t = \|x_0 + te\|$, which decreases toward α_0 as t decreases towards 0. (Otherwise change e to

$-e$.) Lemma 7 gives that $x_0 + te \in B - B$. We now have $N(\alpha_0) = \|Kx_0\| = \lim_{t\to 0+} \|K(x + te)\| \geqq \lim_{t\to 0+} N(\alpha_t) = \lim_{\alpha\to\alpha_0+0} N(\alpha) \geqq N(\alpha_0)$, and the continuity from the right is proved. Part (c) above now gives that N is continuous.

Now to consider (g): put

$$\kappa = \inf\{\|Kx\| : x \in B_0, \|x\| = 1\} \quad \text{and} \quad \kappa' = \lim_{\alpha\to\infty} (N(\alpha)/\alpha).$$

Since $B_0 \subset \{x : \alpha x \in B - B\}$ we have $\kappa' \leqq \kappa$. Suppose that $\kappa' < \kappa$. Then there should exist a y_n such that $\|y_n\| = 1$, $ny_n \in B - B$ and $\|Ky_n\| \leqq \kappa'' < \kappa$. Then Lemma 6 gives that there should exist a subsequence converging to $y_0 \in B_0$. But $\|y_0\| = 1$ and $\|Ky_0\| \leqq \kappa'' < \kappa$ gives a contradiction, which shows (g).

THEOREM 4. *A subset $M \subset H$ is convex, symmetric and boundedly compact if and only if $M = M_0 \oplus M'$, where M_0 is a finite-dimensional subspace of M and M' is convex, symmetric and compact.*

Proof. As in Lemmas 5, 6 and 7, we see that $M_0 = \{x \in H : nx \in M \text{ for all } n \in Z^+\}$ is a finite-dimensional subspace and that $M + M_0 = M$. Let M' be the orthogonal projection of M on the orthogonal complement of M_0. We see that M' becomes convex, symmetric and closed. Since $M_0 + M = M$, we get $M = M_0 \oplus M'$ and $M' \subset M$. If there were arbitrarily large elements in M', then as in Lemma 6, one can show that $M' \cap M_0 \neq \emptyset$, which contradicts $M' \perp M_0$. Hence M' is closed and bounded, and since it is a subset of M, it is compact. Conversely, let $M = M_0 \oplus M'$, where M_0 and M' are as in the assumption. It is evident that M is convex, symmetric and closed. To every $r > 0$ there is an $s > 0$ such that $E_r \cap M \subset (E_s \cap M_0) \oplus M'$, since M' is bounded. Now both terms of the direct sum are compact, and therefore so is $E_r \cap M$. Hence M is boundedly compact.

3. Examples.

Example 1. If B is a finite-dimensional subspace of H, we have $B - B = B_\infty = B_0 = B$, which is boundedly compact. Such constraints are used when one knows that the solution of the integral equation is a linear combination of functions from some finite set of functions, for instance, a polynomial of degree $\leqq 26$.

Example 2. If B is compact, so is $B - B$. Such a constraint is, for instance, $B = \{x \in H : |(x, e_k)| \leqq \alpha_k\}$, where $\{e_k\}_{k=1}^\infty$ is a complete orthonormal basis and $\sum_{k=1}^\infty \alpha_k^2 < \infty$.

Example 3. If $H = L^2(a, b)$, we put $B_\omega = \{f \in L^2(a, b)$: f is absolutely continuous and $f' \in L^2(a, b)$ with $\|f'\| \leqq \omega\}$. We have that $B_\omega - B_\omega = B_{2\omega}$, which is boundedly compact. To see this, let $E_r = \{f \in L^2(a, b) : \|f\| \leqq r\}$, $C^{te} = \{f \in L^2(a, b) : f$ is constant$\}$ and let I be the compact operator defined by $Ig(t) = \int_a^t g(s)\, ds$. This gives $B_\omega = C^{te} \oplus IE_\omega$, where C^{te} is finite-dimensional and IE_ω is compact. According to Theorem 4, B_ω is boundedly compact.

One can also use higher derivatives to construct similar constraints. Such constraints have been used by Phillips [3] and Twomey [4] and similar methods have been used by Tikhonov [5] and [6].

We have seen that if B is symmetric, then $B - B = 2B$ and the conditions "B is boundedly compact" and "$B - B$ is boundedly compact" are equivalent. Example 4 shows that this is not always the case when B is not symmetric.

Example 4. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis and put B equal to the closed convex hull of $\{0, 2e_1, 2e_1 + e_2, 3e_1 + e_3, \cdots, ke_1 + e_k, \cdots\}$. If $x \in E_r \cap B$, then $0 \leqq (x, e_k) \leqq r/\sqrt{1+k^2}$. Therefore $E_r \cap B$ is a closed subset of a Hilbert cube, and hence $E_r \cap B$ is compact. But $ne_1 = \lim_{k\to\infty}(n/k)(ke_1 + e_k) \in B$, so $B - B$ contains $\{e_k\}_{k=1}^{\infty}$ and is therefore not boundedly compact.

Next we give an example which shows that in Theorem 2 it is essential that B is convex.

Example 5. Let $\{e_k\}_{k=0}^{\infty}$ be an orthonormal basis in H and put

$$y_n = \sum_{i=1}^{n} (e_0 + e_i) \quad \text{and} \quad y_0 = 0.$$

Then put $B = \bigcup_{n=1}^{\infty} [y_n, y_{n-1}]$, where $[x, y] = \{z \in H : z = \alpha x + (1-\alpha)y, 0 \leqq \alpha \leqq 1\}$. We can also write $B = \{y = \int_1^t (e_0 + e_{[s]})\, ds : t \geqq 0\}$. Now B is closed, but B is not convex since $0 \in B$, $2e_0 + e_1 + e_2 \in B$, but $e_0 + \frac{1}{2}(e_1 + e_2) \notin B$. $B - B$ is not boundedly compact since $(B - B) \cap E_r$ contains $r(e_0 e_k)/\sqrt{2}$ for all k if $r \leqq 1$. However, one can show that if K is one-to-one and compact, then $K^{-1}|_B$ is uniformly continuous. Thus the necessity part of Theorem 2 is false. However, $B - B$ boundedly compact is sufficient even if B is not convex, and we can use the same proof.

4. How about Banach spaces? In the previous paragraphs we have shown how to get a "solution" f which is close to the true solution f_0 when we have adequate knowledge of f_0. The solution f has the property that Kf is close to Kf_0 in the L^2-sense. In some problems this is perhaps not the natural norm, so in this paragraph we shall investigate which results in the foregoing paragraphs are valid if $K: E \to F$ is a one-to-one compact linear map and E and F are Banach spaces, for instance, $L^1[a, b]$ or $C[a, b]$.

First we assume that F is still a Hilbert space. If we examine the proofs, we see that the Hilbert space property of E was only needed in Lemma 4, which was used for the necessity part of Theorem 1, which we used in the proof of the necessity part of Theorem 2. Example 6 below will show that the necessity part of Theorem 1 is false for general Banach spaces. However, the necessity part of Theorem 2 can be proved directly.

THEOREM 2'. *If $K^{-1}|_{KB}$ is uniformly continuous, then $B - B$ is boundedly compact.*

Proof. In the proof of Theorem 2, we saw that $K^{-1}|_{K(B-B)}$ is uniformly continuous, too, so it is sufficient to prove that B is boundedly compact.

Since $K^{-1}|_{KB}$ is uniformly continuous and $B \cap E_r$ is closed, then $K(B \cap E_r)$ is closed and thus compact. This gives that $B \cap E_r = K^{-1}|_{KB}(K(B \cap E_r))$ is compact.

We give some examples.

Example 6. Let $E = c_0 = \{x = \{x_i\}_{i=1}^{\infty} : x_i \in R \text{ and } \lim_{i\to\infty} x_i = 0\}$ with $\|x\| = \sup_i |x_i|$ and $B = \{x \in E : 1 \geqq x_1 \geqq x_2 \geqq \cdots\}$. If $K: E \to l^2$ is defined by $K\{x_i\}_{i=1}^{\infty} = \{\lambda_i x_i\}_{i=1}^{\infty}$ with $\lambda_i \neq 0$ and $\sum_{i=1}^{\infty} |\lambda_i|^2 < \infty$, then K is a one-to-one compact linear map. We prove that $K^{-1}|_{KB}$ is continuous by proving that pointwise convergence in B is equivalent with norm convergence: if $x^{(k)}$ and $x \in B$, then

$$\|x^{(k)} - x\| \leqq \max_{i \leqq N} |x_i^{(k)} - x_i| + |x_N^{(k)}| + |x_N|,$$

from which the assertion follows. But B is not boundedly compact since it contains $(1, 0, \cdots), (1, 1, 0, 0, 0, \cdots), (1, 1, 1, 0, 0, \cdots), \cdots$.

Example 7. Let $E = L^1[a, b]$ and $B_\omega = \{f \in E\colon \operatorname{Var} f \leqq \omega\}$. In order to see that B_ω is boundedly compact, we write $B_\omega = C^{te} \oplus B_\omega^0$, where C^{te} = the constant functions of E and $B_\omega^0 = \{f \in B_\omega\colon f$ is left-continuous and $\lim_{t \to a} f(t) = 0\}$. As before, it is enough to prove that B_ω^0 is compact.

There is a one-to-one correspondence between B_ω^0 and the Borel measures of $[a, b]$ with variation less than or equal to ω, and this correspondence is given by $f(t) = \mu[a, t)$ and $\operatorname{Var} f = \|\mu\| = |\mu|[a, b]$. Let $f_i \in B_\omega^0$ and μ_i be the corresponding measures. We can assume that μ_i converges weakly to μ with $\|\mu\| \leqq \omega$, i.e.,

$$\lim_{i \to \infty} \int_a^b \varphi \, d(\mu_i - \mu) = 0 \quad \text{for every } \varphi \in C[a, b].$$

Put

$$h(t, s) = \begin{cases} 1, & s < t, \\ & t \leqq s, \end{cases}$$

$$g_\varepsilon(t, s) = \begin{cases} 1, & s \leqq t - \varepsilon, \\ \dfrac{1}{2} - \dfrac{1}{2\varepsilon}(s - t), & \|x\| \leqq 1\}. \\ 0 & t + \varepsilon \leqq s, \end{cases}$$

and $f(t) = \int_a^b h(t, s)\, d\mu(s) = \mu[a, t)$. Now g_ε is a continuous function, and we have

$$\begin{aligned} \|f_i - f\| &= \int_a^b \left| \int_a^b h(t, s)\, d(\mu - \mu_i)(s) \right| dt \\ &\leqq \int_a^b \left| \int_a^b g_\varepsilon(t, s)\, d(\mu_i - \mu)(s) \right| dt + \int_a^b \int_a^b |h(t, s) - g_\varepsilon(t, s)|\, dt\, d|\mu_i - \mu|(s). \end{aligned}$$

The second term is less than or equal to $\varepsilon \cdot 2\omega$, and the first tends to zero for each fixed ε, from the dominated convergence theorem. Thus f_i converges to f and the compactness of B_ω^0 is proved.

Example 8. Let $E = C[a, b]$ with the supremum norm and $B_\omega = \{f \in E\colon |f(t) - f(s)| \leqq \omega |t - s|\}$, which is boundedly compact. Again put $B_\omega = C^{te} \oplus B_\omega^0$. The Arzela–Ascoli theorem gives that B_ω^0 is compact.

Example 9. Let $E = C[a, b]$ and $B_\omega = \{f \in E\colon f$ is absolutely continuous with $\int_a^b |f'|^2 \, dt \leqq \omega^2\}$, which is boundedly compact. This follows from Example 3 and the fact the the supremum norm $\|\cdot\|_\infty$ and the L^2-norm $\|\cdot\|_2$ are equivalent on B_ω:

$$\|f\|_2^2 = \int_a^b |f(t)|^2 \, dt \leqq \|f\|_\infty^2 \cdot |b - a|,$$

$$\begin{aligned} \|f\|_\infty^2 &= |f(t_0)|^2 = |f(t_0)^2| = \left| \int_a^{t_0} 2f(s) \cdot f'(s)\, ds \right| \\ &\leqq 2 \int_a^b |f(s)| \cdot |f'(s)|\, ds \leqq 2\|f\|_2 \cdot \|f'\|_2 \leqq 2\omega \|f\|_2 . \end{aligned}$$

Example 10. Let $E = L^1[a, b]$ and $B_{(\alpha)} = \{f \in E : |\hat{f}(n)| \leqq \alpha_n\}$, where $\sum_{n=-\infty}^{\infty} \alpha_n < \infty$. Then $B_{(\alpha)} \subset C[a, b]$ is a compact set even in $C[a, b]$: let $f_k \in B_{(\alpha)}$. We can assume than $\lim_{k \to \infty} \hat{f}_k(n) = \gamma_n$, where $|\gamma_n| \leqq \alpha_n$. Put

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \gamma_n e^{int}.$$

We have

$$|f_k(t) - f(t)| \leqq \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} |\hat{f}_k(n) - \gamma_n| \leqq \frac{1}{2\pi} \sum_{n=-N}^{N} |\hat{f}_k(n) - \gamma_n| + \frac{1}{\pi} \sum_{|n|>N} \alpha_n < \varepsilon$$

if k is large enough.

The Hilbert space property of F was only used in Theorem 3 which is valid[1] even in rotund (i.e., strictly convex) and reflexive Banach spaces. If F is not rotund—for example, L^1 or L^∞— then the closest point, if it exists, is not unique. However if M is boundedly compact, the existence of a nearest point in M is easily proved, and if x is near M, all points which are nearest points to x are close to each other. Thus if $M = KB$ with $K^{-1}|_{KB}$ (uniformly) continuous, then $\hat{K}$ is perhaps not single-valued on the complement of KB but "(uniformly) continuous" on KB, which is enough to solve the integral equation problem.

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[1] The necessity part is only proved under additional assumptions such as "M is weakly closed", "M is locally compact", or "the nearest point projection is both continuous and weakly continuous".

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PROPERTIES OF LIRON'S POLYNOMIALS*

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Abstract. For real nonzero k, put

$$S_m(k) = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_n^{-2m-2},$$

where α_n runs through the nonzero roots of $\tan \alpha = k\alpha$. Liron [5] found a generating function for $S_m(k)$ and showed that $S_m(k) = (k - 1)^{-m-1}P_{m+1}(k)$, where $P_{m+1}(k)$ is a polynomial in k of degree $m + 1$. Carlitz [1] showed that $P_{m+1}(k)$ has coefficients that are closely related to tangent coefficients of higher order. In the present paper additional properties of $P_{m+1}(k)$ are derived. In particular it is shown that $P_{m+1}(k)$ can be expressed simply in terms of $P_m(k)$, $P'_m(k)$ and $P'_{m+1}(k)$, and this result leads to a simple recurrence formula for the coefficients. Also formulas for $P'_{m+1}(1)$ and $P''_{m+1}(1)$ are found, and Carlitz's result that $(3m + 3)!P_{m+1}(k)/(m + 1)!$ has integral coefficients is sharpened.

1. Introduction. For real nonzero k, put

$$S_m(k) = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_n^{-2m-2}, \tag{1.1}$$

where α_n runs through the nonzero roots of $\tan \alpha = k\alpha$. Liron [5] showed that

$$\sum_{m=0}^{\infty} S_m(1)t^{2m} = \frac{3}{2t^2} + \frac{\sin t}{2(t \cos t - \sin t)} \tag{1.2}$$

and pointed out that

$$S_m(1) = \sigma_{2m+2}(3/2),$$

the Rayleigh function of order 3/2. The early history of the Rayleigh function can be found in [7, p. 502]. In more recent years it has been studied by Kishore [4] and Lorch [6], among others. It should perhaps be mentioned that for $m > 0$,

$$S_m(1) = \frac{(-1)^m 3 \cdot 2^{2m+1}}{(2m + 2)!} V_{2m+2}, \tag{1.3}$$

where $V_0, V_1, V_2, \cdots$ are the van der Pol numbers discussed by the present writer in [2].

For $k \neq 1$, Liron showed that

$$\sum_{m=0}^{\infty} S_m(k)t^{2m} = \frac{1}{2kt^2} + \frac{1}{2}\left(1 - \frac{k - 1}{k^2t^2}\right)\frac{k \sin t}{kt \cos t - \sin t}. \tag{1.4}$$

He also showed that

$$S_m(k) = (k - 1)^{-m-1}P_{m+1}(k), \tag{1.5}$$

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where $P_{m+1}(k)$ is a polynomial of degree $m + 1$ in k with rational coefficients, and

$$P_{m+1}(1) = 3^{-m-1}. \tag{1.6}$$

Carlitz [1] then proved that

$$S_m(k) = \frac{U^{(1)}_{2m+1}}{2(2m+1)!} + (m+1)\sum_{r=1}^{m+1}(k-1)^{-r}\frac{U^{(r)}_{2m+r+2}}{r(2m+r+2)!} \tag{1.7}$$

for $k \neq 1$, $m \geqq 1$, where $U_n^{(r)}$ is defined by

$$(\tan t - t)^r = \sum_{n=3r}^{\infty} U_n^{(r)}\frac{t^n}{n!}. \tag{1.8}$$

Carlitz also proved that

$$\frac{(3m+3)!}{(m+1)!}P_{m+1}(k) \tag{1.9}$$

has integral coefficients.

In the present paper additional properties of $P_{m+1}(k)$ are found. In particular we prove that for $m > 0$,

$$3P_{m+1}(k) = k^2P_m(k) + \frac{k-1}{m+1}P'_{m+1}(k) - \frac{k^2(k-1)}{m}P'_m(k)$$

and this result leads to a simple recurrence formula for the coefficients of $P_{m+1}(k)$. In fact the coefficients of k^n, $0 \leqq n \leqq 4$, are computed in this paper and expressed in terms of Bernoulli numbers. We also prove that if $q_1, \cdots, q_s$ are all the odd primes less than $2m + 4$ and if

$$e_i = \left[\frac{2m+2}{q_i - 1}\right], \qquad i = 1, \cdots, s,$$

then

$$2\left(\prod_{i=1}^{s} q_i^{e_i}\right)P_{m+1}(k)$$

has integral coefficients. Finally, we derive the following formulas:

$$P'_{m+1}(1) = \tfrac{2}{5}(\tfrac{1}{3})^m(1+m), \qquad m > 0,$$

$$P''_{m+1}(1) = \tfrac{2}{525}(\tfrac{1}{3})^{m-1}(42m^2 + 43m + 1), \qquad m > 1.$$

2. A sequence of rational numbers related to $S_m(k)$. We begin by defining rational numbers $H_n(k)$ which generalize the van der Pol numbers V_n defined by

$$\frac{x^3/6}{x(e^x+1) - 2(e^x-1)} = \sum_{n=0}^{\infty}\frac{V_n}{n!}x^n.$$

Equations (2.9) and (2.14) in this section will show that the numbers $H_n(k)$ are closely related to $S_m(k)$ and $S'_m(k)$. They will be used to prove some of the results in §§3 and 4.

For real k, $k \neq 0$ or 1, define $H_n(k, a)$ by means of

$$\frac{(2k-2)xe^{ax}}{kx(e^x+1)-2(e^x-1)} = \sum_{n=0}^{\infty} H_n(k, a)\frac{x^n}{n!}, \tag{2.1}$$

and let $H_n(k, 0) = H_n(k)$. Thus

$$\frac{(2k-2)x}{kx(e^x+1)-2(e^x-1)} = \sum_{n=0}^{\infty} H_n(k)\frac{x^n}{n!}, \tag{2.2}$$

and

$$H_n(k, a) = \sum_{r=0}^{n}\binom{n}{r}H_r(k)a^{n-r}. \tag{2.3}$$

It follows from (2.1) that

$$H_n(k, 1-a) = (-1)^n H_n(k, a)$$

and so

$$H_n(k, 1) = (-1)^n H_n(k). \tag{2.4}$$

If we multiply the left side of (2.1) by $kx(e^x+1) - 2(e^x-1)$ we see that

$$\begin{aligned}(n+1)kH_n(k, a+1) + (n+1)kH_n(k, a)&\\ -2H_{n+1}(k, a+1) + 2H_{n+1}(k, a) &= (2k-2)(n+1)a^n,\end{aligned} \tag{2.5}$$

and letting $a = 0$ and replacing n by $2m$ we have

$$H_{2m}(k) = \frac{-2}{(2m+1)k}H_{2m+1}(k), \qquad m > 0. \tag{2.6}$$

Now by (2.1) and (2.4) we have

$$\frac{(2k-2)x(e^x-1)}{kx(e^x+1)-2(e^x-1)} = -2\sum_{m=0}^{\infty} H_{2m+1}(k)\frac{x^{2m+1}}{(2m+1)!}, \tag{2.7}$$

and thus,

$$\frac{\sin t}{kt\cos t - \sin t} = \sum_{m=0}^{\infty}\frac{(-1)^{m-1}2^{2m+1}}{(k-1)(2m+1)!}H_{2m+1}(k)t^{2m}. \tag{2.8}$$

Comparing (2.6) and (2.8) with (1.4), we see that for $m > 0$,

$$S_m(k) = \frac{(-1)^m 2^{2m-1}}{k-1}\left(\frac{k^2 H_{2m}(k)}{(2m)!} + \frac{4(k-1)H_{2m+2}(k)}{(2m+2)!}\right). \tag{2.9}$$

In fact it follows from Carlitz's examination of (1.4) that for $m > 0$,

$$H_{2m}(k) = (-1)^m 2^{-2m}(2m)!\sum_{r=1}^{m}\frac{U_{2m+r}^{(r)}}{(2m+r)!}(k-1)^{-r}, \tag{2.10}$$

where $U_n^{(r)}$ is defined by (1.8). It follows that

$$H_n(k) = (k-1)^{-[n/2]}Q_n(k), \qquad n > 0, \tag{2.11}$$

where $Q_n(k)$ is a polynomial in k of degree $[(n-1)/2]$. It is clear from (2.6) that $Q_n(0) = 0$ if n is odd, $n > 1$.

If we multiply both sides of (2.2) by $kx(e^x + 1) - 2(e^x - 1)$ and compare coefficients of x, we derive the recurrence formula

$$H_0(k) = 1,$$

$$(n+1)H_n(k) = -\sum_{r=0}^{n-1} \binom{n+1}{r} \frac{(n+1-r)k - 2}{2(k-1)} H_r(k). \tag{2.12}$$

If we replace n by $2m$ in (2.5) and consider the two cases $a = 0$, $a = -1$, we derive, for $m > 0$,

$$H_{2m}(k) = \frac{1}{2} - \frac{1}{2(k-1)(2m+1)} \sum_{r=0}^{m-1} \binom{2m+1}{2r} [(2m+1-2r)k - 2] H_{2r}(k). \tag{2.13}$$

Using either (2.12) or (2.13) we have

$$H_1(k) = -\frac{1}{2}, \qquad H_2(k) = -\frac{1}{6(k-1)},$$

$$H_3(k) = \frac{k}{4(k-1)}, \qquad H_4(k) = \frac{6k-1}{30(k-1)^2}.$$

Though the numbers $H_n(k)$ may be of some interest in their own right, we shall mainly be concerned with their relationship to $S_m(k)$ and $P_{m+1}(k)$. Of particular interest are (2.9) and, for $m > 0$,

$$\frac{k-1}{m+1} S'_m(k) = \frac{(-1)^m 2^{2m+2}}{(2m+2)!} H_{2m+2}(k), \tag{2.14}$$

which we now prove. By (2.1) and (2.4) we have

$$\frac{t \cos t}{kt \cos t - \sin t} = \sum_{m=0}^{\infty} \frac{(-1)^m 2^{2m}}{(k-1)(2m)!} H_{2m}(k) t^{2m} \tag{2.15}$$

and Carlitz has proved

$$\frac{t \cos t - \sin t}{kt \cos t - \sin t} = (k-1) \sum_{m=0}^{\infty} S'_m(k) \frac{t^{2m+2}}{m+1}. \tag{2.16}$$

Comparing (2.8), (2.15) and (2.16), and using (2.6), we have (2.14). This can also be proved easily from (1.7) and (2.10).

3. The coefficients of $P_{m+1}(k)$. Differentiating (1.5) with respect to k we have

$$S'_m(k) = -(m+1)(k-1)^{-m-2}P_{m+1}(k) + (k-1)^{-m-1}P'_{m+1}(k), \tag{3.1}$$

and by (2.9), (2.14) and (3.1) we have for $m > 0$,

$$3P_{m+1}(k) = k^2P_m(k) - \frac{k^2(k-1)}{m}P'_m(k) + \frac{k-1}{m+1}P'_{m+1}(k). \tag{3.2}$$

It follows that

$$P_{m+1}(1) = 3^{-1}P_m(1) = 3^{-2}P_{m-1}(1) = \cdots = 3^{-m-1},$$

a result also proved by both Liron and Carlitz.

We can derive from (3.2) a recurrence formula for the coefficients of $P_{m+1}(k)$. Let

$$P_{m+1}(k) = A_{m+1,0} + A_{m+1,1}k + \cdots + A_{m+1,m+1}k^{m+1}. \tag{3.3}$$

It follows from (3.2) that, for $r > 0$,

$$\begin{aligned} mrA_{m+1,r} = {} & -m(3m+4-r)A_{m+1,r-1} + (m+1)(r-2)A_{m,r-2} \\ & + (m+1)(m-r+3)A_{m,r-3}. \end{aligned} \tag{3.4}$$

Here it is understood that $A_{m+1,r} = 0$ if $r < 0$.

It is possible, by means of (3.4) and a value for $A_{m+1,0}$, to express $A_{m+1,r}$ in terms of Bernoulli numbers. We recall that the Bernoulli numbers $B_0, B_1, B_2, \cdots$ can be defined by means of

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

or by means of the formula

$$B_0 = 1,$$

$$\sum_{r=0}^{n} \binom{n}{r} B_r = B_n, \qquad n > 1.$$

These numbers are well known and have been extensively studied. The first 60 are listed in [3, p. 234].

To find a value for $A_{m+1,0}$ we use (2.11) and (2.12). If we define $Q_0(k) = H_0(k) = 1$, we have

$$\sum_{r=0}^{n} \binom{n+1}{r} (-1)^{[r/2]} Q_r(0) = 0$$

and so

$$Q_{2n}(0) = (-1)^n B_{2n}.$$

It follows from (2.9) that

$$A_{m+1,0} = \frac{-2^{2m+1}}{(2m+2)!} B_{2m+2}, \tag{3.5}$$

as was proved by Liron in another way.

Thus by (3.4) and (3.5) we have

$$A_{m+1,1} = \frac{3 \cdot 2^{2m}}{(2m+1)!} B_{2m+2},$$

$$A_{m+1,2} = \frac{-3 \cdot 2^{2m-1}(3m+2)}{(2m+1)!} B_{2m+2},$$

$$A_{m+1,3} = \frac{(3m+1)(3m+2)2^{2m-1}}{(2m+1)!} B_{2m+2} + \frac{2^{2m}(m+1)}{3 \cdot (2m)!} B_{2m},$$

$$A_{m+1,4} = \frac{-3m(3m+1)(3m+2)2^{2m-3}}{(2m+1)!} B_{2m+2} - \frac{2^{2m-1}(m+1)}{(2m-1)!} B_{2m}.$$

It follows that

$$m!r!A_{m+1,r} = \sum_{n=1}^{[(r+1)/2]} \frac{E_n B_{2m+4-2n}}{(2m+4-2n)!},$$

where $E_1, E_2, \cdots$ are integers.

Formula (3.4) can also be obtained in the following way. From (2.14) and (3.1) we have

$$\frac{(-1)^m 2^{2m+1}}{(2m+1)!} Q_{2m+2}(k) + (m+1)P_{m+1}(k) = (k-1)P'_{m+1}(k). \tag{3.6}$$

We differentiate (3.6) $r-1$ times to obtain

$$(k-1)P^{(r)}_{m+1}(k) = (m-r+2)P^{(r-1)}_{m+1}(k) + \frac{(-1)^m 2^{2m+1}}{(2m+1)!} Q^{(r-1)}_{2m+2}(k). \tag{3.7}$$

If we multiply (2.9) by $(k-1)^{m+1}$, differentiate $r-1$ times, and let $k=0$ we have

$$\frac{(-1)^m 2^{2m+1}}{(2m+2)!} Q^{(r-1)}_{2m+2}(0) = P^{(r-1)}_{m+1}(0) + \frac{(-1)^{m-1} 2^{2m}}{(2m)!} \binom{r-1}{2} Q^{(r-3)}_{2m}(0). \tag{3.8}$$

Substituting (3.8) into (3.7), with $k=0$, gives us

$$P^{(r)}_{m+1}(0) = -(3m+4-r)P^{(r-1)}_{m+1}(0) - (2m+2) \sum_{s=1}^{[(r-1)/2]} \frac{(r-1)!}{(r-2s-1)!} P^{(r-2s-1)}_{m-s+1}(0), \tag{3.9}$$

and so we have

$$rA_{m+1,r} = -(3m+4-r)A_{m+1,r-1} - (2m+2) \sum_{s=1}^{[(r-1)/2]} A_{m-s+1,r-2s-1}. \tag{3.10}$$

If we subtract $((r-2)/m)A_{m,r-2}$ from $(r/(m+1))A_{m+1,r}$, using (3.10), we obtain (3.4).

We note that (2.10) gives us the formula

$$\begin{aligned} A_{m+1,r} &= \binom{m+1}{r} \frac{(-1)^{m+1-r}}{2(2m+1)!} U^{(1)}_{2m+1} \\ &\quad + (m+1) \sum_{s=1}^{m+1-r} \binom{m+1-s}{r} \frac{(-1)^{m+1-s-r}}{s(2m+2+s)!} U^{(s)}_{2m+2+s}, \end{aligned} \tag{3.11}$$

where $U_n^{(s)}$ is defined by (1.8). Thus from the series

$$\tan t = \sum_{n=1}^{\infty} T_{2n-1} \frac{t^{2n-1}}{(2n-1)!},$$

where

$$T_{2n-1} = \frac{(-1)^{n+1}2^{2n}(2^{2n}-1)}{2n} B_{2n},$$

we have

$$A_{m+1,m+1} = \frac{T_{2m+1}}{2(2m+1)!},$$

$$A_{m+1,m} = \frac{-(m+1)T_{2m+1}}{2(2m+1)!} + \frac{(m+1)T_{2m+3}}{(2m+3)!},$$

and by using the formula

$$\frac{rU_n^{(r+1)}}{n!} = \frac{U_{n+1}^{(r)}}{n!} - \frac{2rU_{n-1}^{(r)}}{(n-1)!} - \frac{rU_{n-2}^{(r-1)}}{(n-2)!} \tag{3.12}$$

which was proved by Carlitz, it is possible to compute the coefficients $A_{m+1,r}$ in terms of Bernoulli numbers.

4. Prime divisors of the coefficients. In this section we sharpen the result that (1.9) has integral coefficients. We shall first prove theorems for the polynomials $Q_{2m}(k)$ and then, using (2.9), look at the corresponding theorems for $P_{m+1}(k)$. We begin by stating as a lemma a well-known result concerning prime divisors of factorials.

LEMMA 4.1. *Let q be a prime number and let $v_q(n)$ be the exponent of the highest power of q dividing $n!$ If*

$$n = a_0 + a_1 q + \cdots + a_r q^r, \qquad 0 \leqq a_i < q,$$

$$S = a_0 + a_1 + \cdots + a_r,$$

then

$$v_q(n) = \frac{n-S}{q-1}.$$

THEOREM 4.1. *Let $q_1, \cdots, q_s$ be all the odd primes less than $2m+2$ and let $c_i = [2m/(q_i - 1)]$, $i = 1, \cdots, s$. Then*

$$\frac{2^{2m} q_1^{c_1} \cdots q_s^{c_s}}{(2m)!} Q_{2m}(k)$$

has integral coefficients.

Proof. The proof is by induction on m. We first verify the theorem for $m = 0, 1, 2$ using the values of $H_{2m}(k)$ given in §2. Assuming the theorem is true for all values of Q_{2r}, $2r < 2m$, we multiply (2.13) by

$$\frac{2^{2m} q_1^{c_1} \cdots q_s^{c_s}}{(2m)!} (k-1)^m.$$

By Lemma 4.1 we see that

$$\left[\frac{2m}{q-1}\right] - \left[\frac{2r}{q-1}\right] \geq v_q(2m+1-2r),$$
$$2m-2r-1 \geq v_2(2m+1-2r),$$

and Theorem 4.1 follows.

COROLLARY. *Let q be an odd prime. If $2m = hq + t$, $0 \leqq h \leqq q-2$, $0 \leqq t < q-1-h$, then $Q_{2m}(k)$ has coefficients that are integral* (mod q).

We shall use the following definition. If $f(x)$ and $g(x)$ are polynomials with rational coefficients and q is a prime, we define $f(x) \equiv g(x) \pmod q$ if the coefficients of $f(x)$ and $g(x)$ are integral (mod q) and if the corresponding coefficients are congruent (mod q).

THEOREM 4.2. *If q is an odd prime and $1 \leqq m < q+1$, then*

$$\frac{q^m}{[m(p-1)]!} Q_{m(q-1)}(k) \equiv (-1)^m(k-1)^{m(q-3)/2} \pmod q.$$

Proof. The proof is by induction on m. For $m = 1$ we have, by (2.13) and Theorem 4.1,

$$qQ_{q-1}(k) \equiv (k-1)^{(q-3)/2}H_0(k) \equiv (k-1)^{(q-3)/2} \pmod q.$$

Assume the theorem holds for all $Q_{n(q-1)}(k)$, $1 \leq n < m$. It is easily seen by Lemma 4.1 that for $m < q+1$ and $r < m(q-1)$, $m - [r/(q-1)] = v_q(m(q-1) - r + 1)$ if and only if $r = (m-1)(q-1)$. Thus by (2.13) we have

$$\frac{q^m}{[m(q-1)]!} Q_{m(q-1)}(k) \equiv -\frac{(k-1)^{(q-3)/2}q^{m-1}}{[(m-1)(q-1)]!} Q_{(m-1)(q-1)}$$
$$\equiv (-1)^m(k-1)^{m(q-3)/2} \pmod q.$$

Using (2.9) we can now write down the corresponding theorems for $P_{m+1}(k)$.

THEOREM 4.3. *If $q_1, \cdots, q_s$ are all the odd primes less than $2m+4$ and if $e_i = [(2m+2)/(q_i-1)]$, $i = 1, \cdots, s$, then*

$$2q_1^{e_1} \cdots q_s^{e_s} P_{m+1}(k)$$

has integral coefficients.

COROLLARY. *Let q be an odd prime. If $2m+2 = hq+t$, $0 \leqq h \leqq q-3$, $0 \leq t < q-1-h$, then $(2m+2)!P_{m+1}(k)$ has coefficients that are integral* (mod q).

THEOREM 4.4. *If q is an odd prime and $1 \leqq m < q+1$, then*

$$2 \cdot q^m P_{m(q-1)/2}(k) \equiv (-1)^{(mq+m-2)/2}(k-1)^{m(q-3)/2} \pmod q.$$

Theorems 4.3 and 4.4 tell us that if $1 \leqq m < q+1$, then $m(q-1)/2$ is the smallest value of n such that $P_n(k)$ has at least one coefficient with denominator divisible by exactly q^m. Furthermore, the coefficients of $P_{m(q-1)/2}(k)$ having this divisibility property are the coefficients of k^r for all r such that $0 \leqq r \leqq m(q-3)/2$ and

$$\binom{m(q-3)/2}{r}$$

is not divisible by q.

At times Theorem 4.3 is an improvement on Carlitz's result that (1.9) has integral coefficients. For example (1.9) says

$$2^8 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 P_5(k)$$

has integral coefficients while Theorem 4.3 says

$$2 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 P_5(k)$$

has integral coefficients. We can combine the two results in the following way. Let q be an odd prime and let

$$\begin{aligned} 3m + 3 &= a_0 + a_1 q + \cdots + a_n q^n, && 0 \leqq a_i < q, \\ m + 1 &= b_0 + b_1 q + \cdots + b_n q^n, && 0 \leqq b_i < q, \\ S_1 &= a_0 + a_1 + \cdots + a_n, \\ S_2 &= b_0 + b_1 + \cdots + b_n. \end{aligned}$$

Define

$$f(m+1) = \begin{cases} \left[\dfrac{2m+2}{q-1}\right] & \text{if } S_2 \geqq S_1, \\ \dfrac{2m+2-S_1+S_2}{q-1} & \text{if } S_2 < S_1. \end{cases}$$

Then $q^{f(m+1)} P_{m+1}(k)$ has coefficients that are integral (mod q).

5. Formulas for $P'_{m+1}(1)$ and $P''_{m+1}(1)$. It follows from (1.7) that

$$P_{m+1}^{(m+1-r)}(1) = \frac{(m+1)\,(m+1-r)!}{r(2m+r+2)!} U_{2m+r+2}^{(r)},$$

where $P_{m+1}^{(m+1-r)}(1)$ is the $m+1-r$ derivative of $P_{m+1}(k)$ evaluated at $k = 1$. Unfortunately it seems to be a very tedious procedure to find explicit formulas for $U_{2m+r+2}^{(r)}$ from (3.12). In this section we use another method to find $P'_{m+1}(1)$ and $P''_{m+1}(1)$. The following recurrence formula, which is due to Liron, will be useful:

$$(5.1) \quad \sum_{m=0}^{n} \frac{(-1)^m[k(2n-2m+1)-1]}{(2n-2m+1)!} S_m(k) = \frac{n+1}{(2n+3)!}[k(2n+3)-1].$$

Let $m > 1$. By (1.5) and (5.1),

$$(5.2) \qquad P_{m+1}(k+1) = \frac{3k+2}{3!} P_m(k+1) - \frac{(5k+4)k}{5!} P_{m-1}(k+1) + k^2 g(k),$$

where $g(k)$ is a polynomial in k with rational coefficients. We rewrite (5.2) in the form

$$(5.3) \quad P_{m+1}(k+1) \equiv \frac{3k+2}{3!} P_m(k+1) - \frac{(5k+4)k}{5!} P_{m-1}(k+1) \pmod{k^2}.$$

Using (1.6) we then have

$$\begin{aligned} P_{m+1}(k+1) &\equiv \tfrac{1}{3}P_m(k+1) + \tfrac{2}{5}(\tfrac{1}{3})^m k \\ &\equiv (\tfrac{1}{3})^{m-1}P_2(k+1) + \tfrac{2}{5}(\tfrac{1}{3})^m(m-1)k \\ &\equiv (\tfrac{1}{3})^{m+1} + \tfrac{2}{5}(\tfrac{1}{3})^m(m+1)k \pmod{k^2}. \end{aligned}$$

This implies that for $m > 0$

$$P'_{m+1}(1) = \tfrac{2}{5}(\tfrac{1}{3})^m (m+1). \tag{5.4}$$

Similarly for $m > 2$,

$$\begin{aligned} P_{m+1}(k+1) &\equiv \frac{3k+2}{3!} P_m(k+1) - \frac{(5k+4)k}{5!} P_{m-1}(k+1) \\ &\quad + \frac{(7k+6)k^2}{7!} P_{m-2}(k+1) \\ &\equiv \left(\frac{1}{3}\right)^{m+1} + \frac{2}{5}\left(\frac{1}{3}\right)^m (m+1)k \\ &\quad + \left(\frac{1}{3}\right)^{m-1} \frac{k^2}{525}(42m^2+43m+1) \pmod{k^3}. \end{aligned}$$

This implies that for $m > 1$,

$$P''_{m+1}(1) = (\tfrac{1}{3})^{m-1} \tfrac{2}{525}(42m^2+43m+1). \tag{5.5}$$

6. $P_{m+1}(k)$ for $0 \leqq m \leqq 4$. In this section we list the first five polynomials $P_{m+1}(k)$. We note that $P_1(k)$, $P_2(k)$ and $P_3(k)$ were computed by Liron.

$$\begin{aligned} P_1(k) &= \tfrac{1}{2}k - \tfrac{1}{6}, \\ P_2(k) &= \tfrac{1}{6}k^2 - \tfrac{1}{15}k + \tfrac{1}{90}, \\ P_3(k) &= \tfrac{1}{15}k^3 - \tfrac{4}{105}k^2 + \tfrac{1}{105}k - \tfrac{1}{945}, \\ P_4(k) &= \tfrac{17}{630}k^4 - \tfrac{58}{2835}k^3 + \tfrac{11}{1575}k^2 - \tfrac{2}{1575}k + \tfrac{1}{9450}, \\ P_5(k) &= \tfrac{31}{2835}k^5 - \tfrac{323}{31,185}k^4 + \tfrac{422}{93,555}k^3 - \tfrac{1}{891}k^2 + \tfrac{1}{6237}k - \tfrac{1}{93,555}. \end{aligned}$$

We conjecture that the signs of the coefficients alternate; i.e.,

$$(-1)^{m+1+r}A_{m+1,r} > 0$$

for $r = 0, \cdots, m+1$. We can verify this by our computations in §3 for $r = 0, \cdots, 4$ and $r = m$, $r = m+1$ by using the following properties of the Bernoulli numbers [3, pp. 240–246]:

$$(-1)^{n+1}B_{2n} > 0, \qquad n > 0,$$

$$\frac{24(2n+1)(2n+2)}{(2\pi)^4} < \left|\frac{B_{2n+2}}{B_{2n}}\right| < \frac{(2n+1)(2n+2)}{(2\pi)^2}.$$

7. An analogous problem and concluding remarks. We note that the methods of this paper could be used on the polynomial $T_m(k)$ defined by

$$T_m(k) = \frac{1}{2} \sum_{n=0}^{\infty} \beta_n^{-2m-2},$$

where β_n runs through the roots of $k \cot \beta + \beta = 0$. Both Liron and Carlitz discuss this polynomial. To use the method of this paper, define $G_n(k)$ by

$$\frac{4k}{2k(e^x + 1) - x(e^x - 1)} = \sum_{n=0}^{\infty} G_n(k) \frac{x^n}{n!}.$$

It can be shown that formulas analogous to (2.9) and (2.14) hold. It does not appear that any new information is obtained by using this method, however. A recurrence formula for the coefficients can be derived, but it is equivalent to formula (2.5) in [1].

In conclusion we remark that many of the properties of the numbers $H_n(k)$ discussed in this paper are not really necessary to prove the new results concerning $P_{m+1}(k)$. For example, if we define $H_{2m}(k)$ by means of (2.10), then (2.14) follows from (1.7), and (2.9) follows from Carlitz's work. This is all that is necessary to prove (3.2) and (3.4). Also formula (5.1) can be used to prove the theorems in §4, with the possible exception of the fact that $2P_{m+1}(k)$ has coefficients that are integral (mod 2). As indicated by the results in §2, however, the numbers $H_n(k)$ are closely related to the Bernoulli and van der Pol numbers, and therefore may be of some interest in their own right. The writer hopes they are an interesting alternate approach to the study of $P_{m+1}(k)$.

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GENERALIZED INVERSES IN REPRODUCING KERNEL SPACES: AN APPROACH TO REGULARIZATION OF LINEAR OPERATOR EQUATIONS*

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Abstract. In this paper a study of generalized inverses of linear operators in reproducing kernel Hilbert spaces (RKHS) is initiated. Explicit expressions for generalized inverses and minimal-norm solutions of linear operator equations in RKHS are obtained in several forms. The relation between the regularization operator of the equation $Af = g$ and the generalized inverse of the operator A in RKHS is demonstrated. In particular, it is shown that they are the same if the range of the operator is closed in an appropriate RKHS. Finally, properties of the regularized pseudosolutions in this setting are studied.

It is shown that this approach provides a natural and effective setting for regularization problems when the operator maps one RKHS into another.

1. Introduction. Let X and Y be Hilbert spaces and let A be a linear operator on a domain $\mathscr{D}(A) \subset X$ into Y. The operator A is said to have a generalized inverse $A^\dagger$ on a domain $\mathscr{D}(A^\dagger) \subset Y$ if for each $y \in \mathscr{D}(A^\dagger)$, $\inf\{\|Ax - y\| : x \in X\} = \|AA^\dagger y - y\|$ and $\|A^\dagger y\|$ is smaller than the norm of any other element $u \in X$ at which the preceding infimum is attained. It is well known and can be easily shown that if A is a bounded operator, or if A is a densely defined closed operator, then $A^\dagger$ exists on $\mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, where $\mathscr{R}(A)$ is the range of A. The domain $\mathscr{D}(A^\dagger)$ in this case is a dense subset of Y and $A^\dagger$ is unbounded unless $\mathscr{R}(A)$ is closed in Y. A compact operator with infinite-dimensional range is a prototype of an operator for which $\mathscr{R}(A)$ is not closed.

To impart continuity to $A^\dagger$ when $\mathscr{R}(A)$ is not closed in Y, one might consider subsets X', Y' of X, Y, respectively, equipped with topologies which are not equivalent to those of X and Y, and such that the generalized inverse of A, when viewed as an operator from Y' to X', exists and is bounded. The topologies of X' and Y' are required to be induced by inner products, and must be amenable to the original setting of the operator equation $Ax = y$, so that questions of least squares solvability and related approximation schemes are still meaningful in a wide context.

One objective of this paper is to show, when X and Y are $\mathscr{L}_2$-spaces of square-integrable real-valued functions, that the topology of reproducing kernel spaces is an appropriate topology for the goal stated above, and thereby to initiate a systematic study of generalized inverses of linear operators acting between two reproducing kernel Hilbert spaces. This study has strong interface with the problem of regularization of (ill-posed or poorly-conditioned) linear operator equations. This brings us to another objective of this paper, which is to provide a new approach to regularization in the context of RKHS.

At present there are several approaches to the investigation and regularization of ill-posed problems. These are discussed briefly in our report [10], which forms an earlier draft of this paper and contains an extensive bibliography on these

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approaches. In this paper we present another approach to regularization based on the notion of least squares solution of minimal norm and on regularization operators in RKHS. Our approach coincides in philosophy with some of the known approaches cited in [5], [16], [10] (in the sense that we change the notion of the solution and consider the problem in new spaces), even though it differs sharply in technical details. We exploit (in an optimal way) the geometry of RKHS and obtain results which are the best possible in this context. The basic results of this paper are stated in Theorems 3.1, 4.1, 4.2, 5.1, 5.2 and 6.1. Applications of this approach to rates of convergence of approximate solutions will appear elsewhere [11], [12].

To our knowledge this is the first time that generalized inverses of linear operators and reproducing kernels are used *simultaneously* in the same context. It is befitting to mention here that the concepts of a generalized inverse (of a matrix) and RKHS both go back to the work of E. H. Moore [7].

2. Generalized inverses, reproducing kernel spaces, and pseudosolutions of linear operator equations. Let X and Y be two Hilbert spaces over the real scalars and let A be a linear operator on $\mathscr{D}(A) \subset X$ into Y. Let $\mathscr{R}(A)$, $\mathscr{N}(A)$ and A^* denote, respectively, the range, nullspace and adjoint of A. The orthogonal compliment of a subspace S is denoted by $S^\perp$; the closure of S is denoted by $\bar{S}$ and the orthogonal projector on a closed subspace $\mathscr{M}$ is denoted by $P_{\mathscr{M}}$.

We consider the linear operator equation

$$Ax = y. \tag{2.1}$$

DEFINITION 2.1. An element $u \in X$ is said to be a *least squares solution* of (2.1) if $\inf\{\|Ax - y\| : x \in X\} = \|Au - y\|$. If the set S_y of all least squares solutions of (2.1) for a given $y \in Y$ has an element v of minimal norm, then v is called a *pseudosolution* of (2.1).

DEFINITION 2.2. The operator equation (2.1) is said to be *well-posed* (relative to the spaces X and Y) if for each $y \in Y$, (2.1) has a unique pseudosolution which depends continuously on y; otherwise the equation is said to be *ill-posed*.

Obviously (2.1) has a least squares solution for a given $y \in Y$ if and only if there exists an element $w \in \mathscr{R}(A)$ which is closest to y. From this it follows immediately that (2.1) has a least squares solution if and only if $P_{\overline{\mathscr{R}(A)}}y \in \mathscr{R}(A)$, or equivalently $y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$. For such y, it is easy to see that the set S_y has a unique element of minimal norm if and only if $P_{\overline{\mathscr{N}(A)}}u \in \mathscr{N}(A)$ for some $u \in S_y$ (in which case this is also true for each $x \in S_y$). Thus a pseudosolution of (2.1) exists if and only if

$$y \in A(\mathscr{D}(A) \cap \mathscr{N}(A)^\perp) \oplus \mathscr{R}(A)^\perp. \tag{2.2}$$

In what follows we shall primarily be interested in the cases when A is a *closed* linear operator on a dense domain $\mathscr{D}(A) \subset X$, *or* when A is a *bounded* linear operator on X. In either of these cases, since $\mathscr{N}(A)$ is closed, condition (2.2) reduces to the condition

$$y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp. \tag{2.3}$$

The (linear) map which associated with each y satisfying (2.3) a unique pseudosolution defines the *generalized inverse* of A, which is denoted by $A^\dagger$.

For each $y \in \mathscr{D}(A^\dagger)$, we thus have $S_y = A^\dagger y \oplus \mathscr{N}(A)$. Note that in our setting, $A^\dagger$ is a densely defined operator.

We summarize in the following proposition equivalent properties of the generalized inverse (see Nashed [8]).

PROPOSITION 2.1. *Each of the following sets of conditions characterizes the generalized inverse $A^\dagger$ of a bounded or a densely defined closed operator*:

(a) $AA^\dagger A = A$ *on* $\mathscr{D}(A)$, $A^\dagger A A^\dagger = A^\dagger$ *on* $\mathscr{D}(A^\dagger)$, $AA^\dagger = P_{\overline{\mathscr{R}(A)}}|\mathscr{D}(A^\dagger)$ *and* $A^\dagger A = P_{\mathscr{N}(A)^\perp}|\mathscr{D}(A)$, *where the vertical bar denotes the restriction of the projector to the indicated domain.*

(b) $A^\dagger$ *is the unique linear extension of* $\{A|\mathscr{N}(A)^\perp\}^{-1}$ *to* $\mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$ *so that* $\mathscr{N}(A^\dagger) = \mathscr{R}(A)^\perp$.

(c) *For* $y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, $A^\dagger y$ *is the unique solution of minimal norm of the "normal" equation* $A^*Ax = A^*y$, *provided* $\mathscr{R}(A) \subset \mathscr{D}(A^*)$.

(d) *For* $y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, $A^\dagger y$ *is the unique solution of minimal norm of the "projectional" equation* $Ax = P_{\overline{\mathscr{R}(A)}}y$.

PROPOSITION 2.2 *The following statements are equivalent for A as above*:

(a) *The operator equation* (2.1) *is well-posed in* (X, Y).

(b) *A has a closed range in Y.*

(c) $A^\dagger$ *is a bounded operator on Y into X.*

Proof. (a) implies that $\mathscr{D}(A^\dagger) = Y$ and thus from (2.3), $\mathscr{R}(A) = \overline{\mathscr{R}(A)}$. Statement (c) follows from (b) using Proposition 2.1(b) and the closed graph theorem. That (c) implies (a) is obvious.

Convention 2.1. In this paper we encounter on several occasions a composition of two operators, say A and B, where B is unbounded and densely defined but AB is bounded. In all such cases we shall assume that AB has already been *extended* as usual (i.e., by continuity) to the *closure* of the domain of B. An example is the composition $AA^\dagger$ when $\mathscr{R}(A)$ is a nonclosed subspace. Then $\mathscr{D}(A^\dagger)$ is dense, but $AA^\dagger$ is bounded and can be extended to $\overline{\mathscr{R}(A)} \oplus \mathscr{R}(A)^\perp$, even though $A^\dagger$ cannot (see also part (a) of Proposition 2.1).

When $\mathscr{R}(A)$ is not closed, the problem of finding least squares solutions of (2.1) is ill-posed relative to the spaces X, Y. An ill-posed problem relative to (X, Y) may be recast in some cases as a well-posed problem relative to new spaces $X' \subset X$ and $Y' \subset Y$, with topologies on X' and Y' which are different respectively from the topologies on X and Y. From the point of regularization, the topologies on X' and Y' should not be too restrictive and must lend themselves to requirements which are satisfied by a wide class of admissible solutions of pseudosolutions. This is precisely the point which we exploit in connection with the topologies on reproducing kernel Hilbert spaces.

A Hilbert space $\mathscr{H}$ of real-valued functions defined on a set S is said to be a *reproducing kernel Hilbert space* (RKHS) if all the evaluation functionals $f \to f(s)$ for $f \in \mathscr{H}$ and $s \in S$ are continuous. In this case there exists, by the Riesz representation theorem, a unique element in $\mathscr{H}$ (call it Q_s) such that

$$\langle f, Q_s \rangle = f(s), \qquad f \in \mathscr{H}. \tag{2.4}$$

The *reproducing kernel* (RK) is defined by

$$Q(s, s') := \langle Q_s, Q'_s \rangle, \qquad s, s' \in S. \tag{2.5}$$

Let $\mathscr{H}_Q$ denote the RKHS with reproducing kernel Q, and denote the inner product and norm in $\mathscr{H}_Q$ by $\langle \cdot, \cdot \rangle_Q$ and $\| \cdot \|_Q$, respectively. Note that $Q(s, s')$ $(\equiv Q_s(s'))$ is a nonnegative definite symmetric kernel on $S \times S$, and that $\{Q_s, s \in S\}$ spans $\mathscr{H}_Q$ since $\langle Q_s, f \rangle_Q = 0$, $s \in S$, implies $f(s) = 0$. For properties of reproducing kernel spaces, see Aronszajn [1], Shapiro [15, Chap. 6] and Parzen [13].

If S is a bounded interval (or if S is an unbounded interval but $\iint Q^2(s, s')\, ds\, ds' < \infty$), and $Q(s, s')$ is continuous on $S \times S$ (the only case we shall consider here), then it is easy to show that $\mathscr{H}_Q$ is a space of continuous functions. Note also that $\mathscr{L}_2[S]$ is *not* an RKHS since the evaluation functionals are not continuous.

An RKHS $\mathscr{H}_Q$ with RK Q determines a self-adjoint Hilbert–Schmidt operator (also denoted by Q) on $\mathscr{L}_2[S]$ to $\mathscr{L}_2[S]$ by

$$(Qf)(s) = \int_S Q(s, s') f(s')\, ds', \qquad f \in \mathscr{L}_2[S]. \tag{2.6}$$

Since $Q(s, s')$ is assumed to be continuous, then by the theorems of Mercer, Hilbert and Schmidt [14, pp. 242–246], the operator Q has an $\mathscr{L}_2[S]$-complete orthonormal system of eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ with $\lambda_i \geqq 0$ and $\sum_{i=1}^{\infty} \lambda_i < \infty$ (thus Q is a *trace-class* operator; see [2, Chap. XI.9] or [3, Chap. 2]); also $Q(s, s')$ has the uniformly convergent Fourier expansions

$$Q(s, s') = \sum_{i=1}^{\infty} \lambda_i \phi_i(s) \phi_i(s')$$

and

$$Qf = \sum_{i=1}^{\infty} \lambda_i (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i, \tag{2.7}$$

where $(\cdot, \cdot)_{\mathscr{L}_2[S]}$ is the inner product in $\mathscr{L}_2[S]$.

It is well known (see, for example, [17]) that

$$\mathscr{H}_Q = \{f : f \in \mathscr{L}_2[S], \sum_{i=1}^{\infty} \lambda_i^{-1} (f, \phi_i)^2_{\mathscr{L}_2[S]} < \infty\},$$

where the notational convention $0/0 = 0$ is being adopted, and

$$\langle f_1, f_2 \rangle_Q = \sum_{i=1}^{\infty} \lambda_i^{-1} (f_1, \phi_i)_{\mathscr{L}_2[S]} (f_2, \phi_i)_{\mathscr{L}_2[S]}.$$

The operator Q has a well-defined symmetric square root $Q^{1/2}$ which is a Hilbert–Schmidt operator ([14, pp. 242–246] or [3, Chap. 2]):

$$Q^{1/2} f = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i. \tag{2.8}$$

Thus, since $\mathscr{N}(Q) = \mathscr{N}(Q^{1/2})$,

$$\mathscr{H}_Q = \mathrm{Q}^{1/2}(\mathscr{L}_2[S]) = Q^{1/2}(\mathscr{L}_2[S] \ominus \mathscr{N}(Q)).$$

$(Q^{1/2})^\dagger$ has the representation

$$(Q^{1/2})^\dagger f = \sum_{i=1}^{\infty} (\sqrt{\lambda_i})^\dagger (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i \tag{2.9}$$

on $\mathscr{H}_Q \oplus \mathscr{H}_Q^\perp$ ($\perp$ in $\mathscr{L}_2[S]$), where, for θ a real number, $\theta^\dagger = \theta^{-1}$, $\theta \neq 0$; $\theta^\dagger = 0$, $\theta = 0$. Similarly $Q^\dagger$ has the representation

$$Q^\dagger f = \sum_{i=1}^{\infty} \lambda_i^\dagger (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i \tag{2.10}$$

on its domain.

For any operator Q on $\mathscr{L}_2[S]$ induced by an RK $Q(s, s')$, as in (2.6) we shall adopt the notational conventions

$$Q^{-1/2} := (Q^{1/2})^\dagger \quad \text{and} \quad Q^{-1} := Q^\dagger. \tag{2.11}$$

We have the relations

$$\|f\|_Q = \inf\{\|p\|_{\mathscr{L}_2[S]}, p \in \mathscr{L}_2[S], f = Q^{1/2}p\}, \qquad f \in \mathscr{H}_Q,$$

$$\langle f_1, f_2 \rangle_Q = (Q^{-1/2} f_1, Q^{-1/2} f_2)_{\mathscr{L}_2[S]}, \qquad f_1, f_2 \in \mathscr{H}_Q,$$

and, if $f_1 \in \mathscr{H}_Q$ and $f_2 \in \mathscr{H}_Q$ with $f_2 = Q\rho$ for some $\rho \in \mathscr{L}_2[S]$, then

$$\langle f_1, f_2 \rangle_Q = (f_1, \rho)_{\mathscr{L}_2[S]}. \tag{2.12}$$

3. Relationship between generalized inverses in RKHS and $\mathscr{L}_2$-spaces. We are now ready to explore properties of the generalized inverse of a linear operator between two RK spaces. In the remainder of this paper we let $X = \mathscr{L}_2[S]$ and $Y = \mathscr{L}_2[T]$ denote the Hilbert spaces of square-integrable real-valued functions on the closed, bounded intervals S and T, respectively. Let A be a linear operator from X into Y. Let $\subset$ denote point set inclusion only, and suppose that A has the following properties:

$$\mathscr{H}_Q \subset \mathscr{D}(A) \subset X, \tag{3.1}$$

where $\mathscr{H}_Q$ is an RKHS with continuous RK on $S \times S$;

$$A(\mathscr{H}_Q) = \mathscr{H}_{\tilde{R}} \subset \mathscr{H}_R \subset Y, \tag{3.2}$$

where $\mathscr{H}_{\tilde{R}}$ and $\mathscr{H}_R$ are RKHS with continuous RK's on $T \times T$; and

$$\mathscr{N}(A) \text{ in } \mathscr{H}_Q \text{ is closed in } \mathscr{H}_Q. \tag{3.3}$$

We emphasize in particular that the space $\mathscr{H}_{\tilde{R}}$ is not necessarily closed in the topology of $\mathscr{H}_R$.

Let $A^\dagger_{(X,Y)}$ denote the generalized inverse of A, when A is considered as a map from X into Y, and let $A^\dagger_{(Q,R)}$ denote the generalized inverse of A when A is considered as a map from $\mathscr{H}_Q$ into $\mathscr{H}_R$. Now the topologies in (X, Y) are not the same as the topologies in $(\mathscr{H}_Q, \mathscr{H}_R)$. Thus the generalized inverses $A^\dagger_{(X,Y)}$ and $A^\dagger_{(Q,R)}$ have distinct continuity properties in general. We shall now develop the relation between $A^\dagger_{(Q,R)}$ and certain (X, Y) and (Y, Y) generalized inverses. In the sequel, the operators $R: Y \to Y$ and $R^{1/2}: Y \to Y$ are defined from the RK of $\mathscr{H}_R$ analogous to Q and $Q^{1/2}$; see (2.7) and (2.8). We continue the notational convention of (2.11), that is, $R^{-1} = R^\dagger = R^\dagger_{(Y,Y)}$ and $R^{-1/2} = (R^{1/2})^\dagger_{(Y,Y)}$.

THEOREM 3.1. *Under assumptions* (3.1)–(3.3), *let* $y \in \mathscr{D}(A^\dagger_{(Q,R)})$, *i.e.,* $y \in \mathscr{H}_{\tilde R} \oplus \mathscr{H}_{\tilde R}^\perp$ ($\perp$ *in* $\mathscr{H}_R$). *Then*

$$y \in \mathscr{D}(Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}) \tag{3.4}$$

and

$$A^\dagger_{(Q,R)}y = Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}y. \tag{3.5}$$

Proof. The (maximal) domain of $A^\dagger_{(Q,R)}$ is $\mathscr{H}_{\tilde R} \oplus \mathscr{H}_{\tilde R}^\perp$ ($\perp$ in $\mathscr{H}_R$). Denote the operator $Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}$ by L. We first show that $\mathscr{D}(A^\dagger_{(Q,R)}) \subset \mathscr{D}(L)$. Let $\tilde A = R^{-1/2}AQ^{1/2}$. The operator $\tilde A$ is defined over all of X since $\mathscr{D}(Q^{1/2}) = X$, $Q^{1/2}(X) = \mathscr{H}_Q$, $\mathscr{D}(A) \supset \mathscr{H}_Q$, $A(\mathscr{H}_Q) = \mathscr{H}_{\tilde R} \subset \mathscr{H}_R$ and $\mathscr{H}_R \subset \mathscr{D}(R^{-1/2})$. Also $\mathscr{R}(\tilde A) = \tilde A(X) = R^{-1/2}(\mathscr{H}_{\tilde R}) \subset R^{-1/2}(\mathscr{H}_R) \subset Y$. Thus $\mathscr{D}(\tilde A^\dagger_{(X,Y)}) = \mathscr{R}(\tilde A) \oplus \mathscr{R}(\tilde A)^\perp$ ($\perp$ in Y), and

$$\mathscr{D}(\tilde A^\dagger_{(X,Y)}) = R^{-1/2}(\mathscr{H}_{\tilde R}) \oplus (R^{-1/2}(\mathscr{H}_{\tilde R}))^\perp \quad (\perp \text{ in } Y).$$

We now show

$$y \in \mathscr{H}_{\tilde R} \quad \text{implies} \quad y \in \mathscr{D}(L), \tag{3.6}$$

$$y \in \mathscr{H}_{\tilde R}^\perp \quad (\perp \text{ in } \mathscr{H}_R) \quad \text{implies} \quad y \in \mathscr{D}(L). \tag{3.7}$$

To prove (3.6), let $y \in \mathscr{H}_{\tilde R}$. Then $R^{-1/2}y \in \mathscr{D}(\tilde A^\dagger_{(X,Y)})$, so $y \in \mathscr{D}(\tilde A^\dagger_{(X,Y)}R^{-1/2})$, which implies $y \in \mathscr{D}(L)$ since $\mathscr{R}(\tilde A^\dagger_{(X,Y)})$ is contained in X, the domain of $Q^{1/2}$. To prove (3.7), let $y \in \mathscr{H}_{\tilde R}^\perp$ ($\perp$ in $\mathscr{H}_R$). This means that $y \in \mathscr{D}(R^{-1/2})$ and $\langle y, g\rangle_R = 0$ for all $g \in \mathscr{H}_{\tilde R}$. But for each $g \in \mathscr{H}_{\tilde R}$ there exists a unique $\psi \in Y \ominus \mathscr{N}(R^{1/2})$ such that $g = \tilde R^{1/2}\psi$. Thus $(R^{-1/2}y, R^{-1/2}\tilde R^{1/2}\psi)_Y = 0$ for all $\psi \in Y \ominus \mathscr{N}(\tilde R^{1/2})$. Thus $R^{-1/2}y$ is orthogonal to $R^{-1/2}(\mathscr{H}_{\tilde R})$ in Y, so that $R^{-1/2}y \in (R^{-1/2}(\mathscr{H}_{\tilde R}))^\perp$, $\perp$ in Y, $y \in \mathscr{D}(\tilde A^\dagger_{(X,Y)}R^{-1/2})$ and hence in $\mathscr{D}(Q^{1/2}\tilde A^\dagger_{(X,Y)}R^{-1/2})$.

Now we prove (3.5). For $y \in \mathscr{D}(A^\dagger_{(Q,R)})$, let $z = A^\dagger_{(Q,R)}y$. Then z is the unique element of minimal $\mathscr{H}_Q$-norm in the set

$$\mathscr{S} = \{u : \|Au - y\|_R = \inf_{x \in \mathscr{H}_R} \|Ax - y\|_R\}. \tag{3.8}$$

Let $x = Q^{1/2}p$ for $p \in X$, and let $\tilde y = R^{-1/2}y$. Let

$$W = \{w : \|\tilde A w - \tilde y\|_Y = \inf_{p \in X} \|\tilde A p - \tilde y\|_Y\}.$$

Then also

$$\begin{aligned} W &= \{w : \|R^{-1/2}AQ^{1/2}w - R^{-1/2}y\|_Y = \inf_{p\in X} \|R^{-1/2}AQ^{1/2}p - R^{-1/2}y\|_Y\} \\ &= \{w : \|AQ^{1/2}w - y\|_R = \inf_{p \in X} \|AQ^{1/2}p - y\|_R\}. \end{aligned} \tag{3.9}$$

Let v be the element of minimal X-norm in W. Then $v = \tilde A^\dagger_{(X,Y)}\tilde y = \tilde A^\dagger_{(X,Y)}R^{-1/2}y$. On the other hand, upon comparing (3.8) and (3.9) we have $z = Q^{1/2}v = Q^{1/2}\tilde A^\dagger_{(X,Y)}R^{-1/2}y$. Thus $z = Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}y$ and

$$A^\dagger_{(Q,R)}y = Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{X,Y}R^{-1/2}y,$$

which is the desired result.

COROLLARY 3.1. *If* $A(\mathscr{H}_Q) = \mathscr{H}_R$, *then* $A^{\dagger}_{(Q,R)}$ *is bounded.*

Proof. This follows from Proposition 2.2, or directly from (3.4)–(3.5).

It should be noted that an operator A may satisfy the assumption of Corollary 3.1 while failing to have a closed range in the space Y. This is, for example, the case if A is a Hilbert–Schmidt linear integral operator (with nondegenerate kernel) on X. It is this observation which makes RKHS useful in the context of regularization and approximation of ill-posed linear operator equations. An application of Theorem 3.1 is given in §5.

4. Explicit representations of minimal-norm solutions of linear operator equations in reproducing kernel spaces. We assume that $\mathscr{H}_Q$ is chosen so that

(4.1) the linear functionals $\{\mathscr{E}_t : t \in T\}$ defined by $\mathscr{E}_t f = (Af)(t)$ are continuous in $\mathscr{H}_Q$.

Then by the Riesz representation theorem, there exists $\{\eta_t,\ t \in T\} \in \mathscr{H}_Q$ such that

$$(Af)(t) = \langle \eta_t, f\rangle_Q, \qquad t \in T, \quad f \in \mathscr{H}_Q. \tag{4.2}$$

By (2.4), η_t is explicitly given by

$$\eta_t(s) = \langle \eta_t, Q_s\rangle = (AQ_s)(t). \tag{4.3}$$

($\eta_t(s)$ is readily obtained in a more explicit form from (4.3) if A is a differential or integral operator.)

Let $R(t, t')$ be the nonnegative definite kernel on $T \times T$ given by

$$R(t, t') = \langle \eta_t, \eta_{t'}\rangle_Q, \qquad t, t' \in T. \tag{4.4}$$

Let $\mathscr{H}_R$ be the RKHS with RK R given by (4.4). Let R_t be the element of $\mathscr{H}_R$ defined by $R_t(t') = R(t, t')$, and let $\langle \cdot, \cdot \rangle_R$ be the inner product in $\mathscr{H}_R$. Let V be the closure of the span of $\{\eta, t \in T\}$ in $\mathscr{H}_Q$. Now $\{R_t, t \in T\}$ spans $\mathscr{H}_R$, and by the properties of RKHS, we have

$$\langle \eta_t, \eta_{t'}\rangle_Q = R(t, t') = \langle R_t, R_{t'}\rangle_R. \tag{4.5}$$

Thus there is an isometric isomorphism between the subspace V and $\mathscr{H}_R$, generated by the correspondence

$$\eta_t \in V \sim R_t \in \mathscr{H}_R. \tag{4.6}$$

Then $f \in V \sim g \in \mathscr{H}_R$ if and only if $\langle \eta_t, f\rangle_Q = g(t) = \langle R_t, g\rangle_R$, $t \in T$, i.e., if and only if $g(t) = (Af)(t)$, $t \in T$. Thus $A(\mathscr{H}_Q) = A(V) = \mathscr{H}_R$. The nullspace of A in $\mathscr{H}_Q$ is $\{f : f \in \mathscr{H}_Q, \|Af\|_R = 0\}$. Since

$$\langle \eta_t, f\rangle_Q = 0, \quad t \in T \qquad \text{and} \qquad f \in \mathscr{H}_Q \Rightarrow f \in V^{\perp},$$

and $f \in V$ implies $\|f\|_Q = \|Af\|_V$, it follows that the nullspace of A in $\mathscr{H}_Q$ is $V^{\perp}$ ($\perp$ in $\mathscr{H}_Q$). Hence (4.1) entails that the nullspace of $A : \mathscr{H}_Q \to \mathscr{H}_R$ in $\mathscr{H}_Q$ is always closed, irrespective of the topological properties of $A : X \to Y$.

We list the following table of corresponding sets and elements, under the correspondence $\sim$ of (4.6), where the entries on the left are in $\mathscr{H}_Q$:

$$\begin{aligned} V &\sim \mathscr{H}_R, \\ f &\sim g, \\ \eta_t &\sim R_t, \\ P_V Q_s &\sim \eta_s^*. \end{aligned} \tag{4.7}$$

Here P_V is the projector from $\mathscr{H}_Q$ onto the (closed) subspace V, $g(t) = \langle \eta_t, f \rangle_Q$, $t \in T$, and $\eta_s^* = AQ_s = A(P_V Q_s)$, i.e.,

$$\eta_s^*(t) = \langle \eta_t, P_V Q_s \rangle_Q = \eta_t(s). \tag{4.8}$$

We have the following theorem.

THEOREM 4.1. *Let A and $\mathscr{H}_Q$ satisfy* (4.1), *and let R be given by* (4.5), *where η_t is defined by* (4.2). *Let $\eta_s^* = AQ_s$. Then, for $g \in \mathscr{H}_R$,*

$$(A^\dagger_{(Q,R)} g)(s) = \langle \eta_S^*, g \rangle_R, \qquad s \in S.$$

Proof. Let $\hat{f}$ be the element in $\mathscr{H}_Q$ of minimal $\mathscr{H}_Q$-norm which satisfies $Af = g$, that is, $\hat{f} = A^\dagger_{(Q,R)} g$. Then $\hat{f} \in V$ and $g \sim \hat{f}$. Also $\eta_s^* \sim P_V Q_s$. Thus

$$\hat{f}(s) = \langle Q_s, \hat{f} \rangle_Q = \langle P_V Q_s, \hat{f} \rangle_Q = \langle \eta_s^*, g \rangle_R.$$

We next obtain another operator representation of $A^\dagger_{(Q,R)}$.

THEOREM 4.2. *Suppose*

(i) *$\mathscr{D}(A^*)$ is dense in Y, where A^* is the adjoint of A considered as an operator from X to Y;*

(ii) *A and $\mathscr{H}_Q$ satisfy* (4.1);

(iii) *$\mathscr{H}_Q$ and $\mathscr{H}_R = A(\mathscr{H}_Q)$ possess continuous RK's.*

Then, for $g \in \mathscr{H}_R$,

$$(A^\dagger_{(Q,R)} g)(s) = (QA^*(AQA^*)^\dagger_{(Y,Y)} g)(s), \qquad s \in S.$$

Proof. First we show that $R = AQA^*$. This follows by observing that, for $g \in \mathscr{D}(A^*)$, (4.2), (2.12), (4.7) and the isomorphism between V and $\mathscr{H}_R$ give

$$\begin{aligned} (AQA^*g)(t) = \langle \eta_t, QA^*g \rangle_Q &= (\eta_t, A^*g)_X \\ &= (A\eta_t, g)_Y = (R_t, g)_Y \\ &= \int_T R(t, t') g(t')\, dt', \qquad t \in T. \end{aligned}$$

Thus, AQA^* coincides with the bounded operator R on $\mathscr{D}(A^*)$ and hence by extension on Y. We write $(AQA^*)^\dagger_{(Y,Y)} = R^{-1}$. Next, suppose $g \in R(\mathscr{D}(A^*))$, and let $\rho = R^{-1} g$. Then, since $g = R\rho$, Theorem 4.1 and (2.12) give

$$\begin{aligned} (A^\dagger_{(Q,R)} g)(s) &= \langle \eta_s^*, g \rangle_R = (\eta_s^*, \rho)_Y = (AQ_s, \rho)_Y \\ &= (Q_s, A^*\rho)_X = (QA^*\rho)(s) = (QA^*(AQA^*)^\dagger_{(Y,Y)} g)(s), \qquad s \in S. \end{aligned}$$

It can be shown easily that if $\mathscr{D}(A^*)$ is dense in Y, then $R(\mathscr{D}(A^*))$ is dense in $\mathscr{H}_R$. Thus (4.9) extends to all $g \in \mathscr{H}_R$.

DEFINITION 4.1. Let $A:X \to Y$. The *pseudocondition number* of A (relative to the norms of X and Y) is

$$\gamma(A;X,Y):=\sup_{\substack{x\neq 0\\ x\in\mathscr{D}(A)}}\frac{\|Ax\|_Y}{\|x\|_X}\cdot\sup_{\substack{y\neq 0\\ y\in\mathscr{D}(A^\dagger)}}\frac{\|A^\dagger y\|_X}{\|y\|_Y}.$$

The equation $Af = g$ is said to be *poorly conditioned* in the spaces X, Y if the number $\gamma(A;X,Y)$ is much greater than one. Note that $1 \leqq \gamma(A;X,Y)$; for ill-posed problems, γ is not finite.

Suppose $\mathscr{H}_Q$ is an RKHS with $\mathscr{H}_Q \subset \mathscr{D}(A)$, and A and $\mathscr{H}_Q$ satisfy (4.1) with $A(\mathscr{H}_Q) = \mathscr{H}_R$, R given by (4.4). Then $\gamma(A;\mathscr{H}_Q,\mathscr{H}_R) = 1$. To see this, write $x \in \mathscr{H}_Q$ in the form $x = x_1 + x_2$, where $x_2 \in V^\perp$. Then $Ax = Ax_1 = y_1$ and $\|y_1\|_R = \|x_1\|_Q$. Thus

$$\gamma(A;\mathscr{H}_Q,\mathscr{H}_R)=\sup_{x\neq 0}\frac{\|y_1\|_R}{\|x\|_Q}\cdot\sup_{y_1\neq 0}\frac{\|x_1\|_Q}{\|y_1\|_R}=1.$$

On the other hand, the number $\gamma(A;X,Y)$ may be large. Thus the casting of the operator equation $Af = g$ in the reproducing kernel spaces $\mathscr{H}_Q$, $\mathscr{H}_R$ always leads to a well-conditioned (indeed, optimally-conditioned) problem.

5. Regularization of pseudosolutions in reproducing kernel spaces. In this section we study properties of *regularized pseudosolutions* (in RKHS) f_λ of the operator equation $Af = g$, where g is not necessarily in the range of the operator A. By a regularized pseudosolution we mean a solution to the variational problem: Find f_λ in $\mathscr{H}_Q$ to minimize

$$\phi_g(f) = \|g - Af\|_P^2 + \lambda\|f\|_Q^2, \tag{5.1}$$

where $\mathscr{H}_Q$ is an RKHS in the domain of A, $\|\cdot\|_P$ denotes the norm in an RKHS $\mathscr{H}_P$ with RK P, $\mathscr{H}_P \subset Y$, $\phi_g(f)$ is assigned the value $+\infty$ if $g - Af \notin \mathscr{H}_P$, and $\lambda > 0$. We suppose A and $\mathscr{H}_Q$ satisfy (4.1), hence $A(\mathscr{H}_Q) = \mathscr{H}_R$, where $\mathscr{H}_R$ possesses an RK. As before, A may be unbounded, invertible, or compact considered as an operator from $X(=\mathscr{L}_2[S])$ to $Y(=\mathscr{L}_2[T])$. It is assumed that g possesses a (not necessarily unique) representation $g = g_0 + \xi$, for some $g_0 \in A(\mathscr{H}_Q)$ and $\xi \in \mathscr{H}_P$. ξ may be thought of as a "disturbance."

For $\lambda > 0$, let $\mathscr{H}_{\lambda P}$ be the RKHS with RK $\lambda P(t,t')$, where $P(t,t')$ is the RK on $T \times T$ associated with $\mathscr{H}_p$. We have $\mathscr{H}_P = \mathscr{H}_{\lambda P}$ and

$$\|\cdot\|_P^2 = \lambda\|\cdot\|_{\lambda P}^2. \tag{5.2}$$

Let $R(\lambda) = R + \lambda P$, and let $\mathscr{H}_{R(\lambda)}$ be the RKHS with RK $R(\lambda) = R(\lambda;t,t')$. According to Aronszajn [1, p. 352], $\mathscr{H}_{R(\lambda)}$ is the Hilbert space of functions of the form

$$g = g_0 + \xi, \tag{5.3}$$

where $g_0 \in \mathscr{H}_R$ and $\xi \in \mathscr{H}_P$. Following Aronszajn [1], we note that this decomposition is *not* unique unless $\mathscr{H}_R$ and $\mathscr{H}_P$ have no element in common except the zero element. The norm in $\mathscr{H}_{R(\lambda)}$ is given by

$$\|g\|_{R(\lambda)}^2 = \min\{\|g_0\|_R^2 + \|\xi\|_{\lambda P}^2 : g_0 \in \mathscr{H}_R, \xi \in \mathscr{H}_P, g_0 + \xi = g\}, \tag{5.4}$$

where, however, the g_0 and ξ attaining the minimum in (5.4) are easily shown to be unique by the strict convexity of the norm.

Consider now the problem of finding $f_\lambda \in \mathscr{H}_Q$ to minimize $\phi_g(f)$ in (5.1), for $g \in \mathscr{H}_{R(\lambda)}$. Then $g - Af_\lambda$ must be in $\mathscr{H}_P$ and it is obvious that $f_\lambda \in V$, the orthogonal complement of the nullspace of A in $\mathscr{H}_Q$. For any $f \in V$, $\|f\|_Q = \|Af\|_R$ by the isometric isomorphism between V and $\mathscr{H}_R$, and (5.1) may be written in the equivalent form: Find $f_\lambda \in V$ to minimize

$$\lambda\|Af\|_R^2 + \|g - Af\|_P^2. \tag{5.5}$$

Comparing (5.4) and (5.5) with the aid of (5.2), we see that g_0 and ξ attaining the minimum on the right-hand side of (5.4) are related to the solution f_λ, of the minimization problem (5.5), by

$$g_0 = Af_\lambda \quad \text{and} \quad \xi = g - Af_\lambda.$$

In the following theorem, we give a representation of the solution f_λ.

THEOREM 5.1. *Suppose $\mathscr{D}(A^*)$ is dense in Y, $\mathscr{H}_Q \subset \mathscr{D}(A)$ and A and $\mathscr{H}_Q$ satisfy* (4.1). *Suppose $\mathscr{H}_Q$, $\mathscr{H}_R(=A(\mathscr{H}_Q))$ and $\mathscr{H}_P \subset Y$ all have continuous RK's. Then, for $g \in \mathscr{H}_{R(\lambda)}$, the unique minimizing element $f_\lambda \in \mathscr{H}_Q$ of the functional $\phi_g(f)$ is given by*

$$\langle \eta_s^*, g\rangle_{R(\lambda)} = f_\lambda(s) = (QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g)(s), \qquad s \in S, \tag{5.6}$$

where $\eta_s^* = AQ_s$.

Proof. First, our assumptions give that $AQA^* + \lambda P(=R + \lambda P)$ is a well-defined positive definite operator on Y. We demonstrate, for

$$g \in (AQA^* + \lambda P)(\mathscr{D}(A^*)),$$

that

$$f_\lambda = QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g. \tag{5.7}$$

Now, $g - Af_\lambda = \lambda P(AQA^* + \lambda P)^\dagger_{(Y,Y)}g \in \mathscr{H}_P$, so that this demonstration will be effected if we show that

$$\phi_g(f_\lambda) < \phi_g(f_\lambda + \delta)$$

for any $\delta \in \mathscr{H}_Q$, with $\|\delta\|_Q \neq 0$.

But

$$\begin{aligned}\phi_g(f_\lambda + \delta) &= \|\lambda P(AQA^* + \lambda P)^\dagger_{(Y,Y)}g\|_P^2 - 2\lambda((AQA^* + \lambda P)^\dagger_{(Y,Y)}g, A\delta)_Y + \|A\delta\|_P^2 \\ &\quad + \lambda\|QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g\|_Q^2 + 2\lambda(A^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g, \delta)_X \\ &\quad + \|\delta\|_Q^2 \\ &= \phi_g(f_\lambda) + \|A\delta\|_P^2 + \|\delta\|_Q^2 > \phi_g(f_\lambda), \qquad \delta \neq 0.\end{aligned}$$

We next show that, for $g \in (AQA^* + \lambda P)(\mathscr{D}(A^*))$, that

$$\langle \eta_s^*, g\rangle_{R(\lambda)} = (QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g)(s).$$

Let $(AQA^* + \lambda P)^\dagger g = \rho \in \mathscr{D}(A^*)$. Then using (2.12) with Q replaced by $R(\lambda)$ gives

$$\langle \eta_s^*, g\rangle_{R(\lambda)} = (\eta_s^*, \rho)_Y = (AQ_s, \rho)_Y = (Q_s, A^*\rho)_X$$
$$= (QA^*\rho)(s) = (QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)} g)(s).$$

Thus we have proved (5.6) for $g \in (AQA^* + \lambda P)(\mathscr{D}(A^*))$.

We next show that $QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)} \equiv QA^*(R + \lambda P)^{-1}$ defines a bounded linear operator from $\mathscr{H}_{R(\lambda)}$ to $\mathscr{H}_Q$. If $g \in \mathscr{H}_{R(\lambda)}$, then $(R + \lambda P)^{-1/2} g \Rightarrow p \in Y \ominus \mathscr{N}(R + \lambda P)$ and

$$Q^{1/2}A^*(R + \lambda P)^{-1} g = Q^{1/2}A^*(R + \lambda P)^{-1/2} p \in Y,$$

since

$$\|Q^{1/2}A^*(R + \lambda P)^{-1/2} p\|_Y = \|R^{1/2}(R + \lambda P)^{-1/2} p\|_Y \leqq \|p\|_Y.$$

Therefore

$$QA^*(R + \lambda P)^{-1} g \in \mathscr{H}_Q, \qquad g \in \mathscr{H}_{R(\lambda)}.$$

But

$$\|QA^*(R + \lambda P)^{-1} g\|_Q = Q^{1/2}A^*(R + \lambda P)^{-1/2} p\|_Y \leqq \|p\|_Y = \|g\|_{R(\lambda)}.$$

It can be shown that $(R + \lambda P)(\mathscr{D}(A^*))$ is dense in $\mathscr{H}_{R(\lambda)}$, so that the right-hand equality in (5.6) extends to all $g \in \mathscr{H}_{R(\lambda)}$, and the left-hand equality obviously extends by the continuity of the inner product.

We call the (linear) mapping which assigns (by Theorem 5.1) to each $g \in \mathscr{H}_{R(\lambda)}$ the unique minimizing element f_λ the *regularization operator* of the equation $Af = g$.

The most useful situations occur, of course, when $\mathscr{H}_R$ is strictly contained in $\mathscr{H}_{R(\lambda)}$. For example, $\mathscr{H}_R$ may be a dense subset of Y in the Y-topology and $\mathscr{H}_{R(\lambda)}$ a bigger dense subset. We discuss this case further in §6. On the other hand, if $\mathscr{H}_R^\perp$ (in Y) is not empty, then P may be chosen so that the closure of $\mathscr{H}_P$ in the Y-topology equals $\mathscr{H}_R^\perp$ in Y. Then $\mathscr{H}_P \cap \mathscr{H}_R = \{0\}$, $\mathscr{H}_{\lambda P}$ and $\mathscr{H}_R$ are orthogonal subspaces of $\mathscr{H}_{R(\lambda)}$ (see [1]), and the decomposition (5.3) is unique. In this case we have the following theorem which shows that the regularization operator is indeed a generalized inverse in an appropriate RKHS.

THEOREM 5.2. *If $\mathscr{H}_P \cap \mathscr{H}_R = \{0\}$, then the minimizing element f_λ of* (5.1) *is the solution to the problem: Find $f \in \mathscr{S}$ to minimize*

$$\|f\|_Q, \tag{5.8}$$

where

$$\mathscr{S} = \{f : f \in \mathscr{H}_Q, \|g - Af\|_{R(\lambda)} = \inf_{h \in \mathscr{H}_Q} \|g - Ah\|_{R(\lambda)}\}. \tag{5.9}$$

Proof. We first note that if $\mathscr{H}_P \cap \mathscr{H}_R = \{0\}$, then also $\mathscr{H}_{\lambda P} \cap \mathscr{H}_R = \{0\}$ and the decomposition $g = g_0 + \xi$ with $g_0 \in \mathscr{H}_R$ and $\xi \in \mathscr{H}_{\lambda P}$ is unique, with

$$g_0 = R(R + \lambda P)^{-1} g \quad \text{and} \quad \xi = \lambda P(R + \lambda P)^{-1} g.$$

This decomposition is also independent of λ in this case, $PR = RP = 0$, and $R(R + \lambda P)^{-1}$ is the restriction of the projection onto $\mathscr{N}(R)^\perp$ in Y to the domain

$R^{1/2}Y \oplus P^{1/2}Y$. We have

$$\|g - Af\|^2_{R(\lambda)} = \|g_0 + \xi - Af\|^2_{R(\lambda)} = \|g_0 - Af\|^2_R + \|\xi\|^2_{\lambda R}.$$

Thus since $g_0 \in A(\mathscr{H}_Q)$, $\inf\{\|g - Af\|_{R(\lambda)}: f \in \mathscr{H}_Q\} = \|\xi\|_{\lambda P}$ and $\mathscr{S} = \{f : f \in \mathscr{H}_Q, Af = g_0\}$. Hence $f_\lambda = A^\dagger_{(Q,R)}g_0 = QA^*R^{-1}g_0 = QA^*(R + \lambda P)^{-1}g$.

Remark 5.1. In our setting we have

$$A(\mathscr{H}_Q) = \mathscr{H}_R \subset \mathscr{H}_{R(\lambda)} \subset Y.$$

Replacing $\mathscr{H}_{\tilde{R}}$ and $\mathscr{H}_R$ in (3.2) by $\mathscr{H}_R$ and $\mathscr{H}_{R(\lambda)}$, respectively, we get from (3.5):

$$A^\dagger_{(Q,R(\lambda))}y = Q^{1/2}[(R + \lambda P)^{-1/2}AQ^{1/2}]^\dagger_{(X,Y)}(R + \lambda P)^{-1/2}y \tag{5.10}$$

for $y \in \mathscr{D}(A^\dagger_{(Q,R(\lambda))})$; see (3.4.).

It is helpful to remember that the topology on $\mathscr{H}_R$ is not, in general, the restriction of the topology of $\mathscr{H}_{R(\lambda)}$, with the notable exception of the case $\mathscr{H}_R \cap \mathscr{H}_P = \{0\}$. In [11] the authors provide a concrete example arising in the approximate solution of boundary value problems where $\mathscr{H}_R$ is not a closed subspace of $\mathscr{H}_R$.

If $\mathscr{H}_R \cap \mathscr{H}_P = \{0\}$, then $\mathscr{H}_R$ is a closed subspace of $\mathscr{H}_{R(\lambda)}$ and (by Theorem 5.2)

$$A^\dagger_{(Q,R(\lambda))} = QA^*(R + \lambda P)^{-1}. \tag{5.11}$$

Note that in this case, the generalized inverse and the regularization operator coincide.

If $\mathscr{H}_R = A(\mathscr{H}_Q)$ is not closed in $\mathscr{H}_{R(\lambda)}$, then the regularization operator and the generalized inverse are different. Also, the right-hand sides of (5.10) and (5.11) are not the same: (5.11) has maximal domain $\mathscr{H}_{R(\lambda)}$, while (5.10) has maximal domain $\mathscr{H}_R \oplus \mathscr{H}_R^\perp$ ($\perp$ in $\mathscr{H}_{R(\lambda)}$).

6. Properties of f_λ when $\mathscr{H}_R \subset \mathscr{H}_P$. Rates of convergence of f_λ to the generalized inverse. In this section we note some properties of f_λ as $\lambda \to 0$ when $\mathscr{H}_R \subset \mathscr{H}_P$. If $g \in \mathscr{H}_R = A(\mathscr{H}_Q)$, then we have $f_\lambda \to A^\dagger_{(Q,R)}g$ as $\lambda \to 0$; here we may say something about the rate of convergence if certain additional conditions are satisfied (compare also with Ivanov and Kudrinskii [4]). However, g may not be in the domain of $A^\dagger_{(Q,R)}$. This situation can occur if, for example, $\mathscr{H}_R$ is dense in $\mathscr{H}_{R(1)}$. In this case, $\lim_{\lambda\to 0} \|f_\lambda\|_Q = \infty$.

THEOREM 6.1. *Let $g = Af_0 + \xi_0$, where $f_0 \in V$, $\xi_0 \in \mathscr{H}_P$, and suppose that $\mathscr{H}_R \subset \mathscr{H}_P$. Then*

(i) *$B = P^{-1/2}R^{1/2}$ is a bounded operator on $Y = \mathscr{L}_2[T]$;*

(ii) *if $\xi_0 = 0$ and $\|(B^*B)^{-1}R^{-1/2}(Af_0)\|_{\mathscr{L}_2[T]} < \infty$, then*

$$\|A^\dagger_{(Q,R)}g - f_\lambda\|^2_Q = O(\lambda^2);$$

(iii) *if $\xi_0 = 0$ and $\|(B^*B)^{-1/2}R^{-1/2}(Af_0)\|_{\mathscr{L}_2[T]} < \infty$, then*

$$\|A^\dagger_{(Q,R)}g - f_\lambda\|^2_Q = O(\lambda);$$

(iv) *if $\xi_0 \notin \mathscr{H}_R$, then $\lim_{\lambda\to 0} \|f_\lambda\|_Q = \infty$.*

Here inverses indicated by $^-$ are the generalized inverses in the geometry of $\mathscr{L}_2$-spaces.

Proof. Assertion (i) follows from the fact that $\mathscr{H}_R = R^{1/2}(\mathscr{L}_2[T])$ and $\mathscr{H}_P = P^{1/2}(\mathscr{L}_2[T])$. If $\mathscr{H}_R \subset \mathscr{H}_P$, then $R^{1/2}(\mathscr{L}_2[T]) \subset P^{1/2}(\mathscr{L}_2[T])$, so that $P^{-1/2}R^{1/2}$ is bounded. To prove assertions (ii) and (iii), we note that since $A(\mathscr{H}_Q) = \mathscr{H}_R$, $R^{-1/2}(Af_0)$ is a well-defined element ϕ of $\mathscr{L}_2[T]$, and after some computation, we obtain that if $\xi_0 = 0$, then

$$\begin{aligned}\|A^{\dagger}_{(Q,R)}g - f_\lambda\|_R &= \|(I - R^{1/2}(R + \lambda P)^{-1}R^{1/2})\phi\|_{\mathscr{L}_2[T]}\\ &= \|\lambda(B^*B + \lambda I)^{-1}\phi\|_{\mathscr{L}_2[T]}\\ &= \lambda\|(B^*B + \lambda I)^{-1}\phi\|_{\mathscr{L}_2[T]}\\ &\leqq \lambda\|(B^*B)^{-1}\phi\|_{\mathscr{L}_2[T]}.\end{aligned}$$

If $\|(B^*B)^{-1}R^{-1/2}Af_0)\|_{\mathscr{L}_2[T]} = m < \infty$, then

$$\|A^{\dagger}_{(Q,R)}g - f_\lambda\|_R \leq \lambda m,$$

thus proving assertion (ii).

Assertion (iii) follows by noting that $\lambda(B^*B + \lambda I)^{-1} \leqq I$ in the sense of positive definiteness; thus $\lambda(B^*B + \lambda I)^{-1} \leqq \lambda^{1/2}(B^*B + \lambda I)^{-1/2}$. Hence,

$$\begin{aligned}\lambda^2\|B^*B + \lambda I)^{-1}\phi\|^2_{\mathscr{L}_2[T]} &\leqq \lambda\|(B^*B + \lambda I)^{-1/2}\phi\|^2_{\mathscr{L}_2[T]}\\ &\leqq \lambda\|(B^*B)^{-1/2}\phi\|^2_{\mathscr{L}_2[T]},\end{aligned}$$

giving assertion (iii).

To see (iv), we observe that

$$\|QA^*(AQA^* + \lambda P)^{-1}\xi\|_Q = \|Q^{1/2}A^*(AQA^* + \lambda P)^{-1}\xi\|_{\mathscr{L}_2[S]}.$$

Since $\xi \in \mathscr{H}_P$, we have $\xi = P^{1/2}\theta$ for some $\theta \in \eta(P)^\perp$ ($\perp$ in $\mathscr{L}_2[T]$). Then

$$\|Q^{1/2}A^*(AQA^* + \lambda P)^{-1}P^{1/2}\theta\|_{\mathscr{L}_2[S]} = \|(BB^*)^{1/2}(BB^* + \lambda I)^{-1}\theta\|_{\mathscr{L}_2[S]}.$$

If $\{\lambda_\nu, \phi_\nu\}_{\nu=1}^{\infty}$ are the eigenvalues and eigenfunctions of the bounded positive operator BB^*, then

$$\|(BB^*)^{1/2}(BB^* + \lambda I)^{-1}\theta\|^2_{\mathscr{L}_2[S]} = \sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{(\lambda_\nu + \lambda)^2}(\phi_\nu, \theta)^2_{\mathscr{L}_2[S]}.$$

Since $\xi \notin \mathscr{H}_R$, $P^{1/2}\theta$ is not in the domain of $R^{-1/2}$ and θ is not in the domain of B^{-1}. Thus

$$\|(BB^*)^{-1/2}\theta\|^2_{\mathscr{L}_2[S]} = \sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu}(\phi_\nu, \theta)^2_{\mathscr{L}_2[S]} = \infty$$

and

$$\lim_{\lambda\to 0}\|(BB^*)(BB^* + \lambda I)^{-1}\theta\|_{\mathscr{L}_2[S]} = \infty.$$

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